รายงานการวิจัย

ปัญหาค่าขอบสำหรับสมการเชิงผลต่างอันดับเศษส่วนและ
เงื่อนไขขอบผลรวมไม่เฉพาะที่

Boundary Value Problem for Fractional Difference Equation
with Nonlocal - Sum Boundary Condition

โดย

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บทคัดย่อ

ส่วนที่ 1

โครงการวิจัย	ปัญหาค่าขอบสำหรับสมการเชิงผลต่างอันดับเศษส่วนและเงื่อนไขขอบผลรวมไม่เฉพาะที่ Boundary Value Problem for Fractional Difference Equation with Nonlocal - Sum Boundary Condition
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ส่วนที่ 2

งานวิจัยนี้ ศึกษาการหาเงื่อนไขเพียงพอสำหรับการมีอยู่จริงของคำตอบของปัญหาค่าขอบผลรวมไม่เฉพาะที่ สำหรับสมการเชิงผลต่างอันดับเศษส่วน ในรูป

$$\Delta_C^{\alpha} u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta}u)(t+\alpha-2)), \ t \in \mathbb{N}_{0,T},$$
$$u(\alpha-2) = y(u),$$
$$u(T+\alpha) = \Delta^{-\gamma} g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3),$$

เมื่อ $1<\alpha\le 2$, $0<\beta\le 1$, $2<\gamma\le 3$, Δ_C^{α} คือผลต่างอันดับเศษส่วนของคาปูโตอันดับ α, ให้ฟังก์ชันต่อเนื่อง $g \in C(U,U)$ และ $g \in C\left(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}^{+} \cap U\right)$ กำหนดฟังก์ชัน $f : \mathbb{N}_{\alpha-2,T+\alpha} \times U \to U$ สำหรับ $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0,\infty)$ เมื่อ

$$(\Psi^{\beta}u)(t) := [\Delta^{\beta}\varphi u](t+\beta) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-\beta-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} \varphi(t, s+\beta)\, u(s+\beta)$$

ในการศึกษาครั้งนี้ จะแสดงการมีอยู่จริงของคำตอบโดยใช้ทฤษฎีจุดตรึง Banach และทฤษฎีจุดตรึงของ Schaefer พร้อมทั้งยกตัวอย่างประกอบ

ABSTRACT

In this research, we consider the sufficient conditions for the existence of solutions of a discrete fractional boundary value problem of fractional difference equations In this paper we consider a Caputo fractional sum-difference equations with fractional sum boundary value conditions of the form

$$
\begin{aligned}
&\Delta_C^{\alpha} u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta} u)(t+\alpha-2)), \quad t \in \mathbb{N}_{0,T}, \\
&u(\alpha-2) = y(u), \\
&u(T+\alpha) = \Delta^{-\gamma} g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3),
\end{aligned}
$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $2 < \gamma \leq 3$, Δ_C^{α} is the Caputo fractional difference operater order α, $y \in C(U, U)$ and $g \in C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}^{+} \cap U)$ are given functions, $f : \mathbb{N}_{\alpha-\mathbb{2},\mathbb{T}+\alpha} \times \mathbb{U} \times \mathbb{U} \to \mathbb{U}$, for $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0, \infty)$,

$$
(\Psi^{\beta} u)(t) := [\Delta^{-\beta} \varphi\, u](t+\beta) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-\beta-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} \varphi(t, s+\beta)\, u(s+\beta).
$$

Our goal is to establish some criteria of existence for the boundary problems with nonlocal-sum boundary condition, using the Banach fixed point theorem and the Schaefer's fixed point theorem. Finally, we present some examples to show the importance of these results.

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Soawaluk Chasreechai
Thanin Sitthiwirattham

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Chapter 1

Introduction

Mathematicians have employed this fractional calculus in recent years to model and solve various applied problems. In particular, fractional calculus is a powerful tool for the processes which appears in nature, e.g. biology, ecology and other areas, can be found in [16] and [17] and the references therein. The continuous fractional calculus has received increasing attention within the last ten years or so, and the theory of fractional differential equations has been a new important mathematical branch due to its extensive applications in various fields of science, such as physics, mechanics, chemistry, engineering, etc. Although the discrete fractional calculus has seen slower progress, within the recent several years, a lot of papers have appeared, which has helped to build up some of the basic theory of this area, see [1]-[15] and references cited therein.

At present, there is a development of boundary value problems for fractional difference equations which shows an operation of the investigative function. The study may also have another function which is related to our interested one. These creations are incorporating with nonlocal conditions which are both extensive and more complex, for instance

Agarwal *et* al. [1] investigated the existence of solutions for two fractional boundary value problems

$$\begin{cases} \Delta^{\mu}_{\mu-2}x(t) = g(t+\mu-1, x(t+\mu-1), \Delta x(t+\mu-1)), \quad t \in \mathbb{N}_{\nvdash,+\nvDash}, \\ x(\mu-2) = 0, \quad x(\mu+b+1) = \sum_{k=\mu-1}^{\alpha} x(k), \end{cases} \tag{1.1}$$

where $1 < \mu \leq 2$ and $g \in C(\mathbb{N}_{\mu-1,\mu+b+1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function.

$$\begin{cases} \Delta^{\mu}_{\mu-3}x(t) = g(t+\mu-2, x(t+\mu-2)), \quad t \in \mathbb{N}_{0,b+3}, \\ x(\mu-3) = x(\mu+b+1) = 0, \quad x(\alpha) = \sum_{k=\gamma}^{\beta} x(k), \end{cases} \tag{1.2}$$

where $2 < \mu \leq 3$, $\alpha, \beta, \gamma \in \mathbb{N}_{\mu-2,\mu+b}$, and $g \in C(\mathbb{N}_{\mu-2,\mu+b+1} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ is a given function.

Kang *et* al. [3] obtained sufficient conditions for the existence of positive solutions for a nonlocal boundary value problem

$$\begin{cases} -\Delta^{\mu} y(t) = \lambda h(t+\mu-1)\, f(y(t+\mu-1)), \quad t \in \mathbb{N}_{0,b}, \\ y(\mu-2) = \Psi(y), \qquad y(\mu+b) = \Phi(y), \end{cases} \tag{1.3}$$

where $1 < \mu \leq 2$, $f \in C([0,\infty),[0,\infty))$, $h \in C(\mathbb{N}_{\mu-1,\mu+b-1},[0,\infty))$ are given functions and $\Psi, \Phi : \mathbb{R}^{b+3} \to \mathbb{R}$ are given functionals.

Sitthiwirattham [15] examined a Caputo fractional sum boundary value problem with a p-Laplacian of the form

$$\begin{cases} \Delta_C^{\alpha}[\phi_p(\Delta_C^{\beta} x)](t) = f(t+\alpha+\beta-1, x(t+\alpha+\beta-1)), \quad t \in \mathbb{N}_{0,T}, \\ \Delta_C^{\beta} x(\alpha-1) = 0, \\ x(\alpha+\beta+T) = \rho \Delta^{-\gamma} x(\eta+\gamma), \end{cases} \tag{1.4}$$

where $0 < \alpha, \beta \leq 1$, $1 < \alpha+\beta \leq 2$, $0 < \gamma \leq 1$, $\eta \in \mathbb{N}_{\alpha+\beta-1,\alpha+\beta+T-1}$, ρ is a constant, $f : \mathbb{N}_{\alpha+\beta-2,\alpha+\beta+T} \times \mathbb{R} \to \mathbb{R}$ is a continuous function and ϕ_p is the p-Laplacian operator.

In this research we consider the nonlinear discrete fractional boundary value problem of the form In this paper we consider a Caputo fractional sum-difference equations with fractional sum boundary value conditions of the form

$$\begin{aligned} &\Delta_C^{\alpha} u(t) = f(t+\alpha-1, u(t+\alpha-1), (\Psi^{\beta} u)(t+\alpha-2)), \quad t \in \mathbb{N}_{0,T}, \\ &u(\alpha-2) = y(u), \\ &u(T+\alpha) = \Delta^{-\gamma} g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3), \end{aligned} \tag{1.5}$$

where $1 < \alpha \leq 2$, $0 < \beta \leq 1$, $2 < \gamma \leq 3$, Δ_C^{α} is the Caputo fractional difference operator order α. For $u \in \mathbb{R}$, $y \in C(U,U)$ and $g \in C(\mathbb{N}_{\alpha-2,T+\alpha}, \mathbb{R}^{+} \cap U)$ are given functions, $f : \mathbb{N}_{\alpha-2,T+\alpha} \times U \times U \to U$, and for $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0,\infty)$,

$$(\Psi^{\beta} u)(t) := [\Delta^{-\beta} \varphi\, u](t+\beta) = \frac{1}{\Gamma(\beta)} \sum_{s=\alpha-\beta-2}^{t-\beta} (t-\sigma(s))^{\underline{\beta-1}} \varphi(t, s+\beta)\, u(s+\beta).$$

In this research project, we will study the existence and uniqueness of solutions of a class of boundary value problems for Caputo fractional difference equations with nonlocal-sum boundary value condition.

The research project is organized as follows: In Chapter 2, for the convenience of the reader, we cite some definitions and fundamental results on fixed point theorems, difference equations, fractional difference equations. Some auxiliary lemmas, needed in the proofs of our main results are presented in Chapter

3 section 1. Chapter 3 section 2 contains the existence and uniqueness results for the problem (1.5) which are shown by applying Banach's contraction principle and Schaefer's fixed point theorem. Finally, some examples illustrating the applicability of our results are presented in Chapter 3 section 3.

Chapter 2

Basic Concepts and Preliminaries

The aim of this chapter is to give some definitions and properties of the Metric spaces and Banach spaces, difference equations, fractional difference equations.

2.1 Metric Spaces and Banach Spaces

Definition 2.1.1 A metric space is a pair (X, d), where X is a set and d is a metric on X(or distance function on X), that is, a real valued function defined on $X \times X$ such that for all $x, y, z \in X$ we have:
(i) $d(x, y) \geqslant 0$
(ii) $d(x, y) = 0$ if and only if $x = y$
(iii) $d(x, y) = d(y, x)$ (symmetry)
(iv) $d(x, y) \leqslant d(x, z) + d(z, y)$ (triangle inequality).

Definition 2.1.2 A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be convergent if there is an $x \in X$ such that

$$\lim_{n\to\infty} d(x_n, x) = 0 \tag{2.1}$$

x is called the limit of $\{x_n\}$ and we write

$$\lim_{n\to\infty} x_n = x \tag{2.2}$$

$$\text{or, simple,} \quad x_n \to x \tag{2.3}$$

we say that $\{x_n\}$ converges to x. If $\{x_n\}$ is not convergent, it is said to be divergent.

Definition 2.1.3 A sequence $\{x_n\}$ in a metric space $X = (X, d)$ is said to be Cauchy if for every $\varepsilon > 0$ there is an $N(\varepsilon) \in N$ such that $d(x_m, x_n) < \varepsilon$ for every $m, n \geqslant N(\varepsilon)$.

Definition 2.1.4 A metric space (X, d) is said to be complete if every Cauchy sequence in X converges.

Definition 2.1.5 Given a point $x_0 \in X$ and a real number $r > 0$, we define two types of sets:

(a) **Open Ball** $B_r(x_0) = \{x \in X | d(x, x_0) < r\}$

(b) **Closed Ball** $\bar{B}_r(x_0) = \{x \in X | d(x, x_0) < r\}$.

Definition 2.1.6 A subset M of a metric space X is said to be open if it contains a ball about each of its points. A subset K of X is said to be closed if its complement (in X) is open. The closure of M denoted by $\bar{M}$ is the smallest closed set containing M.

Definition 2.1.7 Let X be a linear space (or vector space). A norm on X is a real-valued function $\|\cdot\|$ on X such that the following conditions are satisfied by all members x and y of X and each scalar α:
(i) $\|x\| \geqslant 0$ and $\|x\| = 0$ if and only if $x = 0$
(ii) $\|\alpha x\| = |\alpha| \|x\|$
(iii) $\|x + y\| \leqslant \|x\| + \|y\|$ (triangle inequality).
The ordered pair $(X, \|\cdot\|)$ is called a normed space or normed vector space or normed linear space.

Definition 2.1.8 Let X be normed space. The metric induced by the norm of X is the metric d on X defined by the formula $d(x, y) = \|x - y\|$ for all $x, y \in X$. The norm topology of X is the topology obtained from this metric.

Definition 2.1.9 A Banach norm or complete norm is a norm that induces a complete metric. A normed space is a Banach space or B-space or complete normed space if its norm is a Banach norm.

Definition 2.1.10 A subset A of a vector space X is said to be convex if $x, y \in A$ implies

$$\{z \in X | z = \alpha x + (1 - \alpha) y, \quad 0 \leq \alpha \leq 1\} \subset A.$$

Definition 2.1.11 A metric space X is said to be compact if every sequence in X has a convergent subsequence. A subset M of X is said to be compact if M is compact considered as a subspace of X.

Definition 2.1.12 Let X and Y be normed spaces. An operator $T : X \to Y$ is called compact linear operator(or completely continuous linear operator) if T is linear and if for every bonded subset M of X, the image $T(M)$ is relatively compact, that is, the closure $\bar{T}(M)$ is compact.

2.2 Fixed Points Theorems

A wide range of problems in nonlinear analysis may be presented in the form of an abstract equation
$$Lu = Nu,$$
where $L : X \to Z$ is a linear operator and $N : Y \to Z$ is a nonlinear operator defined in appropriate normed spaces $X \subset Y$ and Z. For example, the Dirichlet problem fits into this setting, with $Lu := u''$ and $Nu := f(\cdot, u)$. In this case, N is defined, for example, over the set of continuous functions, but L requires twice differentiable functions. Together with the boundary conditions, this yields the following possible choice of X, Y and Z:

$$\begin{aligned} X &= \{u \in C^2([0,1]) : u(0) = u(1) = 0\}, \\ Y &= \{u \in C([0,1]) : u(0) = u(1) = 0\}, \\ Z &= C([0,1]). \end{aligned}$$

In some cases, one may just consider the restriction of N to X and try to find zeros of the function $F : X \to Z$ given by $Fu = Lu - Nu$; however, in many situations this approach is not enough, and a different analysis is required. In particular, the previous Dirichlet problem is an example of so-called nonresonant problems since the operator $L : X \to Z$ is invertible: for each $\varphi \in Y$, the problem $u''(t) = \varphi(t)$ has a unique solution $u \in X$. Thus, the functional equation $Lu = Nu$ is transformed into a fixed point problem:
$$u = L^{-1}Nu.$$

2.2.1 Banach fixed point theorem

Several abstract tools have been developed to deal with problems of this kind; in this chapter, we begin with one of the most popular fixed point theorems in complete metric spaces the contraction mapping theorem. Let X and Y be metric spaces. A mapping $T : X \to Y$ is called a contraction if it is globally Lipschitz with constant $\alpha < 1$. In other wards, $T : X \to Y$ is a contraction if there exists $\alpha < 1$ such that $d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2)$ for all $x_1, x_2 \in X$. Note that used the same d for the distance in both X and Y; this is not a problem, in particular, because we shall only consider the case $X = Y$.

Theorem 2.2.1 *Let X be a complete metric space, and let $T : X \to X$ be a contraction. Then T has a unique fixed point $\hat{x}$. Furthermore, if x_0 is an arbitrary pint of X and a sequence is defined iteratively by $x_{n+1} = T(x_n)$, then $\hat{x} = \lim_{n\to\infty} x_n$.*

The contraction mapping theorem allows a simple and direct proof of the Picard existence and uniqueness theorem. In this case, we want to solve the functional equation

$$x(t) = x_0 + \int_0^t f(s, x(s))ds,$$

so the "obvious "fixed point operator is

$$Tx(t) := x_0 + \int_0^t f(s, x(s))ds.$$

We only need to find an appropriate complete metric space X such that $T : X \to X$ is well defined and contractive.

To this end, let us first consider constants $\hat{\delta}, r > 0$, such that $K \subset \Omega$, where

$$K := [t_0 - \hat{\delta}, t_0 + \hat{\delta}] \times \bar{B}_r(x_0).$$

Next, define $M := \|f_{|K|}\|_\infty$ and L as the Lipschitz constant of f over K, and let

$$X := \{x \in C[t_0 - \delta, t_0 + \delta], \mathbb{R}^n) : x(t) \in \bar{B}_r(x_0) \text{ for all } t.$$

for some $\delta \leq \bar{\delta}$ to be established. In other words, X is just the closed ball of radius r centered in x_0 in the space $C([t_0 - \delta, t_0 + \delta], \mathbb{R}^\ltimes)$, equipped with the usual metric

$$d(x, y) = \max_{t \in [t_0 - \delta, t_0 + \delta]} |x(t) - y(t)|.$$

It is clear that $T : X \to C([t_0 - \delta, t_0 + \delta], \mathbb{R}^\ltimes)$ is well defined and, moreover,

$$|Tx(t) - x_0| = \left| \int_{t_0}^t f(s, x(s))ds \right| \leq M\delta.$$

Choosing $\delta \leq \frac{r}{M}$, it follows that X is an invariant set, i.e. $T(X) \subset X$. On the other hand for $x, y \in X$, then

$$d(Tx, Ty) = \max_{t \in [t_0 - \delta, t_0 + \delta]} \left| \int_{t_0}^t (f(s, x(s)) - f(s, y(s)))ds \right| \leq \delta L d(x, y)$$

Hence, it suffices to take $\delta < \min\left\{\frac{r}{M}, \frac{1}{L}\right\}$, and then $T : X \to X$ is a contraction.

Although the Banach theorem ensures that the fixed point is unique, an extra step is needed for the uniqueness invoked in Theorem 2.2.1, since, in principle, for the same δ there might be other solutions that abandon the ball $\bar{B}_r(x_0)$. One possible line of reasoning is as follows: suppose y is another solution and fix $\bar{\delta} \in (0, \delta]$ such that $|y(t)| \leq r$ for $t \in [t_0 - \bar{\delta}, t_0 + \bar{\delta}]$. Next, redefine the space X accordingly, so the operator T has a unique fixed point and thus $x = y$

on $[t_0 - \bar{\delta}, t_0 + \bar{\delta}]$. This proves only local uniqueness, in the sense that two solutions must coincide in a neighborhood of t_0. But now the same existence and (local)uniqueness result does the rest of the job: suppose two solutions x and y are defined over an open interval I containing t_0; then the set $J := \{t \in I : x(t) = y(t)\}$ is closed in I and nonempty. Moreover, if $t_1 \in J$, then $x(t_1) = y(t_1)$, and hence $x = y$ in a neighborhood of t_1. This shows that J is open and, consequently, $J = I$.

Theorem 2.2.2 Let X be a complete metric spaces, and let $T : X \to X$ be a mapping. If $T^n := T \circ T \circ \cdots T$ (n times) is a contraction for some positive integer n, then T has a unique fixed point.

Theorem 2.2.3 Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be continuous and globally Lipschitz with respect to x with constant L. Then for any $(t_0, x_0) \in (a, b) \times \mathbb{R}^n$ the unique solution of the problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = x_0 \end{cases}$$

is defined over $[a, b]$.

The contraction mapping theorem is an efficient tool for proving existence and uniqueness, although its application might also be quite restrictive. The assumption that f is globally Lipschitz is already strong; furthermore, we have required the Lipschitz constant to be small. Nevertheless, there are many cases in which this assumption can be relaxed. In Picard's fundamental theorem, this was easy: only a local Lipschitz assumption was required since we were looking for local solutions; in other situations, the global Lipschitz condition may be avoided if one is able to obtain a priori bounds for the solutions, as we saw in the first chapter. But, still, one must prove that the fixed point operator is contractive: this explains why the Lipschitz constant must be small. The smallness assumption can be dropped when the operator has some other properties, such as monotonicity.

2.2.2 Schaefers fixed point theorem

Theorem 2.2.4 *(Arzelá-Ascoli Theorem) A set of function in $C[a, b]$ with the sup norm, is relatively compact if and only it is uniformly bounded and equicontinuous on $[a, b]$.*

Theorem 2.2.5 *If a set is closed and relatively compact then it is compact.*

Theorem 2.2.6 *(Schaefer's fixed point theorem) Assume that X is a banach space and that $T : X \to X$ is continuous compact mapping. Moreover assume* *that the set*

$$\bigcup_{0\le\lambda\le 1}\{x\in X : x=\lambda T(x)\}$$

is bounded. Then T has a fixed point.

2.3 Difference Equations

Difference equations usually describe the evolution of certain phenomena over the course of time. For example, if a certain population has discrete generations, the size of the $(n+1)$ st generation $x(n+1)$ is a function of the n th generation $x(n)$. This relation expresses itself in the difference equation

$$x(n+1) = f(x(n)).$$

We may look at this problem from another point of view. Starting from a point x0, one may generate the sequence

$$x_0, f(x_0), f(f(x_0)), f(f(f(x_0))), ...$$

For convenience we adopt the notation

$$f_2(x_0) = f(f(x_0)), f_3(x_0) = f(f(f(x_0))), \text{ etc.}$$

2.3.1 The Difference Operator

In this section we will begin a systematic study of difference equations. Many of the calculations involved in solving and analyzing these equations can be simplified by use of the difference calculus, a collection of mathematical tools quite similar to the differential calculus. The present chapter briefly surveys the most important aspects of the difference calculus. It is not essential to memorize all the formulas presented here, but it is useful to have an overview of the available techniques and to observe the differences and similarities between the difference and the differential calculus. Just as the differential operator plays the central role in the differential calculus, the difference operator is the basic component of calculations involving finite differences

Definition 2.3.1 Let $y(t)$ be a function of a real or complex variable t. The difference operator : Δ, is defined by

$$\Delta y(t) = y(t+1) - y(t).$$

For the most part, we will take the domain of y to be a set of consecutive integers such as the natural numbers $N = 1, 2, 3, .$ However, sometimes it is useful to choose a continuous set of t values such as the interval $[0, \infty)$ or the complex plane as the domain. The step size of one unit used in the definition is not really a restriction. Consider a difference operation with a step size $h > 0$ –say, $z(s + h) - z(s)$. Let $y(t) = z(th)$.

Then

$$\begin{aligned} z(s+h) - z(s) &= z(th + h) - z(th) \\ &= y(t+1) - y(t) \\ &= \Delta y(t). \end{aligned}$$

Occasionally we will apply the difference operator to a function of two or more variables. In this case, a subscript will be used to indicate which variable is to be shifted by one unit. For example,

$$\Delta_t te^n = (t + 1)e^n - te^n = e^n,$$

while

$$\Delta_n te^n = te^{n+1} - te^n = t(e - 1)e^n.$$

Higher order differences are defined by composing the difference operator with itself. The second order difference is

$$\begin{aligned} \Delta^2 y(t) &= \Delta(\Delta y(t)) \\ &= \Delta\left(y(t+1) - y(t)\right) \\ &= (y(t+2) - y(t+1)) - (y(t+1) - y(t)) \\ &= y(t+2) - 2y(t+1) + y(t). \end{aligned}$$

The following formula for the n th order difference is readily verified by induction:

$$\Delta^n y(t) = \sum_{k=0}^{n} (-1)^k C(n, k)\, y(t + n - k).$$

2.3.2 The summation

To make effective use of the difference operator, we introduce in this section its fight inverse operator, which is sometimes called the indefinite sum.

Definition 2.3.2 *An indefinite sum (or "antidifference") of $y(t)$, denoted $\sum y(t)$, is any function so that*

$$\Delta[\sum y(t)] = y(t)$$

for all t in the domain of y.

Definition 2.3.3 *If $z(t)$ is an indefinite sum of $y(t)$, then every indefinite sum of $y(t)$ is given by*

$$\sum y(t) = y(t) + C(t)$$

where $C(t)$ has the same domain as $y(t)$ and $\Delta C(t) = 0$.

Since the discrete case will be the most important case, we state the following corollary.

Corollary 2.3.4 *Let $y(t)$ be defined on a set of the type $\{a, a+1, a+2, ...\}$, where a is any real number, and let $z(t)$ be an indefinite sum of $y(t)$. Then every indefinite sum of $y(t)$ is given by*

$$\sum y(t) = y(t) + C,$$

Where C is an arbitrary constant.

There is a useful formula for computing definite sums, which is analogous to the fundamental theorem of calculus:

Theorem 2.3.5 *If z_n is an indefinite sum of y_n, then*

$$\sum_{k=m}^{n-1} y_k = z_n - z_m.$$

The next theorem gives a version of the summation by parts method for definite sums.

Theorem 2.3.6 *If $m < n$, then*

$$\sum_{k=m}^{n-1} y_k \Delta z_k = [y_k - z_k]_m^n - \sum_{k=m}^{n-1} (\Delta y_k)\, z_{k+1}.$$

2.4 Fractional Difference Equations

In this section, we introduce notations, definitions, and lemmas which is used in the main results.

Definition 2.4.1 *The gamma function :* $\Gamma(z)$*, which is defined by*

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt,$$

if the real part of z is positive. Formally applying integration by parts, we have

$$\begin{aligned}\Gamma(z+1) &= \int_0^\infty e^{-t}t^{z}dt \\ &= \left[-e^{-t}t^z\right]_0^\infty - \int_0^\infty (-e^{-t})zt^{z-1}dt \\ &= z\int_0^\infty e^{-t}t^{z-1}dt,\end{aligned}$$

so that $\Gamma(z)$ satisfies the difference equation

$$\Gamma(z+1) = z\Gamma(z)$$

We see that the gamma function extends the factorial to most of the complex plane. Graphed for real values of x in the interval $(\infty, -\infty)$, $x \notin \{0, -1, -2, ...\}$.

Definition 2.4.2 *We define the generalized falling function by*

$$t^{\underline{\alpha}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}$$

for any t *and* α*, for which the right-hand side is defined. If* $t+1-\alpha$ *is a pole of the gamma function and* $t+1$ *is not a pole, then* $t^{\underline{\alpha}} = 0$.

It is understood that the definition of $t^{\underline{\alpha}}$ is given only for those values of t and α that make the formulas meaningful. For example, $(-2)^{\underline{-3}}$ is not defined (division by zero), and $(\frac{1}{2})^{\underline{\frac{3}{2}}}$ is meaningless because $\Gamma(0)$ is undefined.

Definition 2.4.3 *The* α*-order fractional sum of a function* f*, for* $\alpha > 0$*, is defined by*

$$\Delta^{-\alpha}f(t) = \Delta^{-\alpha}f(t;a) := \frac{1}{\Gamma(\alpha)}\sum_{s=a}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}f(s),$$

for $t \in \{a+\alpha, a+\alpha+1, \ldots\} := \mathbb{N}_{a+\alpha}$ *and* $\sigma(s) = s+1$.

We also define the α*-order Caputo fractional difference for* $\alpha > 0$ *by*

$$\Delta_C^{\alpha} f(t) := \Delta^N \Delta_C^{\alpha-N} f(t),$$

where $t \in \mathbb{N}_{a+N-\alpha}$ *and* $N \in \mathbb{N}$ *is chosen so that* $0 \leq N-1 < \alpha \leq N$.

Lemma 2.4.4 *Let* t *and* α *be any numbers for which* $t^{\underline{\alpha}}$ *and* $t^{\underline{\alpha-1}}$ *are defined. Then*

$$\Delta t^{\underline{\alpha}} = \alpha t^{\underline{\alpha-1}}.$$

Lemma 2.4.5 *If* $t \leq r$, *then* $t^{\underline{\alpha}} \leq r^{\underline{\alpha}}$ *for any* $\alpha > 0$.

Lemma 2.4.6 *Let* $0 \leq N-1 < \alpha \leq N$. *Then*

$$\Delta^{-\alpha} \Delta_C^{\alpha} y(t) = y(t) + C_0 + C_1 t + \ldots + C_{N-1} t^{\underline{N-1}},$$

for some $C_i \in \mathbb{R}$, *with* $1 \leq i \leq N$.

To define the solution of the boundary value problem (1.5) we need the following lemma which deals with linear variant of the boundary value problem (1.5) and gives a representation of the solution.

Chapter 3

Existence of solutions for Caputo fractional difference equations

3.1 An auxiliary lemma

Lemma 3.1.1 *Let* $1 < \alpha \leq 2$, $2 < \gamma \leq 3$, $y : C(\mathbb{N}_{\alpha-2,T+\alpha}, U) \to U$ *and* $h \in C(\mathbb{N}_{\alpha-1,T+\alpha-1}, U)$ *be given. Then the problem*

$$\begin{cases} \Delta_C^{\alpha} u(t) = h(t+\alpha-1), \quad t \in \mathbb{N}_{0,T}, \\ u(\alpha-2) = y(u), \\ u(T+\alpha) = \Delta^{-\gamma} g(T+\alpha+\gamma-3)\, u(T+\alpha+\gamma-3), \end{cases} \tag{3.1}$$

has the unique solution

$$\begin{aligned} u(t) = & \left(1 - \frac{t^{\underline{1}}}{T+\alpha}\right) y(u) + \frac{t^{\underline{1}}}{T+\alpha} A(u) \\ & - \frac{t^{\underline{1}}}{(T+\alpha)\Gamma(\alpha)} \sum_{s=\alpha-1}^{T+\alpha-1} (T+2\alpha-1-\sigma(s))^{\underline{\alpha-1}} h(s) \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha-1}^{t-1} (t+\alpha-1-\sigma(s))^{\underline{\alpha-1}} h(s), \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} & A(u) \\ = \; & -\frac{y(u)\sum_{s=\alpha-2}^{T+\alpha-3} g(s)\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}\left(1-\frac{s^{\underline{1}}}{T+\alpha}\right)}{\frac{1}{T+\alpha}\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}}\,g(s)\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}-\Gamma(\gamma)} \\ & -\frac{\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha-3}\sum_{\xi=\alpha-1}^{s-1} g(s)\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}(s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}\,h(\xi)}{\frac{1}{T+\alpha}\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}}g(s)\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}-\Gamma(\gamma)} \\ & +\frac{\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha-3}\sum_{\xi=\alpha-1}^{T+\alpha-1} g(s)\,s^{\underline{1}}\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}(T+2\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}\,h(\xi)}{\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}}\,g(s)\,(T+\alpha+\gamma-3+\sigma(s))^{\underline{\gamma-1}}-\Gamma(\gamma)}. \end{aligned} \tag{3.3}$$

Proof. Using Lemma 2.4.6, a general solution for (3.1) can be written in the form

$$u(t) = C_0 + C_1 t^{\underline{1}} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\,h(s+\alpha-1), \tag{3.4}$$

for $t \in \mathbb{N}_{\alpha-2,\alpha+T}$. Applying the first boundary condition of (3.1) implies

$$C_0 = y(u).$$

So,

$$u(t) = y(u) + C_1 t^{\underline{1}} + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-\sigma(s))^{\underline{\alpha-1}}\,h(s+\alpha-1). \tag{3.5}$$

The second condition of (3.1) implies

$$\begin{aligned} u(T+\alpha) &= y(u) + C_1(T+\alpha) + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\,h(s+\alpha-1) \\ &= \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha-3}(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}g(s)\,u(s). \end{aligned}$$

A constant C_1 can be obtained by solving the above equation, so

$$\begin{aligned} C_1 &= \frac{1}{(T+\alpha)\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha-3}(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}g(s)\,u(s) - \frac{y(u)}{T+\alpha} \\ &\quad -\frac{1}{(T+\alpha)\Gamma(\alpha)}\sum_{s=0}^{T}(T+\alpha-\sigma(s))^{\underline{\alpha-1}}\,h(s+\alpha-1). \end{aligned}$$

Substituting a constant C_1 into (3.5), we get

$$\begin{aligned} u(t) &= \left(1-\frac{t^{\underline{1}}}{T+\alpha}\right) y(u)+\frac{t^{\underline{1}}}{T+\alpha}\left[\frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha-3}(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}g(s)\,u(s)\right. \\ &\quad\left.-\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha-1}^{T+\alpha-1}(T+2\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s)\right] \\ &\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=\alpha-1}^{t-1}(t+\alpha-1-\sigma(s))^{\underline{\alpha-1}}h(s). \end{aligned} \tag{3.6}$$

Let $A(u)=\dfrac{1}{\Gamma(\gamma)}\displaystyle\sum_{s=\alpha-2}^{T+\alpha-3}(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}g(s)\,u(s)$. Then

$$\begin{aligned} A(u) &= \frac{1}{\Gamma(\gamma)}\sum_{s=\alpha-2}^{T+\alpha-3} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}\left\{\left(1-\frac{s^{\underline{1}}}{T+\alpha}\right) y(u)\right. \\ &\quad+\frac{s^{\underline{1}}}{T+\alpha}\left[A(u)-\frac{1}{\Gamma(\alpha)}\sum_{\xi=\alpha-1}^{T+\alpha-1}(T+2\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}h(\xi)\right] \\ &\quad\left.+\frac{1}{\Gamma(\alpha)}\sum_{\xi=\alpha-1}^{s-1}(s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}h(\xi)\right\}, \end{aligned}$$

by calculating, we get that $A(u)$ on (3.3).
Substituting $A(u)$ into (3.6), we obtain (3.2). □

3.2 Existence results

Now we are in a position to establish the main results. First, we are going to deal with problems (1.5).

For $U \subseteq \mathbb{R}$, let $(U, \|\cdot\|)$ be a Banach space and $\mathcal{C} = C(\mathbb{N}_{\alpha-2,T+\alpha}, U)$ denote the Banach space of all continuous functions from $\mathbb{N}_{\alpha-2,T+\alpha} \to U$ endowed with a topology of uniform convergence with the norm denoted by $\|\cdot\|_{\mathcal{C}}$. For this purpose, we consider the operator $F : \mathcal{C} \to \mathcal{C}$ by

$$\begin{aligned}(Fu)(t) &= \left(1 - \frac{t^{\underline{1}}}{T+\alpha}\right) y(u) + \frac{t^{\underline{1}}}{T+\alpha} A(u) \\ &\quad - \frac{t^{\underline{1}}}{(T+\alpha)\Gamma(\alpha)} \sum_{s=\alpha-1}^{T+\alpha-1} (T+2\alpha-1-\sigma(s))^{\underline{\alpha-1}} f(s, u(s), (\Psi^\beta u)(s-1)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha-1}^{t-1} (t+\alpha-1-\sigma(s))^{\underline{\alpha-1}} f(s, u(s), (\Psi^\beta u)(s-1)), \qquad (3.7)\end{aligned}$$

where

$$\begin{aligned}&A(u) \\ = \; & \frac{1}{\frac{1}{T+\alpha}\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} - \Gamma(\gamma)} \times \qquad (3.8) \\ & \Bigg[-y(u) \sum_{s=\alpha-2}^{T+\alpha-3} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} \left(1 - \frac{s^{\underline{1}}}{T+\alpha}\right) \\ & - \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha-3} \sum_{\xi=\alpha-1}^{s-1} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} (s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}} \times \\ & f(\xi, u(\xi), (\Psi^\beta u)(\xi-1)) + \frac{1}{(T+\alpha)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha-3} \sum_{\xi=\alpha-1}^{T+\alpha-1} s^{\underline{1}} g(s) \times \\ & (T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} (T+2\alpha-1-\sigma(\xi))^{\underline{\alpha-1}} f(\xi, u(\xi), (\Psi^\beta u)(\xi-1)) \Bigg].\end{aligned}$$

It is easy to see that the problem (1.5) has solutions if and only if the operator F has fixed points.

Theorem 3.2.1 *Assume that $f : \mathbb{N}_{\alpha-2,T+\alpha} \times U \times U \to U$ is continuous and maps bounded subsets of $\mathbb{N}_{\alpha-2,T+\alpha} \times U \times U$ in to relatively compact subsets of U, $\varphi : \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha} \to [0, \infty)$ is continuous with $\varphi_0 = \max\{\varphi(t-1, s) : (t, s) \in \mathbb{N}_{\alpha-2,T+\alpha} \times \mathbb{N}_{\alpha-2,T+\alpha}\}$ and $y : \mathcal{C} \to U$ is a given functional. In addition, suppose that*

(H_1) *There exist constants* $\tau_1, \tau_2 > 0$ *such that for each* $t \in \mathbb{N}_{\alpha-2,\alpha+T}$ *and* $u, v \in \mathcal{C}$

$$|f(t,u(t),(\Psi^\beta u)(t-1)) - f(t,v(t),(\Psi^\beta v)(t-1))| \le \tau_1|u-v| + \tau_2|(\Psi^\beta u) - (\Psi^\beta v)|.$$

(H_2) *There exists a constant* $\mu > 0$ *such that for each* $u, v \in \mathcal{C}$

$$|y(u) - y(v)| \le \mu \|u - v\|_{\mathcal{C}}.$$

(H_3) *For each* $t \in \mathbb{N}_{\alpha-2,\alpha+T}$

$$0 < g(t) < K \quad \text{and} \quad K\,[T + \alpha - \gamma(2-\alpha) - 3] - (T+\alpha)\,\Gamma(\gamma+2)\,\Gamma(T) > 0.$$

(H_4) $\Theta := \mu\Omega + \Lambda\left(\tau_1 + \tau_2 \frac{\varphi_0 (T+\beta+2)^{\beta}}{\Gamma(\beta+1)}\right) < 1,$

where

$$\Omega = 2 + \frac{K\,[\gamma(T+2)+3]\,\Gamma(T+\gamma)}{K\,[T+\alpha-\gamma(2-\alpha)-3] - (T+\alpha)\,\Gamma(\gamma+2)\,\Gamma(T)}, \tag{3.9}$$

$$\begin{aligned}\Lambda = {} & \frac{K\,\Gamma(T+\gamma-2)}{T\,\Gamma(\alpha+1)\Gamma(T-2)\Big[K\,[T+\alpha-\gamma(2-\alpha)-3] - (T+\alpha)\,\Gamma(\gamma+2)\,\Gamma(T)\Big]} \times \\ & \Big[(\gamma+1)T(T-1)(T-2)(T+\alpha-2) + [T + \alpha(\alpha+\gamma) - 3]\,\Gamma(T+\alpha)\Big] \\ & + \frac{2\,\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)}.\end{aligned} \tag{3.10}$$

Then the problem (1.5) *has a unique solution on* $\mathbb{N}_{\alpha-2,\alpha+T}$.

Proof. We will show that F is a contraction. For any $u, v \in \mathcal{C}$ and for each $t \in \mathbb{N}_{\alpha-2,\alpha+T}$, we have

$$\begin{aligned} & |(Fu)(t) - (Fv)(t)| \\ \le {} & \left|1 - \frac{t^{\underline{1}}}{T+\alpha}\right| |y(u) - y(v)| + \frac{t^{\underline{1}}}{T+\alpha}\,|A(u) - A(v)| + \frac{t^{\underline{1}}}{(T+\alpha)\Gamma(\alpha)} \times \\ & \sum_{s=\alpha-1}^{T+\alpha-1} (T-1-\sigma(s))^{\underline{\alpha-1}}\,|f(s,u(s),(\Psi^\beta u)(s)) - f(s,v(s),(\Psi^\beta v)(s-1))| \\ & + \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}}\,|f(s,u(s),(\Psi^\beta u)(s)) - f(s,v(s),(\Psi^\beta v)(s-1))| \end{aligned}$$

$$
\begin{aligned}
< \;& \mu\|u-v\|_{\mathcal{C}}\left[1+\frac{t^{\underline{1}}}{T+\alpha}\right] \\
&+\frac{t^{\underline{1}}}{\left|\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}}\, g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}-(T+\alpha)\Gamma(\gamma)\right|}\times \\
&\left[\mu\|u-v\|_{\mathcal{C}}\sum_{s=\alpha-2}^{T+\alpha-3} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}\left(1-\frac{s^{\underline{1}}}{T+\alpha}\right)\right. \\
&+\frac{\left[\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\right]}{\Gamma(\alpha)}\sum_{s=\alpha}^{T+\alpha-3}\sum_{\xi=\alpha-1}^{s-1} g(s)(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}\times \\
&(s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}+\frac{\left[\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\right]}{(T+\alpha)\Gamma(\alpha)}\times \\
&\left.\sum_{s=\alpha}^{T+\alpha-3}\sum_{\xi=\alpha-1}^{T+\alpha-1} s^{\underline{1}}\, g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}}(T+2\alpha-1-\sigma(\xi))^{\underline{\alpha-1}}\right] \\
&+\frac{t^{\underline{1}}\left[\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\right]}{(T+\alpha)\Gamma(\alpha)}\sum_{s=\alpha-1}^{T+\alpha-1}(T+2\alpha-1-\sigma(s))^{\underline{\alpha-1}} \\
&+\frac{\left[\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\right]}{\Gamma(\alpha)}\sum_{s=\alpha-1}^{t-1}(t+\alpha-1-\sigma(s))^{\underline{\alpha-1}} \\
< \;& 2\mu\|u-v\|_{\mathcal{C}}+\frac{\mu\|u-v\|_{\mathcal{C}}\,\frac{2K\Gamma(T+\gamma+3)}{\gamma\Gamma(T+3)}}{\frac{K[T+\alpha-\gamma(2-\alpha)-3]}{\gamma(\gamma+1)\,\Gamma(T)}-(T+\alpha)\,\Gamma(\gamma)} \\
&+\frac{\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\left[\frac{K\,[\gamma(T+2)+3]\,\Gamma(T+\gamma)}{\gamma(\gamma+1)\Gamma(T)}+\frac{K\,\Gamma(T+\alpha(\alpha+\gamma)-3)\,\Gamma(T+\alpha)\,\Gamma(T+\gamma-2)}{\gamma(\gamma+1)\,\Gamma(\alpha+1)\Gamma(T-2)\Gamma(T+1)}\right]}{\frac{K[T+\alpha-\gamma(2-\alpha)-3]}{\gamma(\gamma+1)\,\Gamma(T)}-(T+\alpha)\,\Gamma(\gamma)} \\
&+\frac{2\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\,\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)} \\
= \;& \mu\|u-v\|_{\mathcal{C}}\left\{2+\frac{K\;[\gamma(T+2)+3]\;\Gamma(T+\gamma)}{K\,[\,T+\alpha-\gamma(2-\alpha)-3\,]-(T+\alpha)\,\Gamma(\gamma)\,\Gamma(T)}\right\} \\
&+\left(\left(\tau_1+\tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\|u-v\|_{\mathcal{C}}\right)\times \\
&\left\{\frac{K\,\Gamma(T+\gamma-2)}{T\,\Gamma(\alpha+1)\Gamma(T-2)\left[K\,[\,T+\alpha-\gamma(2-\alpha)-3\,]-(T+\alpha)\,\Gamma(\gamma+2)\,\Gamma(T)\right]}\times\right. \\
&\left[(\gamma+1)T(T-1)(T-2)(T+\alpha-2)+[\,T+\alpha(\alpha+\gamma)-3\,]\;\Gamma(T+\alpha)\right] \\
&\left.+\frac{2\,\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)}\right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \|u-v\|_{\mathcal{C}}\left\{\mu\Omega + \Lambda\left(\tau_1 + \tau_2\frac{\varphi_0(T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\right\} \\
&= \|u-v\|_{\mathcal{C}}\,\Theta \\
&\leq \|u-v\|_{\mathcal{C}}.
\end{aligned}
$$

Consequently, F is a contraction. Therefore, by the Banach fixed point theorem, we get that F has a fixed point which is a unique solution of the problem (1.5) on $t \in \mathbb{N}_{\alpha-2,\alpha+T}$. □

Theorem 3.2.2 *Assume that* $f : \mathbb{N}_{\alpha-2,T+\alpha} \times U \times U \to U$ *is continuous and maps bounded subsets of* $\mathbb{N}_{\alpha-2,T+\alpha} \times U \times U$ *in to relatively compact subsets of* U *and* $y : \mathcal{C} \to U$ *is a given functional. In addition, suppose that* (H_3) *holds, and*

(H_5) *There exists a constant* $L_1 > 0$ *such that for each* $t \in \mathbb{N}_{\alpha-2,\alpha+T}$ *and* $u \in \mathcal{C}$

$$|f(t, u(t), (\Psi^{\beta}u)(t-1))| \leq L_1.$$

(H_6) *There exists a constant* $L_2 > 0$ *such that for each* $u \in \mathcal{C}$

$$|y(u)| \leq L_2.$$

Then the problem (1.5) *has at least one solution on* $\mathbb{N}_{\alpha-2,\alpha+T}$.

Proof. We will use the Schaefer's fixed point theorem to prove this result. Let F be the operator defined in (3.7). It is clearly to see that $F : \mathcal{C} \to \mathcal{C}$ is completely continuous. So, it remains to show that the set

$$\mathcal{E} = \{u \in \mathcal{C} \,:\, u = \lambda F u \quad \text{for some} \quad 0 < \lambda < 1\}$$

is bounded.

Let $u \in \mathcal{E}$, then $u(t) = \lambda(Fu)(t)$ for some $0 < \lambda < 1$. Thus, for each

$t \in \mathbb{N}_{\alpha-2,\alpha+T}$, we have

$$\begin{aligned}
u(t) &= \lambda(Fu)(t) \; < \; (Fu)(t) \\
&\leq |y(u)| \cdot \left|1 - \frac{t^{\underline{1}}}{T+\alpha}\right| + \frac{t^{\underline{1}}}{\sum_{s=\alpha-2}^{T+\alpha-3} s^{\underline{1}} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} - (T+\alpha)\,\Gamma(\gamma)} \times \\
&\quad \Bigg| - y(u) \sum_{s=\alpha-2}^{T+\alpha-3} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} \left(1 - \frac{s^{\underline{1}}}{T+\alpha}\right) \\
&\quad - \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha-3} \sum_{\xi=\alpha-3-1}^{s-1} g(s)\,(T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} (s+\alpha-1-\sigma(\xi))^{\underline{\alpha-1}} \times \\
&\quad |f(\xi, u(\xi), (\Psi^{\beta}u)(\xi-1))| + \frac{1}{(T+\alpha)\Gamma(\alpha)} \sum_{s=\alpha}^{T+\alpha-3} \sum_{\xi=\alpha-1}^{T+\alpha-1} s^{\underline{1}} g(s) \times \\
&\quad (T+\alpha+\gamma-3-\sigma(s))^{\underline{\gamma-1}} (T+2\alpha-1-\sigma(\xi))^{\underline{\alpha-1}} \, |f(\xi, u(\xi), (\Psi^{\beta}u)(\xi-1))| \Bigg| \\
&\quad + \frac{t^{\underline{1}}}{(T+\alpha)\Gamma(\alpha)} \sum_{s=\alpha-1}^{T+\alpha-1} (T+2\alpha-1-\sigma(s))^{\underline{\alpha-1}} \, |f(s, u(s), (\Psi^{\beta}u)(s-1))| \\
&\quad + \frac{1}{\Gamma(\alpha)} \sum_{s=\alpha-1}^{t-1} (t+\alpha-1-\sigma(s))^{\underline{\alpha-1}} \, |f(s, u(s), (\Psi^{\beta}u)(s-1))| \\
&< 2\,L_2 + L_1 \frac{2\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)} \\
&\quad + \frac{L_2 \frac{K\,[\gamma(T+2)+3]\,\Gamma(T+\gamma)}{\gamma(\gamma+1)\Gamma(T)} + L_1 \left[\frac{K\,[\gamma(T+2)+3]\,\Gamma(T+\gamma)}{\gamma(\gamma+1)\Gamma(T)} + \frac{K\,\Gamma(T+\alpha(\alpha+\gamma)-3)\,\Gamma(T+\alpha)\,\Gamma(T+\gamma-2)}{\gamma(\gamma+1)\,\Gamma(\alpha+1)\Gamma(T-2)\Gamma(T+1)}\right]}{\frac{K[T+\alpha-\gamma(2-\alpha)-3]}{\gamma(\gamma+1)\,\Gamma(T)} - (T+\alpha)\,\Gamma(\gamma)} \\
&= L_2\Omega + L_1\,\Lambda,
\end{aligned}$$

which implies that for each $t \in \mathbb{N}_{\alpha-2,\alpha+T}$, we have

$$\|u\|_{\mathcal{C}} \leq L_2\Omega + L_1\,\Lambda,$$

where Ω and Λ are defined on (3.3)−(3.4). This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Schaefer's fixed point theorem, we conclude that F has a fixed point which is a solution of the problem (1.5). □

3.3 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 3.3.1 Consider the following fractional sum boundary value problem

$$\begin{aligned}
\Delta^{\frac{3}{2}}u(t) &= \frac{e^{-(t+\frac{1}{2})}}{5\left(t+\frac{201}{2}\right)^2}\cdot\frac{|u|+1}{1+\sin^2 u}+\sum_{s=-1}^{t-1}\left(t-\frac{1}{2}-\sigma(s)\right)^{\underline{-1/2}}\times \\
&\quad \frac{\arctan\left[\cos^2\left(t-\frac{3}{2}\right)\pi\right]\, e^{-2\left|s-t+\frac{7}{2}\right|}}{2000\sqrt{\pi}\,\left(t+\frac{19}{2}\right)^2}u(s+\frac{1}{2}), \quad t\in\mathbb{N}_{0,4}, \\
u\left(-\frac{1}{2}\right) &= \frac{|u|}{(100\,e)^3}\sin^2|\pi u| \qquad (3.11)\\
u\left(\frac{11}{2}\right) &= \Delta^{-\frac{11}{4}}\,u\left(\frac{17}{8}\right)\left[1000e+200\cos^2\left(\frac{17}{8}\right)\right].
\end{aligned}$$

Here $\alpha=\frac{3}{2}, T=4,\ \beta=\frac{1}{2},\ y(u)=\frac{|u|}{(100\,e)^3}\sin^2|\pi u|,\ \gamma=\frac{11}{4},\ g(t)=1000e+200\cos^2 t,\ \ \varphi(t-1,s+\beta)=\frac{e^{-2|s-t+4|}}{2000\sqrt{\pi}}$ and

$$f(t,u(t),(\Psi^{\beta}u)(t-1))=\frac{e^{-t}}{5(t+100)^2}\cdot\frac{|u|+1}{1+\sin^2 u}+\left[\frac{\arctan\left[\cos^2(t-2)\pi\right]}{(t+9)^2}\right][\Delta^{-\frac{1}{2}}\varphi\, u](t+\tfrac{1}{2}).$$

Let $t\in\mathbb{N}_{\frac{1}{2},\frac{9}{2}}$, we have

$$\begin{aligned}
&|f(t,u(t),(\Psi^{\frac{1}{2}}u)(t-1))-f(t,v(t),(\Psi^{\frac{1}{2}}u)(t-1))| \\
&\le \tfrac{4}{404010}|u-v|+\tfrac{22}{2527}|(\Psi^{\frac{1}{2}}u)-(\Psi^{\frac{1}{2}}v)|,
\end{aligned}$$

so (H_1) holds with $\tau_1=\frac{4}{404010}$, $\tau_2=\frac{22}{2527}$, and we have $\varphi_0=\frac{1}{2000\,e^8\sqrt{\pi}}$,

$$|y(u)-y(v)|=\left|\tfrac{|u|}{(100\,e)^3}\sin^2|\pi u|-\tfrac{|v|}{(100\,e)^3}\sin^2|\pi v|\right|\le\tfrac{1}{(100\,e)^3}\|u-v\|_{\mathcal{C}},$$

so (H_2) holds with $\mu=\frac{1}{(100\,e)^3}$.

Since $1000e\le g(t)\le 1000e+200=K$, we have

$$K\,[\,T+\alpha-\gamma(2-\alpha)-3\,]-(T+\alpha)\,\Gamma(\gamma+2)\,\Gamma(T)\approx 547.011>0$$

then (H_3) is satisfied.

Also, we have

$$\Omega\approx 47128.501 \ \text{ and } \ \Lambda\approx 14297.052.$$

We can show that

$$\begin{aligned}
&\mu\Omega + \left(\tau_1 + \tau_2 \frac{\varphi_0 (T+\beta+2)^{\underline{\beta}}}{\Gamma(\beta+1)}\right)\left[\Lambda + \frac{2\Gamma(T+\alpha+1)}{\Gamma(\alpha+1)\Gamma(T+1)}\right] \\
= &\ \frac{1}{(100\,e)^3}(47128.501) + (14297.052)\left[\frac{4}{404010} + \left(\frac{22}{2527}\right)\frac{(\frac{13}{2})^{\underline{\frac{1}{2}}}}{2000\,e^8\sqrt{\pi}\,\Gamma(\frac{3}{2})}\right] \\
\approx &\ 0.144 < 1.
\end{aligned}$$

Hence, by Theorem 3.2.1, the boundary value problem (3.3.1) has a unique solution. □

Example 3.3.2 Consider the following fractional sum boundary value problem

$$\begin{aligned}
\Delta^{\frac{3}{2}} u(t) &= \frac{\left(t+\frac{1}{2}\right)^{\frac{1}{2}}\, e^{-3\left(t+\frac{1}{2}\right)}}{1 + \left(t+\frac{1}{2}\right)\,|1+\cos^2(u+\pi)|} + \sum_{s=-1}^{t-1}\left(t - \frac{1}{2} - \sigma(s)\right)^{\underline{-1/2}} \times \\
&\quad \frac{\left(t+\frac{3}{2}\right)\left(s+\frac{1}{2}\right)\, e^{-\left(s+\frac{1}{2}\right)^2}}{\sqrt{\pi}\,\left|\left(t+\frac{3}{2}\right)\left(s+\frac{1}{2}\right) - \left(t+\frac{1}{2}\right)\right|}\, u\left(s+\frac{1}{2}\right), \quad t \in \mathbb{N}_{0,3}, \\
u\left(-\frac{1}{2}\right) &= \frac{|u^2+2|^{\underline{1-|u^2+2|}}}{\pi + u^2}, \\
u\left(\frac{9}{2}\right) &= \Delta^{-\frac{8}{3}}\, u\left(\frac{11}{6}\right)\left(12\,e + \sin\left(\frac{11}{6}\right)\right)^2.
\end{aligned} \tag{3.12}$$

Here $\alpha = \frac{3}{2}$, $T = 3$, $\beta = \frac{1}{2}$, $y(u) = \frac{|u^2+2|^{\underline{1-|u^2+2|}}}{\pi+u^2}$, $\gamma = \frac{8}{3}$ and $g(t) = (12e + \sin t)^2$, $\varphi(t-1, s+\beta) = \frac{(t+1)\left(s+\frac{1}{2}\right)e^{-\left(s+\frac{1}{2}\right)^2}}{\left|(t+1)\left(s+\frac{1}{2}\right)-t\right|}$ and

$$\begin{aligned}
f(t, u(t), (\Psi^{\beta} u)(t-1)) &= \frac{\sqrt{t}\, e^{-3t}}{1 + t\,|1+\cos^2(u+\pi)|} \\
&\quad + \sum_{s=-1}^{t-\frac{3}{2}} (t-1-\sigma(s))^{\underline{-1/2}}\, \varphi\left(t-1, s+\frac{1}{2}\right) u\left(s+\frac{1}{2}\right).
\end{aligned}$$

Clearly for $t \in \mathbb{N}_{\frac{1}{2},\frac{7}{2}}$, we have

$$\varphi_0 < \frac{\left|(t+1)\left(s+\frac{1}{2}\right) - t\right| + t}{\left|(t+1)\left(s+\frac{1}{2}\right) - t\right|} < 1 + \frac{1}{\left|\left(s+\frac{1}{2}\right) - 1\right|} \leq 3,$$

$$|f(t, u(t), (\Psi^{\frac{1}{2}} u)(t))| \leq \left| \frac{\sqrt{t}}{1+t} + \frac{\varphi_0 (\frac{9}{2})^{\underline{-1/2}}}{\Gamma(\frac{3}{2})} \right| < 2.566 = L_1,$$

$$\begin{aligned} |y(u)| &= \frac{\Gamma(|u^2+2|+1)}{(\pi+u^2)\,\Gamma(2|u^2+2|)} < \frac{1}{\pi} = L_2, \\ (12e)^2 \le g(t) \le (12e+1)^2 &= K, \\ \text{and} \quad \frac{1}{6}\left(12e+1\right)^2 - \frac{9}{2}\Gamma\left(\frac{14}{3}\right)\Gamma\left(3\right) &= 55.974 > 0. \end{aligned}$$

Hence, the conditions $(H_3), (H_5), (H_6)$ of Theorem 3.2.2 are satisfied, and consequently boundary value problem (3.3.2) has at least one solution. □

Chapter 4

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