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Original Article

On m-bi-hyperideals in semihyperrings

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Abstract

We introduce the notion of m-bi-hyperideals of a semihyperring as a generalization of bi-hyperideals, and we investigate some of its properties. Moreover, we characterize regular semihyperrings using m-bi-hyperideals.

Keywords: semihyperring, bi-hyperideal, m-bi-hyperideal, (m, n)-quasi hyperideal

1. Introduction

The notion of quasi-ideals was introduced for semirings without zero and proved some properties on semirings using quasi-ideals, see in (Iséki, 1958). Shabir, Ali, and Batool (2004) characterized semirings by their quasi-ideals. Good and Hughes (1952) introduced the concept of bi-ideals of semigroups. In 1970, Lajos and Szász (1970) introduced the concept of bi-ideals in associative rings. Quasi-ideals are a generalization of left or right ideals, while bi-ideals are a generalization of quasi-ideals. In 2018, Munirand Shafiq (20 18) introduced the idea of m-bi-ideals in semirings as a generalization of bi-ideals.

The concept of hyperstructures was introduced by Marty (1934) in the 8th Congress of Scandinavian Mathematicians. There are many authors expanded the concept of hyperstructures which appears in Corsini (1993), Corsini and Leoreanu (2003), Davvaz and Leoreanu-Fotea (2007), Vougiouklis (1994). The notion of a semihyperring, which both the sum and the product are hyperoperations, was defined by Vougiouklis (1990) as a generalization of a semiring. The concept of bi-hyperideals in semihyperrings, as a generalization of bi-ideals in semirings, was studied by Huang, Yin and Zhan (2013) and Omidi and Davvaz (2016, 2017). In 2017, Omidi and Davvaz (2017) introduced the notion of (m, n)-quasi-hyperideals in ordered semihyperrings and investigated some of its properties. In this paper, we introduce the concept of m-bi-hyperideals of semihyperrings as a generalization of bi-hyperideals, and we investigate some of its properties. Then, we characterize regular semihyperrings by m-bi-hyperideals.

2. Preliminaries

Let X be a nonempty set. A mapping $\circ : X \times X \to \mathcal{P}^*(X)$, where $\mathcal{P}^*(X)$ denotes the set of all nonempty subsets of X, is called a *hyperoperation* on X (Corsini, 1993; Corsini & Leoreanu, 2003; Davvaz & Leoreanu-Fotea, 2007; Vougiouklis, 1994). The couple $(X,\circ)$ is called a *hypergroupoid*. If $A, B \in \mathcal{P}^*(X)$ and $x \in X$, then we denote

$$A \circ B = \bigcup_{a\in A, b\in B} a \circ b\,, A \circ x = A \circ \{x\} \text{ and } x \circ B = \{x\} \circ B.$$

A hypergroupoid $(X,\circ)$ is called a *semihypergroup* if for every $x, y, z \in X$, $(x \circ y) \circ z = x \circ (y \circ z)$, which means that

$$\bigcup_{u\in x\circ y} u \circ z = \bigcup_{v\in y\circ z} x \circ v.$$

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A triple $(S,+,\cdot)$ is called a *semihyperring* (Vougiouklis, 1990) if it satisfies the following conditions:

(i) $(S,+)$ is a semihypergroup;

(ii) $(S,\cdot)$ is a semihypergroup;

(iii) $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$, for all $x,y,z\in S$.

A semihyperring $(S,+,\cdot)$ is said to be *additively commutative* if $a+b=b+a$, for all $a,b\in S$. An element $0\in S$ is said to be an *absorbing zero* if $a+0=0+a=\{a\}$ and $a\cdot 0=0\cdot a=\{0\}$, for all $a\in S$. An element e of a semihyperring $(S,+,\cdot)$ is said to be *the multiplicative identity* if $x\in e\cdot x$ and $x\in x\cdot e$, for all $x\in S$. A nonempty subset T of a semihyperring $(S,+,\cdot)$ is said to be a *subsemihyperring* of S if T is closed under both the hyperaddition $+$ and the hypermultiplication $\cdot$.

A nonempty subset I of a semihyperring $(S,+,\cdot)$ is called a *left* (resp. *right*) *hyperideal* of S if it satisfies $I+I\subseteq I$ and $S\cdot I\subseteq I$ (resp. $I\cdot S\subseteq I$). If I is both a left and a right hyperideal of S, then I is called a *hyperideal* of S. Throughout this paper, we assume that $(S,+,\cdot)$ is an additively commutative semihyperring with an absorbing zero 0.

For more convenient, we write S for a semihyperring $(S,+,\cdot)$, AB for $A\cdot B$ and ab for $a\cdot b$, for any nonempty subsets A and B of S and $a,b\in S$. Now, we present the definitions of quasi-hyperideals, bi-hyperideals and (m,n)-quasi-hyperideals of semihyperrings, see for example in Davvaz and Omidi (2016), Haung, Yin, and Zhan (2013), Kar and Purkait (2017), Omidi and Davvaz (2016), Omidi and Davvaz (2017).

A nonempty subset Q of a semihyperring S is called a *quasi-hyperideal* of S if it satisfies $Q+Q\subseteq Q$ and $(SQ)\cap(QS)\subseteq Q$. A subsemihyperring B of a semihyperring S is called a *bi-hyperideal* of S if it satisfies $BSB\subseteq B$. Every left (resp. right) hyperideal of a semihyperring S is a quasi-hyperideal of S. We note that each quasi-hyperideal of a semihyperring S is a bi-hyperideal of S.

For a semihyperring S and $m\in\mathbb{N}$, we denote $S^m=SSS\cdots S$ (m times). If a semihyperring S contains the multiplicative identity, then $S^m=S$. A subsemihyperring Q of a semihyperring S is called an (m,n)-*quasi-hyperideal* (Omidi & Davvaz, 2017) of S if it satisfies $(S^mQ)\cap(QS^n)\subseteq Q$, where m and n are positive integers.

It is clear that a quasi-hyperideal Q of a semihyperring S is a $(1,1)$-quasi hyperideal of S. In addition, an (m,n)-quasi-hyperideal of a semihyperring S is a (k,l)-quasi-hyperideal of S, for all $k\geq m$ and $l\geq n$. Furthermore, any (m,n)-quasi hyperideal of a semihyperring S need not to be a quasi-hyperideal of S (Omidi & Davvaz, 2017).

For any nonempty subsets A and B of a semihyperring S, we denote

$\Sigma A=\{\,t\in S\mid t\in\sum_{i\in I}a_i,\ a_i\in A$ and I is a finite subset of $\mathbb{N}\,\}$,

$\Sigma AB=\{\,t\in S\mid t\in\sum_{i\in I}a_ib_i,\ a_i\in A,\ b_i\in B$ and I is a finite subset of $\mathbb{N}\,\}$,

$\Sigma a=\Sigma\{a\}$, for any $a\in S$ and

$\sum_{i\in\emptyset}a_i=\{0\}$, for every $a_i\in S$.

Remark 1. For any nonempty subsets A and B of a semihyperring S, the following statements hold:

(i) $\Sigma(\Sigma A)=\Sigma A$;

(ii) $\Sigma A\subseteq\Sigma A+\Sigma B$;

(iii) $(\Sigma A)(\Sigma B)\subseteq\Sigma AB$;

(iv) $\Sigma(A+B)\subseteq\Sigma A+\Sigma B$.

3. Properties of m-bi-hyperideals

In this section, we introduce the concept of an m-bi-hyperideal of a semihyperring. Then, we investigate some properties of an m-bi-hyperideal in a semihyperring. Moreover, we study characterizations of regular semihyperrings by their m-bi-hyperideals.

Definition 3.1. A subsemihyperring B of a semihyperring S is called an *m-bi-hyperideal* of S if it satisfies $BS^mB \subseteq B$, where m is a positive integer.

We note that every bi-hyperideal of a semihyperring is a 1-bi-hyperideal. For any nonempty subset A of a semihyperring S, we denote by $\langle A\rangle_B$ the smallest m-bi-hyperideal of S containing A. Now, the m-bi-hyperideal $\langle A\rangle_B$ is called *the m-bi-hyperideal of S generated by A*. If $A = \{a\}$, then we define $\langle a\rangle_B = \langle\{a\}\rangle_B$. Then we have the following lemma.

Lemma 3.2. Let A be a nonempty subset of a semihyperring S. Then

$$\langle A\rangle_B = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA.$$

Proof. Let $M = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA$. Clearly, $A \subseteq M$. Since S is additively commutative, M is closed under the hyperaddition. Next, by Remark 1, we have

$$\begin{aligned}
M^2 &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA)^2 \\
&\subseteq \Sigma AA + \Sigma AA^2 + \cdots + \Sigma AA^m + \Sigma AA^{m+1} + \Sigma AAS^mA \\
&\quad + \Sigma A^2A + \Sigma A^2A^2 + \cdots + \Sigma A^2A^m + \Sigma A^2A^{m+1} + \Sigma A^2AS^mA \\
&\quad + \cdots + \Sigma A^mA + \Sigma A^mA^2 + \cdots + \Sigma A^mA^m + \Sigma A^mA^{m+1} + \Sigma A^mAS^mA \\
&\quad + \Sigma A^{m+1}A + \Sigma A^{m+1}A^2 + \cdots + \Sigma A^{m+1}A^m + \Sigma A^{m+1}A^{m+1} + \Sigma A^{m+1}AS^mA \\
&\quad + \Sigma AS^mAA + \Sigma AS^mAA^2 + \cdots + \Sigma AS^mAA^m + \Sigma AS^mAA^{m+1} + \Sigma AS^mAAS^mA \\
&\subseteq \Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA \\
&\subseteq M.
\end{aligned}$$

Thus, M is a subsemihyperring of S. By Remark 1, we have

$$\begin{aligned}
MS^mM &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA) \\
&\qquad S^m(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA) \\
&\subseteq \Sigma AS^mA \\
&\subseteq M.
\end{aligned}$$

Hence, M is an m-bi-hyperideal of S. Finally, let K be any m-bi-hyperideal of S containing A. It follows that ΣA, ΣA^2, ..., ΣA^m, ΣA^{m+1} and ΣAS^mA are nonempty subsets of K. We obtain that $M \subseteq K$. Therefore, M is the m-bi-hyperideal of S generated by A, that is, $\langle A\rangle_B = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma A^{m+1} + \Sigma AS^mA$.

In a particular of Lemma 3.2, if $A = \{a\}$ then we have the following corollary.

Corollary 3.3. Let S be a semihyperring and $a \in S$. Then

$$\langle a\rangle_B = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma a^{m+1} + \Sigma aS^ma.$$

Theorem 3.4. Every bi-hyperideal of a semihyperring is also an m-bi-hyperideal, for each $m \in \mathbb{N}$.

Proof. Assume that B is a bi-hyperideal of a semihyperring S. Then B is a subsemihyperring of S. Since $S^m \subseteq S$, $BS^mB \subseteq BSB \subseteq B$. Hence, B is an m-bi-hyperideal of S.

The converse of Theorem 3.4 is not true, that is, any m-bi-hyperideal of a semihyperring S need not to be a bi-hyperideal of S. In the following example, we apply the semiring defined in Example 3.3 in (Munir & Shafiq, 2018) to construct a semihyperring.

Example 3.5. Let $S = \left\{ \begin{bmatrix} 0 & u & v & w \\ 0 & 0 & x & y \\ 0 & 0 & 0 & z \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, u,v,w,x,y,z \in \mathbb{N} \cup \{0\} \right\}$. Then $(S,+,\cdot)$ is a semiring under usual the matrix addition and the matrix multiplication. For every $A, B \in S$, we define $A \leq B$ iff $a_{ij} \leq b_{ij}$, where $i,j \in \{1,2,3,4\}$. Next, we define the hyperoperations $\oplus$ and $\odot$ on S by letting $A, B \in S$,

$A \oplus B = \{X \in S \mid X \leq A + B\}$ and

$A \odot B = \{X \in S \mid X \leq A \cdot B\}$.

We can show that $(S, \oplus, \odot)$ is a semihyperring. Now, let

$$B = \left\{ \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{bmatrix} \middle| \, a,b \in \mathbb{N} \cup \{0\} \right\}.$$

It is not difficult to check that B is a subsemihyperring of S. Then B is a 2-bi-hyperideal of S, that is, $B \odot S^2 \odot B \subseteq B$, but $B \odot S \odot B \nsubseteq B$.

Theorem 3.6. Let B_1 and B_2 be an m_1-bi-hyperideal and an m_2-bi-hyperideal of a semihyperring S, respectively. Then $B_1 \cap B_2$ is an m-bi-hyperideal of S, where $m = \max\{m_1, m_2\}$.

Proof. Since $0 \in B_i$ for all $i \in \{1,2\}$, $B_1 \cap B_2 \neq \emptyset$. It is not difficult to show that $B_1 \cap B_2$ is a subsemihyperring of S. Then $(B_1 \cap B_2)S^m(B_1 \cap B_2) \subseteq B_1 S^{m_1} B_1 \subseteq B_1$ and $(B_1 \cap B_2)S^m(B_1 \cap B_2) \subseteq B_2 S^{m_2} B_2 \subseteq B_2$. This implies that $(B_1 \cap B_2)S^m(B_1 \cap B_2) \subseteq B_1 \cap B_2$. Therefore, $B_1 \cap B_2$ is an m-bi-hyperideal of S.

Theorem 3.7. Let S be a semihyperring with the multiplication identity. Then ΣBT and ΣTB are m-bi-hyperideals of S, for any m-bi-hyperideal B and nonempty subset T of S.

Proof. Assume that B is an m-bi-hyperideal and T is a nonempty subset of S. Obviously, ΣBT is closed under the hyperaddition. Since S has the multiplicative identity and by Remark 1, we have $(\Sigma BT)(\Sigma BT) \subseteq \Sigma BTBT \subseteq \Sigma BSBT = \Sigma BS^m BT \subseteq \Sigma BT$. It follows that ΣBT is a subsemihyperring of S. Now, by Remark 1, we have

$(\Sigma BT)S^m(\Sigma BT) \subseteq \Sigma BTS^m BT \subseteq \Sigma BSS^m BT = \Sigma BS^{m+1} BT \subseteq \Sigma BS^m BT \subseteq \Sigma BT$.

Hence, ΣBT is an m-bi-hyperideal of S. Similarly, we can show that ΣTB is an m-bi-hyperideal of S.

Corollary 3.8. Let B_1 be an m_1-bi-hyperideal and B_2 be an m_2-bi-hyperideal of a semihyperring S with the multiplicative identity. Then $\Sigma B_1 B_2$ is an m-bi-hyperideal of S, where $m = \max\{m_1, m_2\}$.

Theorem 3.9. Every (m_1, m_2)-quasi-hyperideal of a semihyperring S is an m-bi-hyperideal of S, where $m = \max\{m_1, m_2\}$.

Proof. Assume that B is an (m_1, m_2)-quasi-hyperideal of a semihyperring S. Let $m = \max\{m_1, m_2\}$. So, we have $BS^m B \subseteq SS^m B = S^{m+1} B \subseteq S^{m_1} B$ and $BS^m B \subseteq BS^m S = BS^{m+1} \subseteq BS^{m_2}$. That is, $BS^m B \subseteq (S^{m_1} B) \cap (BS^{m_2}) \subseteq B$. Hence, B is an m-bi-hyperideal of S.

The converse of Theorem 3.9 does not hold, namely, every m-bi-hyperideal of a semihyperring S need not to be an (m_1, m_2)-quasi-hyperideal of S, where $m = \max\{m_1, m_2\}$, as the following example shows.

Example 3.10. By the semihyperring $(S, \oplus, \odot)$ defined in Example 3.5, the set B is a 2-bi-hyperideal of S, but not a $(2,1)$-quasi-hyperideal of S, since $(S^2 \odot B) \cap (B \odot S) \nsubseteq B$.

Theorem 3.11. Let Q_1 be an (m_1, n_1)-quasi-hyperideal and Q_2 be an (m_2, n_2)-quasi-hyperideal of a semihyperring S with the multiplicative identity. Then $\Sigma Q_1 Q_2$ is an m-bi-hyperideal of S, where $m = \max\{m_1, n_1, m_2, n_2\}$.

Proof. By Theorem 3.9, Q_1 is an m_1'-bi-hyperideal and Q_2 is an m_2'-bi-hyperideal, where $m_1' = \max\{m_1, n_1\}$ and $m_2' = \max\{m_2, n_2\}$, respectively. By Corollary 3.8, $\Sigma Q_1 Q_2$ is an m-bi-hyperideal of S, where $m = \max\{m_1', m_2'\}$.

A subsemihyperring A of a semihyperring S is called an *m-left* (resp. *n-right*) *hyperideal* (Omidi & Davvaz, 2017) of S if it satisfies $S^m A \subseteq A$ (resp. $AS^n \subseteq A$), where m (resp. n) is a positive integer.

For any nonempty subset A of a semihyperring S, we denote by $\langle A \rangle_L$ and $\langle A \rangle_R$ as the smallest m-left hyperideal and the smallest n-right hyperideal of S containing A, respectively. Now, the m-left hyperideal $\langle A \rangle_L$ is called *the m-left hyperideal of S generated by A* and the n-right hyperideal $\langle A \rangle_R$ is called *the n-right hyperideal of S generated by A*. If $A = \{a\}$, then we will write $\langle a \rangle_L$ and $\langle a \rangle_R$ instead of $\langle \{a\} \rangle_L$ and $\langle \{a\} \rangle_R$, respectively. Then we have the following lemma.

Lemma 3.12. Let A be a nonempty subset of a semihyperring S. Then the following statements hold:

(*i*) $\langle A \rangle_L = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A$;

(*ii*) $\langle A \rangle_R = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^n + \Sigma AS^n$.

Proof. (*i*): Let $I = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A$. Clearly, $A \subseteq I$. Then I is closed under the hyperaddition, since S is additively commutative. Next, by Remark 1, we have

$$\begin{aligned}
I^2 &= (\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A)^2 \\
&\subseteq \Sigma AA + \Sigma AA^2 + \cdots + \Sigma AA^m + \Sigma AS^m A \\
&\quad + \Sigma A^2 A + \Sigma A^2 A^2 + \cdots + \Sigma A^2 A^m + \Sigma A^2 S^m A \\
&\quad + \cdots + \Sigma A^m A + \Sigma A^m A^2 + \cdots + \Sigma A^m A^m + \Sigma S A^{mm} A \\
&\quad + \Sigma S^m AA + \Sigma S^m AA^2 + \cdots + \Sigma S^m AA^m + \Sigma S^m AS^m A \\
&\subseteq \Sigma A^2 + \Sigma A^3 + \cdots + \Sigma A^m + \Sigma S^m A \\
&\subseteq I.
\end{aligned}$$

We obtain that I is a subsemihyperring of S. Now, by Remark 1, we have $S^m I = S^m(\Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A) \subseteq \Sigma S^m A \subseteq I$. Hence, I is an m-left hyperideal of S containing A. Let J be any m-left hyperideal of S containing A. Thus, ΣA, ΣA^2, ..., ΣA^m and $\Sigma S^m A$ are nonempty subsets of J. This implies that $I \subseteq J$. Therefore, I is the m-left hyperideal of S generated by A, that is, $\langle A \rangle_L = \Sigma A + \Sigma A^2 + \cdots + \Sigma A^m + \Sigma S^m A$.

(*ii*): The proof is similar to (*i*).

In particular of Lemma 3.12, if $A = \{a\}$ then we have the following corollary.

Corollary 3.13. Let S be a semihyperring and $a \in S$. Then the following statements hold:

(*i*) $\langle a \rangle_L = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma S^m a$;

(*ii*) $\langle a \rangle_R = \Sigma a + \Sigma a^2 + \cdots + \Sigma a^n + \Sigma a S^n$.

Theorem 3.14. Every m-left (resp. n-right) hyperideal of a semihyperring S is an m-bi-hyperideal (resp. n-bi-hyperideal) of S.

Proof. Assume that A is an m-left hyperideal of a semihyperring S. Cleary, A is a subsemihyperring of S. Now, $AS^m A \subseteq AA \subseteq A$. Hence, A is an m-bi-hyperideal of S. For an n-right hyperideal of S, the proof is similar.

In general, an m-bi-hyperideal need not to be an m-left or m-right hyperideal of a semihyperring. This follows from the following example.

Example 3.15. Let $S = \{a, b, c, d\}$. Define two hyperoperations $+$ and $\cdot$ on S as follows:

$+$	a	b	c	d
a	$\{a\}$	$\{b\}$	$\{c\}$	$\{d\}$
b	$\{b\}$	$\{b\}$	$\{b\}$	$\{b\}$
c	$\{c\}$	$\{b\}$	$\{a, c\}$	$\{c, d\}$
d	$\{d\}$	$\{b\}$	$\{a, c, d\}$	$\{a, c\}$

$\cdot$	a	b	c	d
a	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
b	$\{a\}$	$\{b\}$	$\{a, c\}$	$\{a\}$
c	$\{a\}$	$\{a\}$	$\{a\}$	$\{a\}$
d	$\{a\}$	$\{a, d\}$	$\{a\}$	$\{a\}$

Then $(S, +, \cdot)$ is a semihyperring (Huang, Yin, & Zhan, 2013). Let $A = \{a, b\}$. We have that A is an m-bi-hyperideal of S, but it is not an m-left hyperideal of S, since $S^m A = \{a, b, d\} \nsubseteq A$.

Theorem 3.16. Let L and R be an m_l-left hyperideal andan m_r-right hyperideal of a semihyperring S, respectively. Then $L \cap R$ is an m-bi-hyperideal of S, where $m = \max\{m_l, m_r\}$.

Proof. It is easy to show that $L \cap R$ is a subsemihyperring of S. By Theorem 3.14, L is an m_l-bi-hyperideal and R is an m_r-bi-hyperideal of S. By Theorem 3.6, $L \cap R$ is an m-bi-hyperideal of S, where $m = \max\{m_l, m_r\}$.

A semihyperring S is called *regular* (Davvaz & Omidi, 2016; Huang, Yin, & Zhan, 2013) if for each $a \in S$, there exists $x \in S$ such that $a \in axa$.

Lemma 3.17. Let S be a semihyperring. The following conditions are equivalent:

(*i*) S is regular;

(*ii*) $a \in aSa$, for all $a \in S$;

(*iii*) $A \subseteq ASA$, for all $\emptyset \neq A \subseteq S$.

Theorem 3.18. Let S be a regular semihyperring. Then (m_1, m_2)-quasi-hyperideals and m-bi-hyperideals coincide in S, where $m = \max\{m_1, m_2\}$.

Proof. Let $m = \max\{m_1, m_2\}$. It is sufficient to show that every m-bi-hyperideal is an (m_1, m_2)-quasi-hyperideal of S. Let B be an m-bi-hyperideal of S. Then B is a subsemihyperring of S. Let $a \in (S^{m_1}B) \cap (BS^{m_2})$. By Lemma 3.17, we have $a \in aSa \subseteq (BS^{m_2})S(S^{m_1}B) = BS^{m_1+m_2+1}B \subseteq BS^m B \subseteq B$. It follows that $(S^{m_1}B) \cap (BS^{m_2}) \subseteq B$. Hence, B is an (m_1, m_2)-quasi-hyperideal of S.

Theorem 3.19. Let S be a semihyperring and m_1, $m_2 \in \mathbb{N}$ such that $m = \max\{m_1, m_2\}$. Then the following statements are equivalent:

(*i*) S is regular;

(*ii*) $R \cap L = RS^m L$, for any m_1-left hyperideal L and m_2-right hyperideal R of S;

(*iii*) $B = \Sigma BS^m B$, for each m-bi-hyperideal B of S.

Proof. (*i*) ⇒ (*ii*): Let L be an m_1-left hyperideal and R be an m_2-right hyperideal of S. Then $RS^m L \subseteq RS^{m+1} \subseteq RS^{m_2} \subseteq R$ and $RS^m L \subseteq S^{m+1}L \subseteq S^{m_1}L \subseteq L$. This implies that $RS^m L \subseteq R \cap L$. Let $a \in R \cap L$. Since S is regular, there exists $x \in S$ such that $a \in axa$. We obtain that

$$a \in axa \subseteq axaxa \subseteq \cdots \subseteq a\left(\underbrace{xa \cdots ax}_{m\ times}\right)a \subseteq RS^m L.$$

Hence, $R \cap L \subseteq RS^m L$. Therefore, $R \cap L = RS^m L$.

(ii) ⇒ (iii): Let B be an m-bi-hyperideal of S. Obviously, $\Sigma BS^mB \subseteq B$. By assumption, Remark 1 and Lemma 3.12, we have $B \subseteq \langle B\rangle_R \cap \langle B\rangle_L = \langle B\rangle_R S^m \langle B\rangle_L = (\Sigma B + \Sigma B^2 + \cdots + \Sigma B^{m_2} + \Sigma BS^{m_2})S^m(\Sigma B + \Sigma B^2 + \cdots + \Sigma B^{m_1} + \Sigma S^{m_1}B) \subseteq \Sigma BS^mB$ $\subseteq \Sigma B \subseteq B$. Consequently, $B = \Sigma BS^mB$.

(iii) ⇒ (i): Let $a \in S$. By assumption, Remark 1 and Corollary 3.3, we have $a \in \langle a\rangle_B = \Sigma\langle a\rangle_B S^m \langle a\rangle_B = \Sigma(\Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma a^{m+1} + \Sigma aS^ma)S^m(\Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma a^{m+1} + \Sigma aS^ma) \subseteq \Sigma aS^ma \subseteq aSa$. By Lemma 3.17, S is regular.

Theorem 3.20. A semihyperring S is regular if and only if $B \cap L \subseteq BSL$, for every n-left hyperideal L and m-bi hyperideal B of S.

Proof. Assume that S is a regular semihyperring. Let L be an n-left hyperideal and B be an m-bi-hyperideal of S. Let $a \in B \cap L$. Since S is regular, there exists $x \in S$ such that $a \in axa$. Also, $a \in BSL$. Hence, $B \cap L \subseteq BSL$. Conversely, let $a \in S$. By assumption, Remark 1, Corollary 3.3 and Corollary 3.13, we obtain $a \in \langle a\rangle_B \cap \langle a\rangle_L \subseteq \langle a\rangle_B S \langle a\rangle_L = (\Sigma a + \Sigma a^2 + \cdots + \Sigma a^m + \Sigma a^{m+1} + \Sigma aS^ma)S(\Sigma a + \Sigma a^2 + \cdots + \Sigma a^n + \Sigma S^na) \subseteq aSa$. By Lemma 3.17, S is regular.

The proof of the following theorem is similar to the proof of Theorem 3.20.

Theorem 3.21. A semihyperring S is regular if and only if $R \cap B \subseteq RSB$, for any n-right hyperideal R and m-bi-hyperideal B of S.

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