Complete Research Reports

Convergence theorems of hybrid methods and its numerical experiment for equilibrium problem for two semigroups operators in Hilbert space

By

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2561

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Abstract

The main objective of this work is to modified two hybrid projection algorithm. First, we prove the srtongly convergence to common fixed points of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive mappings, second, we prove strongly convergence of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive semigroups. Our main results extended and improved the results of I. Inchan and S. Plubtieng [4] and Q. L. Dong, S. He and Y. J. Cho [1].

List of Contenets

CHAPTER I
INTRODUCTION

Let H be a real Hilbert Space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Recall also that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [13]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [3, 7, 9, 10, 11]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [5, 11].

Very recently, Takahashi, Takeuchi and Kubota [12] prove the following strong convergence theorems by hybrid method foe nonexpansive mappings and nonexpansive semigroup in Hilbert space.

Theorem 1.1.1. *[12] Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.1.2. *[12] Let H be a Hilbert space and C be a nonempty closed* *convex subset of H. Let $\mathcal{T} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive mapping semigroup on C such that $F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $0 \leq \alpha_n \leq a < 1$, $0 < \lambda_n < \infty$ for all $n \in \mathbb{N}$ and $\lambda_n \to \infty$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

In 2008, Inchan and Plubtieng [4] modified Ishikawa iteration process for two asymptotically nonexpansive mappings, for C is a nonempty closed convex subset of a Hilbert space H, let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$.

The second modificstion Ishikawa iteration process for two asymptotically nonexpansive semigroups. for C is a nonempty closed convex subset of a Hilbert space H, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t) z_n dt \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} S(t) x_n dt \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}} x_0, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\widetilde{\theta}_n = (1-\alpha_n)[(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](diamC)^2 \to 0$(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt$ and $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$ and $\widetilde{t}_n \to \infty, \widetilde{s}_n \to \infty$.

Then, prove that both iteration converges strongly to common fixed points of two asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups, respectively.

In 2015, Dong, He and Cho [1], introduce a hybrid algorithm. Let T and S be two nonexpansive mappings into itselt such that $F(T) \cap F(S) \neq \emptyset$, the sequence generated as follows:

$$\begin{cases} x_0 \in C \\ y_n = \alpha_n x_n + (1-\alpha_n) T x_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n) x_n] + (1-\beta_n) S y_n, \\ C_n = \{z \in C : \sigma \|z_n - z\|^2 + (1-\sigma)\|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0, \end{cases} \tag{1.5}$$

for each $n \geq 0$, where $\alpha_n, \beta_n \in [0,1]$, $\delta \in [0,), \gamma_n \in [0,], \sigma \in (0,1)$. Then prove that $\{x_n\}$ converge in norm to $P_{F(T) \cap F(S)} x_0$.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.1.3. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x-y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.1.4. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x-y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ *we have* $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ *then* $\mathcal{T}$ *is asymptotically nonexpansive semigroup. If we let* $s = 1$ *we have* $\frac{2+2s}{1+2s} = \frac{4}{3} \nless 1$*, then* $\mathcal{T}$ *is not necessary nonexpansive semigroup.*

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

Inspired and motivate by above, the purpose of this paper to introduce two algorithms. The first of this work we extend the results of Dong, He and Cho [1] for S and T are two asymptotically nonexpansive mappings then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{1.6}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\theta_n = [(1-\beta_n)(s_n^2 - 1) + (1-\alpha_n)(t_n^2 - 1) + (1-\beta_n)s_n^2(1-\alpha_n)(t_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $t_n \to 1$ and $s_n \to 1$ as $n \to \infty$).

The second of this work we extend the results of Dong, He and Cho [1] for

$\mathcal{T}$ is an asymptotically nonexpansive semigroup then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0 \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{1.7}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\widetilde{\theta}_n = [(1-\beta_n)(\widetilde{s}_n^2 - 1) + (1-\alpha_n)(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{s}_n^2(1-\alpha_n)(\widetilde{t}_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt \to 1$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt \to 1$ as $n \to \infty$).

CHAPTER II
BASIC RESEARCH

2.1 Usefull Lemmas

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Definition 2.1.1. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Definition 2.1.2. Let H be a real Hilbert Space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self mapping T of C is a contraction if

$$\|T(x) - Tf(y)\| \leq \alpha\|x - y\|$$

for some $\alpha \in (0, 1)$ and T is a nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$, and T is asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that

$$\|T^n x - T^n y\| \leq k_n\|x - y\|$$

for all $n \geq 1$ and $x, y \in C$.

Definition 2.1.3. A one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [13]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s+t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t \|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Lemma 2.1.4. *Let H be a real Hilbert space, then the following hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle, \forall x, y \in H;$

(ii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, t \in [0, 1], \forall x, y \in H.$

Lemma 2.1.5. *[5] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:*

(i) $x_n \rightharpoonup z$*; and*

(ii) $\limsup_{t\to\infty}\limsup_{n\to\infty}\|T(t)x_n - x_n\| = 0$,

then $z \in Fix(\mathcal{T})$.

Lemma 2.1.6. *[5] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \ge 0$,*

$$\limsup_{u\to\infty}\limsup_{t\to\infty}\sup_{x\in C}\|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.1.7. *[6] Let T be an asymptotically nonexpansive mapping defined on a bounded convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x \in F(T)$.*

Lemma 2.1.8. *[8] Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \le \|u - q\|$$

for all $n \ge 1$, then $x_n \to q$.

Lemma 2.1.9. *[5] Let C be a nonempty bounded closed convex subset of a Hilbert space H and $\mathfrak{T} = \{T(t) : 0 \le t < \infty\}$ be an asymptotically nonexpansive semigroups on C. If $\{x_n\}$ is a sequence in C satisfying the properties*

1. $x_n \rightharpoonup z$,

2. $\limsup_{t\to\infty}\limsup_{n\to\infty}\|T(t)x_n - x_n\|$,

then $z \in F(\mathfrak{T})$.

CHAPTER III
MAIN RESULTS

In this section we introduce two theorems, first Theorem we prove the strong convergence theorem of the algorithm (3.8) into $P_{F(T)\cap F(S)}x_0$. The second Theorem we prove the strong convergence of modified hybrid iterative method (3.20) into $P_{\Im}x_0$.

3.1 Convergence theorem for asymptotically nonexpansive mappings

The first of this work we extend the results of Dong, He and Cho [1] for S and T are two asymptotically nonexpansive mappings then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0 \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n \\ z_{n+1} = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{3.8}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\theta_n = [(1-\beta_n)(s_n^2-1) + (1-\alpha_n)(t_n^2-1) + (1-\beta_n)s_n^2(1-\alpha_n)(t_n^2-1)](diamC)^2 \to 0$ as $n \to \infty$(here $t_n \to 1$ and $s_n \to 1$ as $n \to \infty$).

Theorem 3.1.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T, S : C \to C$ be two asymptotically nonexpansive mappings with the* *sequences* $\{t_n\}$ *and* $\{s_n\}$*, respectively, such that* $F(T) \cap F(S) \neq \emptyset$*. Assume that* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $[0, 1]$ *such that* $\alpha_n, \beta_n \leq 1 - \delta$ *for some* $\delta \in (0, 1]$*. Then the sequence* $\{x_n\}$ *generated by (3.8) converges in norm to* $P_{F(T)\cap F(S)} x_0$*.*

Proof. Putting $t_\infty \sup\{t_n : n \geq 1\} < \infty$ and $s_\infty \sup\{s_n : n \geq 1\} < \infty$. We first show by induction that $F(T) \cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. It obvious that $F(T) \cap F(S) \subseteq C_1$. Suppose that $F(T) \cap F(S) \subseteq C_k$ for each $k \in \mathbb{N}$. Let $u \in F(T) \cap F(S) \subseteq C_k$, then from Lemma 2.1.4, we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\
&= \|\alpha_k (x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\
&= \alpha_k \|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\
&\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\
&\leq \alpha_k \|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - \|x_k - u\|^2 + \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\alpha_k)\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2. \qquad (3.9)
\end{aligned}$$

Simolarly, we note that from Lemma 2.1.4 and (3.9), we have

$$
\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)S^k y_k - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(S^k y_k - u)\|^2 \\
&= \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leq \beta_k[\gamma_k\|y_k - u\|^2 + (1-\gamma_k)\|x_k - u\|^2] + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|y_k - u\|^2 \\
&\leq \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2] \\
&\leq \beta_k\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\beta_k)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2\|x_k - u\|^2 + (1-\beta_k)s_k^2(1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(s_k^2-1)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \qquad (3.10)
\end{aligned}
$$

From (3.9) and (3.10), we obtain that

$$
\begin{aligned}
\|z_k - u\|^2 + \|y_k - u\|^2 &\leq \|x_k - u\|^2 + (1-\beta_k)(s_k^2-1)\|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \\
&\quad +\|x_k - u\|^2 + (1-\beta_k)s_k^2(1-\alpha_k)(t_k^2-1)\|x_k - u\|^2 \\
&\leq 2\|x_k - u\|^2 + [(1-\beta_k)(s_k^2-1) + 2(1-\alpha_k)(t_k^2-1) \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)](diamC)^2 \\
&= 2\|x_k - u\|^2 + \theta_k \qquad (3.11)
\end{aligned}
$$

where $\theta_k = [(1-\beta_k)(s_k^2-1)+2(1-\alpha_k)(t_k^2-1)+(1-\beta_k)s_k^2(1-\alpha_k)](diamC)^2 \to 0$ as $n \to \infty$. It follows that $u \in C_{k+1}$ and then $F(T)\cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex foe all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for each $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subseteq C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z_m \in C_k$ and $\|z_k - z_m\|^2 + \|y_k - z_m\|^2 \leq 2\|z_m - x_k\|^2 + \theta_k$. From Lemma 2.1.4, we have

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - z_m + z_m - z\|^2 + \|y_k - z_m + z_m - z\|^2 \\
&\leq [\|z_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle z_k - z_m, z_m - z\rangle] \\
&\quad +[\|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle] \\
&= \|z_k - z_m\|^2 + \|y_k - z_m\|^2 + 2\|z_m - z\|^2 \\
&\quad +2(\|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|) \\
&\leq 2\|x_k - z_m\|^2 + \theta_k \\
&\quad +2(\|z_m - z\|^2 + \|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|).
\end{aligned}$$

Taking $m \to \infty$, it follows that

$$\|z_k - z\|^2 + \|y_k - z\|^2 \leq 2\|x_k - z\|^2 + \theta_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subseteq C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|z_k - x\|^2 + \|y_k - x\|^2 \leq$

$2\|x_k - x\|^2 + \theta_k$ and $\|z_k - y\|^2 + \|y_k - y\|^2 \leq 2\|x_k - y\|^2 + \theta_k$. Hence

$$
\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - (\alpha x + (1-\alpha)y)\|^2 + \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(z_k - x) + (1-\alpha)(z_k - y)\|^2 + \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|z_k - x\|^2 + (1-\alpha)\|z_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\quad +\alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(\|z_k - x\|^2 + \|y_k - x\|^2) + (1-\alpha)(\|z_k - y\|^2 + \|y_k - y\|^2) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(2\|x_k - x\|^2 + \theta_k) + (1-\alpha)(2\|x_k - y\|^2 + \theta_k) - 2\alpha(1-\alpha)\|x - y\|^2 \\
&= 2[\alpha\|x_k - x\|^2 + (1-\alpha)\|x_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2] \\
&\quad +\alpha\theta_k + (1-\alpha)\theta_k \\
&= 2\|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k \\
&= 2\|x_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= 2\|x_k - z\|^2 + \theta_k.
\end{aligned}
$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y \rangle \geq 0 \tag{3.12}$$

for all $y \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$
\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u \rangle = -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - u \rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}
$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \tag{3.13}$$

for all $u \in \Finv(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \tag{3.14}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we claim that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From (3.14), we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since, $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Next, we now claim that $\lim_{n\to\infty}\|Tx_n - x_n\| = 0 = \lim_{n\to\infty}\|Sx_n - x_n\|$. Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leq 2\|x_n - x_{n+1}\|^2 + \theta_n.$$

From $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty}\|y_n - x_{n+1}\|, \tag{3.15}$$

which yields

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty, \tag{3.16}$$

and then we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

By definition of y_n, we have $y_n - x_n = (1 - \alpha_n)(T^n x_n - x_n)$, we obtain

$$\|T^n x_n - x_n\| = \frac{1}{1 - \alpha_n}\|y_n - x_n\|.$$

Since $\alpha_n \leq 1 - \delta$, then we have

$$\|T^n x_n - x_n\| \to 0, \tag{3.17}$$

as $n \to \infty$. From $z_n = \beta_n[\gamma_n y_n + (1 - \gamma_n)x_n] + (1 - \beta_n)S^n y_n$, we have

$$\begin{aligned} \|S^n y_n - z_n\| &= \frac{\beta_n}{1 - \beta_n}\|\gamma_n(y_n - z_n) + (1 - \gamma_n)(x_n - z_n)\| && (3.18) \\ &\leq \frac{1}{1 - \beta_n}(\gamma_n\|y_n - z_n\| + (1 - \gamma_n)\|x_n - z_n\|), && (3.19) \end{aligned}$$

which yields

$$\|S^n y_n - z_n\| \to 0 \text{ as } n \to \infty,$$

as $n \to \infty$ and so

$$\begin{aligned}\|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1 + t_\infty)\|x_n - x_{n+1}\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Similarly, we have

$$\|Sx_n - x_n\| \to 0 \text{ as } n \to \infty.$$

By Lemma 2.1.7, and boundedness of $\{x_n\}$, we have $\emptyset \neq \omega_w(x_n) \subset F(T) \cap F(S)$. Since $z_0 = P_{F(T)\cap F(S)}x_0, z_0 \in F(T) \cap F(S) \subset C$ and Lemma 2.1.8, guarantees the strong convergence of $\{x_n\}$ to $P_{F(T)\cap F(S)}x_0$. This completes the proof. □

3.2 Convergence theorem for asymptotically nonexpansive semigroups

The second of this work we extend the results of Dong, He and Cho [1] for $\mathcal{T}$ is an asymptotically nonexpansive semigroup then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0 \\ y_n = \alpha_n x_n + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt \\ z_n = \beta_n[\gamma_n y_n + (1 - \gamma_n)x_n] + (1 - \beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \widetilde{\theta}_n\} \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{3.20}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\widetilde{\theta_n} = [(1-\beta_n)(\widehat{s}_n^2 - 1) + (1-\alpha_n)(\widehat{t}_n^2 - 1) + (1-\beta_n)\widehat{s}_n^2(1-\alpha_n)(\widehat{t}_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $\widehat{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt \to 1$ and $\widehat{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt \to 1$ as $n \to \infty$).

We prove strongly convergence of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive semigroups with converge to common fixed points $P_{F(T)\cap F(S)}$.

Theorem 3.2.1. *Let H be a Hilbert space and let C be a nonempty colsed bounded subset of H. Let $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathfrak{S} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathfrak{F} = F(\mathfrak{T}) \cap F(\mathfrak{S}) \neq \emptyset$ and let $x_0 \in C$. Let $C_1 = C, x_1 = P_{C_1}x_0$ and $\{x_n\}$ be a sequence generated by (3.20) with satisfies $\alpha_n, \beta_n, \gamma_n \in [0,1]$, $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t_n \to \infty, s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}x_0$.*

Proof. First observe that $\mathfrak{F} \subseteq C_n$ for all $n \in \mathbb{N}$. For $\mathfrak{F} \subset C = C_1$ is obvious. Suppose that $\mathfrak{F} \subset C_k$ for each $k \in \mathbb{N}$. Let $u \in \mathfrak{F} \subset C_k$. Then we have

$$
\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u)\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} \|T(t)x_k - u\| dt)^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T\|x_k - u\| dt)^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T dt)^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2. \qquad (3.21)
\end{aligned}
$$

By Lemma 2.1.4 again, we have

$$
\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u)\|^2 \\
&\leq \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&\leq \beta_k\|\gamma_k(y_k - u) + (1-\gamma_k)(x_k - u)\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} \|S(t)y_k - u\|dt)^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S\|y_k - u\|dt)^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S dt)^2\|y_k - u\|^2 \\
&\leq \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] \\
&\leq \beta_k\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 + (1-\beta_k)\widetilde{s_k}^2\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(\widetilde{s_k}^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \qquad (3.22)
\end{aligned}
$$

From (3.21) and (3.22), we obtain that

$$\begin{aligned}
\|z_k - u\|^2 + \|y_k - u\|^2 \;\leq\; & \|x_k - u\|^2 + (1-\beta_k)(\widetilde{s_k}^2 - 1)\|x_k - u\|^2 \\
& +(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
& +(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 + \|x_k - u\|^2 \\
& +(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2. \\
& +(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + \|x_k - u\|^2 \\
& +(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
\leq\; & 2\|x_k - u\|^2 + [(1-\beta_k)(\widetilde{s_k}^2 - 1) + 2(1-\alpha_k)(\widetilde{t_k}^2 - 1) \\
& +(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)](diamC)^2 \\
=\; & 2\|x_k - u\|^2 + \widetilde{\theta}_k,
\end{aligned}$$

where $\widetilde{\theta}_k = [(1-\beta_k)(\widetilde{s}_k^2 - 1) + 2(1-\alpha_k)(\widetilde{t}_k^2 - 1) + (1-\beta_k)\widetilde{s}_k^2(1-\alpha_k)(\widetilde{t}_k^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $\widetilde{t}_k = \frac{1}{t_k}\int_0^{t_k} L_t^T dt \to 1$ and $\widetilde{s}_k \frac{1}{s_k}\int_0^{s_k} L_t^S dt \to 1$ as $k \to \infty$). It follows that $u \in C_{k+1}$ and then $F(T) \cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. Again, by using the same argument in the proof in Theorem 3.1.1, we can show that C_n is closed and convex foe all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for each $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subseteq C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z \in C_k$ and $\|z_k - z_m\|^2 \leq \|z_m - x_k\|^2 + \widetilde{\theta}_k$.

From Lemma 2.1.4, we have

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - z_m + z_m - z\|^2 + \|y_k - z_m + z_m - z\|^2 \\
&\leq [\|z_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle z_k - z_m, z_m - z\rangle] \\
&\quad +[\|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle] \\
&= \|z_k - z_m\|^2 + \|y_k - z_m\|^2 + 2\|z_m - z\|^2 \\
&\quad +2(\|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|) \\
&\leq 2\|x_k - z_m\|^2 + \widetilde{\theta}_k \\
&\quad +2(\|z_m - z\|^2 + \|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|).
\end{aligned}$$

Taking $m \to \infty$, it follows that

$$\|z_k - z\|^2 + \|y_k - z\|^2 \leq 2\|x_k - z\|^2 + \widetilde{\theta}_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subseteq C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|z_k - x\|^2 + \|y_k - x\|^2 \leq 2\|x_k - x\|^2 + \widetilde{\theta}_k$ and $\|z_k - y\|^2 + \|y_k - y\|^2 \leq 2\|x_k - y\|^2 + \widetilde{\theta}_k$. Hence

$$
\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - (\alpha x + (1-\alpha)y)\|^2 + \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(z_k - x) + (1-\alpha)(z_k - y)\|^2 + \|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|z_k - x\|^2 + (1-\alpha)\|z_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\quad +\alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(\|z_k - x\|^2 + \|y_k - x\|^2) + (1-\alpha)(\|z_k - y\|^2 + \|y_k - y\|^2) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(2\|x_k - x\|^2 + \widetilde{\theta}_k) + (1-\alpha)(2\|x_k - y\|^2 + \widetilde{\theta}_k) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&= 2[\alpha\|x_k - x\|^2 + (1-\alpha)\|x_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2] + \alpha\widetilde{\theta}_k \\
&\quad +(1-\alpha)\widetilde{\theta}_k \\
&= 2\|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \widetilde{\theta}_k \\
&= 2\|x_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= 2\|x_k - z\|^2 + \widetilde{\theta}_k.
\end{aligned}
$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \tag{3.23}$$

for all $y \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$
\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}
$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \tag{3.24}$$

for all $u \in \Finv(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \tag{3.25}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty}\|x_n - x_0\|$ exists. Next, we claim that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. From (3.25), we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since, $\lim_{n\to\infty}\|x_n - x_0\|$ exists, we have $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$. Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leq 2\|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n.$$

From $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty}\|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty}\|y_n - x_{n+1}\|, \tag{3.26}$$

which yields

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty, \tag{3.27}$$

and then we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

We now claim that

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

Indeed, by definition of y_n and $x_{n+1} \subset C_n$ we have

$$\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \; = \; \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.28}$$

From $z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt$, we have

$$\begin{aligned}
\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| &= \frac{\beta_n}{1-\beta_n}\|\gamma_n(y_n - z_n) + (1-\gamma_n)(x_n - z_n\| \\
&\leq \frac{\beta_n}{1-\beta_n}(\gamma_n\|y_n - z_n\| + (1-\gamma_n)\|x_n - z_n\|) \\
&\to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| &\leq \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - \frac{1}{s_n}\int_0^{s_n} S(t)y_n dt\| \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\leq \frac{1}{s_n}\int_0^{s_n} \|S(t)x_n - S(t)y_n\| dt + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| \\
&\quad + \|z_n - x_n\| \\
&\leq \frac{1}{s_n}\int_0^{s_n} L_t^S dt \|x_n - y_n\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\leq \widetilde{s}_n \|x_n - y_n\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

For all $0 \leq r < \infty$, we note that

$$
\begin{aligned}
\|S(r)x_n - x_n\| &\leq \|S(r)x_n - S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt)\| \\
&\quad + \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\
&\quad + \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \\
&\leq (L_\infty + 1)\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \\
&\quad + \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\
&:= (L_\infty + 1)A_n^S(r) + B_n^S(r),
\end{aligned}
$$

where $A_n^S := \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\|$ and $B_n^S := \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\|$.

By Lemma 2.1.6, we have $\limsup_{n\to\infty} A_n^S(r) = 0 = \limsup_{n\to\infty} B_n^S(r)$. We can deduce that for all $0 \le r < \infty$,

$$\begin{aligned} \|T(r)x_n - x_n\| &\le \|T(r)x_n - T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt)\| \\ &\quad + \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\| \\ &\quad + \|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \\ &\le (L_\infty + 1)\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \\ &\quad + \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\| \\ &:= (L_\infty + 1)A_n^T(r) + B_n^T(r), \end{aligned}$$

where $A_n^T(r) := \|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\|$ and $B_n^T(r) := \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\|$.
By Lemma 2.1.6, we have $\limsup_{n\to\infty} A_n^T(r) = 0 = \limsup_{n\to\infty} B_n^T(r)$. Then we obtain

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

We note by Lemma **??**, that every weak limit of $\{x_n\}$ is a member of $\mathfrak{F}$. From $x_n \rightharpoonup z \in P_{\mathfrak{F}}x_0$, we have $x_0 - x_n \rightharpoonup x_0 - z_0$, from H satisfies the Kadec-Klee property, it follows that

$$x_0 - x_n \to x_0 - z_0.$$

So, we have

$$\|x_n - z_0\| = \|x_n - x_0 - (z_0 - z_0)\| \to 0 \text{ as } n \to \infty.$$

Hence, $x_n \to z_0$. This complete the proof. □

CHAPTER IV
CONCLUSIONS

From chapter 3 we have 4 theorems for submitted to thai journals of mathematics.

4.1 Outputs Results

Theorem 4.1.1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $T, S : C \to C$ *be two asymptotically nonexpansive mappings with the sequences* $\{t_n\}$ *and* $\{s_n\}$*, respectively, such that* $F(T) \cap F(S) \neq \emptyset$*. Assume that* $\{\alpha_n\}, \{\beta_n\}$ *and* $\{\gamma_n\}$ *are the sequences in* $[0, 1]$ *such that* $\alpha_n, \beta_n \leq 1-\delta$ *for some* $\delta \in (0, 1]$*. Then the sequence* $\{x_n\}$ *generated by (3.8) converges in norm to* $P_{F(T)\cap F(S)}x_0$*.*

Theorem 4.1.2. *Let* H *be a Hilbert space and let* C *be a nonempty colsed bounded subset of* H*. Let* $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ *and* $\mathfrak{S} = \{S(t) : 0 \leq t < \infty\}$ *be two asymptotically nonexpansive semigroups on* C *such that* $\mathfrak{F} = F(\mathfrak{T}) \cap F(\mathfrak{S}) \neq \emptyset$ *and let* $x_0 \in C$*. Let* $C_1 = C, x_1 = P_{C_1}x_0$ *and* $\{x_n\}$ *be a sequence generated by (3.20) with satisfies* $\alpha_n, \beta_n, \gamma_n \in [0, 1]$*,* $0 \leq \alpha_n \leq a < 1$ *and* $0 < b \leq \beta_n \leq c < 1$ *for all* $n \in \mathbb{N} \cup \{0\}$ *and* $t_n \to \infty, s_n \to \infty$*. Then* $\{x_n\}$ *converges strongly to* $z_0 = P_{\mathfrak{F}}x_0$*.*

4.2 Output 1 paper

1. Hybrid iterative method for two asymptotically nonexpansive semigroup in Hilbert spaces, Submitted to **Thai Journal of Mathematics**.

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APPENDIX

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Hybrid iterative methods for two asymptotically nonexpansive semigroups in Hilbert spaces

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Abstract : The main objective of this work is to modified two hybrid projection algorithm. First, we prove the srtongly convergence to common fixed points of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive mappings, second, we prove strongly convergence of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive semigroups. Our main results extended and improved the results of I. Inchan and S. Plubtieng [4] and Q. L. Dong, S. He and Y. J. Cho [1].
2000 Mathematics Subject Classification : 46C05, 47D03, 47H09, 47H10, 47H20.

1 Introduction

Let H be a real Hilbert Space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [2] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$.

Recall also that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said

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to be a (continuous) Lipschitian semigroup on C (see, e. g., [13]) if the following conditions are satisfied:

(*i*) $T(0)x = x, x \in C$

(*ii*) $T(s+t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(*iii*) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(*iv*) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t \|x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [3, 7, 9, 10, 11]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [5, 11].

Very recently, Takahashi, Takeuchi and Kubota [12] prove the following strong convergence theorems by hybrid method foe nonexpansive mappings and nonexpansive semigroup in Hilbert space.

Theorem 1.1. *[12] Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases} \tag{1.1}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Theorem 1.2. *[12] Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let $\mathcal{T} = \{T(s) : 0 \leq s < \infty\}$ be a one-parameter nonexpansive mapping semigroup on C such that $F(\mathcal{T}) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)\frac{1}{\lambda_n}\int_0^{\lambda_n} T(s)u_n ds, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|, \\ u_{n+1} = P_{C_{n+1}}x_0, n \in \mathbb{N}, \end{cases} \tag{1.2}$$

where $0 \leq \alpha_n \leq a < 1$, $0 < \lambda_n < \infty$ for all $n \in \mathbb{N}$ and $\lambda_n \to \infty$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(\mathcal{T})}x_0$.

In 2008, Inchan and Plubtieng [4] modified Ishikawa iteration process for two asymptotically nonexpansive mappings, for C is a nonempty closed convex subset of a Hilbert space H, let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n z_n \\ z_n = \beta_n x_n + (1-\beta_n)S^n x_n \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\} \\ x_{n+1} = P_{C_{n+1}} x_0, n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $\theta_n = (1-\alpha_n)[(t_n^2 - 1) + (1-\beta_n)t_n^2(s_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$.

The second modificstion Ishikawa iteration process for two asymptotically nonexpansive semigroups. for C is a nonempty closed convex subset of a Hilbert space H, $\mathcal{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathcal{S} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathcal{F} = F(\mathcal{T}) \cap F(\mathcal{S}) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t) z_n dt \\ z_n = \beta_n x_n + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} S(t) x_n dt \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \widetilde{\theta_n}\} \\ x_{n+1} = P_{C_{n+1}} x_0, n \in \mathbb{N}, \end{cases} \tag{1.4}$$

where $\widetilde{\theta_n} = (1-\alpha_n)[(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{t}_n^2(\widetilde{s}_n^2 - 1)](diamC)^2 \to 0$(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt$ and $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}$ and $\widetilde{t}_n \to \infty, \widetilde{s}_n \to \infty$.

Then, prove that both iteration converges strongly to common fixed points of two asymptotically nonexpansive mappings and asymptotically nonexpansive semigroups, respectively.

In 2015, Dong, He and Cho [1], introduce a hybrid algorithm. Let T and S be two nonexpansive mappings into itselt such that $F(T) \cap F(S) \neq \emptyset$, the sequence generated as follows:

$$\begin{cases} x_0 \in C \\ y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)Sy_n, \\ C_n = \{z \in C : \sigma\|z_n - z\|^2 + (1-\sigma)\|y_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, n \geq 0, \end{cases} \tag{1.5}$$

for each $n \geq 0$, where $\alpha_n, \beta_n \in [0,1]$, $\delta \in [0,), \gamma_n \in [0,], \sigma \in (0,1)$. Then prove that $\{x_n\}$ converge in norm to $P_{F(T) \cap F(S)} x_0$.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.3. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x - y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.4. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x - y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \not< 1$, then $\mathcal{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

Inspired and motivate by above, the purpose of this paper to introduce two algorithms. The first of this work we extend the results of Dong, He and Cho [1] for S and T are two asymptotically nonexpansive mappings then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0, \\ y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n, \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{1.6}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\theta_n = [(1-\beta_n)(s_n^2 - 1) + (1-\alpha_n)(t_n^2 - 1) + (1-\beta_n)s_n^2(1-\aleph_n)(t_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $t_n \to 1$ and $s_n \to 1$ as $n \to \infty$).

The second of this work we extend the results of Dong, He and Cho [1] for $\mathcal{T}$ is an asymptotically nonexpansive semigroup then we consider

$$\begin{cases} x_0 \in C = C_1, x_1 = P_{C_1}x_0 \\ y_n = \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(t)z_n dt \\ z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt \\ C_{n+1} = \{z \in C_n : \|z_n - z\|^2 + \|y_n - z\|^2 \leq 2\|x_n - z\|^2 + \widetilde{\theta_n}\} \\ x_{n+1} = P_{C_{n+1}}x_0, n \geq 0, \end{cases} \tag{1.7}$$

where $\alpha_n, \beta_n, \gamma_n \in [0,1]$ and $\widetilde{\theta_n} = [(1-\beta_n)(\widetilde{s}_n^2 - 1) + (1-\alpha_n)(\widetilde{t}_n^2 - 1) + (1-\beta_n)\widetilde{s}_n^2(1-\alpha_n)(\widetilde{t}_n^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $\widetilde{t}_n = \frac{1}{t_n}\int_0^{t_n} L_t^T dt \to 1$ and $\widetilde{s}_n \frac{1}{s_n}\int_0^{s_n} L_t^S dt \to 1$ as $n \to \infty$).

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space, then the following hold:*
(i) $\|x+y\|^2 \leq \|x\|^2 + 2\langle x, y\rangle + \|y\|^2, \forall x, y \in H$;
(ii) $\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2, t \in [0,1], \forall x, y \in H.$

Lemma 2.2. *[5] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:*
(i) $x_n \rightharpoonup z$; *and*
(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0,$
then $z \in Fix(\mathcal{T})$.

Lemma 2.3. *[5] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.4. *[6] Let T be an asymptotically nonexpansive mapping defined on a bounded convex subset C of a Hilbert space H. If $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$, then $x \in F(T)$.*

Lemma 2.5. *[8] Let C be a nonempty closed convex subset of H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|$$

for all $n \geq 1$, then $x_n \to q$.

3 Main Results

In this section we introduce two theorems, first Theorem we prove the strong convergence theorem of the algorithm (1.6) into $P_{F(T)\cap F(S)}x_0$. The second Theorem we prove the strong convergence of modified hybrid iterative method (1.7) into $P_{\Im}x_0$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $T, S : C \to C$ be two asymptotically nonexpansive mappings with the sequences $\{t_n\}$ and $\{s_n\}$, respectively, such that $F(T) \cap F(S) \neq \emptyset$. Assume that $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are the sequences in $[0,1]$ such that $\alpha_n, \beta_n \leq 1-\delta$ for some $\delta \in (0,1]$. Then the sequence $\{x_n\}$ generated by (1.6) converges in norm to $P_{F(T)\cap F(S)}x_0$.*

Proof. Putting $t_\infty \sup\{t_n : n \geq 1\} < \infty$ and $s_\infty \sup\{s_n : n \geq 1\} < \infty$. We first show by induction that $F(T) \cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. It obvious that $F(T) \cap F(S) \subseteq C_1$. Suppose that $F(T) \cap F(S) \subseteq C_k$ for each $k \in \mathbb{N}$. Let $u \in F(T) \cap F(S) \subseteq C_k$, then from Lemma 2.1, we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)T^k x_k - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(T^k x_k - u)\|^2 \\
&= \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 - \alpha_k(1-\alpha_k)\|x_k - T^k x_k\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|T^k x_k - u\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - \|x_k - u\|^2 + \alpha_k\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\alpha_k)\|x_k - u\|^2 + (1-\alpha_k)t_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2. \qquad (3.1)
\end{aligned}$$

Simolarly, we note that from Lemma 2.1 and (3.1), we have

$$\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)S^k y_k - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(S^k y_k - u)\|^2 \\
&= \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leq \beta_k[\gamma_k\|y_k - u\|^2 + (1-\gamma_k)\|x_k - u\|^2] + (1-\beta_k)\|S^k y_k - u\|^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|y_k - u\|^2 \\
&\leq \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2[\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2] \\
&\leq \beta_k\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + (1-\beta_k)s_k^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 - (1-\beta_k)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2\|x_k - u\|^2 + (1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \qquad (3.2)
\end{aligned}$$

From (3.1) and (3.2), we obtain that

$$\begin{aligned}
\|z_k - u\|^2 + \|y_k - u\|^2 &\leq \|x_k - u\|^2 + (1-\beta_k)(s_k^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + \|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\leq 2\|x_k - u\|^2 + [(1-\beta_k)(s_k^2 - 1) + 2(1-\alpha_k)(t_k^2 - 1) \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)](diamC)^2 \\
&= 2\|x_k - u\|^2 + \theta_k \qquad (3.3)
\end{aligned}$$

where $\theta_k = [(1-\beta_k)(s_k^2 - 1) + 2(1-\alpha_k)(t_k^2 - 1) + (1-\beta_k)s_k^2(1-\alpha_k)](diamC)^2 \to 0$ as $n \to \infty$. It follows that $u \in C_{k+1}$ and then $F(T) \cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$.

Next, we show that C_n is closed and convex foe all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for each $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subseteq C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z_m \in C_k$ and $\|z_k - z_m\|^2 + \|y_k - z_m\|^2 \leq 2\|z_m - x_k\|^2 + \theta_k$. From Lemma 2.1, we have

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - z_m + z_m - z\|^2 + \|y_k - z_m + z_m - z\|^2 \\
&\leq [\|z_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle z_k - z_m, z_m - z\rangle] \\
&\quad +[\|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle] \\
&= \|z_k - z_m\|^2 + \|y_k - z_m\|^2 + 2\|z_m - z\|^2 \\
&\quad +2(\|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|) \\
&\leq 2\|x_k - z_m\|^2 + \theta_k + 2(\|z_m - z\|^2 + \|z_k - z_m\|\|z_m - z\| \\
&\quad +\|y_k - z_m\|\|z_m - z\|).
\end{aligned}$$

Taking $m \to \infty$, it follows that

$$\|z_k - z\|^2 + \|y_k - z\|^2 \leq 2\|x_k - z\|^2 + \theta_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subseteq C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|z_k - x\|^2 + \|y_k - x\|^2 \leq 2\|x_k - x\|^2 + \theta_k$ and $\|z_k - y\|^2 + \|y_k - y\|^2 \leq 2\|x_k - y\|^2 + \theta_k$. Hence

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - (\alpha x + (1-\alpha)y)\|^2 + \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(z_k - x) + (1-\alpha)(z_k - y)\|^2 \\
&\quad +\|\alpha(y_k - x) + (1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|z_k - x\|^2 + (1-\alpha)\|z_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\quad +\alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(\|z_k - x\|^2 + \|y_k - x\|^2) + (1-\alpha)(\|z_k - y\|^2 + \|y_k - y\|^2) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(2\|x_k - x\|^2 + \theta_k) + (1-\alpha)(2\|x_k - y\|^2 + \theta_k) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&= 2[\alpha\|x_k - x\|^2 + (1-\alpha)\|x_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2] \\
&\quad +\alpha\theta_k + (1-\alpha)\theta_k \\
&= 2\|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \theta_k \\
&= 2\|x_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= 2\|x_k - z\|^2 + \theta_k.
\end{aligned}$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \tag{3.4}$$

for all $y \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - u \rangle = -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - u \rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \tag{3.5}$$

for all $u \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. From $x_n = P_{C_n} x_0$ and $x_{n+1} = P_{C_{n+1}} x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1} \rangle \geq 0 \tag{3.6}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned} 0 &\leq \langle x_0 - x_n, x_n - x_{n+1} \rangle = -\langle x_n - x_0, x_n - x_0 \rangle + \langle x_0 - x_n, x_0 - x_{n+1} \rangle \\ &\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|. \end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we claim that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From (3.6), we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\ &= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1} \rangle + \|x_0 - x_{n+1}\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n \rangle - 2\langle x_0 - x_n, x_n - x_{n+1} \rangle \\ &\quad + \|x_0 - x_{n+1}\|^2 \\ &\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\ &= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2. \end{aligned}$$

Since, $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Next, we now claim that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0 = \lim_{n\to\infty} \|Sx_n - x_n\|$. Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leq 2\|x_n - x_{n+1}\|^2 + \theta_n.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty} \|y_n - x_{n+1}\|, \tag{3.7}$$

which yields

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty, \tag{3.8}$$

and then we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

By definition of y_n, we have $y_n - x_n = (1-\alpha_n)(T^n x_n - x_n)$, we obtain

$$\|T^n x_n - x_n\| = \frac{1}{1-\alpha_n}\|y_n - x_n\|.$$

Since $\alpha_n \leq 1-\delta$, then we have

$$\|T^n x_n - x_n\| \to 0, \tag{3.9}$$

as $n \to \infty$. From $z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)S^n y_n$, we have

$$\begin{aligned} \|S^n y_n - z_n\| &= \frac{\beta_n}{1-\beta_n}\|\gamma_n(y_n - z_n) + (1-\gamma_n)(x_n - z_n)\| & (3.10)\\ &\leq \frac{1}{1-\beta_n}(\gamma_n\|y_n - z_n\| + (1-\gamma_n)\|x_n - z_n\|), & (3.11)\end{aligned}$$

which yields

$$\|S^n y_n - z_n\| \to 0 \text{ as } n \to \infty,$$

as $n \to \infty$ and so

$$\begin{aligned} \|Tx_n - x_n\| &\leq \|Tx_n - T^{n+1}x_n\| + \|T^{n+1}x_n - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\ &\leq t_\infty\|x_n - T^n x_n\| + \|T^{n+1}x_{n+1} - x_{n+1}\| + (1+t_\infty)\|x_n - x_{n+1}\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Similarly, we have

$$\|Sx_n - x_n\| \to 0 \text{ as } n \to \infty.$$

By Lemma 2.4, and boundedness of $\{x_n\}$, we have $\emptyset \neq \omega_w(x_n) \subset F(T) \cap F(S)$. Since $z_0 = P_{F(T)\cap F(S)}x_0, z_0 \in F(T)\cap F(S) \subset C$ and Lemma 2.5, guarantees the strong convergence of $\{x_n\}$ to $P_{F(T)\cap F(S)}x_0$. This completes the proof. □

Next Theorem we prove strongly convergence of a sequence $\{x_n\}$ with generated by hybrid projection algorithm of two asymptotically nonexpansive semigroups with converge to common fixed points $P_{F(T)\cap F(S)}$.

Theorem 3.2. *Let H be a Hilbert space and let C be a nonempty colsed bounded subset of H. Let $\mathfrak{T} = \{T(t) : 0 \leq t < \infty\}$ and $\mathfrak{S} = \{S(t) : 0 \leq t < \infty\}$ be two asymptotically nonexpansive semigroups on C such that $\mathfrak{F} = F(\mathfrak{T}) \cap F(\mathfrak{S}) \neq \emptyset$ and let $x_0 \in C$. Let $C_1 = C, x_1 = P_{C_1}x_0$ and $\{x_n\}$ be a sequence generated by (1.7) with satisfies $\alpha_n, \beta_n, \gamma_n \in [0,1]$, $0 \leq \alpha_n \leq a < 1$ and $0 < b \leq \beta_n \leq c < 1$ for all $n \in \mathbb{N}\cup\{0\}$ and $t_n \to \infty, s_n \to \infty$. Then $\{x_n\}$ converges strongly to $z_0 = P_{\mathfrak{F}}x_0$.*

Proof. First observe that $\mathfrak{F} \subseteq C_n$ for all $n \in \mathbb{N}$. For $\mathfrak{F} \subset C = C_1$ is obvious. Suppose that $\mathfrak{F} \subset C_k$ for each $k \in \mathbb{N}$. Let $u \in \mathfrak{F} \subset C_k$. Then we have

$$\begin{aligned}
\|y_k - u\|^2 &= \|\alpha_k x_k + (1-\alpha_k)\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&= \|\alpha_k(x_k - u) + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u)\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)\|\frac{1}{t_k}\int_0^{t_k} T(t)x_k dt - u\|^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} \|T(t)x_k - u\| dt)^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T\|x_k - u\| dt)^2 \\
&\leq \alpha_k\|x_k - u\|^2 + (1-\alpha_k)(\frac{1}{t_k}\int_0^{t_k} L_t^T dt)^2\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2. \qquad (3.12)
\end{aligned}$$

By Lemma 2.1 again, we have

$$\begin{aligned}
\|z_k - u\|^2 &= \|\beta_k[\gamma_k y_k + (1-\gamma_k)x_k] + (1-\beta_k)\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&= \|\beta_k([\gamma_k y_k + (1-\gamma_k)x_k] - u) + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u)\|^2 \\
&\leq \beta_k\|[\gamma_k y_k + (1-\gamma_k)x_k] - u\|^2 + (1-\beta_k)\|\frac{1}{s_k}\int_0^{s_k} S(t)y_k dt - u\|^2 \\
&\leq \beta_k\|\gamma_k(y_k - u) + (1-\gamma_k)(x_k - u)\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} \|S(t)y_k - u\| dt)^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S\|y_k - u\| dt)^2 \\
&\leq \beta_k\gamma_k\|y_k - u\|^2 + \beta_k(1-\gamma_k)\|x_k - u\|^2 + (1-\beta_k)(\frac{1}{s_k}\int_0^{s_k} L_t^S dt)^2\|y_k - u\|^2 \\
&\leq \beta_k\gamma_k[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] + \beta_k(1-\gamma_k)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2[\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2] \\
&\leq \beta_k\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 + (1-\beta_k)\widetilde{s_k}^2\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&= \|x_k - u\|^2 + (1-\beta_k)(\widetilde{s_k}^2 - 1)\|x_k - u\|^2 + (1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&\quad + (1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \qquad (3.13)
\end{aligned}$$

From (3.12) and (3.13), we obtain that

$$\begin{aligned}
\|z_k - u\|^2 + \|y_k - u\|^2 &\leq \|x_k - u\|^2 + (1-\beta_k)(\widetilde{s_k}^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 \\
&\quad +(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2 + \|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(\widetilde{t_k}^2 - 1)\|x_k - u\|^2. \\
&\quad +(1-\beta_k)s_k^2(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 + \|x_k - u\|^2 \\
&\quad +(1-\alpha_k)(t_k^2 - 1)\|x_k - u\|^2 \\
&\leq 2\|x_k - u\|^2 + [(1-\beta_k)(\widetilde{s_k}^2 - 1) + 2(1-\alpha_k)(\widetilde{t_k}^2 - 1) \\
&\quad +(1-\beta_k)\widetilde{s_k}^2(1-\alpha_k)(\widetilde{t_k}^2 - 1)](diamC)^2 \\
&= 2\|x_k - u\|^2 + \widetilde{\theta}_k,
\end{aligned}$$

where $\widetilde{\theta_k} = [(1-\beta_k)(\widetilde{s}_k^2 - 1) + 2(1-\alpha_k)(\widetilde{t}_k^2 - 1) + (1-\beta_k)\widetilde{s}_k^2(1-\alpha_k)(\widetilde{t}_k^2 - 1)](diamC)^2 \to 0$ as $n \to \infty$(here $\widetilde{t}_k = \frac{1}{t_k}\int_0^{t_k} L_t^T dt \to 1$ and $\widetilde{s}_k \frac{1}{s_k}\int_0^{s_k} L_t^S dt \to 1$ as $k \to \infty$). It follows that $u \in C_{k+1}$ and then $F(T) \cap F(S) \subseteq C_n$ for all $n \in \mathbb{N}$. Again, by using the same argument in the proof in Theorem 3.1, we can show that C_n is closed and convex foe all $n \in \mathbb{N}$. It obvious that $C_1 = C$ is closed and convex. Suppose that C_k is closed and convex for each $k \in \mathbb{N}$. Let $\{z_m\}_{m=1}^{\infty} \subseteq C_{k+1} \subseteq C_k$ with $z_m \to z$ as $m \to \infty$. Since C_k is closed and $z_m \in C_{k+1}$, we have $z \in C_k$ and $\|z_k - z_m\|^2 \leq \|z_m - x_k\|^2 + \widetilde{\theta}_k$. From Lemma 2.1, we have

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - z_m + z_m - z\|^2 + \|y_k - z_m + z_m - z\|^2 \\
&\leq [\|z_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle z_k - z_m, z_m - z\rangle] \\
&\quad +[\|y_k - z_m\|^2 + \|z_m - z\|^2 + 2\langle y_k - z_m, z_m - z\rangle] \\
&= \|z_k - z_m\|^2 + \|y_k - z_m\|^2 + 2\|z_m - z\|^2 \\
&\quad +2(\|z_k - z_m\|\|z_m - z\| + \|y_k - z_m\|\|z_m - z\|) \\
&\leq 2\|x_k - z_m\|^2 + \widetilde{\theta}_k + 2(\|z_m - z\|^2 + \|z_k - z_m\|\|z_m - z\| \\
&\quad +\|y_k - z_m\|\|z_m - z\|).
\end{aligned}$$

Taking $m \to \infty$, it follows that

$$\|z_k - z\|^2 + \|y_k - z\|^2 \leq 2\|x_k - z\|^2 + \widetilde{\theta}_k.$$

Then $z \in C_{k+1}$ and hence C_{k+1} is closed. Let $x, y \in C_{k+1} \subseteq C_k$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_k is convex, $z \in C_k$. Thus, we have $\|z_k - x\|^2 + \|y_k - x\|^2 \leq 2\|x_k - x\|^2 + \widetilde{\theta}_k$ and $\|z_k - y\|^2 + \|y_k - y\|^2 \leq 2\|x_k - y\|^2 + \widetilde{\theta}_k$.

Hence

$$\begin{aligned}
\|z_k - z\|^2 + \|y_k - z\|^2 &= \|z_k - (\alpha x + (1-\alpha)y)\|^2 + \|y_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(z_k - x) + (1-\alpha)(z_k - y)\|^2 + \|\alpha(y_k - x) \\
&\quad +(1-\alpha)(y_k - y)\|^2 \\
&= \alpha\|z_k - x\|^2 + (1-\alpha)\|z_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\quad +\alpha\|y_k - x\|^2 + (1-\alpha)\|y_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(\|z_k - x\|^2 + \|y_k - x\|^2) \\
&\quad +(1-\alpha)(\|z_k - y\|^2 + \|y_k - y\|^2) - 2\alpha(1-\alpha)\|x - y\|^2 \\
&\leq \alpha(2\|x_k - x\|^2 + \widetilde{\theta}_k) + (1-\alpha)(2\|x_k - y\|^2 + \widetilde{\theta}_k) \\
&\quad -2\alpha(1-\alpha)\|x - y\|^2 \\
&= 2[\alpha\|x_k - x\|^2 + (1-\alpha)\|x_k - y\|^2 - \alpha(1-\alpha)\|x - y\|^2] \\
&\quad +\alpha\widetilde{\theta}_k + (1-\alpha)\widetilde{\theta}_k \\
&= 2\|\alpha(x_k - x) + (1-\alpha)(x_k - y)\|^2 + \widetilde{\theta}_k \\
&= 2\|x_k - (\alpha x + (1-\alpha)y)\|^2 \\
&= 2\|x_k - z\|^2 + \widetilde{\theta}_k.
\end{aligned}$$

It follows that $z \in C_{k+1}$ and hence C_{k+1} is convex. Therefore, C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. Since $x_n = P_{C_n}x_0$, it follows that

$$\langle x_0 - x_n, x_n - y\rangle \geq 0 \tag{3.14}$$

for all $y \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. So $u \in F(T) \cap F(S)$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - u\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - u\rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - u\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - u\|$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - u\| \tag{3.15}$$

for all $u \in F(T) \cap F(S)$ and $\forall n \in \mathbb{N}$. From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subseteq C_n$, we obtain that

$$\langle x_0 - x_n, x_n - x_{n+1}\rangle \geq 0 \tag{3.16}$$

for all $n \in \mathbb{N}$. So, for all $x_{n+1} \in C_{n+1}$, for $n \in \mathbb{N}$, we have

$$\begin{aligned}
0 &\leq \langle x_0 - x_n, x_n - x_{n+1}\rangle = -\langle x_n - x_0, x_n - x_0\rangle + \langle x_0 - x_n, x_0 - x_{n+1}\rangle \\
&\leq -\|x_n - x_0\|^2 + \|x_0 - x_n\|\|x_0 - x_{n+1}\|.
\end{aligned}$$

This implies that

$$\|x_0 - x_n\|^2 \leq \|x_0 - x_n\|\|x_0 - x_{n+1}\|,$$

and hence

$$\|x_0 - x_n\| \leq \|x_0 - x_{n+1}\|,$$

for all $n \in \mathbb{N}$. Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty} \|x_n - x_0\|$ exists. Next, we claim that $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. From (3.16), we have

$$\begin{aligned}
\|x_n - x_{n+1}\|^2 &= \|(x_n - x_0) + (x_0 - x_{n+1})\|^2 \\
&= \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle + \|x_0 - x_{n+1}\|^2 \\
&= \|x_n - x_0\|^2 - 2\langle x_0 - x_n, x_0 - x_n\rangle - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&\quad + \|x_0 - x_{n+1}\|^2 \\
&\leq \|x_n - x_0\|^2 - 2\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 \\
&= -\|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2.
\end{aligned}$$

Since, $\lim_{n\to\infty} \|x_n - x_0\|$ exists, we have $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$. Since $x_{n+1} \in C_n$, we have

$$\|z_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2 \leq 2\|x_n - x_{n+1}\|^2 + \widetilde{\theta}_n.$$

From $\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0$ and $\theta_n \to 0$ as $n \to \infty$, it follows that

$$\lim_{n\to\infty} \|z_n - x_{n+1}\| = 0 = \lim_{n\to\infty} \|y_n - x_{n+1}\|, \tag{3.17}$$

which yields

$$\|z_n - x_n\| \leq \|z_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty, \tag{3.18}$$

and then we have

$$\|y_n - x_n\| \leq \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty.$$

We now claim that

$$\limsup_{r\to\infty} \limsup_{n\to\infty} \|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty} \limsup_{n\to\infty} \|S(r)x_n - x_n\|.$$

Indeed, by definition of y_n and $x_{n+1} \subset C_n$ we have

$$\left\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\right\| = \frac{1}{1-\alpha_n}\|y_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.19}$$

From $z_n = \beta_n[\gamma_n y_n + (1-\gamma_n)x_n] + (1-\beta_n)\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt$, we have

$$\begin{aligned}
\left\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\right\| &= \frac{\beta_n}{1-\beta_n}\|\gamma_n(y_n - z_n) + (1-\gamma_n)(x_n - z_n\| \\
&\leq \frac{\beta_n}{1-\beta_n}(\gamma_n\|y_n - z_n\| + (1-\gamma_n)\|x_n - z_n\|) \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}$$

It follows that

$$
\begin{aligned}
\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| &\leq \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - \frac{1}{s_n}\int_0^{s_n} S(t)y_n dt\| \\
&\quad +\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\leq \frac{1}{s_n}\int_0^{s_n} \|S(t)x_n - S(t)y_n\| dt \\
&\quad +\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\leq \frac{1}{s_n}\int_0^{s_n} L_t^S dt \|x_n - y_n\| \\
&\quad +\|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\leq \widetilde{s}_n\|x_n - y_n\| + \|\frac{1}{s_n}\int_0^{s_n} S(t)y_n dt - z_n\| + \|z_n - x_n\| \\
&\quad \to 0 \text{ as } n \to \infty.
\end{aligned}
$$

For all $0 \leq r < \infty$, we note that

$$
\begin{aligned}
\|S(r)x_n - x_n\| &\leq \|S(r)x_n - S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt)\| \\
&\quad +\|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\
&\quad +\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \\
&\leq (L_\infty + 1)\|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\| \\
&\quad +\|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\| \\
&:= (L_\infty + 1)A_n^S(r) + B_n^S(r),
\end{aligned}
$$

where $A_n^S := \|\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt - x_n\|$ and $B_n^S := \|S(r)(\frac{1}{s_n}\int_0^{s_n} S(t)x_n dt) - \frac{1}{s_n}\int_0^{s_n} S(t)x_n dt\|$. By Lemma 2.3, we have $\limsup_{n\to\infty} A_n^S(r) = 0 = \limsup_{n\to\infty} B_n^S(r)$. We can deduce that for all $0 \leq r < \infty$,

$$\begin{aligned}
\|T(r)x_n - x_n\| &\leq \|T(r)x_n - T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt)\| \\
&+\|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\| \\
&+\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \\
&\leq (L_\infty + 1)\|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\| \\
&+\|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\| \\
&:= (L_\infty + 1)A_n^T(r) + B_n^T(r),
\end{aligned}$$

where $A_n^T(r) := \|\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt - x_n\|$ and $B_n^T(r) := \|T(r)(\frac{1}{t_n}\int_0^{t_n} T(t)x_n dt) - \frac{1}{t_n}\int_0^{t_n} T(t)x_n dt\|$. By Lemma 2.3, we have $\limsup_{n\to\infty} A_n^T(r) = 0 = \limsup_{n\to\infty} B_n^T(r)$. Then we obtain

$$\limsup_{r\to\infty}\limsup_{n\to\infty}\|T(r)x_n - x_n\| = 0 = \limsup_{r\to\infty}\limsup_{n\to\infty}\|S(r)x_n - x_n\|.$$

We note by Lemma 2.2, that every weak limit of $\{x_n\}$ is a member of $\mathfrak{F}$. From $x_n \rightharpoonup z \in P_{\mathfrak{F}}x_0$, we have $x_0 - x_n \rightharpoonup x_0 - z_0$, from H satisfies the Kadec-Klee property, it follows that

$$x_0 - x_n \to x_0 - z_0.$$

So, we have

$$\|x_n - z_0\| = \|x_n - x_0 - (z_0 - z_0)\| \to 0 \text{ as } n \to \infty.$$

Hence, $x_n \to z_0$. This complete the proof. □

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