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Original Article

Linear dependence of four types of arithmetic functions

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Abstract

General criteria were derived for the linear dependence of arithmetic functions over the complex field as well as several other criteria for arithmetic functions that were solutions of additive, multiplicative, exponential, and logarithmic equations. A number of examples were worked out in order to compare the results with the existing ones.

Keywords: arithmetic functions, linear dependence

1. Introduction

An arithmetic function is a complex-valued function defined over the set of natural numbers $\mathbb{N}$. Let $\mathbf{A}$ be a set of arithmetic functions equipped with the usual addition and the Dirichlet convolution defined for $f_1, f_2 \in \mathbf{A}$ by

$$(f_1 + f_2)(n) := f_1(n) + f_2(n),$$

$$(f_1 * f_2)(n) := \sum_{d|n} f_1(d) f_2(n/d) \quad (n \in \mathbb{N}).$$

It is well-known by Cashwell and Everett (1959) that $(\mathbf{A}, +, *)$ is a unique factorization domain. The identity with respect to $*$ is the arithmetic function I defined by $I(n) = 1$ for $n = 1$ and $I(n) = 0$ for $n > 1$. For $f \in \mathbf{A}$, its Dirichlet inverse, i.e., the inverse with respect to $*$, denoted by f^{-1}, exists if and only if $f(1) \neq 0$. A function $A \in \mathbf{A}$ is said to be *additive* if $A(m+n) = A(m) + A(n)$ $(m, n \in \mathbb{N})$. A function $M \in \mathbf{A}$ is said to be *multiplicative* if $M(mn) = M(m)M(n)$ for all $m, n \in \mathbb{N}$ with $\gcd(m,n) = 1$. An *exponential* function $E \in \mathbf{A}$ is a function satisfying $E(m+n) = E(m)E(n)$ $(m, n \in \mathbb{N})$. A *logarithmic* function $L \in \mathbf{A}$ is a function satisfying $L(mn) = L(m) + L(n)$ $(m, n \in \mathbb{N})$.

In one of our previous studies, the $\mathbb{C}$-linear dependence of three types of arithmetic functions, namely, additive, exponential and logarithmic were investigated. It was found that

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1) two additive functions are always linearly dependent,
2) exponential functions are always linearly independent, and
3) logarithmic functions are linearly dependent if and only if they are algebraically dependent.

The case of multiplicative functions was previously investigated by Kaczorowski, Molteni, and Perelli (1999, 2006). They found that if the multiplicative functions $I, f_1, \ldots, f_n$ are pairwise non-equivalent (recall that two multiplicative arithmetic functions f and g are *equivalent* if $f(p^m) = g(p^m)$ for all $m \in \mathbb{N}$ and all but finitely many primes p), then $f_1, \ldots, f_n$ are $\mathbb{C}$-linearly independent.

Here, we continue our existing investigation. Complementing the results in Komatsu, Laohakosol, and Reungsinsub (2011, 2012) we further investigated the $\mathbb{C}$-linear dependence of arithmetic functions which are solutions of additive equation, multiplicative equation, exponential equation, and logarithmic equation. To this end, the general criteria for linear dependence were proved. For additive functions, we extended one of our earlier results to cover the linearly dependence of general $n\,(\geq 2)$ functions. An alternative proof that exponential functions are always linearly independent was given. For multiplicative functions, conditions for a finite set of nonzero pairwise distinct multiplicative functions to be linearly independent were established. Conditions for linear independence of multiplicative functions based on an old method of Popken (1962) were proved. Finally, a necessary condition for linear independence of a finite set of nonzero pairwise distinct logarithmic arithmetic functions was derived. Several examples illustrating the so-obtained criteria were worked out in order to compare with the existing criteria.

2. Results

Our first result deals with two general criteria for linear (in)dependence.

Theorem 1. Let $f_1, f_2, \ldots, f_n$ be $n\,(\geq 2)$ nonzero, pairwise distinct arithmetic functions. Assume that there exists an index $J \in \{1, \ldots, n\}$ such that $f_J(1) \neq 0$.

1) If there exist distinct $m_1, \ldots, m_{n-1} \in \mathbb{N} \setminus \{1\}$ such that

$$\begin{vmatrix} F_{1,1} & \cdots & F_{J-1,1} & F_{J+1,1} & \cdots & F_{n,1} \\ \vdots & & & & & \\ F_{1,n-1} & \cdots & F_{J-1,n-1} & F_{J+1,n-1} & \cdots & F_{n,n-1} \end{vmatrix} \neq 0,$$

where $F_{i,j} = \left(f_i * f_J^{-1}\right)(m_j)$ for $i = 1, \ldots, n$ and $j = 1, \ldots, n-1$, then $f_1, f_2, \ldots, f_n$ are $\mathbb{C}$-linearly independent.

2) If

$$\begin{vmatrix} F_{1,1} & \cdots & F_{J-1,1} & F_{J+1,1} & \cdots & F_{n,1} \\ \vdots & & & & & \\ F_{1,n-1} & \cdots & F_{J-1,n-1} & F_{J+1,n-1} & \cdots & F_{n,n-1} \end{vmatrix} = 0,$$

for all $m_1, \ldots, m_{n-1} \in \mathbb{N} \setminus \{1\}$ then $f_1, f_2, \ldots, f_n$ are $\mathbb{C}$-linearly dependent.

Proof. The proof can be found in Ponpetch, Laohakosol, and Mavecha (2017).

2.1 Additive functions

In this subsection, we consider additive functions and start with an auxiliary result.

Proposition 2. If $A \in \mathbf{A}$ is a nonzero additive function, then its Dirichlet inverse is given by $A^{-1} = \left(1/A^2(1)\right)\mu A$.

Theorem 3. If $A_1, \ldots, A_n$ are $n\,(\geq 2)$ nonzero pairwise distinct additive arithmetic functions, then they are $\mathbb{C}$-linearly dependent.

Proof. The proof can be found in Ponpetch, Laohakosol, and Mavecha (2017).

2.2 Exponential functions

In this section, we give another proof of the linear independence of exponential functions.

Theorem 4. The nonzero exponential arithmetic functions $E_1,\ldots,E_n$ $(n\geq 2)$ are pairwise distinct if and only if they are $\mathbb{C}$-linearly independent.

Proof. We have proved in Ponpetch, Laohakosol, and Mavecha (2017) that $E_1,\ldots,E_n$ are linearly independent.

Conversely, assume that $E_1,\ldots,E_n$ are not pairwise distinct. Then there are distinct indices $i, j\in\{1,\ldots,n\}$ such that $E_i(m)=E_j(m)$ $(m\in\mathbb{N})$ yielding a linear relation.

2.3 Multiplicative functions

In this section, we present another condition for linear independence of multiplicative functions, and show that without such condition, there are examples of both dependent and independent functions.

Theorem 5. Let $M_1, M_2,\ldots,M_n$ $(n\geq 2)$ be nonzero, pairwise distinct multiplicative arithmetic functions. If there are distinct primes $p_1, p_2,\ldots, p_{n-1}$ such that

$$\left(M_i * M_j^{-1}\right)(p_1 p_2\cdots p_{n-1})\neq 0 \tag{3}$$

for all $i, j\in\{1,\ldots,n\}$ with $i\neq j$, then $M_1, M_2,\ldots,M_n$ are $\mathbb{C}$-linearly independent.

Proof. We prove this theorem by induction on n. For the case $n=2$, suppose on the contrary that M_1, M_2 are $\mathbb{C}$-linearly independent. Then there are complex constants c_1, c_2 not all zero such that

$$c_1M_1+c_2M_2=0. \tag{4}$$

Operating by M_1^{-1} through (4), we get

$$c_1I(m)+c_2\left(M_2 * M_1^{-1}\right)(m)=0 \quad (m\in\mathbb{N}). \tag{5}$$

Replacing m by p_1, the prime p_1 satisfying (3), in (5), we get $c_2\left(M_2 * M_1^{-1}\right)(p_1)=0.$ Using (3), we deduce $c_2=0.$ Putting $c_2=0$ in (4), we get $c_1=0.$

Assume now that the theorem holds up to $n-1$ functions, we prove its validity for n functions. Suppose on the contrary that $M_1, M_2,\ldots,M_n$ $(n\geq 3)$ are $\mathbb{C}$-linearly dependent. Then there are complex constants $c_1, c_2,\ldots, c_n$ not all zero such that

$$c_1M_1+c_2M_2+\cdots+c_nM_n=0. \tag{6}$$

Operating by M_1^{-1} through (6), we get

$$c_1I(m)+c_2F_2(m)+\cdots+c_nF_n(m)=0 \quad (m\in\mathbb{N}), \tag{7}$$

where $F_j=M_j * M_1^{-1}$ $(j=2,\ldots,n)$. Let $p_1,\ldots,p_{n-1}$ be distinct primes satisfying (3) and let

$$V_{p_{n-1}}=\{m\in\mathbb{N}.\,;\,\gcd(m, p_{n-1})=1\}$$

Replacing m by $tp_{n-1}, t\in V_{p_{n-1}}$, in (7), we get

$$c_2F_2(p_{n-1})F_2(t)+\cdots+c_nF_n(p_{n-1})F_n(t)=0. \tag{8}$$

For $j=2,\ldots,n,$ define

$$G_j(m)=\begin{cases}F_j(m) & \text{if } m\in V_{p_{n-1}}\\ 0 & \text{otherwise.}\end{cases}$$

It is easy to check each G_j is multiplicative and so that the relation (8) becomes

$$d_2G_2(m)+\cdots+d_nG_n(m)=0 \quad (m\in\mathbb{N}), \tag{9}$$

where $d_j=c_jF_j(p_{n-1})$ $(j=2,\ldots,n)$. If G_j $(j=2,\ldots,n)$ are zero functions, then $F_j(m)=0$ for $m\in V_{p_{n-1}}$. Since $p_1\in V_{p_{n-1}}$, using the multiplicativity of $M_j * M_1^{-1}$ and (3), we get $0=F_j(p_1)=\left(M_j * M_1^{-1}\right)(p_1)\neq 0,$ which is a contradiction. Thus G_j $(j=2,\ldots,n)$ are nonzero functions. Since $0\neq\left(M_j * M_k^{-1}\right)(p_1)=M_j(p_1)-M_k(p_1)$ for j not equal to k, we have

$$G_j(p_1)=F_j(p_1)=\left(M_j*M_1^{-1}\right)(p_1)=M_j(p_1)-M_1(p_1)$$
$$\neq M_k(p_1)-M_1(p_1)=\left(M_k*M_1^{-1}\right)(p_1)=F_k(p_1)=G_k(p_1)$$

showing that $G_j\ \ (j=2,\dots,n)$ are pairwise distinct. Since

$$\begin{aligned}\left(G_j*G_k^{-1}\right)(p_1\dots p_{n-2})&=\left(G_j*G_k^{-1}\right)(p_1)\cdots\left(G_j*G_k^{-1}\right)(p_{n-2})\\&=\left(F_j*F_k^{-1}\right)(p_1)L\left(F_j*F_k^{-1}\right)(p_{n-2})\\&=\left(F_j(p_1)-F_k(p_1)\right)\cdots\left(F_j(p_{n-2})-F_k(p_{n-2})\right)\\&=\left(M_j(p_1)-M_k(p_1)\right)\cdots\left(M_j(p_{n-2})-M_k(p_{n-2})\right)\\&=\left(M_j*M_k^{-1}\right)(p_1)\cdots\left(M_j*M_k^{-1}\right)(p_{n-2})\neq 0,\end{aligned}$$

for all $j,k\in\{2,\dots,n\},\ j\neq k,$ the multiplicative functions $G_j\ \ (j=2,\dots,n)$ satisfying (3). Thus, the induction hypothesis yields that $G_2,\dots,G_n$ are $\mathbb{C}$-linearly independent, which in turn implies, from (9), that $0=d_j=c_jF_j(p_{n-1})\ (j=2,\dots,n).$ Since $F_j=M_j*M_1^{-1},$ using the multiplicativity and (3), we have $F_j(p_{n-1})\neq 0,$ which shows that $c_j=0\ (j=2,\dots,n).$ Replacing $c_j=0\ (j=2,\dots,n)$ in (6), we get $c_1=0.$

If the condition (3) does not hold, then $M_1,M_2,\dots,M_n$ can either be $\mathbb{C}$-linearly dependent, or independent as seen from the following examples.

Example 6. Consider the four functions $F_1,F_2,F_3,F_4,$ defined, respectively, by

$F_1(1)=F_1(2)=1,\ F_1(3)=F_1(5)=F_1(6)=F_1(10)=2,\ F_1(15)=F_1(30)=4,$
$F_1(n)=0$ for all positive integers $n\neq 1,2,3,5,6,10,15,30.$

$F_2(1)=F_2(3)=1,\ F_2(2)=F_2(5)=F_2(6)=F_2(15)=2,\ F_2(10)=F_2(30)=4,$
$F_2(n)=0$ for all positive integers $n\neq 1,2,3,5,6,10,15,30.$

$F_3(1)=1,\ F_3(2)=3,\ F_3(3)=F_3(5)=2,\ F_3(6)=F_3(10)=6,\ F_3(15)=4,\ F_3(30)=12,$
$F_3(n)=0$ for all positive integers $n\neq 1,2,3,5,6,10,15,30.$

$F_4(1)=1,\ F_4(2)=F_4(5)=2,\ F_4(3)=3,\ F_4(6)=F_4(15)=6,\ F_4(10)=4,\ F_4(30)=12,$
$F_4(n)=0$ for all positive integers $n\neq 1,2,3,5,6,10,15,30.$

It is easily checked that F_1,F_2,F_3,F_4 are multiplicative functions with inverses, for $r\in\mathbb{N},$

$F_1^{-1}(1)=1,\ F_1^{-1}(2^r)=(-1)^r,\ F_1^{-1}(3^r)=F_1^{-1}(5^r)=(-2)^r,\ F_1^{-1}(6^r)=F_1^{-1}(10^r)=(2)^r,$
$F_1^{-1}(15^r)=(4)^r,\ F_1^{-1}(30^r)=(-4)^r,\ F_1^{-1}(n)=0$ for all positive integers $n\neq 1,2^r,$
$3^r,5^r,6^r,10^r,15^r,30^r.$

$F_2^{-1}(1)=1,\ F_2^{-1}(2^r)=F_2^{-1}(5^r)=(-2)^r,\ F_2^{-1}(3^r)=(-1)^r,\ F_2^{-1}(6^r)=F_2^{-1}(15^r)=(2)^r,$
$F_2^{-1}(10^r)=(4)^r,\ F_2^{-1}(30^r)=(-4)^r,\ F_2^{-1}(n)=0$ for all positive integers $n\neq 1,2^r,$
$3^r,5^r,6^r,10^r,15^r,30^r.$

$F_3^{-1}(1)=1,\ F_3^{-1}(2^r)=(-3)^r,\ F_3^{-1}(3^r)=F_3^{-1}(5^r)=(-2)^r,\ F_3^{-1}(6^r)=F_3^{-1}(10^r)=(6)^r,$
$F_3^{-1}(15^r)=(4)^r,\ F_3^{-1}(30^r)=(-12)^r,\ F_3^{-1}(n)=0$ for all positive integers $n\neq 1,2^r,$
$3^r,5^r,6^r,10^r,15^r,30^r.$

$F_4^{-1}(1)=1,\ F_4^{-1}(2^r)=F_4^{-1}(5^r)=(-2)^r,\ F_4^{-1}(3^r)=(-3)^r,\ F_4^{-1}(6^r)=F_4^{-1}(15^r)=(6)^r,$
$F_4^{-1}(10^r)=(4)^r,\ F_4^{-1}(30^r)=(-12)^r,\ F_4^{-1}(n)=0$ for all positive integers $n\neq 1,2^r,$
$3^r,5^r,6^r,10^r,15^r,30^r.$

In this case, the condition (3) does not hold while we have $\mathbb{C}$-linearly dependent relation $F_1 - F_2 + F_3 - F_4 = 0.$

Example 7. Consider the four functions $G_1, G_2, G_3, G_4,$ defined, respectively, by

$$G_1(1)=1,\ G_1(2)=G_1(5)=2,\ G_1(3)=G_1(10)=4,\ G_1(6)=G_1(15)=8,\ G_1(30)=16,$$
$$G_1(n)=0 \text{ for all positive integers } n\neq 1,2,3,5,6,10,15,30.$$

$$G_2(1)=1,\ G_2(2)=3,\ G_2(3)=9,\ G_2(5)=2,\ G_2(6)=27,\ G_2(10)=6,\ G_2(15)=18,$$
$$G_2(30)=54,\ G_2(n)=0 \text{ for all positive integers } n\neq 1,2,3,5,6,10,15,30.$$

$$G_3(1)=1,\ G_3(2)=5,\ G_3(3)=25,\ G_3(5)=2,\ G_3(6)=125,\ G_3(10)=10,\ G_3(15)=50,$$
$$G_3(30)=250,\ G_3(n)=0 \text{ for all positive integers } n\neq 1,2,3,5,6,10,15,30.$$

$$G_4(1)=1,\ G_4(2)=7,\ G_4(3)=49,\ G_4(5)=2,\ G_4(6)=343,\ G_4(10)=14,\ G_4(15)=98,$$
$$G_4(30)=686,\ G_4(n)=0 \text{ for all positive integers } n\neq 1,2,3,5,6,10,15,30.$$

It is easily checked that G_1, G_2, G_3, G_4 are multiplicative functions with inverses, for $r\in\mathbb{N}$,

$$G_1^{-1}(1)=1,\ G_1^{-1}(2^r)=G_1^{-1}(5^r)=(-2)^r,\ G_1^{-1}(3^r)=(-4)^r,\ G_1^{-1}(6^r)=G_1^{-1}(15^r)=(8)^r,$$
$$G_1^{-1}(10^r)=(4)^r,\ G_1^{-1}(30^r)=(-16)^r,\ G_1^{-1}(n)=0 \text{ for all positive integers } n\neq 1,2^r,$$
$$3^r,5^r,6^r,10^r,15^r,30^r.$$

$$G_2^{-1}(1)=1,\ G_2^{-1}(2^r)=(-3)^r,\ G_2^{-1}(3^r)=(-9)^r,\ G_2^{-1}(5^r)=(-2)^r,\ G_2^{-1}(6^r)=(27)^r,$$
$$G_2^{-1}(10^r)=(6)^r,\ G_2^{-1}(15^r)=(18)^r,\ G_2^{-1}(30^r)=(-54)^r,\ G_2^{-1}(n)=0 \text{ for all positive}$$
$$\text{integers } n\neq 1,2^r,3^r,5^r,6^r,10^r,15^r,30^r.$$

$$G_3^{-1}(1)=1,\ G_3^{-1}(2^r)=(-5)^r,\ G_3^{-1}(3^r)=(-25)^r,\ G_3^{-1}(5^r)=(-2)^r,\ G_3^{-1}(6^r)=(125)^r,$$
$$G_3^{-1}(10^r)=(10)^r,\ G_3^{-1}(15^r)=(50)^r,\ G_3^{-1}(30^r)=(-250)^r,\ G_3^{-1}(n)=0 \text{ for all}$$
$$\text{positive integers } n\neq 1,2^r,3^r,5^r,6^r,10^r,15^r,30^r.$$

$$G_4^{-1}(1)=1,\ G_4^{-1}(2^r)=(-7)^r,\ G_4^{-1}(3^r)=(-49)^r,\ G_4^{-1}(5^r)=(-2)^r,\ G_4^{-1}(6^r)=(343)^r,$$
$$G_4^{-1}(10^r)=(14)^r,\ G_4^{-1}(15^r)=(98)^r,\ G_4^{-1}(30^r)=(-686)^r,\ G_4^{-1}(n)=0 \text{ for all}$$
$$\text{positive integers } n\neq 1,2^r,3^r,5^r,6^r,10^r,15^r,30^r.$$

Here, the condition (3) does not hold. We show that G_1, G_2, G_3, G_4 are $\mathbb{C}$-linearly independent. Suppose on the contrary that G_1, G_2, G_3, G_4 are $\mathbb{C}$-linearly dependent. Then there are complex constants $c_1,\ldots,c_4$ not all zero such that

$$c_1G_1(n)+c_2G_2(n)+c_3G_3(n)+c_4G_4(n)=0. \tag{10}$$

Putting $n=1,2,3,6,$ respectively in (10), we get

$$c_1G_1(1)+c_2G_2(1)+c_3G_3(1)+c_4G_4(1)=0$$
$$c_1G_1(2)+c_2G_2(2)+c_3G_3(2)+c_4G_4(2)=0$$
$$c_1G_1(3)+c_2G_2(3)+c_3G_3(3)+c_4G_4(3)=0$$
$$c_1G_1(6)+c_2G_2(6)+c_3G_3(6)+c_4G_4(6)=0.$$

Using the defining values of $G_1, G_2, G_3, G_4,$ the coefficient matrix of the above system is non-singular, and this implies that $c_1=c_2=c_3=c_4=0,$ which is a contradiction.

We proceed now to use a method of Popken (1962) to derive the criterion for linear independence of multiplicative functions. Let S ($\subseteq \mathbb{N}$) be a commutative semi-group in which a unique factorization condition holds. We assume that S has an identity-element 1 and no other unit than 1. By a reduced semi-group S_0, we mean a set of $m \in S$ such that $\gcd(m, x_0) = 1$ for some fixed $x_0 \in S$.

Theorem 8. Let $M_1, M_2, \ldots, M_n$ $(n \geq 2)$ be nonzero, pairwise distinct multiplicative arithmetic functions. Suppose that there exists a semi-group S ($\subseteq \mathbb{N}$) with the properties described above, and there are $c_1(\neq 0), c_2, \ldots, c_n \in \mathbb{C}$ such that

$$c_1 M_1(t) + \cdots + c_n M_n(t) = 0 \quad (t \in S). \tag{11}$$

Then there is at least one suffix $h \in \{2, \ldots, n\}$ such that

$$M_1(m) = M_h(m) \quad (m \in S_0) \tag{12}$$

for some reduced semi-group $S_0 \subseteq S$. Moreover, $M_1 * M_h^{-1}$ vanishes on some reduced semi-group $S_0 \setminus \{1\}$ contained in S.

Proof. We prove by induction on n. For the case $n = 2$, there are $c_1(\neq 0), c_2 \in \mathbb{C}$ such that

$$c_1 M_1(t) + c_2 M_2(t) = 0 \quad (t \in S). \tag{13}$$

Taking $t = 1$ in (13), we get $c_1 = -c_2 \neq 0$. Thus, (13) yields $M_1(t) = M_2(t)$ for all $t \in S$, i.e., (12) holds with $h = 2$ and $S_0 = S$. Assume that the theorem holds up to $n-1$ functions, we next prove its validity for n functions. If $M_1(t) = M_n(t)$ for all $t \in S$, then there is nothing more to prove. Otherwise, $M_1 \neq M_n$ on S, and so there exists $x_1 \in S$ such that $M_1(x_1) \neq M_n(x_1)$. By assumption, we have

$$c_1 M_1(t) + \cdots + c_n M_n(t) = 0 \quad (t \in S). \tag{14}$$

Let $S_1 = \{m \in S; \gcd(m, x_1) = 1\} \subseteq S$. Taking $t = mx_1$, where $m \in S_1$, in (14) and by multiplicitivity, we get

$$c_1 M_1(x_1) M_1(m) + \cdots + c_n M_n(x_1) M_n(m) = 0 \quad (m \in S_1). \tag{15}$$

Taking $t = m$, where $m \in S_1$, in (14) and multiplying by $M_n(x_1)$, we have

$$c_1 M_n(x_1) M_1(m) + \cdots + c_n M_n(x_1) M_n(m) = 0 \quad (m \in S_1). \tag{16}$$

Subtracting (15) and (16), we have

$$c_1 \big(M_1(x_1) - M_n(x_1)\big) M_1(m) + \cdots + c_{n-1} \big(M_{n-1}(x_1) - M_n(x_1)\big) M_{n-1}(m) = 0 \quad (m \in S_1). \tag{17}$$

Since $c_1 \big(M_1(x_1) - M_n(x_1)\big) \neq 0$, the relation (17) is similar to the relation (11) on S_1. By the induction hypothesis, there exists a reduced semi-group $S_2 \subseteq S_1 \subseteq S$ such that $M_1 = M_h$ on S_2 for some $h \in \{2, \ldots, n-1\}$. Define $\phi = M_1 * M_h^{-1}$. Clearly

$$\phi(m) = \left(M_1 * M_h^{-1}\right)(m) = \sum_{d|m} M_1(d) M_h^{-1}\left(\frac{m}{d}\right) \tag{18}$$

for all $m \in S_0 \setminus \{1\}$. Since $\gcd(m, x_0) = 1$, we have $\gcd(d, x_0) = 1$ and $\gcd(m/d, x_0) = 1$, i.e., $d, m/d \in S_0$. The relation (18) becomes

$$\phi(m) = \left(M_1 * M_h^{-1}\right)(m) = \sum_{d|m} M_1(d) M_h^{-1}\left(\frac{m}{d}\right) = \sum_{d|m} M_h(d) M_h^{-1}(\frac{m}{d}) = I(m) = 0$$

which holds for all $m \in S_0 \setminus \{1\}$.

Corollary 9. Let $M_1, M_2, \ldots, M_n$ $(n \geq 2)$ be nonzero, pairwise distinct multiplicative arithmetic functions. For all indices $j = 2, \ldots, n,$ if there exists an $\alpha \in$ such that the relation

$$\left(M_i * M_j^{-1}\right)(p^\alpha) \neq 0 \tag{19}$$

holds for all primes p, then $M_1, M_2, \ldots, M_n$ are $\mathbb{C}$-linearly independent.

Proof. Suppose by the contrary $M_1, M_2, \ldots, M_n$ are $\mathbb{C}$-linearly dependent. Then there are complex constants $c_1, c_2 \ldots, c_n$ not all zero such that

$$c_1 M_1(t) + c_2 M_2(t) + \cdots + c_n M_n(t) = 0 \ \ (t \in \mathbb{N}). \tag{20}$$

Since $S \subseteq \mathbb{N}$, the relation (20) restricts to

$$c_1 M_1(m) + c_2 M_2(m) + \cdots + c_n M_n(m) = 0 \ \ (m \in S). \tag{21}$$

Without loss of generality, assume that $c_1 \neq 0$. By Theorem 8, for some $j = 2, \ldots, n,$ the relation $M_1(m) = M_j(m)$ holds for all m in some reduced semi-group $S_0 = \{m \in S; \gcd(m, x_0) = 1\} \subseteq S,\ x_0 \in S,$ and $M_1 * M_j^{-1}$ vanishes on $S_0 \setminus \{1\}$. Let $x_0 = p_1^{\beta_1} \cdots p_r^{\beta_r} \in S \subseteq \mathbb{N}$, $p_1, \ldots, p_r$ being primes, and $\beta_1, \ldots, \beta_r \in \mathbb{N}$. Choose another prime $q \notin \{p_1, \ldots, p_r\}$. Then for any $\alpha \in \mathbb{N}$, we have $\gcd(q^\alpha, x_0) = 1$, i.e., $q^\alpha \in S_0$. Thus, $\left(M_1 * M_j^{-1}\right)(q^\alpha) = 0,$ which is a contradiction.

We next exhibit by examples that Theorem 5, the result in Kaczorowski, Molteni, and Perelli (2006) and Corollary 9 are somewhat independent of one another by analyzing the case of two multiplicative arithmetic functions f and g. In this case, the corresponding three linearly independent conditions are:

- **A** (Theorem 5) f and g are nonzero pairwise distinct with $\left(f * g^{-1}\right)(p) \neq 0$ for some prime p.
- **B** (Kaczorowski, Molteni, & Perelli 2006) I, f, g are pairwise non-equivalent, i.e., there are infinitely many primes p such that $I(p^m) \neq f(p^m),\ I(p^m) \neq g(p^m)$ and $f(p^m) \neq g(p^m)$ for some $m \in \mathbb{N}$.
- **C$_\alpha$** (Corollary 9) f and g are nonzero pairwise distinct with $\left(f * g^{-1}\right)(p^\alpha) \neq 0$ for all primes p.

Example 10. Consider two multiplicative functions f_1 and g_1 defined by $f_1(n) = n, \quad g_1(n) = n^2 \quad (n \in \mathbb{N})$.

- **A** is true because $\left(f_1 * g_1^{-1}\right)(p) = p - p^2 \neq 0$ for all primes p.

- **B** is true because $I(p^m)=0\neq p^m=f_1(p^m),\ I(p^m)=0\neq p^{2m}=g_1(p^m),\ f_1(p^m)=p^m\neq p^{2m}=g_1(p^m)$ for all primes p and for all $m\in\mathbb{N}$.
- $\mathbf{C}_1$ is true because $\left(f_1 * g_1^{-1}\right)(p)=p-p^2\neq 0$ for all primes p.

The functions f_1 and g_1 are $\mathbb{C}$-linearly independent by any one of the three criteria **A**, **B** , or $\mathbf{C}_1$.

Example 11. Consider two multiplicative functions f_2 and g_2 defined by $f_2(1)=1,\ f_2(2)=3,\ f_2(n)=0$ for $n\neq 1,2$ and $g_2(1)=1,\ g_2(2)=5,\ g_2(n)=0$ for $n\neq 1,2.$

- **A** is true because evaluating at the prime 2, we get $\left(f_2 * g_2^{-1}\right)(2)=3-5=-2\neq 0.$
- **B** does not hold because $f_2(p^m)=g_2(p^m)=0$ for all primes $p\neq 2$ and for all $m\in\mathbb{N}$.
- $\mathbf{C}_\alpha$ does not hold for all $\alpha\in\mathbb{N}$ because $\left(f_2 * g_2^{-1}\right)(p^\alpha)=0$ for all primes $p\neq 2.$

Thus, f_2 and g_2 are $\mathbb{C}$-linearly independent by the criterion **A**, but not by the other two criteria.

Example 12. Consider two multiplicative functions f_3 and g_3 defined by

$$f_3(1)=1,\ f_3(p)=0,\ f_3(2^m)=0\ (m\geq 2),\ f_3(p^m)=p^m\ \ (p\neq 2, m\geq 2),$$
$$g_3(1)=1,\ g_3(p)=0,\ g_3(2^m)=0\ (m\geq 2),\ g_3(p^m)=p^{2m}\ \ (p\neq 2, m\geq 2).$$

- **A** does not hold because $\left(f_3 * g_3^{-1}\right)(p)=0$ for all primes p.
- **B** is true because $I(p^m)=0\neq p^m=f_3(p^m),\ I(p^m)=0\neq p^{2m}=g_3(p^m),\ f_3(p^m)=p^m\neq p^{2m}=g_3(p^m)$ for all primes $p\neq 2$ and for all $m\in\mathbb{N}\setminus\{1\}$.
- $\mathbf{C}_\alpha$ does not hold for all $\alpha\in\mathbb{N}$ because $\left(f_3 * g_3^{-1}\right)(2^\alpha)=0.$

Thus, f_3 and g_3 are $\mathbb{C}$-linearly independent by the criterion **B**, but not by the other two criteria.

Example 13. Consider two multiplicative functions f_4 and g_4 defined by

$$f_4(1)=1,\ f_4(p)=0,\ f_4(2^m)=2\ (m\geq 2),\ f_4(p^m)=p^{2m}\ \ (p\neq 2, m\geq 2),$$
$$g_4(1)=1,\ g_4(p)=0,\ g_4(2^m)=4\ (m\geq 2),\ g_4(p^m)=0\ (p\neq 2, m\geq 2).$$

- **A** does not hold because $\left(f_4 * g_4^{-1}\right)(p)=0-0=0$ for all primes p.
- **B** does not hold because $I(p^m)=g_4(p^m)=0$ for all primes $p\neq 2$ and for all $m\in\mathbb{N}$.
- $\mathbf{C}_2$ is true because $\left(f_4 * g_4^{-1}\right)(p^2)=-g_4(p^2)+g_4^2(p)-f_4(p)g_4(p)+f_4(p^2)=0+0-0+p^4\neq 0$ for all primes $p\neq 2$ and $\left(f_4 * g_4^{-1}\right)(2^2)=-4+0-0+2=-2\neq 0.$

Thus, f_4 and g_4 are $\mathbb{C}$-linearly independent by the criterion $\mathbf{C}_2$, but not by the other two criteria.

Example 14. Consider two multiplicative functions f_5 and g_5 defined by

$$f_5(1)=1,\ f_5(2)=2,\ f_5(p)=0\ (p\neq 2),\ f_5(2^m)=0\ (m\geq 2),\ f_5(3^m)=0\ (m\geq 2),$$
$$f_5(p^m)=p^m\ \ (p\neq 2,3; m\geq 2),$$

$$g_5(1)=1,\ g_5(2)=4,\ g_5(p)=0\ (p\neq 2),\ g_5(2^m)=0\ (m\geq 2),\ g_5(3^m)=0\ (m\geq 2),$$
$$g_5(p^m)=p^{2m}\ \ (p\neq 2,3; m\geq 2).$$

- **A** is true because there is a prime 2 such that $\left(f_5 * g_5^{-1}\right)(2) = 2 - 4 = -2 \neq 0.$
- **B** is true because $I(p^m) = 0 \neq p^m = f_5(p^m),\ I(p^m) = 0 \neq p^{2m} = g_5(p^m),\ f_5(p^m) = p^m \neq p^{2m} = g_5(p^m)$ for all primes $p \neq 2,3$ and for all $m \in \mathbb{N} \setminus \{1\}.$
- $\mathbf{C}_\alpha$ does not hold for all $\alpha \in \mathbb{N}$ because $\left(f_5 * g_5^{-1}\right)(3^\alpha) = 0.$

Thus, f_5 and g_5 are $\mathbb{C}$-linearly independent by the criteria **A** and **B**, but not by $\mathbf{C}_\alpha$.

Example 15. Consider two multiplicative functions f_6 and g_6 defined by

$$f_6(1) = 1,\ f_6(p) = 0\ (p \neq 2),\ f_6(2^m) = 2\ (m \geq 1),\ f_6(p^m) = p^{2m}\ (p \neq 2, m \geq 2),$$
$$g_6(1) = 1,\ g_6(p) = 0\ (p \neq 2),\ g_6(2^m) = 4\ (m \geq 1),\ g_6(p^m) = 0\ (p \neq 2, m \geq 2).$$

- **A** is true because there is a prime 2 such that $\left(f_6 * g_6^{-1}\right)(2) = 2 - 4 = -2 \neq 0.$
- **B** does not hold because $I(p^m) = g_6(p^m) = 0$ for all primes $p \neq 2$ and for all $m \in \mathbb{N}.$
- $\mathbf{C}_2$ is true because $\left(f_6 * g_6^{-1}\right)(p^2) = -g_6(p^2) + g_6^2(p) - f_6(p)g_6(p) + f_6(p^2) = 0 + 0 - 0 + p^4 \neq 0$ for all primes $p \neq 2$ and $\left(f_6 * g_6^{-1}\right)(2^2) = -4 + 4^2 - 2(4) + 2 = 6 \neq 0.$

Thus, f_6 and g_6 are $\mathbb{C}$-linearly independent by the criteria **A** and $\mathbf{C}_2$, but not by **B**.

Example 16. Consider two multiplicative functions f_7 and g_7 defined by

$$f_7(1) = 1,\ f_7(2) = 2,\ f_7(p) = 0\ (p \neq 2),\ f_7(2^m) = 2\ (m \geq 2),\ f_7(p^m) = p^m\ (p \neq 2, m \geq 2),$$
$$g_7(1) = 1,\ g_7(2) = 2,\ g_7(p) = 0\ (p \neq 2),\ g_7(2^m) = 0\ (m \geq 2),\ g_7(p^m) = p^{2m}\ (p \neq 2, m \geq 2).$$

- **A** does not hold because $\left(f_7 * g_7^{-1}\right)(p) = 0$ for all primes p.
- **B** is true because $I(p^m) = 0 \neq p^m = f_7(p^m),\ I(p^m) = 0 \neq p^{2m} = g_7(p^m),\ f_7(p^m) = p^m \neq p^{2m} = g_7(p^m)$ for all primes $p \neq 2$ and for all $m \in \mathbb{N} \setminus \{1\}.$
- $\mathbf{C}_3$ is true because
$$\begin{aligned}\left(f_7 * g_7^{-1}\right)(p^3) &= f_7(p^3) - f_7(p^2)g_7(p) - f_7(p)g_7(p^2) + f_7(p)g_7^2(p) - g_7(p^3)\\ &\quad + 2g_7(p^2)g_7(p) - g_7^3(p)\\ &= p^3 - p^6 \neq 0\end{aligned}$$
for all primes $p \neq 2$ and $\left(f_7 * g_7^{-1}\right)(2^3) = 2 - 4 - 0 + 8 - 0 + 0 - 8 = -2 \neq 0.$

Thus, f_7 and g_7 are $\mathbb{C}$-linearly independent by the criteria **B** and $\mathbf{C}_3$, but not by **A**.

2.4 Logarithmic functions

As for logarithmic functions, we show that subject to an extra condition, they are $\mathbb{C}$-linearly independent, while without such a condition they are $\mathbb{C}$-linearly dependent.

Theorem 17. Let $L_1, \ldots, L_n$ be $n\,(\geq 2)$ nonzero pairwise distinct logarithmic arithmetic functions.

1) If there exist distinct primes $p_1,\ldots,p_n$ such that

$$\begin{vmatrix} L_1(p_1) & L_2(p_1) & \cdots & L_n(p_1) \\ \vdots & & & \\ L_1(p_n) & L_2(p_n) & \cdots & L_n(p_n) \end{vmatrix} \neq 0, \tag{22}$$

then $L_1,\ldots,L_n$ are $\mathbb{C}$-linearly independent.

2) If the condition

$$\begin{vmatrix} L_1(p_1) & L_2(p_1) & \cdots & L_n(p_1) \\ \vdots & & & \\ L_1(p_n) & L_2(p_n) & \cdots & L_n(p_n) \end{vmatrix} = 0 \tag{23}$$

holds for all distinct primes $p_1,\ldots,p_n,$ then $L_1,\ldots,L_n$ are $\mathbb{C}$-linearly dependent.

Proof. **1)** The proof can be found in Ponpetch, Laohakosol, and Mavecha (2017).

2) We first treat the case $n=2.$ From (23) we have

$$c_1L_1(p_1)+c_2L_2(p_1)=0,\ c_1L_1(p_2)+c_2L_2(p_2)=0 \tag{24}$$

for some complex constants c_1, c_2 not all zero. Without loss of generality, assume $c_1 \neq 0.$ Then the system (24) becomes

$$L_1(p_1)=d_2L_2(p_1),\ L_1(p_2)=d_2L_2(p_2) \tag{25}$$

where $d_2=-c_2/c_1 \in \mathbb{C}$. Taking another prime p_j in place of p_1 in (23), we get another system

$$c_1'L_1(p_j)+c_2'L_2(p_j)=0,\ c_1'L_1(p_2)+c_2'L_2(p_2)=0. \tag{26}$$

If $c_1'=0,$ then $L_2=0,$ which is a contradiction. Thus $c_1'\neq 0,$ and rewrite (26) as

$$L_1(p_j)=d_2'L_2(p_j), L_1(p_2)=d_2'L_2(p_2). \tag{27}$$

Subtracting corresponding equations (except the first) in the two systems (25) and (27), we get $d_2=d_2'.$ Hence $L_1(p)=d_2L_2(p)$ for all prime $p,$ implying that $L_1=d_2L_2.$ Now, we proceed to the general case. Assume the result holds up to $n-1$ functions, we use induction to show that it holds for n functions. The vanishing of the determinant (23) infers that their columns are dependent, i.e., there are complex constants $c_1,\ldots,c_n$ not all zero such that

$$c_1L_1(p_1)+c_2L_2(p_1)+\cdots+c_nL_n(p_1)=0,\ldots,c_1L_1(p_n)+c_2L_2(p_n)+\cdots+c_nL_n(p_n)=0. \tag{28}$$

Since not all of $c_1,\ldots,c_n$ are zero, without loss of generality, assume $c_1\neq 0.$ The system (28) becomes

$$L_1(p_1)=d_2L_2(p_1)+\cdots+d_nL_n(p_1),\ldots,L_1(p_n)=d_2L_2(p_n)+\cdots+d_nL_n(p_n), \tag{29}$$

where $d_i=-c_i/c_1\in\mathbb{C}\quad(i=2,\ldots,n).$ Taking another prime p_j in place of p_1 in (28), we get another system

$$c_1'L_1(p_j)+c_2'L_2(p_j)+\cdots+c_n'L_n(p_j)=0,\ldots,c_1'L_1(p_n)+c_2'L_2(p_n)+\cdots+c_n'L_n(p_n)=0. \tag{30}$$

If $c_1'=0,$ from the system (30) leaving the first row we get another homogeneous system of order $n-1.$ If the determinant of the system is $0,$ we are done by the induction hypothesis; otherwise it implies that $c_2'=\cdots=c_n'=0,$ which is a contradiction. If $c_1'\neq 0,$ rewrite (30) as

$$L_1(p_j)=d_2'L_2(p_j)+\cdots+d_n'L_n(p_j),\ldots,L_1(p_n)=d_2'L_2(p_n)+\cdots+d_n'L_n(p_n), \tag{31}$$

where $d_i'=-c_i'/c_1'\in\mathbb{C}\quad(i=2,\ldots,n).$ Subtracting corresponding equations (except the first) in the two systems (29) and (31) leads to the homogeneous system

$$\begin{aligned}&(d_2-d_2')L_2(p_2)+(d_3-d_3')L_2(p_2)+\cdots+(d_n-d_n')L_n(p_2)=0,\ldots,\\&(d_2-d_2')L_2(p_n)+(d_3-d_3')L_2(p_n)+\cdots+(d_n-d_n')L_n(p_n)=0.\end{aligned}$$

If the coefficient matrix of this last system is singular, we return to the lower case. If it is non-singular, then $d_i=d_i'\,(i=2,\ldots,n),$ implying that $L_1(p)=d_2L_2(p)+\cdots+d_nL_n(p)$ for all prime $p,$ and so $L_1=d_2L_2+\cdots+d_nL_n.$

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