วารสารคณิตศาสตร์ Mathematical Journal 64(699) กันยายน – ธันวาคม 2562

โดย สมาคมคณิตศาสตร์แห่งประเทศไทย ในพระบรมราชูปถัมภ์

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ผลเฉลยของสมการไดโอแฟนไทน์ $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$

Integral Solutions of the Diophantine Equation $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$

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วันที่รับบทความ : 24 เมษายน 2562

วันที่แก้ไขบทความ : 22 กรกฎาคม 2562

วันที่ตอบรับบทความ : 9 กันยายน 2562

บทคัดย่อ

ในบทความนี้ เราได้หาผลเฉลยที่เป็นจำนวนเต็มบวก ของสมการไดโอแฟนไทน์ $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$

คำสำคัญ: สมการไดโอแฟนไทน์ ผลเฉลยที่เป็นจำนวนเต็มบาก

ABSTRACT

In this paper, we find positive integral solutions of the Diophantine equation $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$.

Keywords: Diophantine equation, Positive integral solutions

1. Introduction

In 1948, Erdös and Straus [1], [3] have conjectured that for each $n \geq 2$, the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{4}{n}$ has a positive integral solution (x,y,z), where $x \neq y$, $y \neq z$ and $z \neq x$. Later, many researchers studied and solved this Diophantine equation, see [2], [4], [7] and [8].

In 2013, Rabago and Tagle [5] studied on the areas and volume of a certain regular solid and the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$. Sandor [6] offered an elementary approach to the solution of the Diophantine equation $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}$.

In this paper, we try to solve for all positive integral solutions of the Diophantine equation $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$.

2. Method of Analysis

From the Diophantine equation $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}=\frac{1}{2}$, we get $x \geq 3, y \geq 5$ and $z \geq 7$. We are going to split into three cases.

<u>Case I</u> If $x \leq y \leq z$ or $x \leq z \leq y$, then $\frac{1}{2}=\frac{1}{x}+\frac{2}{y}+\frac{3}{z} \leq \frac{6}{x}$. Hence, $3 \leq x \leq 12$.

If $x=3$, then $\frac{2}{y}+\frac{3}{z}=\frac{1}{6}$, i.e., $(y-12)(z-18)=216$. (1)

If $x=4$, then $\frac{2}{y}+\frac{3}{z}=\frac{1}{4}$, i.e., $(y-8)(z-12)=96$. (2)

If $x=5$, then $\frac{2}{y}+\frac{3}{z}=\frac{3}{10}$, i.e., $(3y-20)(z-10)=200$. (3)

If $x=6$, then $\frac{2}{y}+\frac{3}{z}=\frac{1}{3}$, i.e., $(y-6)(z-9)=54$. (4)

If $x=8$, then $\frac{2}{y}+\frac{3}{z}=\frac{3}{8}$, i.e., $(3y-16)(z-8)=128$. (5)

If $x=10$, then $\frac{2}{y}+\frac{3}{z}=\frac{2}{5}$, i.e., $(y-5)(2z-15)=75$. (6)

For (1), the only following cases are possible:

$y-12=1, z-18=216$; $y-12=2, z-18=108$; $y-12=3, z-18=72$;

$y-12=4, z-18=54$; $y-12=6, z-18=36$; $y-12=8, z-18=27$;

$y-12=9, z-18=24$; $y-12=12, z-18=18$; $y-12=18, z-18=12$;

$y-12=24, z-18=9$; $y-12=27, z-18=8$; $y-12=36, z-18=6$;

$y-12=54, z-18=4$; $y-12=72, z-18=3$; $y-12=108, z-18=2$;

$y-12=216, z-18=1$.

Leading to the solutions: $(x,y,z)=(3,13,234), (3,14,126), (3,15,90), (3,16,72), (3,18,54), (3,20,45), (3,21,42), (3,24,36), (3,30,30), (3,36,27), (3,39,26), (3,48,24), (3,66,22), (3,84,21), (3,120,20)$ and $(3,228,19)$.

In the same manner

(2) leads to $(x,y,z)=(4,9,108), (4,10,60), (4,11,44), (4,12,36), (4,14,28), (4,16,24), (4,20,20), (4,24,18), (4,32,16), (4,40,15), (4,56,14)$ and $(4,104,13)$.

(3) leads to $(x,y,z)=(5,7,210), (5,8,60), (5,10,30), (5,15,18), (5,20,15)$ and $(5,40,12)$.

(4) leads to $(x,y,z)=(6,7,63), (6,8,36), (6,9,27), (6,12,18), (6,15,15), (6,24,12), (6,33,11)$ and $(6,60,10)$.

(5) leads to $(x,y,z)=(8,8,24), (8,16,12)$ and $(8,48,9)$.

While, (6) leads to $(x,y,z)=(10,10,15)$ and $(10,20,10)$.

If $x=7$, then $\frac{2}{y}+\frac{3}{z}=\frac{5}{14}$. (7)

If $x=9$, then $\frac{2}{y}+\frac{3}{z}=\frac{7}{18}$. (8)

If $x=11$, then $\frac{2}{y}+\frac{3}{z}=\frac{9}{22}$. (9)

If $x=12$, then $\frac{2}{y}+\frac{3}{z}=\frac{5}{12}$. (10)

If $z \le y$, then (7) gives $\frac{5}{14}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{z}$, that is, $7\le z\le 14$. However, only $z\in\{9,10,14\}$ gives positive integral values of y. Hence, $(x,y,z)=(7,84,9),(7,35,10)$ and $(7,14,14)$.

If $y\le z$, then (7) gives $\frac{5}{14}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{y}$, that is, $7\le y\le 14$. However, only $y\in\{7,8\}$ gives positive integral values of z. Hence, $(x,y,z)=(7,7,42)$ and $(7,8,28)$.

If $z\le y$, then (8) gives $\frac{7}{18}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{z}$, that is, $9\le z\le 12$. However, only $z\in\{9\}$ gives positive integral value of y. Hence, $(x,y,z)=(9,36,9)$.

If $y\le z$, then (8) gives $\frac{7}{18}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{y}$, that is, $9\le y\le 12$. However, only $y\in\{9\}$ gives positive integral value of z. Hence, $(x,y,z)=(9,9,18)$.

If $z\le y$, then (9) gives $\frac{9}{22}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{z}$, that is, $11\le z\le 12$. However, both of them do not give positive integral value of y. Hence, there is no solutions for (9).

If $y\le z$, then (9) gives $\frac{9}{22}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{y}$, that is, $11\le y\le 12$. However, both of them do not give positive integral value of z. Hence, there is no solutions for (9)

If $z\le y$, then (10) gives $\frac{5}{12}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{z}$, that is, $z=12$. Hence, $(x,y,z)=(12,12,12)$.

If $y\le z$, then (10) gives $\frac{5}{12}=\frac{2}{y}+\frac{3}{z}\le\frac{5}{y}$, that is, $y=12$. Hence, $(x,y,z)=(12,12,12)$.

<u>Case II</u> If $y<x\le z$ or $y\le z<x$, then $\frac{1}{2}=\frac{1}{x}+\frac{2}{y}+\frac{3}{z}\le\frac{6}{y}$. Hence, $5\le y\le 12$.

If $y=5$, then $\frac{1}{x}+\frac{3}{z}=\frac{1}{10}$, i.e., $(x-10)(z-30)=300$. (11)

If $y=6$, then $\frac{1}{x}+\frac{3}{z}=\frac{1}{6}$, i.e., $(x-6)(z-18)=108$. (12)

If $y=7$, then $\frac{1}{x}+\frac{3}{z}=\frac{3}{14}$, i.e., $(3x-14)(z-14)=196$. (13)

If $y=8$, then $\frac{1}{x}+\frac{3}{z}=\frac{1}{4}$, i.e., $(x-4)(z-12)=48$. (14)

If $y = 10$, then $\frac{1}{x} + \frac{3}{z} = \frac{3}{10}$, i.e., $(3x - 10)(z - 10) = 100$. (15)

If $y = 12$, then $\frac{1}{x} + \frac{3}{z} = \frac{1}{3}$, i.e., $(x - 3)(z - 9) = 27$. (16)

For (11) the only following cases are possible :

$x - 10 = 1, z - 30 = 300;$ $x - 10 = 2, z - 30 = 150;$ $x - 10 = 3, z - 30 = 100;$

$x - 10 = 4, z - 30 = 75;$ $x - 10 = 5, z - 30 = 60;$ $x - 10 = 6, z - 30 = 50;$

$x - 10 = 10, z - 30 = 30;$ $x - 10 = 12, z - 30 = 25;$ $x - 10 = 15, z - 30 = 20;$

$x - 10 = 20, z - 30 = 15;$ $x - 10 = 25, z - 30 = 12;$ $x - 10 = 30, z - 30 = 10;$

$x - 10 = 50, z - 30 = 6;$ $x - 10 = 60, z - 30 = 5;$ $x - 10 = 75, z - 30 = 4;$

$x - 10 = 100, z - 30 = 3;$ $x - 10 = 150, z - 30 = 2;$ $x - 10 = 300, z - 30 = 1$.

Leading to the solutions : $(x, y, z) = (11,5,330), (12,5,180), (13,5,130), (14,5,105), (15,5,90),$ $(16,5,80), (20,5,60), (22,5,55), (25,5,50), (30,5,45), (35,5,42), (40,5,40), (60,5,36), (70,5,35),$ $(85,5,34), (110,5,33), (160,5,32)$ and $(310,5,31)$.

In the same manner,

(12) leads to $(x, y, z) = (7,6,126), (8,6,72), (9,6,54), (10,6,45), (12,6,36), (15,6,30),$ $(18,6,27), (24,6,24), (33,6,22), (42,6,21),$ $(60,6,20)$ and $(114,6,19)$.

(13) leads to $(x, y, z) = (14,7,21), (21,7,18)$ and $(70,7,15)$.

(14) leads to $(x, y, z) = (10,8,20), (12,8,18), (16,8,16), (20,8,15), (628,8,14)$ and $(52,8,13)$.

(15) leads to $(x, y, z) = (20,10,12)$.

While, (16) has no solutions.

If $y = 9$, then $\frac{1}{x} + \frac{3}{z} = \frac{5}{18}$. (17)

If $y = 11$, then $\frac{1}{x} + \frac{3}{z} = \frac{7}{22}$. (18)

If $x \le z$, then (17) gives $\frac{5}{18} = \frac{1}{x} + \frac{3}{z} \le \frac{4}{x}$, that is, $9 < x \le 14$. However, all values of x do not give positive integral value of z. Hence, there is no solutions for (17).

If $z < x$, then (17) gives $\frac{5}{18} = \frac{1}{x} + \frac{3}{z} < \frac{4}{z}$, that is, $9 \le z \le 14$. However, only $z \in \{11,12\}$ gives positive integral values of x. Hence, $(x,y,z) = (198,9,11)$ and $(36,9,12)$.

If $x \le z$, then (18) gives $\frac{7}{22} = \frac{1}{x} + \frac{3}{z} \le \frac{4}{x}$, that is, $11 < x \le 12$. However, both values of x do not give positive integral value of z. Hence, there is no solutions for (18).

If $z < x$, then (18) gives $\frac{7}{22} = \frac{1}{x} + \frac{3}{z} < \frac{4}{z}$, that is, $11 \le z \le 12$. However, only $z \in \{11\}$ gives positive integral value of x. Hence, $(x,y,z) = (22,11,11)$.

<u>Case III</u> If $z < x \le y$ or $z < y < x$, then $\frac{1}{2} = \frac{1}{x} + \frac{2}{y} + \frac{3}{z} \le \frac{6}{z}$. Hence, $7 \le z \le 12$.

If $z = 7$, then $\frac{1}{x} + \frac{2}{y} = \frac{1}{14}$, i.e., $(x-14)(y-28) = 392$. (19)

If $z = 8$, then $\frac{1}{x} + \frac{2}{y} = \frac{1}{8}$, i.e., $(x-8)(y-16) = 128$. (20)

If $z = 9$, then $\frac{1}{x} + \frac{2}{y} = \frac{1}{6}$, i.e., $(x-6)(y-12) = 72$. (21)

If $z = 10$, then $\frac{1}{x} + \frac{2}{y} = \frac{1}{5}$, i.e., $(x-5)(y-10) = 50$. (22)

If $z = 12$, then $\frac{1}{x} + \frac{2}{y} = \frac{1}{4}$, i.e., $(x-4)(y-8) = 32$. (23)

(19) leads to $(x,y,z) = (15,420,7),(16,224,7),(18,126,7),(21,84,7),(22,77,7),(28,56,7),(42,42,7),(63,36,7),(70,35,7),(112,32,7),(210,30,7)$ and $(406,29,7)$.

(20) leads to $(x,y,z) = (9,144,8),(10,80,8),(12,48,8),(16,32,8),(24,24,8),(40,20,8),(72,18,8)$ and $(136,17,8)$.

(21) leads to $(x,y,z) = (10,30,9),(12,24,9),(14,21,9),(15,20,9),(18,18,9),(24,16,9),(30,15,9),(42,14,9)$ and $(78,13,9)$.

(22) leads to $(x,y,z) = (15,15,10),(30,12,10)$ and $(55,11,10)$.

While, (23) has no solutions.

If $z = 11$ then $\frac{1}{x} + \frac{2}{y} = \frac{5}{22}$. (24)

If $x \le y$, then (24) gives $\frac{5}{22} = \frac{1}{x} + \frac{2}{y} \le \frac{3}{x}$, that is, $12 \le x \le 13$, However, both values of x do not give positive integral value of y. Hence, there is no solution for (24).

If $y < x$ then (24) gives $\frac{5}{22} = \frac{1}{x} + \frac{2}{y} < \frac{3}{y}$ that is, $12 \le y \le 13$. However, both values of y do not give positive integral value of x. Hence, there is no solution for (24).

Suggested Problem

Let a, b, c, p and q be positive integers, we may ask for the positive integral solutions (x, y, z) for the Diophantine equation $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{p}{q}$ where x, y and z are positive integers.

Acknowledgements

I would like to give a thanks to Jatupat for helping submission process and this work is supported by Faculty of Science, Burapha University, Thailand.

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