THE TECHNIQUE FOR SOLVING NONLINEAR MATRIX EQUATIONS INVOLVING GENERALIZED CONTRACTION MAPPINGS VIA KY FAN NORMS AND THOMPSON METRICS

BY

MISS KANOKWAN SAWANGSUP

A DISSERTATION SUBMITTED IN PARTIAL FULFILLMENT OF
THE REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY (MATHEMATICS)
DEPARTMENT OF MATHEMATICS AND STATISTICS
FACULTY OF SCIENCE AND TECHNOLOGY
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ABSTRACT

The purpose of this dissertation is to investigate new contraction mappings in metric spaces and b-metric spaces endowed with binary relations. The main results of this dissertation are divided into two parts. In the first part, we improve the notion of a $\mathcal{Z}$-contraction mapping with respect to a b-simulation function and also prove fixed point results on b-metric spaces endowed with only a transitive relation. Our results can reduce to several important results in the past. We also introduce the concept of an $(F,\gamma)_{\Re}$-contraction mapping, which is improved from weaker conditions on F-contraction mappings in metric spaces endowed with a binary relation. We prove fixed point results for $(F,\gamma)_{\Re}$-contraction mappings and also furnish some examples to demonstrate the benefit of our main results. Furthermore, we introduce the new contraction namely $(\psi,\phi,\Re)$-contraction and prove the fixed point theorem for relation-theoretic $(\psi,\phi,\Re)$-contractions in a metric space endowed with a T-orbital transitivity. We also give an example to show the benefit of our theorems. In the last part, we extend and generalize Ran and

Reurings's results to prove nonlinear matrix equations by giving new notions concerning b-simulation functions via Ky Fan norms. Also, we apply fixed point results for $(F,\gamma)_{\Re}$-contraction mappings and $(\psi,\phi,\Re)$-contraction mappings to prove the existence and uniqueness of a solution of some nonlinear matrix equations and we give some numerical examples to support some results of our applications.

Keywords: b-Simulation function, Lower semicontinuous, Right upper semicontinuous, Thompson metric, Ky Fan norm.

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CHAPTER 1

INTRODUCTION

Nonlinear matrix equations occur in several problems in stability analysis, control theory, system theory, and dynamic programming. Many years ago, several authors used various methods to investigate the existence of solutions of nonlinear matrix equations. One of these methods is to use the fixed point theory. The most well known fixed point theorem is the Banach contraction principle, which is due to Banach [1]. Many generalizations of the Banach contraction mapping principle to partially ordered metric spaces are appeared in [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16].

In 2004, Ran and Reuring [17] solved the existence of a solution of a nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i,$$

where $A_1, A_2, \ldots, A_m$ are arbitrary $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is a continuous order-preserving mapping which maps from the set of all $n \times n$ Hermitian matrices into the set of all $n \times n$ positive definite matrices such that $\mathcal{G}(0) = 0$, using the Ky Fan norm and analogous result of the Banach contraction principle for partially ordered metric spaces. After that, Sawangsup *et. al.* [18, 19, 20] used the Ky Fan norm to solve the same nonlinear matrix equation and analogous results of fixed point results for contractions concerning some control functions in the setting spaces endowed with a binary relation. In [21], Lim used the Thompson metric and the fixed point method to solve the existence and uniqueness of a positive definite solution of a nonlinear matrix equation

$$X - \sum_{i=1}^{m} M_i X^{\delta_i} M_i^* = Q, 0 < |\delta_i| < 1,$$

where Q is an $n \times n$ positive semidefinite matrix and $M_1, M_2, \ldots, M_m$ are $n \times n$ nonsingular matrices or Q is an $n \times n$ positive semidefinite matrix and $M_1, M_2, \ldots, M_m$ are arbitrary $n \times n$ matrices. Several techniques involving fixed point theory are more suitable in solving the existence and uniqueness of several nonlinear matrix equations.

Based on many applications of the existence results in the fixed point theory, many mathematicians introduced and investigated new fixed point results for various generalized contraction mappings. For instance, the interesting fixed point results of a new contraction namely $\mathcal{Z}$-contraction with respect to a simulation function introduced by Khojasteha *et al.* [22]. In the same way, Argoubi *et al.* [23] considered a pair of nonlinear mappings satisfying a contractive condition involving a simulation function in metric spaces endowed with a partial order, which is a generalization of the fixed point theorem for $\mathcal{Z}$-contractions with respect to simulation functions in metric spaces. In [24], Demma *et al.* introduced the notion of a b-simulation function in the setting of b-metric spaces, which is a modifying of the concept of a simulation function, and also proved fixed point results by dealing with such contraction in b-metric spaces. Also, fixed point theorems for a new kind of contractions called F-contractions, are shown by Wardowski [26]. After that, many authors generalized and improved F-contraction in different ways.

The main results of this study consist of three topics as follows:

The first topic, we introduce the notion of the $\mathcal{Z}$-contraction mapping with respect to b-simulation functions under an arbitrary binary relation namely $\mathcal{Z}^b_{\Re}$-contraction mappings. We also introduce the concept of $(F,\gamma)_{\Re}$-contraction mappings which improve the concept of F-contraction mappings in metric spaces endowed with a binary relation and the notion of $(\psi,\phi,\Re)$-contraction mappings, where ψ is not a weak altering distance function, and ϕ is not continuous. Next, we investigate the existence and uniqueness of a fixed point for $\mathcal{Z}^b_{\Re}$-contraction mappings in complete b-metric spaces endowed with a transitive relation and give fixed point results for $(F,\gamma)_{\Re}$-contraction mappings in complete metric spaces endowed with a transitive relation. Moreover, we establish the fixed point theorem for relation-theoretic $(\psi,\phi,\Re)$-contractions in a metric space endowed with a T-orbital transitivity. After that, we give some examples to show the benefit of our theorems.

The second topic, we prove the existence and uniqueness of a solution of the nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i,$$

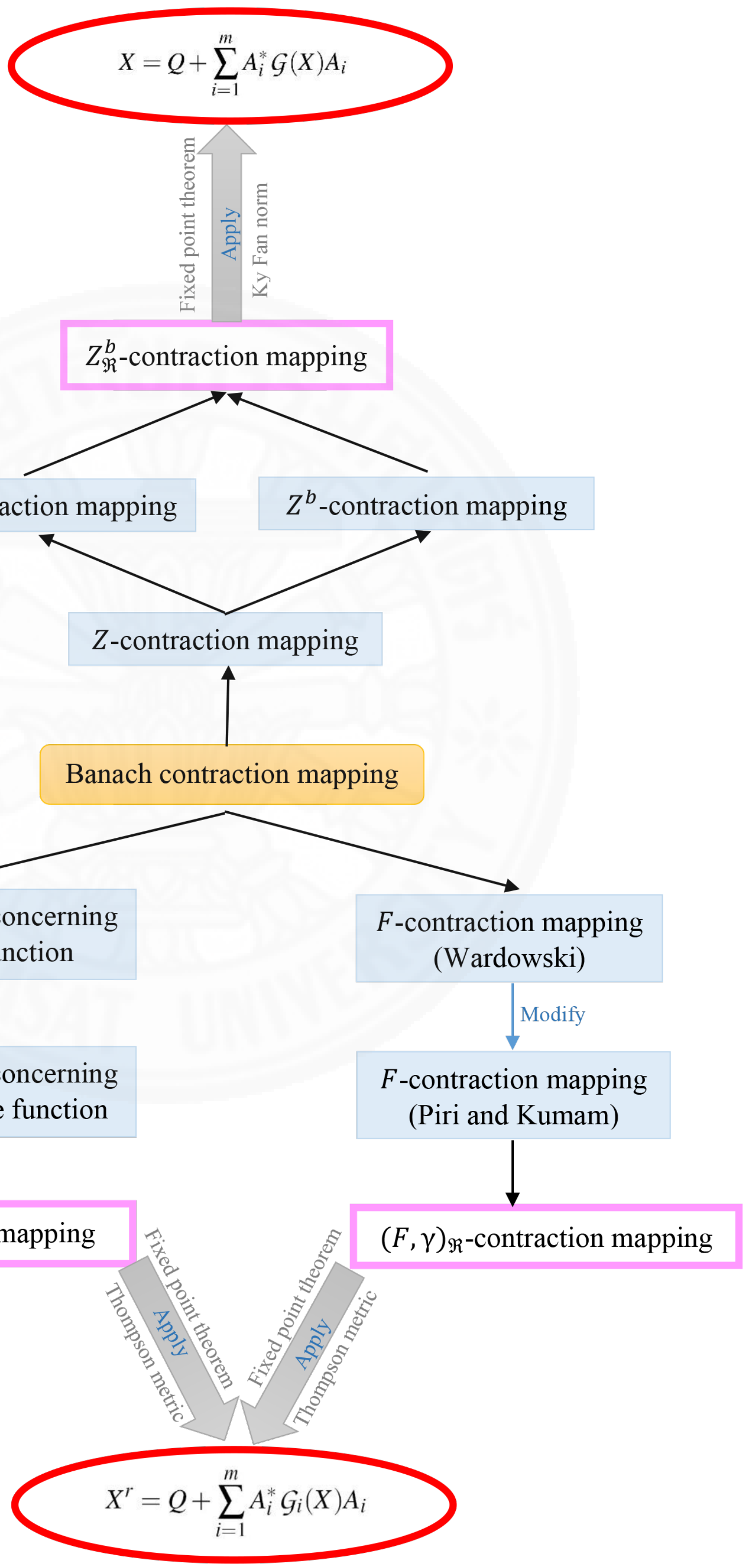
where $A_1, A_2, \ldots, A_m$ are arbitrary $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is a continuous order-preserving mapping which maps from the set of all $n \times n$ Hermitian matrices into the set of all $n \times n$ positive definite matrices such that $\mathcal{G}(0) = 0$ by using fixed point results for $\mathcal{Z}^b_{\Re}$-contraction mappings via Ky Fan norms. Also, we confirm the existence and uniqueness of a definite positive solution of a nonlinear matrix equation by giving numerical examples, which approximate by MATLAB.

The last topic, we apply fixed point results for $(F, \gamma)_{\Re}$-contraction mappings and $(\psi, \phi, \Re)$-contraction mappings via Thompson metrics to solve the nonlinear matrix equation

$$X^r = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i,$$

where $r \geq 1$, $A_1, A_2, \ldots, A_m$ are $n \times n$ nonsingular matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ are continuous order preserving self-mappings on the set of all $n \times n$ positive definite matrices. We also give some numerical examples to confirm the correctness of our applications.

The following diagram shows the motivation and the overall of our main result.



CHAPTER 2

PRELIMINARIES

Throughout this dissertation, we denote by $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}_+$, $\mathbb{Q}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ the set of complex numbers, real numbers, positive real numbers, non-negative real numbers, rational numbers, integers, positive integers, non-negative integers, respectively. Henceforth, let n be a positive integer, X will denote a nonempty set, and $\mathbb{R}^n$ will denote the product set $\underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n-\text{terms}}$.

2.1 Fixed points

Definition 2.1.1. Let X be a nonempty set and let T be a self-mapping on X. A point $x \in X$ is called a *fixed point* of T if and only if $Tx = x$.

Example 2.1.2. Let $X = \mathbb{R}$ and $T : X \to X$ be defined by

$$Tx = 2x + 1$$

for all $x \in X$. Then -1 is a fixed point of T.

Example 2.1.3. Let $X = \mathbb{R}$ and $T : X \to X$ be defined by

$$Tx = -x^2 + 5x$$

for all $x \in X$. Then 0 and 4 are fixed points of T.

Example 2.1.4. Let $X = [0, \infty)$ and $T : X \to X$ be defined by

$$Tx = \frac{1}{2}\ln(x^2 + 1)$$

for all $x \in X$. Then 0 is a fixed point of T.

Example 2.1.5. Let $X = [0, 1]$ and $T : X \to X$ be defined by

$$Tx = x^2 e^{-x} + \frac{1}{3}x^3$$

for all $x \in X$. Then 0 is a fixed point of T.

Example 2.1.6. Let $X = [0,4]$ and $T : X \to X$ be defined by

$$Tx = e^{-x} + 2\sqrt{x}$$

for all $x \in X$. Then f has no a fixed point.

Example 2.1.7. Let $X = \mathbb{R}$ and $T : X \times X \to X \times X$ be defined by

$$T(x,y) = \left(\frac{x}{3} + 2, \frac{y}{4} + 2\right)$$

for all $(x,y) \in X \times X$. Then $\left(3, \frac{8}{3}\right)$ is a fixed point of T.

Example 2.1.8. Let X be the set of 2×2 matrices and $T : X \to X$ be defined by

$$T\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = 3\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for all $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in X$. Then $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ is a fixed point of T.

Example 2.1.9. Let X be the set of 2×2 matrices and $T : X \to X$ be defined by

$$T\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = -2\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} + \begin{pmatrix} 6 & 0 \\ 3 & 9 \end{pmatrix}$$

for all $\begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in X$. Then $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$ is a fixed point of T.

2.2 Fields

Definition 2.2.1. A *field* is a set $\mathbb{F}$ together with two binary operations $+$ and $\cdot$, which are called *addition* and *multiplication*, respectively, satisfying the following axioms. For all $a, b, c \in \mathbb{F}$,

($\mathbb{F}$1) $a + b \in \mathbb{F}$ and $a \cdot b \in \mathbb{F}$;

($\mathbb{F}$2) $a + b = b + a$ and $a \cdot b = b \cdot a$;

($\mathbb{F}$3) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$;

($\mathbb{F}4$) there exists an element in $\mathbb{F}$, called *additive identity element* and denoted by $0_{\mathbb{F}}$, such that for all $a \in \mathbb{F}$, $a + 0_{\mathbb{F}} = a$;

($\mathbb{F}5$) there exists an element in $\mathbb{F}$, called *multiplicative identity element* and denoted by $1_{\mathbb{F}}$, such that for all $a \in \mathbb{F}$, $a \cdot 1_{\mathbb{F}} = a$;

($\mathbb{F}6$) for each $a \in \mathbb{F}$, there exists an element $-a \in \mathbb{F}$ such that $a + (-a) = 0_{\mathbb{F}}$;

($\mathbb{F}7$) for each $a \in \mathbb{F}$ with $a \neq 0_{\mathbb{F}}$, there exists an element $a^{-1} \in \mathbb{F}$ such that $a \cdot (a^{-1}) = 1_{\mathbb{F}}$;

($\mathbb{F}8$) the *left distributive law*, $a \cdot (b + c) = a \cdot b + a \cdot c$ and the *right distributive law*, $(b + c) \cdot a = b \cdot a + c \cdot a$ hold.

This field is denoted by $\langle \mathbb{F}, +, \cdot \rangle$.

Example 2.2.2. $\langle \mathbb{C}, +, \cdot \rangle$, $\langle \mathbb{R}, +, \cdot \rangle$ and $\langle \mathbb{Q}, +, \cdot \rangle$ are fields under the usual addition and usual multiplication but $\langle \mathbb{Z}, +, \cdot \rangle$ is not a field because 2 has no multiplicative inverse.

2.3 Vector spaces

Definition 2.3.1. A nonempty set V is said to be a *vector space* over a field $\langle \mathbb{F}, +, \cdot \rangle$ if the vector addition operation $+ : V \times V \to V$ and the scalar multiplication operation $\cdot : \mathbb{F} \times V \to V$ satisfy the following properties: for all $x, y, z \in V$ and $a, b \in \mathbb{F}$,

($V1$) $(x + y) + z = x + (y + z)$;

($V2$) $x + y = y + x$;

($V3$) there is an element $0 \in V$ such that $x + 0 = x$;

($V4$) for each $x \in V$, there exists an element $(-x) \in V$ such that $x + (-x) = 0$;

($V5$) $(ab)x = a(bx)$;

$(V6)$ $a(x+y) = ax+ay$;

$(V7)$ $(a+b)x = ax+bx$;

$(V8)$ $1_{\mathbb{F}}x = x$.

Also, this vector space is denoted as $(V,+,\cdot)$.

Example 2.3.2. Let X be an arbitrary set and $\mathbb{F}$ be any field, and let $F(X,\mathbb{F})$ be the set of all function from X into $\mathbb{F}$. The set $F(X,\mathbb{F})$ is a vector space with the two algebraic operations defined for $f,g \in F(X,\mathbb{F})$ and $k \in \mathbb{F}$ by

$$(f+g)(t) = f(t)+g(t),$$

$$(kf)(t) = kf(t).$$

Then $(F(X,\mathbb{F}),+,\cdot)$ is a vector space over a field $\mathbb{F}$.

Example 2.3.3. Let $x = (x_1,x_2,\ldots,x_n)$ and $y = (y_1,y_2,\ldots,y_n)$ be vectors in $\mathbb{R}^n$. The sum of these two vectors is defined as the vector

$$x+y = (x_1+y_1,x_2+y_2,...,x_n+y_n).$$

For a scalar $k \in \mathbb{R}$, define scalar multiplications, as the vector

$$ax = (ax_1,ax_2,\ldots,ax_n).$$

Then $(\mathbb{R}^n,+,\cdot)$ is a vector space over a field $\mathbb{R}$.

Example 2.3.4. The set of $m \times n$ matrices with entries from a field $\mathbb{F}$ is a vector space, which we denote by $M_{m\times n}(\mathbb{F})$, under the following operations of addition and scalar multiplication: For $A = (a_{ij})_{m\times n}, B = (b_{ij})_{m\times n} \in M_{m\times n}(\mathbb{F})$ and $c \in \mathbb{F}$,

$$A+B = (a_{ij}+b_{ij})_{m\times n} \quad \text{and} \quad cA = (ca_{ij})_{m\times n}.$$

Remark 2.3.5. A real vector space is a vector space whose field of scalars is the field of real numbers.

Remark 2.3.6. A complex vector space is a vector space whose field of scalars is the complex numbers.

2.4 Normed spaces

Definition 2.4.1. Let X be a real (or a complex) vector space. A *norm* on X is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying the following properties for all $x, y \in X$ and for all scalar α:

$(N1)$ $\|x\| = 0$ if and only if $u = 0$;

$(N2)$ $\|\alpha x\| = |\alpha|\|u\|$;

$(N3)$ $\|x+y\| \leq \|x\| + \|y\|$.

Also, the ordered pair $(X, \|\cdot\|)$ is called a *normed space.*

Remark 2.4.2. In a normed space $(X, \|\cdot\|)$, we get $\|x\| \geq 0$ for all $x \in X$.

Example 2.4.3. Let $X = \mathbb{R}^2$ and a function $\|\cdot\| : X \to \mathbb{R}$ be defined by

$$\|x\| = |x_1| + |x_2|$$

for all $x = (x_1, x_2) \in \mathbb{R}^2$. Then $(\mathbb{R}^2, \|\cdot\|)$ is a normed space, and it is called a *taxicab normed space.*

Example 2.4.4. Let $X = \mathbb{R}^n$ and a function $\|\cdot\| : X \to \mathbb{R}$ be defined by

$$\|x\| = \sqrt{\sum_{i=1}^{n} |x_i|^2}$$

for all $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$. Then $(\mathbb{R}^n, \|\cdot\|)$ is a normed space, and it is called an *Euclidean space* on $\mathbb{R}^n$.

Definition 2.4.5. Let X be a normed space and $\{x_n\}$ be a sequence of element of X. The sequence $\{x_n\}$ *converges to* $x \in X$ denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$ if $\lim_{n\to\infty} \|x - x_n\| = 0$, i.e., for every $\varepsilon > 0$, there exists a natural number N such that $\|x - x_n\| < \varepsilon$ for all $n \geq N$.

Definition 2.4.6. Let X be a normed space and $\{x_n\}$ be a sequence of element of X. The sequence $\{x_n\}$ is called a *Cauchy sequence* if $\lim_{n\to\infty} \|x_m - x_n\| = 0$, i.e., for every $\varepsilon > 0$, there exists a natural number N such that $\|x_m - x_n\| < \varepsilon$ for all $m, n \geq N$.

Definition 2.4.7. Let X be a normed space. If every Cauchy sequence is convergent in X, then X is called a *complete normed space* or *Banach space*.

Example 2.4.8. The Euclidean space $\mathbb{R}^n$ is a Banach space.

2.5 Metric spaces

Definition 2.5.1. Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}$ is called a *metric on* X (or *a distance function on* X) if the following conditions hold for all $x, y, z \in X$:

$(M1)$ $d(x,y) = 0$ if and only if $x = y$;

$(M2)$ $d(x,y) = d(y,x)$;

$(M3)$ $d(x,z) \leq d(x,y) + d(y,z)$.

Remark 2.5.2. In a metric space (X,d), we get $d(x,y) \geq 0$ for all $x,y \in X$.

Remark 2.5.3. Let $(X, \|\cdot\|)$ be a normed space. The mapping $d: X \times X \to \mathbb{R}$ given by

$$d(x,y) = \|x - y\|$$

for all $x,y \in X$ defines a metric on X. This metric is called the *metric induced by norm.*

Definition 2.5.4. A nonempty set X equipped with a metric d on X, denoted by (X,d), is called a *metric space.* The elements of metric space (X,d) are called *points.* For fixed $x,y \in X$, we called the nonnegative number $d(x,y)$ that *distance* from x to y.

Example 2.5.5. Let $X = \mathbb{R}$ and $d: X \times X \to \mathbb{R}$ be defined by

$$d(x,y) = |x - y|$$

for all $x, y \in X$. The first two conditions in Definition 2.5.1 are obviously satisfied, and the third follows from the ordinary triangle inequality for real numbers:

$$\begin{aligned} d(x,y) &= |x-y| \\ &= |(x-z)+(z-y)| \\ &\leq |x-z|+|z-y| \\ &= d(x,z)+d(z,y). \end{aligned}$$

Then d is a metric on $\mathbb{R}$ and it is called a *usual metric* on $\mathbb{R}$. Thus $(\mathbb{R}, d)$ is a metric space called a *usual metric space.*

Example 2.5.6. Let X be a nonempty set and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Then d is a metric on X and it is called that a *discrete metric* on X. Also, the pair (X, d) is called a *discrete metric space.*

Example 2.5.7. Let $X = \mathbb{R}^n$, where $n \in \mathbb{N}$, and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{R}^n$ and it is called a *Euclidian metric* on $\mathbb{R}^n$. Also, the pair (X, d) is called an *n-dimensional Euclidian metric space.*

Example 2.5.8. Let $X = \mathbb{C}^n$, where $n \in \mathbb{N}$, and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}|x_i - y_i|^2}$$

for all $x = (x_1, x_2, \ldots, x_n), y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$. Then d is a metric on $\mathbb{C}^n$ and it is called a *Euclidian metric* on $\mathbb{C}^n$. Also, the pair (X, d) is called an *n-dimensional unitary metric space.*

Example 2.5.9. Let $X = \mathbb{R}^2$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = |x_1 - y_1| + |x_2 - y_2|$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then d is a metric on $\mathbb{R}^2$ and it is called a *taxicab metric* on $\mathbb{R}^2$. Thus, (X,d) is a *taxicab metric space.*

Example 2.5.10. Let $X = \mathbb{R}^2$ and define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Then d is a metric on $\mathbb{R}^2$. Thus, (X,d) is a metric space.

Example 2.5.11. Let (X,d) be a metric space, k be a positive real number, and define $d' : X \times X \to \mathbb{R}$ by

$$d'(x,y) = kd(x,y).$$

Then d' is a metric on X. This is called a *dilotion metric* on X, and (X,d') is called a *dilotion metric space.*

Example 2.5.12. Let $X = c := \{\{x_n\} \subseteq \mathbb{R} \text{ (or } \mathbb{C}) : \{x_n\} \text{ is a convergent sequence}\}$ and define $d_c : X \times X \to \mathbb{R}$ by

$$d_c(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

for all $x = \{x_n\}, y = \{x_n\} \in X$. Then (X, d_c) is a metric space.

Example 2.5.13. Let $X = \ell^\infty := \{\{x_n\} \subseteq \mathbb{R}(\text{or } \mathbb{C}) : \{x_n\} \text{ is a bounded sequence}\}$ and define $d_\infty : X \times X \to \mathbb{R}$ by

$$d_\infty(x,y) = \sup_{n \in \mathbb{N}} |x_n - y_n|$$

for all $x = \{x_n\}, y = \{x_n\} \in X$. Then (X, d_∞) is a metric space.

Example 2.5.14. Let $[a,b]$ be a closed interval on $\mathbb{R}$ and $X = B[a,b] := \{x : [a,b] \to \mathbb{R} : x \text{ is a bounded function}\}$ and define $d_\infty : X \times X \to \mathbb{R}$ by

$$d_\infty(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

for all $x, y \in X$. Then (X, d_∞) is a metric space.

Example 2.5.15. Let $[a,b]$ be a closed interval on $\mathbb{R}$ and $X = C[a,b] := \{x : [a,b] \to \mathbb{R} : x \text{ is a continuous function}\}$ and define $d_\infty : X \times X \to \mathbb{R}$ by

$$d_\infty(x,y) = \sup_{t \in [a,b]} |x(t) - y(t)|$$

for all $x,y \in X$. Then (X, d_∞) is a metric space.

Definition 2.5.16. Let (X,d) be a metric space, $x \in X$ and $\epsilon > 0$. We now define a set

$$B_\epsilon(x) := \{y \in X : d(x,y) < \epsilon\},$$

which is called an *open ball* of radius ϵ with center x.

Definition 2.5.17. Let (X,d) be a metric space. A sequence $\{x_n\}$ in X *converges* to $x \in X$, written $x_n \to x$ as $n \to \infty$ or

$$\lim_{n \to \infty} x_n = x,$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$n > N \text{ implies that } d(x_n, x) < \epsilon.$$

That is, $x_n \to x$ if $d(x_n, x) \to 0$ as $n \to \infty$. Equivalently, $x_n \to x$ as $n \to \infty$ if for every open ball $B_\epsilon(x)$ there exists $N \in \mathbb{N}$ such that $x_n \in B_\epsilon(x)$ for all $n > N$.

Definition 2.5.18. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that the mapping $f : X \to Y$ is *continuous at a point* $x_0 \in X$, if for every $\epsilon > 0$ and $x \in X$ there is a $\delta > 0$ such that $d_Y(fx, fx_0) < \epsilon$ whenever $d_X(x, x_0) < \delta$.

Theorem 2.5.19. *A mapping T from a metric space (X, d_X) into a metric space (Y, d_Y) is continuous at a point $x_0 \in X$ if and only if*

$$x_n \xrightarrow{d_X} x_0 \quad \Rightarrow \quad Tx_n \xrightarrow{d_Y} Tx_0.$$

Definition 2.5.20. Let (X,d) be a metric space. The sequence $\{x_n\}$ in X is called a *Cauchy sequence* if it holds that, for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \epsilon$ for $m, n \geq N$.

Lemma 2.5.21. *Let (X,d) be a metric space, $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n,x_m) \to 0$ when $n,m \to \infty$.*

The following lemma will be useful later.

Lemma 2.5.22 ([27])**.** *Let (X,d) be a metric space and $\{x_n\}$ a sequence in X such that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. If $\{x_n\}$ is not Cauchy in X, then there exist $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) > k$ such that*

$$\begin{aligned}\lim_{k\to\infty} d(x_{m(k)},x_{n(k)}) &= \lim_{k\to\infty} d(x_{m(k)},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)+1}) \\ &= \lim_{k\to\infty} d(x_{m(k)-1},x_{n(k)+1}) = \lim_{k\to\infty} d(x_{m(k)+1},x_{n(k)+1}) = \epsilon.\end{aligned}$$

Definition 2.5.23. Let (X,d) be a metric space. If every Cauchy sequence is convergent in X, then X is called a *complete metric space.*

Example 2.5.24. The usual metric space $\mathbb{R}$ is a complete metric space.

Example 2.5.25. The Euclidean metric space $\mathbb{R}^n$ is a complete metric space.

Example 2.5.26. The unitary metric space $\mathbb{C}^n$ is a complete metric space.

Example 2.5.27. The sequence space (ℓ^∞, d_∞) is a complete metric space.

Example 2.5.28. The sequence space (c, d_c) is a complete metric space.

Example 2.5.29. The function space $(C[a,b], d_\infty)$ is a complete metric space.

2.6 b-Metric spaces

In 1989, Bakhtin [28] introduced the concept of a b-metric space as follows:

Definition 2.6.1 ([28, 29])**.** Let X be a nonempty set and $b \geq 1$ be a given real number. A mapping $d : X \times X \to [0,\infty)$ is said to be a *b-metric* on X if the following conditions hold for all $x,y,z \in X$:

(b1) $d(x,y)=0$ if and only if $x=y$;

(b2) $d(x,y)=d(y,x)$;

(b3) $d(x,z)\leq b[d(x,y)+d(y,z)]$.

Also, the triplet (X,d,b) is called a b-metric space with the coefficient $b\geq 1$.

It is clear that the definition of a b-metric space is an extension of a standard metric space. The following are some examples of b-metric spaces.

Example 2.6.2. The set of real numbers endowed with the mapping $d:\mathbb{R}\times\mathbb{R}\to[0,\infty)$ defined by

$$d(x,y)=|x-y|^p$$

for all $x,y\in\mathbb{R}$, where $p\geq 1$ is a real number, is a b-metric space with the coefficient $b=2^{p-1}$.

Example 2.6.3 ([30])**.** The set $l^p(\mathbb{R}):=\{\{x_n\}\in\mathbb{R}:\Sigma_{n=1}^{\infty}|x_n|^p<1\}$, where $0<p<1$, together with the mapping $d:l^p(\mathbb{R})\times l^p(\mathbb{R})\to[0,\infty)$ defined by

$$d(x,y):=\left(\Sigma_{n=1}^{\infty}|x_n-y_n|^p\right)^{1/p},$$

where $x=\{x_n\},y=\{y_n\}\in l^p(\mathbb{R})$, is a b-metric space with the coefficient $b=2^{1/p}>1$.

Example 2.6.4 ([30])**.** The set $L_p[0,1]:=\{x:[0,1]\to\mathbb{R}:\int_0^1|x(t)|^p dt<1\}$, where $0<p<1$, together with the mapping $d:L_p[0,1]\times L_p[0,1]\to[0,\infty)$ defined by

$$d(x,y):=\left(\int_0^1|x(t)-y(t)|^p dt\right)^{1/p},$$

where $x,y\in L_p[0,1]$, is a b-metric space with the coefficient $b=2^{1/p}>1$.

Example 2.6.5 ([30])**.** The set $X=\{0,1,2\}$ with the mapping $d:X\times X\to[0,\infty)$ defined by

$$d(0,0)=d(1,1)=d(2,2)=0$$

$$d(1,0)=d(0,1)=d(2,1)=d(1,2)=1$$

and

$$d(2,0) = d(0,2) = c$$

where c is given real number such that $c \geq 2$. is a b-metric space with the coefficient $b = \frac{c}{2} \geq 1$.

Example 2.6.6 ([31]). Let (X,d) be a metric space. The mapping $d : X \times X \to [0,\infty)$ defined by

$$\rho(x,y) := (d(x,y))^p,$$

where $p \geq 1$ is a real number, is a b-metric on X with the coefficient $b = 2^{p-1} \geq 1$.

Next, we recall the following basic knowledge in b-metric spaces.

Definition 2.6.7 ([32]). Let (X,d) be a b-metric space. The sequence $\{x_n\}$ in X is called:

(a) a*convergent* sequence if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $x_n \to x$ as $n \to \infty$;

(b) a *Cauchy* sequence if and only if $d(x_n,x_m) \to 0$ as $n,m \to \infty$.

Remark 2.6.8. In a b-metric space (X,d), the following assertions hold:

(a) a convergent sequence has a unique limit;

(b) each convergent sequence is Cauchy;

(c) in general, a b-metric is not continuous.

Definition 2.6.9 ([32]). The b-metric space (X,d) is *complete* if every Cauchy sequence in X is convergent in X.

2.7 Binary relations

Definition 2.7.1. Let X be a nonempty set. A subset $\Re$ of $X \times X$ is called a binary relation on X.

Remark 2.7.2. Let $\Re$ be a binary relation on a nonempty set X. Note that for each pair $x, y \in X$, one of the following conditions holds:

(a) $(x,y) \in \Re$, which amounts to saying that "x is $\Re$-*related to* y" or "x *relates to* y *under* $\Re$." Sometimes, we write $x\Re y$ instead of $(x,y) \in \Re$;

(b) $(x,y) \notin \Re$ which means that "x *is not* $\Re$-*related to* y" or "x *does not relate to* y *under* $\Re$."

In this dissertation, for a binary relation on a nonempty set X and $x, y \in X$, we write $x\Re^{\neq}y$ whenever $x\Re y$ and $x \neq y$.

Definition 2.7.3. Let $\Re$ be a binary relation defined on a nonempty set X and $x, y \in X$. We say that x and y are $\Re$-*comparative* if either $(x,y) \in \Re$ or $(y,x) \in \Re$. We denote it by $[x,y] \in \Re$.

Definition 2.7.4 ([33]). Let X be a nonempty set and T a self-mapping on X. A binary relation $\Re$ defined on X is said to be T-*closed* if for any $x, y \in X$,

$$(x,y) \in \Re \Rightarrow (Tx, Ty) \in \Re.$$

Example 2.7.5. Let $X = [0, \infty)$ and $T : X \to X$ be defined by $Tx = x^2$ for all $x \in X$. Define a relation $\Re$ by

$$\Re = \{(x,y) \in X \times X : x \leq y\}.$$

Then $\Re$ is T-closed.

Definition 2.7.6 ([33]). Let X be a nonempty set and $\Re$ a binary relation on a nonempty set X. A sequence $\{x_n\} \subseteq X$ is said to be $\Re$-*preserving* if

$$(x_n, x_{n+1}) \in \Re$$

for all $n \in \mathbb{N}$.

Example 2.7.7. Let $X = \mathbb{R}$ and $\{x_n\} = \{n\}$. Define a relation $\Re$ by

$$\Re = \{(x,y) \in X \times X : x \leq y\}.$$

Then $\{x_n\}$ is $\Re$-preserving since $(x_n, x_{n+1}) = (n, n+1) \in \Re$.

Example 2.7.8. Let $X = \mathbb{R}$ and $\{x_n\} = \left\{\frac{1}{2^n} : n \in \mathbb{N}\right\}$. Define a relation $\Re$ by

$$\Re = \left\{(0,0), \left(0, \frac{1}{2^n}\right), \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right)\right\}.$$

Then $\{x_n\}$ is $\Re$-preserving since $(x_n, x_{n+1}) = \left(\frac{1}{2^n}, \frac{1}{2^{n+1}}\right) \in \Re$.

Definition 2.7.9 ([33])**.** Let (X,d) be a metric space. A binary relation $\Re$ defined on X is said to be *d-self-closed* if whenever $\{x_n\}$ is an $\Re$-preserving sequence and

$$x_n \xrightarrow{d} x \quad \text{as} \quad n \to \infty,$$

then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \Re$ for all $k \in \mathbb{N}$.

Example 2.7.10. A relation $\Re$ in Example 2.7.8 is d-self-closed with the usual metric d.

Definition 2.7.11 ([34])**.** Let X be a nonempty set and $\Re$ a binary relation on X. A subset E of X is said to be $\Re$-*directed* if for each $x, y \in E$, there exists $z \in X$ such that $(x,z) \in \Re$ and $(y,z) \in \Re$.

Definition 2.7.12 ([35])**.** Let X be a nonempty set and $\Re$ a binary relation on X.

(a) The *inverse, transpose* or *dual relation* of $\Re$ denoted by $\Re^{-1}$, is defined by

$$\Re^{-1} = \{(x,y) \in X \times X : (y,x) \in \Re\}.$$

(b) The *symmetric closure* of $\Re$, denoted by $\Re^s$, is defined to be the set $\Re \cup \Re^{-1}$ (i.e., $\Re^s := \Re \cup \Re^{-1}$). Indeed, $\Re^s$ is the smallest symmetric relation on X containing $\Re$.

Proposition 2.7.13 ([33])**.** *For a binary relation $\Re$ defined on a nonempty set X, we have*

$$(x,y) \in \Re^s \Leftrightarrow [x,y] \in \Re.$$

Definition 2.7.14 ([36])**.** Let X be a nonempty set and $\Re$ a binary relation on X. For $x, y \in X$, a *path of length k* (where k is a natural number) in $\Re$ from x to y is a finite sequence $\{z_0, z_1, z_2, ..., z_k\} \subseteq X$ satisfying the following conditions:

(a) $z_0 = x$ and $z_k = y$,

(b) $(z_i, z_{i+1}) \in \Re$ for all $i = 0, 1, 2, \ldots, k-1$.

Note that a path of length k involves $k+1$ elements of X, although they are not necessarily distinct. We denote by $\Upsilon(x, y, \Re)$ the family of all paths in $\Re$ from x to y.

Definition 2.7.15 ([37])**.** Let $\Re$ be a binary relation on a nonempty set X. A subset $Z \subseteq X$ is said to be $\Re$-*connected* if for each $x, y \in Z$, there exists a path in $\Re$ from x to y.

Definition 2.7.16 ([38])**.** Let $\Re$ be a binary relation on a nonempty set X. A sequence $\{x_n\} \subseteq X$ is said to be: $\Re$-*nondecreasing* if $x_n \Re x_{n+1}$ for all $n \in \mathbb{N}$; $\Re$-*increasing* if $x_n \Re^{\neq} x_{n+1}$ for all $n \in \mathbb{N}$.

Remark 2.7.17. Note that the notion $\Re$ is T-closed is equivalent to say that T is $\Re$-nondecreasing.

Definition 2.7.18 ([39])**.** A binary relation $\Re$ defined on a nonempty set X is said to be

(a) *reflexive* if $(x, x) \in \Re$ for all $x \in X$,

(b) *irreflexive* if $(x, x) \notin \Re$ for all $x \in X$,

(c) *symmetric* if $(x, y) \in \Re$ implies $(y, x) \in \Re$,

(d) *antisymmetric* if $(x, y) \in \Re$ and $(y, x) \in \Re$ implies $x = y$,

(e) *transitive* if $(x, y) \in \Re$ and $(y, z) \in \Re$ implies $(x, z) \in \Re$,

(f) *complete, connected* or *dichotomous* if $[x, y] \in \Re$ for all $x, y \in X$.

Definition 2.7.19. A *partially ordered set* is a set X together with a *partial odering* (also called *partial order*), that is, a binary relation which is written $\preceq$ and satisfies the following conditions for any $x, y, z \in X$:

(a) $x \preceq x$ (Reflexive);

(b) $x \preceq y$ and $y \preceq x$, then $x = y$ (Antisymmetric);

(c) $x \preceq y$ and $y \preceq z$, then $x \preceq z$ (Transitive).

Also, the ordered pair $(X, \preceq)$ is called a partially ordered set.

Remark 2.7.20. Note that we can write a partial ordering $\succeq$ on a set X as follows:

$$y \succeq x \Leftrightarrow x \preceq y.$$

for all $x, y \in X$.

Definition 2.7.21 ([40])**.** Let T be a self-mapping on a nonempty set X. A binary relation $\Re$ on X is said to be

(a) T*-transitive* if it is transitive on TX;

(b) T*-orbitally transitive* if it is transitive on the *orbit* $O(x) = \{x, Tx, T^2x, \ldots\}$ of x under T for all $x \in X$.

Remark 2.7.22. Note that the following implication are obvious and the converse is not true in general.

Transitivity $\Longrightarrow$ T-transitivity $\Longrightarrow$ T-orbitally transitivity

Example 2.7.23 ([40])**.** Let $X = \left\{0, \frac{1}{2}, \frac{1}{2^2}, \ldots\right\}$ and a relation $\Re$ be defined by

$$\Re = \{(x,y) \in X \times X : x > y > 0\} \cup \left\{(0,0), \left(0, \frac{1}{4}\right), \left(0, \frac{1}{2^n}\right) : n \geq 4\right\}.$$

Define $T : X \to X$ by $Tx = \frac{1}{2}x$ for all $x \in X$. Then $\Re$ is T-orbitally transitive which is not T-transitive since $\left(0, \frac{1}{4}\right), \left(\frac{1}{4}, \frac{1}{8}\right) \in \Re$ but $\left(0, \frac{1}{8}\right) \notin \Re$.

Definition 2.7.24 ([40])**.** Let $\Re$ be a binary relation on a nonempty set X and $T : X \to X$ be a given mapping. A sequence $\{x_n\}$ on X is said to be: a $(T;\Re)$-*Picard sequence* if it is a Picard sequence, that is, $x_{n+1} = Tx_n = T^n x_0$ for all $n \in \mathbb{N}$ and x_0 is a given point in X, and $x_n \Re x_{n+1}$ for all $n \in \mathbb{N}$; a $(T;\Re)$-*increasing-Picard sequence* if it is a Picard sequence and $x_n \Re^{\neq} x_{n+1}$ for all $n \in \mathbb{N}$.

Proposition 2.7.25 ([40]). *Let (X,d) be a metric space endowed with a binary relation $\Re$ and T be a self-mapping on X. Suppose that the following conditions hold:*

(i) $\Re$ is T-closed;

(ii) there exists $x_0 \in X$ such that $x_0 \Re T x_0$.

Then there exists a $(T,\Re)-$Picard sequence based at the initial point x_0.

Definition 2.7.26 ([41]). Let (X,d) be a metric space. A self-mapping T on X is said to be an *orbitally continuous* if for each $x,z \in X$ and any sequence $\{n_i\}$ of positive integers with $\lim_{i\to\infty} T^{n_i}x = z \in X$, we have $\lim_{i\to\infty} TT^{n_i}x = Tz$.

Definition 2.7.27 ([42]). Let (X,d) be a metric space endowed with a binary relation $\Re$. A self-mapping T on X is said to be $\Re$-*continuous* if $Tx_n \to Tx$ for all sequence $\{x_n\} \subseteq X$ such that $x_n \to x$ and $x_n \Re x_m$ for all n,m with $n < m$.

Definition 2.7.28 ([40]). Let (X,d) be a metric space endowed with a binary relation $\Re$. A self-mapping T on X is said to be *orbitally* $\Re$-*continuous* if for all $x,z \in X$ and any sequence $\{n_i\}$ of positive integers, we have $T^{n_i}x \to z$ and $T^{n_i}x \Re T^{n_{i+1}}x$ (for all $i \in \mathbb{N}$) imply $TT^{n_i}x \to Tz$.

The following implications show that the concept of orbitally $\Re-$continuity is weaker than $\Re$-continuity, orbitally continuity and continuity.

Continuity	$\Rightarrow$	Orbitally continuity
$\Downarrow$		$\Downarrow$
$\Re$-continuity	$\Rightarrow$	Orbitally $\Re$-continuity

Definition 2.7.29 ([42]). Let (X,d) be a metric space equipped with a binary relation $\Re$. A subset $B \subseteq X$ is said to be $(\Re,d)$-*increasingly regular* if for every $\Re-$increasing sequence $\{x_n\} \subseteq X$ such that $x_n \to x \in X$ as $n \to \infty$, we have $x_n \Re x$ for all n.

Definition 2.7.30 ([38]). Let (X,d) be a metric space. A subset $G \subseteq X$ is said to be *precomplete* if each Cauchy sequence $\{x_n\} \subseteq G$ converges to some $x \in X$.

Definition 2.7.31 ([38]). Let (X,d) be a metric space endowed with a binary relation $\Re$. A subset $G \subseteq X$ is said to be $(\Re,d)$-*increasingly precomplete* if each $\Re$-increasing Cauchy sequence $\{x_n\} \subseteq G$ converges to some $x \in X$.

Note that every precomplete subset of X is $(\Re,d)$-increasingly precomplete whatever the binary relation $\Re$.

2.8 Contraction mappings and generalized contraction mappings

2.8.1 Banach contraction mappings

In 1922, Banach [1] introduced a contraction mapping known as the Banach contraction mapping as follows:

Definition 2.8.1 ([1]). Let (X,d) be a metric space. A mapping $T: X \to X$ is said to be *contraction mapping* if there exists $k \in [0,1)$ such that

$$d(Tx,Ty) \le kd(x,y) \tag{2.8.1}$$

for all $x,y \in X$. The constant k is called *Banach constant.*

Example 2.8.2. Let $X = \mathbb{R}$ with the usual metric d. Define a mapping $T: X \to X$ by

$$Tx = \frac{1}{2}x$$

for all $x \in X$. Then T is a contraction mapping with the Banach constant $k = \frac{1}{2}$.

Example 2.8.3. Let $X = [0,1]$ with the usual metric d. Define a mapping $T:$ $X \to X$ by

$$Tx = cx^2$$

for all $x \in X$, where $c \in \left[0, \frac{1}{2}\right)$. Then T is a contraction mapping with the Banach constant $k = 2c$.

Example 2.8.4. Let $X = \mathbb{R}$ with the usual metric d. Define a mapping $T: X \to X$ by

$$Tx = \cos(\cos x)$$

for all $x \in X$. Then T is a contraction mapping with the Banach constant $k = \sin 1$.

The author also established a remarkable fixed point theorem known as Banach contraction principle as follows:

Theorem 2.8.5 ([1])**.** *Let (X,d) be a complete metric space. If $T: X \to X$ is a contraction mapping, then T has a unique fixed point $\omega \in X$, and for each $x \in X$, we have*

$$\lim_{n\to\infty} T^n x = \omega.$$

Moreover, for each $x \in X$, we have

$$d(T^n x, \omega) \leq \frac{k^n}{1-k} d(Tx, x)$$

for all $n \in \mathbb{N}$.

Example 2.8.6. Let $X = \mathbb{R}$ with the usual metric d. Define a mapping $T: X \to X$ by $Tx = \frac{1}{2}x$ for all $x \in X$. Then T is a contraction mapping with the Banach constant $k = \frac{1}{2}$. It follows from Banach contraction principle that T has a unique fixed point. In this case, a point 0 is a unique fixed point of T.

2.8.2 $\mathcal{Z}$-Contraction mappings

In 2015, Khojasteh *et al.* [22] introduced the definition of a *simulation function* as follows.

Definition 2.8.7 ([22])**.** A mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a simulation function if it satisfies the following conditions:

(ζ1) $\zeta(0,0) = 0$;

(ζ2) $\zeta(t,s) < s - t$ for all $t, s > 0$;

(ζ3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n > 0$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

We denote the set of all simulation functions by $\mathcal{Z}$.

Example 2.8.8 ([22]). Let $\zeta_1, \zeta_2, \zeta_3, : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by the following rules:

1. $\zeta_1(t,s) = \psi(s) - \phi(t)$ for all $t,s \in [0,\infty)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are two continuous functions such that $\psi(t) = \phi(t) = 0$ if and only if $t = 0$ and $\psi(t) < t \leq \phi(t)$ for all $t > 0$;

2. $\zeta_2(t,s) = s - \frac{f(t,s)}{g(t,s)}t$ for all $t,s \in [0,\infty)$, where $f,g : [0,\infty) \to [0,\infty)$ are two continuous functions with respect to each variable such that $f(t,s) > g(t,s)$ for all $t,s > 0$;

3. $\zeta_3(t,s) = s - \varphi(s) - t$ for all $t,s \in [0,\infty)$, where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0$ if and only if $t = 0$.

Then ζ_i is a simulation function for all $i = 1,2,3$.

Definition 2.8.9 ([22]). Let (X,d) be a metric space, $T : X \to X$ be a mapping and $\zeta \in \mathcal{Z}$. Then T is called a *$\mathcal{Z}$-contraction mapping* with respect to ζ if the following condition is satisfied

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0 \tag{2.8.2}$$

for all $x,y \in X$.

Remark 2.8.10 ([22]). Note that $\zeta(t,s) < 0$ for all $t,s > 0$. Therefore, if T is a $\mathcal{Z}$-contraction mapping with respect to $\zeta \in \mathcal{Z}$ then

$$d(Tx,Ty) < d(x,y) \tag{2.8.3}$$

for all distinct $x,y \in X$. This means that every $\mathcal{Z}$-contraction mapping is contractive, and therefore T is continuous.

Theorem 2.8.11 ([22]). *Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to ζ. Then T has a unique fixed point u in X and for every $x_0 \in X$ the Picard sequence $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, converges to the fixed point of T.*

The following example supports the Theorem 2.8.11.

Example 2.8.12 ([22]). Let $X = [0,1]$ and $d : X \times X \to \mathbb{R}$ be defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Then (X,d) is a complete metric space. Define a mapping $T : X \to X$ as $Tx = \frac{x}{x+1}$ for all $x \in X$. Then T is not a Banach contraction, but is a $\mathcal{Z}$-contraction with respect to $\zeta \in \mathcal{Z}$, where

$$\zeta(t,s) = \frac{s}{s+1} - t$$

for all $t,s \in [0,\infty)$. Indeed, if $x,y \in X$, then

$$\begin{aligned} \zeta(d(Tx,Ty),d(x,y)) &= \frac{d(x,y)}{1+d(x,y)} - d(Tx,Ty) \\ &= \frac{|x-y|}{1+|x-y|} - \left|\frac{x}{x+1} - \frac{y}{y+1}\right| \\ &= \frac{|x-y|}{1+|x-y|} - \left|\frac{|x-y|}{(x+1)(y+1)}\right| \\ &\geq 0. \end{aligned}$$

Note that, all the conditions of Theorem 2.8.11 are satisfied and so T has a unique fixed point. In this case, a point 0 is a fixed point of T.

Next year, Demma *et al.* [24] gave the definition of a b-simulation function as follows:

Definition 2.8.13 ([24]). Let $b \geq 1$ be a given real number. A mapping $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ is called a *b-simulation function* if it satisfies the following conditions:

(ξ_1) $\xi(t,s) < s - t$ for all $t,s > 0$;

(ξ_2) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq b \lim_{n\to\infty} t_n < \infty,$$

then

$$\limsup_{n\to\infty} \xi(bt_n, s_n) < 0.$$

We denote the set of all b-simulation functions by $\mathcal{Z}^b$.

Remark 2.8.14. Note that every simulation function is a b-simulation function with $b = 1$.

We give an example of a b-simulation function as follows:

Example 2.8.15. Let $b \geq 1$ and $\xi : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be defined by $\xi_(t,s) = \psi(s) - \phi(t)$, where $\phi, \psi : [0,\infty) \to [0,\infty)$ are functions such that $\psi(t) < t \leq \phi(t)$ for all $t > 0$ and ψ is continuous and nondecreasing. Then ξ satisfies conditions (ξ_1) and (ξ_2) in the Definition 2.8.13 as follows:

(ξ_1) Let $s,t > 0$. Then

$$\xi(t,s) = \psi(s) - \phi(t) < s - t.$$

(ξ_2) Let $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that

$$0 < \lim_{n\to\infty} t_n \leq \liminf_{n\to\infty} s_n \leq \limsup_{n\to\infty} s_n \leq b \lim_{n\to\infty} t_n < \infty.$$

Then

$$\begin{aligned}
\limsup_{n\to\infty} \xi(bt_n, s_n) &= \limsup_{n\to\infty} [\psi(s_n) - \phi(bt_n)] \\
&\leq \limsup_{n\to\infty} \psi(s_n) - \liminf_{n\to\infty} \phi(bt_n) \\
&\leq \psi\left(\limsup_{n\to\infty} s_n\right) - \liminf_{n\to\infty} bt_n \\
&< \limsup_{n\to\infty} s_n - \liminf_{n\to\infty} bt_n \\
&\leq 0.
\end{aligned}$$

Therefore, ξ is a b-simulation function.

Definition 2.8.16 ([24])**.** Let (X,d,b) be a b-metric space, $T : X \to X$ be a mapping and $\xi \in \mathcal{Z}^b$. Then T is called a *$\mathcal{Z}$-contraction mapping with respect to* a *b-simulation functions* ξ if

$$\xi(bd(Tx,Ty), d(x,y)) \geq 0 \tag{2.8.4}$$

for all $x, y \in X$.

Theorem 2.8.17 ([24])**.** *Let (X,d,b) be a complete b-metric space and let $T : X \to X$ be a mapping. If T is a $\mathcal{Z}$-contraction mapping with respect to a b-simulation function ξ. Then T has a unique fixed point.*

2.8.3 F-Contraction mappings

In 2012, Wardowski [26] introduced the concept of an F-contraction mapping as follows:

Definition 2.8.18 ([26])**.** Let (X,d) be a metric space and $T : X \to X$ be a mapping. Then T is called an *F-contraction mapping* if there exists $\tau > 0$ such that

$$\forall x,y \in X \quad [d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))], \tag{2.8.5}$$

where $F : \mathbb{R}^+ \to \mathbb{R}$ is a mapping satisfying:

$(F1)$ F is strictly increasing, i.e. for all $\alpha,\beta \in \mathbb{R}^+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$;

$(F2)$ for each sequence $\{\alpha_n\} \subseteq \mathbb{R}^+$, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

$(F3)$ there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$;

Example 2.8.19 ([26])**.** Let $F_1, F_2, F_3, : \mathbb{R}^+ \to \mathbb{R}$ be defined by the following rules:

1. $F_1(\alpha) = \ln\alpha$ for all $\alpha \in \mathbb{R}^+$;
2. $F_2(\alpha) = \alpha + \ln\alpha$ for all $\alpha \in \mathbb{R}^+$;
3. $F_3(\alpha) = \frac{-1}{\sqrt{\alpha}}$ for all $\alpha \in \mathbb{R}^+$.

Then F_i is a function satisfying the conditions $(F1)-(F3)$ for all $i = 1,2,3$.

Remark 2.8.20 ([26])**.** From $(F1)$ and 2.8.5 it is easy to conclude that every F-contraction T is a contractive mapping, i.e.

$$d(Tx,Ty) < d(x,y), \tag{2.8.6}$$

for all $x,y \in X$ with $Tx \neq Ty$. Thus every F-contraction is a continuous mapping.

Moreover, Wardowski used concept of an F-contraction mapping to consider the existence and uniqueness of fixed point in complete metric spaces and given an example as follows:

Theorem 2.8.21 ([26]). *Let (X,d) be a complete metric space and let $T:X\to X$ be an F-contraction mapping. Then T has a unique fixed point in $x^*\in X$ and for every $x^*\in X$ the sequence $\{T^n x\}$ converges to x^*.*

Example 2.8.22 ([26]). Consider the sequence $\{S_n\}$ as follows:

$$S_n = 1+2+\cdots+n = \frac{n(n+1)}{2}$$

for all $n\in\mathbb{N}$. Let $X=\{S_n : n\in\mathbb{N}\}$ and $d: X\times X\to\mathbb{R}$ be defined by $d(x,y)=|x-y|$ for all $x,y\in X$. Then (X,d) is a complete metric space. Define the mapping $T:X\to X$ by

$$T(S_1)=S_1 \text{ and } T(S_n)=S_{n-1} \text{ for all } n>1.$$

The mapping T is not the Banach contraction mapping. Indeed, we get

$$\lim_{n\to\infty}\frac{d(T(S_n),T(S_1))}{d(S_n,S_1)} = \lim_{n\to\infty}\frac{S_{n-1}-1}{S_n-1}=1.$$

On the other side taking $F(\alpha)=\alpha+\ln\alpha$ for all $\alpha\in\mathbb{R}^+$, we obtain that T is an F-contraction mapping with $\tau=1$. To see this, let us consider the following calculations: First, observe that

$$\forall m,n\in\mathbb{N}[T(S_m)\neq T(S_n)\Leftrightarrow((m>2\wedge n=1)\vee(m>n>1))].$$

For every $m\in\mathbb{N}$, $m>2$ we have

$$\begin{aligned}\frac{d(T(S_m),T(S_1))}{d(S_m,S_1)}e^{d(T(S_m),T(S_1))-d(S_m,S_1)} &= \frac{S_{m-1}-1}{S_m-1}e^{S_{m-1}-S_m}\\ &= \frac{m^2-m-2}{m^2+m-2}e^{-m}\\ &< e^{-m}\\ &< e^{-1}.\end{aligned}$$

For every $m,n \in \mathbb{N}, m > n > 1$ the following holds

$$\begin{aligned}\frac{d(T(S_m),T(S_n))}{d(S_m,S_n)}e^{d(T(S_m),T(S_n))-d(S_m,S_n)} &= \frac{S_{m-1}-S_{n-1}}{S_m-S_n}e^{S_n-S_{n-1}+S_{m-1}-S_m}\\ &= \frac{m+n-1}{m+n+1}e^{n-m}\\ &< e^{n-m}=e^{-1}.\end{aligned}$$

Clearly S_1 is a fixed point of T.

In 2014, Piri and Kumam [43] used the following condition instead of the conditions $(F2)$ and $(F3)$ in Definition 2.8.18:

$(F2')$ $\inf F = -\infty$;

$(F3')$ F is continuous on $(0,\infty)$.

We denote by $\mathfrak{F}$ the set of all functions satisfying the conditions $(F1)$, $(F2')$ and $(F3')$.

Example 2.8.23 ([43])**.** Let $F_1, F_2, F_3, : \mathbb{R}^+ \to \mathbb{R}$ be defined by the following rules:

1. $F_1(\alpha) = -\frac{1}{\alpha}$ for all $\alpha \in \mathbb{R}^+$;
2. $F_2(\alpha) = -\frac{1}{\alpha} + \alpha$ for all $\alpha \in \mathbb{R}^+$;
3. $F_3(\alpha) = \frac{1}{1-e^{\alpha}}$ for all $\alpha \in \mathbb{R}^+$.

Then F_i is a function satisfying the conditions $(F1)$, $(F2')$ and $(F3')$ for all $i = 1,2,3$.

Theorem 2.8.24 ([43])**.** *Let (X,d) be a complete metric space and let $T : X \to X$ be a mapping. Suppose $F \in \mathfrak{F}$ and there exists $\tau > 0$ such that*

$$\forall x,y \in X \quad [d(Tx,Ty) > 0 \Rightarrow \tau + F(d(Tx,Ty)) \leq F(d(x,y))]. \tag{2.8.7}$$

Then T has a unique fixed point in $x^* \in X$ and for every $x^* \in X$ the sequence $\{T^n x\}$ converges to x^*.

2.8.4 Contraction mappings concerning weak altering distance functions

We recall the definition of an altering distance function, which was introduced by Khan *et al.* [25].

Definition 2.8.25. A function $\psi : [0,\infty) \to [0,\infty)$ is said to be *altering distance function* if it satisfies the following conditions:

(ψ1) ψ is continuous and nondecreasing;

(ψ2) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.8.26. Define $\psi_1, \psi_2, \psi_3, \psi_4 : [0,\infty) \to [0,\infty)$ by

$$\psi_1(t) = t^2,\ \psi_2(t) = t,\ \psi_3(t) = te^t,\ \psi_4(t) = \ln(1+t)$$

for all $t \geq 0$. Then ψ_i is a function satisfying the conditions (ψ1) and (ψ2) for all $i = 1,2,3,4$.

In 2012, Yan *et al.* [44] discussed some results on the existence and uniqueness of a fixed point in partially ordered metric spaces by using the concept of an altering distance function as follows.

Theorem 2.8.27 ([44]). *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Suppose that $T : X \to X$ is a continuous and nondecreasing mapping such that*

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)),$$

for all $x,y \in X$ with $x \succeq y$,where ψ is an altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is continuous function with the condition: $\psi(t) > \phi(t)$ for all $t > 0$. If there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$, then T has a fixed point.

Later, Sawangsup and Sintunavarat [19] introduced the following control functions which is a weaker version of altering distance functions.

Definition 2.8.28 ([19]). A function $\psi : [0,\infty) \to [0,\infty)$ is said to be a weak altering distance function if it satisfies the following conditions:

($\psi 1'$) ψ is lower semicontinuous and nondecreasing;

($\psi 2$) $\psi(t) = 0$ if and only if $t = 0$.

Example 2.8.29 ([19]). Define $\psi_1, \psi_2, \psi_3 : [0,\infty) \to [0,\infty)$ by

$$\psi_1(t) = \begin{cases} \ln(1+t) & \text{if } t \leq 1 \\ t & \text{if } t > 1 \end{cases},$$

$$\psi_2(t) = \begin{cases} t^2 & \text{if } t \leq 1 \\ e^t - 1 & \text{if } t > 1 \end{cases},$$

$$\psi_3(t) = \begin{cases} \frac{t^2}{2} & \text{if } t \leq 1 \\ t^2 & \text{if } t > 1 \end{cases}.$$

Then ψ_1, ψ_2 and ψ_3 are weak altering distance functions because ψ_1, ψ_2 and ψ_3 are lower semicontinuous and nondecreasing. Moreover, $\psi_i(t) = 0$ if and only if $t = 0$ for all $i = 1,2,3$.

The authors in [19] also used this concept to prove the following fixed point results in metric spaces endowed with a transitive relation, which is a generalization of fixed point results of Yan *et.al.* [44].

Theorem 2.8.30 ([19]). *Let (X,d) be a complete metric space and $\Re$ be a transitive relation on X. Suppose that $T : X \to X$ is a continuous mappings and $\Re$ is T-closed such that*

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)), \tag{2.8.8}$$

for all $x,y \in X$ with $(x,y) \in \Re$,where ψ is a weak altering distance function and $\phi : [0,\infty) \to [0,\infty)$ is a right upper semicontinuous function such that $\psi(t) > \phi(t)$ for all $t > 0$. If $X(T;\Re) := \{x \in X : (x,Tx) \in \Re\}$ is a nonempty set, then T has a fixed point.

2.9 Positive definite matrices and positive semidefinite matrices

In this dissertation, we will use the following matrix notations: $M(n)$ denotes the set of all $n \times n$ complex matrices, $\mathbb{C}^{n\times 1}$ is a set of all $n \times 1$ vectors. The entry in the i^{th} row and j^{th} column of the matrix A is denoted by, a_{ij} or $[A]_{ij}$.

Definition 2.9.1. The transpose of $m \times n$ matrix A is defined to be the $n \times m$ matrix A^T obtained by interchanging rows and columns in A. More precisely, if $A = [a_{ij}]$, then $[A^T]_{ij} = a_{ji}$. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{pmatrix}.$$

Definition 2.9.2. For $A = [a_{ij}]$, the conjugate matrix is defined to be $\overline{A} = [\overline{a}_{ij}]$, and the *conjugate transpose* of A is defined to be $\overline{A}^T = \overline{A^T}$. From now on, $\overline{A}^T$ will be denoted by A^*, so $[A^*]_{ij} = \overline{a_{ji}}$. Sometimes the matrix A^* is called the *adjoint* of A. For example,

$$\begin{pmatrix} 1-4i & i & 2 \\ 3 & 2+i & 0 \end{pmatrix}^* = \begin{pmatrix} 1+4i & 3 \\ -i & 2-i \\ 2 & 0 \end{pmatrix}.$$

Definition 2.9.3. A matrix $A \in M(n)$ is said to be a *Hermitian matrix* whenever $A = A^*$.

Now, we let $H(n)$ denote the set of Hermitian matrices.

Example 2.9.4. $A = \begin{pmatrix} 2 & 2+2i \\ 2-2i & 4 \end{pmatrix}$ is a Hermitian matrix because

$$\bar{A} = \begin{pmatrix} 2 & 2-2i \\ 2+2i & 4 \end{pmatrix} \text{ and so } A^* = \bar{A}^T = \begin{pmatrix} 2 & 2+2i \\ 2-2i & 4 \end{pmatrix} = A.$$

Example 2.9.5. $A = \begin{pmatrix} 8 & 11i \\ -11i & 0 \end{pmatrix}$ is a Hermitian matrix because

$$\bar{A} = \begin{pmatrix} 8 & -11i \\ 11i & 0 \end{pmatrix} \text{ and so } A^* = \bar{A}^T = \begin{pmatrix} 8 & 11i \\ -11i & 0 \end{pmatrix} = A.$$

Definition 2.9.6. For an $n \times n$ matrix A, scalars λ and vectors $x_{n\times 1} \neq 0$ satisfying $Ax = \lambda x$ are called *eigenvalues* and *eigenvectors* of A, respectively.

Theorem 2.9.7. *If $A \in M(n)$ is a Hermitian matrix, then all eigenvalues of A are real numbers.*

Example 2.9.8. Let $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$. Since $A = A^*$, we get A is a Hermitian matrix. It is easy to see that 1 and 4 are eigenvalues of A.

Example 2.9.9. Let $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. Since $A = A^*$, we get A is a Hermitian matrix. It is easy to see that 0 and 2 are eigenvalues of A.

Lemma 2.9.10. *For an $m \times n$ complex matrix A, the nonzero eigenvalues of A^*A and AA^* are equal and positive.*

Example 2.9.11. Let $A = \begin{pmatrix} i & 2+i & 3 \\ 0 & i & -3i \end{pmatrix}$. Then $A^* = \begin{pmatrix} i & 0 \\ 2+i & i \\ 3 & 3i \end{pmatrix}$ and so

$$AA^* = \begin{pmatrix} 15 & 1+7i \\ 1-7i & 10 \end{pmatrix} \text{ and } A^*A = \begin{pmatrix} 1 & 1-2i & -3i \\ 1+2i & 6 & 3-3i \\ 3i & 3+3i & 18 \end{pmatrix}.$$

Therefore, eigenvalues of AA^* are 5 and 20 and eigenvalues of A^*A are 0, 5 and 20. Note that all nonzero eigenvalues of A^*A and AA^* are equal and positive.

Definition 2.9.12. Hermitian matrix A is called a *positive definite matrix* if

$$x^*Ax > 0$$

for every nonzero $x \in \mathbb{C}^{n\times 1}$.

Definition 2.9.13. Hermitian matrix A is called a *positive semidefinite matrix* if

$$x^*Ax \geq 0$$

for every $x \in \mathbb{C}^{n\times 1}$.

Theorem 2.9.14. *For a Hermitian matrix A, the following statements are equivalent:*

(a) *A is positive definite;*

(b) *all eigenvalues of A are positive;*

(c) *$A = B^*B$ for some nonsingular*[1] *matrix B;*

(d) *$A = B^2$ for some positive definite B. Such a B is unique. We write $B = A^{\frac{1}{2}}$ and call it the (positive definite) square root of A.*

Let denote $P(n)$ by the set of all positive definite matrices.

Example 2.9.15. From Example 2.9.8, we see that A is a Hermitian and eigenvalues of A are $1, 4$, which are positive. Therefore, A is a positive definite matrix.

Theorem 2.9.16. *For a Hermitian matrix A, the following statements are equivalent:*

(a) *A is positive semidefinite;*

(b) *all eigenvalues of A are nonnegative;*

(c) *$A = B^*B$ for some matrix B;*

(d) *$A = B^2$ for some positive semidefinite B. Such a B is unique. We write $B = A^{\frac{1}{2}}$ and call it the (positive semidefinite) square root of A.*

Let denote $H^+(n)$ by the set of all positive definite matrices.

Example 2.9.17. From Example 2.9.9, we see that A is a Hermitian matrix and eigenvalues of A are $0, 2$, which are nonnegative. Therefore, A is a positive semidefinite matrix.

---

[1] An $n \times n$ matrix A is called *nonsingular* or *invertible* if there exists an $n \times n$ matrix B such that

$$AB = I_n = BA.$$

Any matrix B with the above property is called an *inverse* of A and it is denoted by A^{-1}.

Corollary 2.9.18. *If $A \in M(n)$ is a positive semidefinite matrix, then so is each A^k, $k = 1,2,...$*

Proposition 2.9.19. *The sum of positive semidefinite matrices is a positive semidefinite matrix. Also, if A is a positive semidefinite matrix and $a > 0$, then aA is also a positive semi-definite matrix.*

Example 2.9.20. Let $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & -i \\ 0 & i & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Since $A = A^*$ and $B = B^*$, we get A and B are Hermitian matrices. Moreover, eigenvalues of A are $2,2,4$ and eigenvalues of B are $1,2,2$. Therefore, A and B are positive definite matrices. Since

$$A+B = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 5 & -i \\ 0 & i & 5 \end{pmatrix},$$

which $A+B = (A+B)^*$ and eigenvalues of $A+B$ are $3,4,6$. Therefore, $A+B$ is a positive semidefinite matrix. If $a = 2$, then

$$2A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 6 & -2i \\ 0 & 2i & 6 \end{pmatrix}.$$

Note that $2A = (2A)^*$ and eigenvalues of $2A$ are $4,4,8$. Therefore, $2A$ is also positive semidefinite matrix.

Theorem 2.9.21. *If $A \in M(n)$ is a positive semidefinite matrix, and S is any $n \times m$ matrix, then S^*AS is a positive semidefinite matrix.*

Example 2.9.22. Let $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix}$. Now, we have A is a positive semidefinite matrix and so

$$S^*AS = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 1+2i \\ 1-2i & 1 \end{pmatrix}.$$

Since $S^*AS = (S^*AS)^*$, we get S^*AS is a Hermitian matrix. That is, eigenvalues of S^*AS are 0 and 6. This means that S^*AS is a positive semidefinite matrix.

Notation 2.9.23. *Let* $A,B \in H(n)$*. We write*

- $0 \preceq A$ *if* A *is a positive semidefinite matrix;*
- $0 \prec A$ *if* A *is a positive definite matrix.*
- $A \preceq B$ *if* $B - A$ *is a positive semidefinite matrix;*
- $A \prec B$ *if* $B - A$ *is a positive definite matrix.*

Note that $(H(n), \preceq)$ is a partially ordered set.

Theorem 2.9.24. *Let* $A,B \in H(n)$ *and let* S *be an* $n \times m$ *matrix. If* $A \succeq B$*, then* $S^*AS \succeq S^*BS$*.*

Example 2.9.25. Let $A = \begin{pmatrix} 12 & 12i \\ -12i & 12 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 5i \\ -5i & 5 \end{pmatrix}$. Then, A and B are Hermitian matrices and so

$$A - B = \begin{pmatrix} 7 & 7i \\ -7i & 7 \end{pmatrix},$$

which eigenvalues of $A - B$ are 0 and 14. Thus, $A - B$ are positive semidefinite. This means that $A \succeq B$. If $S = \begin{pmatrix} -2i & 3 \\ 4 & 1+3i \end{pmatrix}$, then

$$S^*AS - S^*BS = \begin{pmatrix} 28 & 14 \\ 14 & 7 \end{pmatrix},$$

which eigenvalues of $S^*AS - S^*BS$ are 0 and 35. Therefore, $S^*AS - S^*BS$ is also positive semidefinite. This means that $S^*AS \succeq S^*BS$.

2.10 Ky Fan norms and Thompson metrics

Definition 2.10.1. For $m \times n$ complex matrix A, the nonzero *singular values* of A are the positive square roots of the nonzero eigenvalues of A^*A (and AA^*).

Example 2.10.2. From Example 2.9.11, the nonzero eigenvalues of A^*A (and AA^*) are 5 and 20. Therefore, the nonzero singular values of A are $\sqrt{5}$ and $2\sqrt{5}$.

Proposition 2.10.3. *If A is a Hermitian matrix, then its singular values are the absolute values of its nonzero eigenvalues.*

Example 2.10.4. From the Hermitian matrix A in Example 2.9.8, we get

$$A^*A = \begin{pmatrix} 6 & 5+5i \\ 5-5i & 11 \end{pmatrix}$$

and so nonzero eigenvalues of A^*A are 1 and 16. Therefore, nonzero singular values of A are $\sqrt{1} = 1$ and $\sqrt{16} = 4$, which are absolute values of nonzero eigenvalues of A ($|1| = 1$ and $|4| = 4$).

Proposition 2.10.5. *Let A be an $n \times n$ matrix, then $\sum_{i=1}^{n} \lambda_i = tr(A)$, where λ_i are eigenvalues of A.*

Example 2.10.6. Let $A = \begin{pmatrix} 2 & 1+i \\ 1-i & 3 \end{pmatrix}$. From Example 2.9.8, we know that eigenvalues of A are 1 and 4. That is, $\lambda_1 = 1$, $\lambda_2 = 4$. Therefore, $\sum_{i=1}^{2} \lambda_i = 1+4 = 5 = 2+3 = tr(A)$.

Example 2.10.7. Let $A = \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$. From Example 2.9.9, we know that eigenvalues of A are 0 and 2. Assume that $\lambda_1 = 0$, $\lambda_2 = 2$. Therefore, $\sum_{i=1}^{2} \lambda_i = 0+2 = 2 = 1+1 = tr(A)$.

Definition 2.10.8.

1. The spectral norm $\|\cdot\| : M_{m\times n}(\mathbb{C}) \to \mathbb{R}$ is defined by

$$\|A\| = \sqrt{\lambda^+(A^*A)}$$

for all $A \in M_{m\times n}(\mathbb{C})$, where $\lambda^+(A^*A)$ is the largest eigenvalue of A^*A.

2. The Ky Fan norm (or trace norm) $\|\cdot\|_{tr} : M(n) \to \mathbb{R}$ is defined by

$$\|A\|_{tr} = \sum_{j=1}^{n} s_j(A)$$

for all $A \in M(n)$, where $s_j(A), j = 1,2,\ldots,n$ are the singular values of A.

Remark 2.10.9 ([17])**.** The set $H(n)$ endowed with the trace norm is a complete metric space.

Lemma 2.10.10 ([17])**.** *Let $A \succeq 0$ and $B \succeq 0$ be $n \times n$ matrices. Then*

$$0 \leq tr(AB) \leq \|A\| tr(B),$$

where $\|A\|$ is the spectral norm of a matrix A.

Example 2.10.11. Let $A = \begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 4i \\ -4i & 4 \end{pmatrix}$. It is easy to see that $A \succeq 0$ and $B \succeq 0$ and $tr(B) = 8$. Since

$$AA^* = \begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix} \begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix} = \begin{pmatrix} 18 & 18i \\ -18i & 18 \end{pmatrix},$$

which eigenvalues of AA^* are 0 and 36. Thus, $\|A\| = \sqrt{\lambda^+(A^*A)} = \sqrt{36} = 6$. Since

$$AB = \begin{pmatrix} 3 & 3i \\ -3i & 3 \end{pmatrix} \begin{pmatrix} 4 & 4i \\ -4i & 4 \end{pmatrix} = \begin{pmatrix} 24 & 24i \\ -24i & 24 \end{pmatrix},$$

we get $tr(AB) = 24 + 24 = 48$. Therefore,

$$0 \leq 48 = tr(AB) = (6)(8) = \|A\| tr(B).$$

Lemma 2.10.12 ([46])**.** *If $A \in H(n)$ satisfies $A \prec I$, then $\|A\| < 1$.*

Example 2.10.13. Let $A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Since $A = A^*$, we have $A \in H(n)$. Moreover,

$$I - A = \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 0.8 & 0 \\ 0 & 0 & 0.7 \end{pmatrix},$$

which eigenvalues of $I - A$ are 0.5, 0.8 and 0.7. This implies that $A \prec I$. Consider

$$A^*A = \begin{pmatrix} 0.25 & 0 & 0 \\ 0 & 0.04 & 0 \\ 0 & 0 & 0.09 \end{pmatrix},$$

which eigenvalues of $I - A$ are 0.04, 0.09 and 0.25. Therefore,

$$\|A\| = \sqrt{\lambda^+(A^*A)} = \sqrt{0.25} = 0.5 < 1.$$

Definition 2.10.14 ([21])**.** We use $d_T(,)$ as the Thompson metric on $P(n)$, which is defined by

$$d_T(A,B) = \log\{\max\{\alpha,\beta\}\}, \tag{2.10.1}$$

where $\alpha = \inf\{\delta : A \leq \delta B\} = \lambda^+(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})$, the maximum eigenvalue of $B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and $\beta = \inf\{\delta : A \leq \delta B\} = \lambda^+(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})$, the maximum eigenvalue of $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$.

Remark 2.10.15 ([21])**.** The set $P(n)$ is complete with respect to Thompson metric d_T.

The following properties of the Thompson metric for positive definite Hermitian matrices in the form of a lemma will be useful later.

Lemma 2.10.16 ([47])**.** *Let d_T be the Thompson metric on $P(n)$. Then the following assertions hold:*

(i) $d_T(A,B) = d_T(A^{-1},B^{-1}) = d_T(MAM^*, MBM^*)$ *for all* $A,B \in P(n)$ *and nonsingular matrix* M*;*

(ii) $d_T(A^r,B^r) \leq |\, r \,|\, d_T(A,B)$ *for all* $A,B \in P(n)$ *and* $r \in [-1,1]$*;*

(iii) $d_T(A+B,C+D) \leq \max\{d_T(A,C), d_T(B,D)\}$ *for all* $A,B,C,D \in P(n)$*. In particular,* $d_T(A+B,C+D) \leq d_T(B,D)$*.*

Lemma 2.10.17 ([48])**.** *Let* $A,B \in H(n)$*. IF* $A \succ B \succ 0$ *(or* $A \succeq B \succ 0$*), then* $A^\alpha \succ B^\alpha \succ 0$ *(or* $A^\alpha \succeq B^\alpha \succ 0$*) for all* $\alpha \in [0,1)$ *and* $0 \prec A^\alpha \prec B^\alpha$ *(or* $0 \prec A^\alpha \preceq B^\alpha$*) or all* $\alpha \in [-1,0)$*.*

CHAPTER 3

FIXED POINT RESULTS

In this chapter, the existence and uniqueness of a fixed point of new contraction mappings in the setting of metric spaces and b-metric spaces endowed with binary relations are given. Our investigation will be divided to three sections.

3.1 Fixed point results for $\mathcal{Z}^b_{\Re}$-contraction mappings

According to the concept of $\mathcal{Z}$-contraction mappings with respect to b-simulation functions, we will improve such mappings under an arbitrary binary relation $\Re$ namely $\mathcal{Z}^b_{\Re}$-contraction mappings and investigate the existence and uniqueness of a fixed point of $\mathcal{Z}^b_{\Re}$-contraction mappings in b-metric spaces. Now, we first give the definition of a b-d-self-closed as follows:

Definition 3.1.1. Let (X,d,b) be a b-metric space. A binary relation $\Re$ defined on X is called *b-d-self-closed* if whenever $\{x_n\}$ is an $\Re$-preserving sequence and $x_n \longrightarrow x$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with $[x_{n_k}, x] \in \Re$ for all $k \in \mathbb{N}$.

Let us denote $\mathcal{Z}^b$ by the set of all b-simulation functions. Now we introduce the concept of a $\mathcal{Z}$-contraction mapping with respect to b-simulation functions under an arbitrary binary relation $\Re$ as follows:

Definition 3.1.2. Let (X,d,b) be a b-metric space, $T : X \to X$ be a mapping and $\xi \in \mathcal{Z}^b$. If the following condition holds:

$$\xi(bd(Tx,Ty), d(x,y)) \geq 0 \quad \text{for all} \quad x,y \in X \quad \text{with} \quad (x,y) \in \Re, \tag{3.1.1}$$

then T is called a *$\mathcal{Z}$-contraction mapping with respect to b-simulation functions ξ under an arbitrary binary relation* $\Re$. But for the sake of simplicity, we call only *$\mathcal{Z}^b_{\Re}$-contraction mapping.*

First, we give the following useful proposition concerning the contractive condition of $\mathcal{Z}^b_{\Re}$-contraction mappings.

Proposition 3.1.3. *Let (X,b,d) be a b-metric space, $\Re$ be a binary relation on X, T be a self-mapping on X and $\xi \in \mathcal{Z}^b$. Then the following contractive conditions are equivalent:*

(*i*) $\xi(bd(Tx,Ty),d(x,y)) \geq 0, \quad \forall x,y \in X$ *with* $(x,y) \in \Re$,

(*ii*) $\xi(bd(Tx,Ty),d(x,y)) \geq 0, \quad \forall x,y \in X$ *with* $[x,y] \in \Re$.

Proof. The implication $(ii) \Rightarrow (i)$ is trivial. Conversely, assume that (i) holds. Take $x,y \in X$ with $[x,y] \in \Re$. If $(x,y) \in \Re$, then (ii) directly follows from (i). Now, suppose that $(y,x) \in \Re$, then using the symmetry of d and (i), we get

$$\xi(bd(Tx,Ty),d(x,y)) = \xi(bd(Ty,Tx),d(y,x)) \geq 0.$$

This shows that $(i) \Rightarrow (ii)$. This completes the proof. □

The following lemma is needed to establish the fixed point result endowed with a transitive relation.

Lemma 3.1.4. *Let (X,b,d) be a b-metric space, $\Re$ be a transitive relation on X, T be a self-mapping on X. Suppose that the following conditions hold:*

(*i*) *$X(T;\Re)$ is nonempty;*

(*ii*) *$\Re$ is T-closed;*

(*iii*) *there is a $\xi \in \mathcal{Z}^b$ such that T is a $\mathcal{Z}^b_{\Re}$-contraction mapping with respect to $\xi \in \mathcal{Z}^b$.*

If $\{x_n\}$ is a Picard sequence defined by $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$, where $x_0 \in X(T;\Re)$ and $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$, then the following assertions holds:

(*a*) $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$;

(*b*) $\{x_n\}$ *is a bounded sequence;*

(*c*) $\{x_n\}$ *is a Cauchy sequence.*

Proof. Since $(x_0, Tx_0) \in \Re$, using the T-closedness of $\Re$, we get

$$(Tx_0, T^2x_0), (T^2x_0, T^3x_0), ..., (T^nx_0, T^{n+1}x_0), ... \in \Re \tag{3.1.2}$$

and so $(x_n, x_{n+1}) \in \Re$ for all $n \in \mathbb{N}_0$. By the condition (iii) and (ξ_1), we have

$$\begin{aligned} 0 &\leq \xi(bd(T^nx_0, T^{n+1}x_0), d(T^{n-1}x_0, T^nx_0)) \\ &< d(T^{n-1}x_0, T^nx_0) - bd(T^nx_0, T^{n+1}x_0), \end{aligned}$$

for all $n \in \mathbb{N}$. The above inequality show that for each $n \in \mathbb{N}$

$$bd(x_n, x_{n+1}) < d(x_{n-1}, x_n),$$

which implies that $\{d(x_n, x_{n+1})\}$ is a monotonically decreasing sequence of positive real numbers. So, there exists some $c \geq 0$ such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = c. \tag{3.1.3}$$

Assume that $c > 0$. Setting $s_n := d(x_{n-1}, x_n)$, and $t_n := d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$, by the assumption (iii) and (ξ_2), we get

$$0 \leq \limsup_{n\to\infty} \xi(bd(x_n, x_{n+1}), d(x_{n-1}, x_n)) < 0,$$

which is a contraction. Therefore, $c = 0$. This implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{3.1.4}$$

Next, we show that the conclusion (b) holds. Suppose that $\{x_n\}$ is not a bounded sequence. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(x_{n_k}, x_{n_{k+1}}) > 1 \tag{3.1.5}$$

and

$$d(x_{n_k}, x_m) \leq 1 \tag{3.1.6}$$

for $n_k \leq m \leq n_{k+1}-1$. By the inequalities (3.1.5), (3.1.6), and the triangle inequality of a b-metric space, we have

$$\begin{aligned} 1 &< d(x_{n_k}, x_{n_{k+1}}) \\ &\leq bd(x_{n_k}, x_{n_{k+1}-1}) + bd(x_{n_{k+1}-1}, x_{n_{k+1}}) \\ &\leq b + bd(x_{n_{k+1}-1}, x_{n_{k+1}}). \end{aligned} \tag{3.1.7}$$

Letting $k \to \infty$ in (3.1.7) and using the fact in (a), we get

$$1 \leq \liminf_{k\to\infty} d(x_{n_k}, x_{n_{k+1}}) \leq \limsup_{k\to\infty} d(x_{n_k}, x_{n_{k+1}}) \leq b. \tag{3.1.8}$$

Again, from the assumption (iii) together with (ξ_1) and the triangle inequality of a b-metric space, we have

$$\begin{aligned} bd(x_{n_k}, x_{n_{k+1}}) &\leq d(x_{n_k-1}, x_{n_{k+1}-1}) \\ &\leq bd(x_{n_k-1}, x_{n_k}) + bd(x_{n_k}, x_{n_{k+1}-1}) \\ &\leq bd(x_{n_k-1}, x_{n_k}) + b. \end{aligned} \tag{3.1.9}$$

Letting $k \to \infty$ in the inequality (3.1.9) and using (3.1.8), we obtain that there exists

$$\lim_{k\to\infty} d(x_{n_k}, x_{n_{k+1}}) = 1 \tag{3.1.10}$$

and

$$\lim_{k\to\infty} d(x_{n_k-1}, x_{n_{k+1}-1}) = b. \tag{3.1.11}$$

Putting $t_k =: d(x_{n_k}, x_{n_{k+1}})$ and $s_k := d(x_{n_k-1}, x_{n_{k+1}-1})$ in the condition (ξ_2), we get

$$0 \leq \limsup_{k\to\infty} \xi(bd(x_{n_k}, x_{n_{k+1}}), d(x_{n_k-1}, x_{n_{k+1}-1})) < 0, \tag{3.1.12}$$

which is contradict. Therefore, $\{x_n\}$ is a bounded sequence.

Finally, we show that $\{x_n\}$ is a Cauchy sequence. Denote

$$C_n = \sup\{d(x_i, x_j) : i, j \geq n\} \tag{3.1.13}$$

for all $n \in \mathbb{N}$. It follows from the fact that $\{x_n\}$ is a bounded sequence. Then $C_n < \infty$ for all $n \in \mathbb{N}$. Since $\{C_n\}$ is a positive decreasing sequence, there exists some $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C. \tag{3.1.14}$$

If $C>0$, then for every $k\in\mathbb{N}$, there exist n_k, m_k such that $m_k>n_k\geq k$ and

$$C_k-\frac{1}{k}<d(x_{n_k},x_{m_k})\leq C_k.$$

Therefore,

$$\lim_{n\to\infty} d(x_{n_k},x_{m_k})=C. \tag{3.1.15}$$

Since $\Re$ is a transitive relation, we get $(x_{n_k-1},x_{m_k-1})\in\Re$. This implies that

$$\xi(bd(x_{n_k},x_{m_k}),d(x_{n_k-1},x_{m_k-1}))\geq 0. \tag{3.1.16}$$

From (ξ_1) and (3.1.13), we obtain

$$bd(x_{n_k},x_{m_k})\leq d(x_{n_k-1},x_{m_k-1})\leq C_{k-1}. \tag{3.1.17}$$

Letting $k\to\infty$ in the inequality (3.1.17) and using (3.1.15), we get

$$bC\leq \liminf_{k\to\infty} d(x_{n_k-1},x_{m_k-1})\leq \limsup_{k\to\infty} d(x_{n_k-1},x_{m_k-1})\leq C. \tag{3.1.18}$$

If $b>1$, the inequality (3.1.18) implies that $C=0$. Suppose that $b=1$. Putting $t_k:=d(x_{n_k},x_{m_k})$ and $s_k:=d(x_{n_k-1},x_{m_k-1})$ in the condition (ξ_2), we get

$$0\leq \limsup_{k\to\infty} \xi(bd(x_{n_k},x_{m_k}),d(x_{n_k-1},x_{m_k-1}))<0, \tag{3.1.19}$$

which is contradict. Therefore, $C=0$ and so for each $b\geq 1$, we obtain

$$\lim_{n\to\infty} C_n=0.$$

Hence, $\{x_n\}$ is a Cauchy sequence. This completes the proof. □

Now we establish the fixed point theorem of $\mathcal{Z}^b_{\Re}$-contraction mappings as follows:

Theorem 3.1.5. *Let (X,b,d) be a complete b-metric space, $\Re$ be a transitive relation on X and T be a self-mapping on X. Suppose that the following conditions hold:*

(*i*) *$X(T;\Re)$ is nonempty;*

(ii) $\Re$ *is* T*-closed;*

(iii) *there is a* $\xi \in \mathcal{Z}^b$ *such that* T *is a* $\mathcal{Z}^b_{\Re}$*-contraction mapping with respect to* $\xi \in \mathcal{Z}^b$*;*

(iv) T *is continuous.*

Then T *has a fixed point. Moreover, for each* $x_0 \in X(T;\Re)$*, the Picard sequence* $\{x_n\}$ *in* X *defined by* $x_n = Tx_{n-1}$ *for all* $n \in \mathbb{N}$*, converges to a fixed point of* T*.*

Proof. Let x_0 be an arbitrary point in $X(T;\Re)$. Put $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}_0$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N}_0$, then x_{n^*} is a fixed point of T. Thus we will assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$. By Lemma 3.1.4, we have $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. By condition (iv), we have $Tx_n \to Tx^*$ as $n \to \infty$. This implies that, $Tx^* = x^*$. This completes the proof. □

Theorem 3.1.6. *Theorem 3.1.5 also holds if we replace hypothesis* (iii) *by the following one*

(v) $\Re$ *is b-d-self-closed.*

Proof. The arguments of the proof of theorem 3.1.5 prove that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ with

$$[x_{n_k}, x^*] \in \Re \tag{3.1.20}$$

for all $k \in \mathbb{N}$. Suppose by contradiction that x^* is not a fixed point of T. That is, $x^* \neq Tx^*$ and so $d(x^*, Tx^*) > 0$. Since $x_{n_k} \to x^*$ as $k \to \infty$, there is $k_1 \in \mathbb{N}$ such that

$$d(x_{n_k}, x^*) < d(x^*, Tx^*) \tag{3.1.21}$$

for all $k \geq k_1$. In particular, $x_{n_k} \neq Tx^*$ for all $k \geq k_1$. So

$$d(Tx_{n_k}, Tx^*) = d(x_{n_k+1}, Tx^*) > 0 \tag{3.1.22}$$

for all $k \geq k_1$. Since $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, it is impossible the condition that there exists $k_2 \in \mathbb{N}$ such that $x_{n_k} = x^*$ for all $k \geq k_2$. So there exists a subsequence $\{x_{n_{\sigma(k)}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{\sigma(k)}} \neq x^*$ for all $k \in \mathbb{N}$. Next, let $k_3 \in \mathbb{N}$ such that $\sigma(k_3) \geq k_1$. By (3.1.21) and (3.1.22), we get

$$d(x_{n_{\sigma(k)}}, x^*) > 0 \tag{3.1.23}$$

and

$$d(Tx_{n_{\sigma(k)}}, Tx^*) > 0. \tag{3.1.24}$$

for all $k \geq k_3$. Using (ξ_1), and Proposition 3.1.3, we get

$$0 \leq \xi(bd(Tx_{n_{\sigma(k)}}, Tx^*), d(x_{n_{\sigma(k)}}, x^*)) < d(x_{n_{\sigma(k)}}, x^*) - bd(Tx_{n_{\sigma(k)}}, Tx^*).$$

for all $k \geq k_3$. This implies that

$$bd(Tx_{n_{\sigma(k)}}, Tx^*) \leq d(x_{n_{\sigma(k)}}, x^*)$$

for all $k \geq k_3$. Using the triangle inequality and the symmetry of b-metric d, we have

$$\begin{aligned} d(Tx^*, x^*) &\leq bd(Tx^*, x_{n_{\sigma(k)+1}}) + bd(x_{n_{\sigma(k)+1}}, x^*) \\ &= bd(x_{n_{\sigma(k)+1}}, Tx^*) + bd(x_{n_{\sigma(k)+1}}, x^*) \\ &\leq d(x_{n_{\sigma(k)}}, x^*) + bd(x_{n_{\sigma(k)+1}}, x^*) \end{aligned} \tag{3.1.25}$$

for all $k \geq k_3$. Letting $k \to \infty$ in the inequality (3.1.25), we get $d(Tx^*, x^*) = 0$. That is, $Tx^* = x^*$. This completes the proof. □

Remark 3.1.7. Note that the transitivity of $\Re$ is sufficient to guarantee the existence of a fixed point of $\mathcal{Z}^b_{\Re}$-contraction mappings in b-metric space.

The following theorem guarantees the uniqueness of the fixed point in Theorems 3.1.5 (resp. Theorem 3.1.6).

Theorem 3.1.8. *In addition to the hypothesis of Theorem 3.1.5 (resp.Theorem 3.1.6), suppose that the following condition holds:*

(u) $\Upsilon(x,y,\Re)$ *is nonempty for all* $x,y \in \mathrm{Fix}(T) := \{z \in X : z \text{ is a fixed point of } T\}$.

Then T *has a unique fixed point.*

Proof. To prove uniqueness, assume x^*, y^* are two fixed points of T such that $x^* \neq y^*$. Since $\Upsilon(x,y,\Re)$ is nonempty, for all $x,y \in X$, there exists a path (say $\{z_0, z_1, z_2, ..., z_k\}$) of some finite length k in $\Re$ from x to y so that

$$z_0 = x^*,\ z_k = y^*,\ (z_i, z_{i+1}) \in \Re \quad \text{for each} \quad i = 0,1,2,\ldots,k-1.$$

As $\Re$ is transitive, we have

$$(z_0, z_k) \in \Re$$

for all $i = 0,1,2,\ldots,k-1$ and for all $n \in \mathbb{N}$. Therefore,

$$0 \leq \zeta(bd(Tz_0, Tz_k), d(z_0, z_k)) < d(z_0, z_k) - bd(Tz_0, Tz_k) = d(x^*, y^*) - bd(x^*, y^*).$$

That is,

$$bd(x^*, y^*) = bd(T^n x^*, T^n y^*) < d(x^*, y^*)$$

which is a contradiction. This implies that T has a unique fixed point. □

Remark 3.1.9. Note that we use the result in Theorem 3.1.8 to derive a criterion for the existence of fixed points in some cases wherein several results in [24, 28] cannot be guaranteed the existence of fixed points.

3.2 Fixed point results for $(F,\gamma)_\Re$-contraction mappings

In this section, we prove the existence and uniqueness of a fixed point of $(F,\gamma)_\Re$-contraction mappings in metric spaces. We start our consideration by giving the important concepts of two new control functions.

Definition 3.2.1. Let $\mathcal{F}$ be the set of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following conditions:

(F_2) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;

$(F_{3'})$ F is lower semicontinuous.

Example 3.2.2. Define a mapping $F:\mathbb{R}^+\to\mathbb{R}$ by

1. $F(\alpha)=\begin{cases}\frac{-1}{\alpha} & \text{if } \alpha\le 3\\ \frac{-1}{(\alpha+1)} & \text{if } \alpha>3\end{cases}$,

2. $F(\alpha)=\begin{cases}\frac{-1}{\alpha}+\alpha & \text{if } \alpha\le 2.8\\ 2\alpha-3 & \text{if } \alpha>2.8\end{cases}$,

3. $F(\alpha)=\begin{cases}\frac{-1}{1-e^{\alpha}} & \text{if } \alpha\le 2\\ \ln(\alpha-1) & \text{if } \alpha>2\end{cases}$.

Definition 3.2.3. Let Γ be the set of all functions $\gamma:\mathbb{R}^+\to\mathbb{R}$ satisfying the following conditions:

(γ_1) for each sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ of positive numbers, $\lim_{n\to\infty}\alpha_n=0$ if and only if $\lim_{n\to\infty}\gamma(\alpha_n)=-\infty$;

(γ_2) γ is right upper semicontinuous.

Example 3.2.4. Define a mapping $\gamma:\mathbb{R}^+\to\mathbb{R}$ by

1. $\gamma(\alpha)=\begin{cases}\frac{-1}{\alpha} & \text{if } \alpha<4.6\\ \cos\alpha & \text{if } \alpha\ge 4.6\end{cases}$,

2. $\gamma(\alpha)=\begin{cases}\ln\left(\frac{\alpha}{3}+\sin\alpha\right) & \text{if } \alpha<3.2\\ \sin\alpha & \text{if } \alpha\ge 3.2\end{cases}$,

3. $\gamma(\alpha)=\begin{cases}\ln\alpha & \text{if } \alpha<5\\ \ln(\alpha+\cos\alpha) & \text{if } \alpha\ge 5\end{cases}$.

Note that every continuous function is lower semicontinuous and right upper semicontinuous and so families $\mathcal{F}$ and Γ are larger than the family of functions of Imdad *et. al.* in [49].

Now, we introduce a new contraction mapping concerning control functions in $\mathcal{F}$ and Γ in metric spaces endowed with a binary relation as follows:

Definition 3.2.5. Given a metric space (X,d) and a binary relation $\Re$ on X, let

$$\mathcal{A} = \{ (x,y) \in \Re : d(Tx,Ty) > 0 \}.$$

A self-mapping T on X is said to be $(F,\gamma)_{\Re}$-*contraction mapping* if there exists $F \in \mathcal{F}$, $\gamma \in \Gamma$ and $\tau > 0$ such that

$$\tau + F(d(Tx,Ty)) \leq \gamma(d(x,y)) \tag{3.2.1}$$

for all $(x,y) \in \mathcal{A}$.

The following theorem shows the existence and uniqueness of a fixed point for $(F,\gamma)_{\Re}$-contraction mappings.

Theorem 3.2.6. *Let (X,d) be a complete metric space, let $\Re$ be a transitive relation on X and let T be a self-mapping on X. Suppose that the following conditions hold:*

(a) *$X(T;\Re)$ is nonempty;*

(b) *$\Re$ is T-closed;*

(c) *T is continuous;*

(d) *T is an $(F,\gamma)_{\Re}$-contraction mapping with $F(\alpha) > \gamma(\alpha)$ for all $\alpha > 0$.*

Then T has a fixed point. Moreover, for each $x_0 \in X(T;\Re)$, the Picard sequence $\{T^n x_0\}$ is convergent to the fixed point of T.

Proof. Let x_0 be an arbitrary point in $X(T;\Re)$ and $x_n = Tx_{n-1} = T^n x_0$ for all $n \in \mathbb{N}_0$. If $x_{n^*} = x_{n^*+1}$ for some $n^* \in \mathbb{N}_0$, then x_{n^*} is a fixed point of T and the proof is completed. So we assume that

$$x_n \neq x_{n+1} \tag{3.2.2}$$

for all $n \in \mathbb{N}_0$ and so $d(Tx_n, Tx_{n+1}) > 0$ for all $n \in \mathbb{N}_0$. Since $(x_0, Tx_0) \in \Re$, using that $\Re$ is T-closed, we get

$$(x_n, x_{n+1}) \in \Re \tag{3.2.3}$$

for all $n \in \mathbb{N}_0$. Thus $(x_n, x_{n+1}) \in \mathcal{A}$ for all $n \in \mathbb{N}_0$. Since T is an $(F,\gamma)_{\Re}$-contraction mapping, we have

$$F(d(x_n, x_{n+1})) = F(d(Tx_{n-1}, Tx_n)) \le \gamma(d(x_{n-1}, x_n)) - \tau \tag{3.2.4}$$

for all $n \in \mathbb{N}$. Denote $a_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N}_0$. From (3.2.4), we obtain

$$F(a_n) \le \gamma(a_{n-1}) - \tau < F(a_{n-1}) - \tau \le \gamma(a_{n-2}) - 2\tau \le \cdots \le \gamma(a_0) - n\tau \tag{3.2.5}$$

for all $n \in \mathbb{N}$. From (3.2.5), we obtain $\lim_{n\to\infty} F(a_n) = -\infty$, which together with (F_2), we have

$$\lim_{n\to\infty} a_n = 0. \tag{3.2.6}$$

From (3.2.2) and (3.2.6), we get $x_n \ne x_m$ for all $n, m \in \mathbb{N}_0$ with $n \ne m$. Now, we will show that $\{x_n\}$ is a Cauchy sequence in X. Suppose by contradiction that $\{x_n\}$ is not a Cauchy sequence. By Lemma 2.5.22 and (3.2.6), there exist $\epsilon > 0$ and two subsequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $n(k) > m(k) > k$, such that

$$\lim_{k\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon.$$

Since $\Re$ is transitive, we have $(x_{n(k)-1}, x_{m(k)-1}) \in \Re$. Applying the condition (d), we have

$$\tau + F(d(x_{n(k)}, x_{m(k)})) \le \gamma(d(x_{n(k)-1}, x_{m(k)-1}))$$

and so

$$\begin{aligned} \tau + \liminf_{k\to\infty} F(d(x_{n(k)}, x_{m(k)})) &\le \liminf_{k\to\infty} \gamma(d(x_{n(k)-1}, x_{m(k)-1})) \\ &\le \limsup_{k\to\infty} \gamma(d(x_{n(k)-1}, x_{m(k)-1})). \end{aligned}$$

Thus,

$$\tau + F(\epsilon) \le \gamma(\epsilon) < F(\epsilon),$$

which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence. By the completeness of X, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. So it follows from the continuity of the mapping T that $x_{n+1} = Tx_n \to Tx^*$ as $n \to \infty$. This implies that $Tx^* = x^*$, that is, x^* is a fixed point of T. This completes the proof. $\square$

Now, we give an example to illustrate the utility of Theorem 3.2.6.

Example 3.2.7. Let $X = [0,\infty)$ and $d : X \times X \to [0,\infty)$ be the Euclidean metric defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Thus (X,d) is a complete metric space. Consider the sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ defined as

$$\alpha_n = \frac{n}{3}(n+1)(n+2) \qquad \text{for all } n \in \mathbb{N}.$$

Define a binary relation $\Re$ on X by

$$\Re := \{(1,1)\} \cup \{(1,\alpha_i) : i \in \mathbb{N}\} \cup \{(\alpha_i,\alpha_j) : i < j \text{ for all } i,j \in \mathbb{N}\}.$$

So $\Re$ is transitive relation. Define a mapping $T : X \to X$ by

$$Tx = \begin{cases} x & \text{if } 0 \le x \le 1, \\ \lceil \ln x \rceil & \text{if } 1 \le x \le \alpha_1, \\ \left(\frac{x-\alpha_1}{\alpha_2-\alpha_1}\right)+1 & \text{if } \alpha_1 \le x \le \alpha_2, \\ \frac{\alpha_{n-1}(\alpha_{n+1}-x)+\alpha_n(x-\alpha_n)}{\alpha_{n+1}-\alpha_n} & \text{if } \alpha_i \le x \le \alpha_{i+1} \text{ for all } n \in \mathbb{N}-\{1,2\}. \end{cases}$$

It is easy to see that T is continuous and $\Re$ is T-closed. Now we show that T is an $(F,\gamma)_{\Re}$-contraction mapping with $\tau = 2$ and $F,\gamma : \mathbb{R}^+ \to \mathbb{R}$ which are defined by

$$F(\alpha) = \begin{cases} \frac{-1}{\alpha} + \frac{4}{5}\alpha & \text{if } \alpha \le 1.1, \\ \frac{-1}{\alpha} + \alpha & \text{if } \alpha > 1.1, \end{cases} \quad \text{and} \quad \gamma(\alpha) = \begin{cases} \frac{-1}{\alpha} + \frac{1}{3}\alpha & \text{if } \alpha < 6.5, \\ \frac{-2}{\alpha} + \alpha & \text{if } \alpha \ge 6.5. \end{cases}$$

Let $(x,y) \in \mathcal{A} = \{(x,y) \in \Re : d(Tx,Ty) > 0\}$. So we have to consider into four cases.

Case I: if $x = 1$ and $y = \alpha_2$. Then $d(x,y) = 7$ and $d(Tx,Ty) = 1$ and so

$$2 + F(d(Tx,Ty)) = 2 - \frac{1}{d(Tx,Ty)} + \frac{4}{5}d(Tx,Ty) \le -\frac{2}{d(x,y)} + d(x,y) = \gamma(d(x,y)).$$

Case II: if $x=1$ and $y=\alpha_i$ for all $i>2$. Then $d(x,y)=|1-\alpha_i|\geq 19$ and $d(Tx,Ty)=|1-\alpha_{i-1}|\geq 7$ for all $i>2$ and so

$$\begin{aligned} 2|1-\alpha_{i-1}|-|1-\alpha_i| &< 2|1-\alpha_{i-1}| \\ &< |1-\alpha_i||1-\alpha_{i-1}| \\ &< |1-\alpha_i||1-\alpha_{i-1}|\left(|1-\alpha_i|-|1-\alpha_{i-1}|-2\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & 2+\frac{2}{|1-\alpha_i|}-\frac{1}{|1-\alpha_{i-1}|}\leq|1-\alpha_i|-|1-\alpha_{i-1}| \\ \Rightarrow \quad & 2-\frac{1}{|1-\alpha_{i-1}|}+|1-\alpha_{i-1}|\leq-\frac{2}{|1-\alpha_i|}+|1-\alpha_i|. \end{aligned}$$

That is,

$$2+F(d(Tx,Ty)=2-\frac{1}{d(Tx,Ty)}+d(Tx,Ty)\leq-\frac{2}{d(x,y)}+d(x,y)=\gamma(d(x,y)).$$

Case III: if $x=\alpha_1$ and $y=\alpha_2$. Then $d(x,y)=6$ and $d(Tx,Ty)=1$ and so

$$2+F(d(Tx,Ty)=2-\frac{1}{d(Tx,Ty)}+\frac{4}{5}d(Tx,Ty)\leq-\frac{1}{d(x,y)}+\frac{1}{3}d(x,y)=\gamma(d(x,y)).$$

Case IV: if $x=\alpha_i$ and $y=\alpha_j$ for all $i,j\in\mathbb{N}$ with $i<j$ where $(i,j)\neq(1,2)$. Then $d(x,y)=|\alpha_i-\alpha_j|\geq 12$ and $d(Tx,Ty)=|\alpha_{i-1}-\alpha_{j-1}|\geq 6$. So

$$\begin{aligned} 2|\alpha_{i-1}-\alpha_{j-1}|-|\alpha_i-\alpha_j| &< 2|\alpha_{i-1}-\alpha_{j-1}| \\ &< |\alpha_i-\alpha_j||\alpha_{i-1}-\alpha_{j-1}| \\ &< |\alpha_i-\alpha_j||\alpha_{i-1}-\alpha_{j-1}|\left(|\alpha_i-\alpha_j|-|\alpha_{i-1}-\alpha_{j-1}|-2\right) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad & 2+\frac{2}{|\alpha_i-\alpha_j|}-\frac{1}{|\alpha_{i-1}-\alpha_{j-1}|}\leq|\alpha_i-\alpha_j|-|\alpha_{i-1}-\alpha_{j-1}| \\ \Rightarrow \quad & 2-\frac{1}{|\alpha_{i-1}-\alpha_{j-1}|}+|\alpha_{i-1}-\alpha_{j-1}|\leq-\frac{2}{|\alpha_i-\alpha_j|}+|\alpha_i-\alpha_j|. \end{aligned}$$

That is,

$$2+F(d(Tx,Ty)=2-\frac{1}{d(Tx,Ty)}+d(Tx,Ty)\leq-\frac{2}{d(x,y)}+d(x,y)=\gamma(d(x,y)).$$

From *Case I*, *Case II*, *Case III*, and *Case IV*, we can show that

$$2+F(d(Tx,Ty) \leq \gamma(d(x,y))$$

for all $(x,y) \in \mathcal{A}$.

This yields that T is an $(F,\gamma)_{\Re}$-contraction with $\tau = 2$. Moreover, there exists $x_0 = 1 \in X$ such that $(x_0, Tx_0) \in \Re$. This shows that $X(T;\Re)$ is a nonempty set. Therefore, all the conditions of Theorem 3.2.6 are satisfied and so there exists a fixed point of T. In this case, T has infinite fixed points (see Figure 3.1).

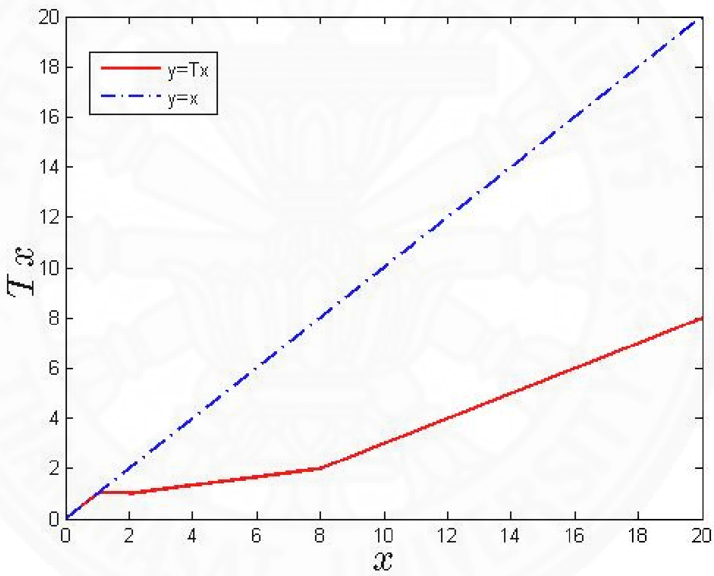


Figure 3.1: Graphs of $y = x$ and $y = Tx$ in Example 3.2.7.

Remark 3.2.8. Note that fixed point theorems concerning F-contraction mappings in [49] can not be used to solve this example since F and γ are not continuous and $F(\alpha) \neq \gamma(\alpha)$ for all $\alpha > 0$.

In the following result, we omit the continuity of T from Theorem 3.2.6.

Theorem 3.2.9. *Theorem 3.2.6 also holds if we replace the hypothesis* (*c*) *by the following one:*

(c') (X,d) *is* $\Re$*-nondecreasing-regular.*

Proof. From the proof of Theorem 3.2.6, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Since $(x_n, x_{n+1}) \in \Re$, it follows from (c') that $(x_n, x^*) \in \Re$ for all $n \in \mathbb{N}$. Now we consider two case depending on $L = \{n \in \mathbb{N} : Tx_n = Tx^*\}$.

Case I: If L is finite, there exist $n_0 \in \mathbb{N}$ such that $Tx_n \neq Tx^*$ for all $n \geq n_0$. So we consider $x_n \neq x^*$, $d(x_n, x^*) > 0$ and $d(Tx_n, Tx^*) > 0$ for all $n \geq n_0$. Since T is $(F,\gamma)_{\Re}$-contraction, we have

$$\tau + F(d(Tx_n, Tx^*)) \leq \gamma(d(x_n, x^*))$$

for all $n \geq n_0$. Since $d(x_n, x^*) \to 0$ as $n \to \infty$, we have $\gamma(d(x_n, x^*)) \to \infty$ as $n \to \infty$ and so $F(d(Tx_n, Tx^*)) \to \infty$ as $n \to \infty$. By (F_2), we have $d(Tx_n, Tx^*) \to 0$ as $n \to \infty$. This implies that $Tx^* = x^*$ and so x^* is a fixed point of T.

Case II: If L is infinite, then $x_{n+1} = Tx_n = Tx^*$ for all $n \in L$. Taking $n \to \infty$, we get $x^* = Tx^*$ and so x^* is a fixed point of T.

Therefore, T has a fixed point. This completes the proof. □

The following theorem guarantees the uniqueness of the fixed point in Theorems 3.2.6 and 3.2.9.

Theorem 3.2.10. *In addition to the hypothesis of Theorem 3.2.6 (respectively, Theorem 3.2.9),* $\Upsilon(x,y,\Re)$ *is nonempty, for all* $x, y \in \mathrm{Fix}(T)$*. Then* T *has a unique fixed point.*

Proof. Suppose that x and y are two distinct fixed points of T. Then $d(Tx,Ty) = d(x,y) > 0$. Since $\Upsilon(x,y,\Re)$ is nonempty, there is a path (say $\{z_0, z_1, z_2, ..., z_k\}$) of some finite length k in $\Re$ from x to y, so that

$$z_0 = x, \quad z_k = y, \quad (z_i, z_{i+1}) \in \Re \quad \text{for each} \quad i = 0, 1, 2, \ldots, k-1.$$

By the transitivity of $\Re$, we get

$$(x, z_1) \in \Re, \ (z_1, z_2) \in \Re, \ldots, \ (z_{k-1}, y) \in \Re \quad \Rightarrow \quad (x, y) \in \Re.$$

The contractivity condition (3.2.1) implies that

$$\tau + F(d(x,y)) = \tau + F(d(Tx,Ty)) \leq \gamma(d(x,y)),$$

Since $F(\alpha) > \gamma(a)$ for all $\alpha > 0$, it follows that

$$\tau + F(d(x,y)) < F(d(x,y)),$$

which is a contradiction because $\tau > 0$. Thus T has a unique fixed point. □

3.3 Fixed point results for $(\psi, \phi, \Re)$-contraction mappings

In this section, we discuss the existence and uniqueness of fixed point of $(\psi, \phi, \Re)$-contraction mappings. So we introduce the following notations.

$$\Psi := \{\psi : [0,\infty) \to [0,\infty) : \psi \text{ is lower semicontinuous and nondecreasing}\},$$

and

$$\Phi := \{\phi : [0,\infty) \to [0,\infty) : \phi \text{ is right upper semicontinuous}\}.$$

Here, it can be pointed out that every continuous function is lower semicontinuous and right upper semicontinuous. So the class Ψ is larger than the family of all weak altering distance functions and the family of all altering distance functions. Now we give some examples of functions including the class of Ψ and the class Φ, respectively.

Example 3.3.1. Define $\psi_1, \psi_2, \psi_3 : [0,\infty) \to [0,\infty)$ by

$$\psi_1(s) = \begin{cases} \ln(1+s)+1 & \text{if } s \leq 1 \\ s+1 & \text{if } s > 1 \end{cases},$$

$$\psi_2(s) = \begin{cases} s^2 & \text{if } s \leq 1 \\ e^s - 1 & \text{if } s > 1 \end{cases},$$

$$\psi_3(s) = \begin{cases} e^s + 1 & \text{if } s \leq 1 \\ 3s + 1 & \text{if } s > 1 \end{cases}.$$

We see that $\psi_1, \psi_2, \psi_3 \in \Psi$. (The graphs of functions ψ_1, ψ_2 and ψ_3 are shown in Figure 3.2).

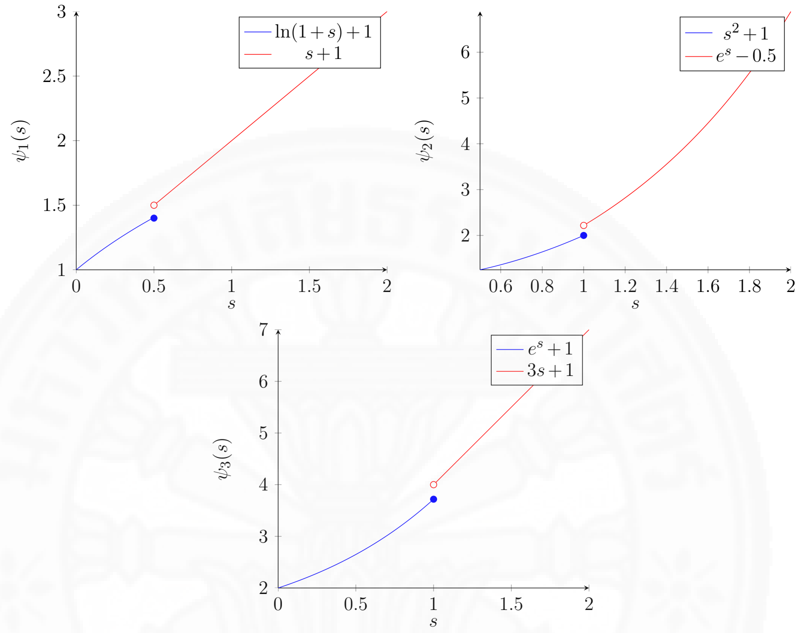


Figure 3.2: Graphs of ψ_1, ψ_2, ψ_3 in Example 3.3.1.

Example 3.3.2. Define $\phi_1, \phi_2, \phi_3 : [0, \infty) \to [0, \infty)$ by

$$\phi_1(s) = \begin{cases} s^2 & \text{if } s < 1 \\ e^s - 1 & \text{if } s \geq 1 \end{cases},$$

$$\phi_2(s) = \begin{cases} \ln(1+s) & \text{if } s < 1 \\ s & \text{if } s \geq 1 \end{cases},$$

$$\phi_3(s) = \begin{cases} \frac{s^2}{2} & \text{if } s < 1 \\ s^2 & \text{if } s \geq 1 \end{cases}.$$

We see that $\phi_1, \phi_2, \phi_3 \in \Phi$. (The graphs of functions ϕ_1, ϕ_2 and ϕ_3 are shown in Figure 3.3).

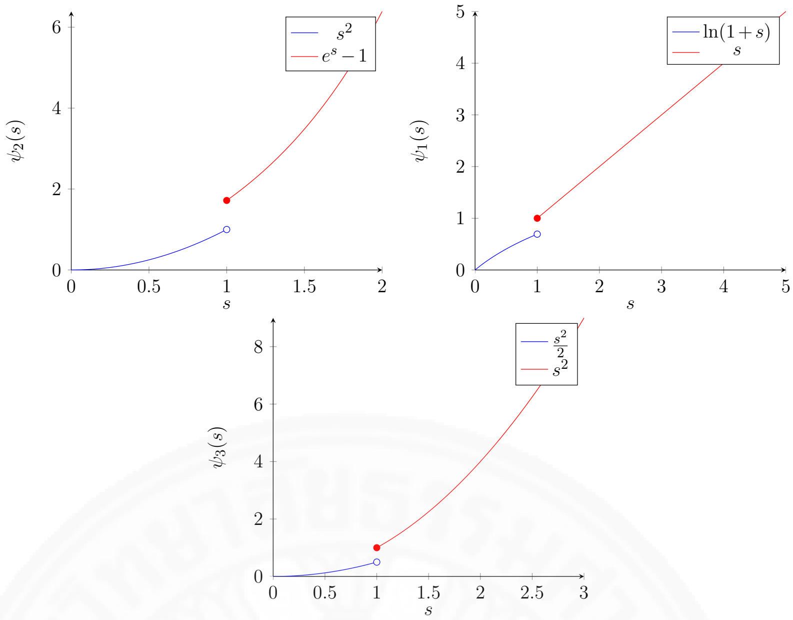


Figure 3.3: Graphs of ϕ_1, ϕ_2, ϕ_3 in Example 3.3.2.

Based on the classes Ψ and Φ, we introduce the notion of a $(\psi, \phi, \Re)$-contraction as follows:

Definition 3.3.3. Let (X, d) be a metric space. A self-mapping T on X is said to be a $(\psi, \phi, \Re)$-*contraction mapping* if there exist $\psi \in \Psi$ and $\phi \in \Phi$ such that

$$\psi(d(Tx, Ty)) \leq \phi(d(x, y))) \tag{3.3.1}$$

for all $x, y \in X$ with $x\Re^{\bowtie} y$ and $Tx\Re^{\bowtie} Ty$.

Now we give a useful proposition which immediate due to the symmetricity of d.

Proposition 3.3.4. *Let (X, d) be a metric space, $\Re$ be a binary relation on X, $T : X \to X$ be a mapping, $\psi \in \Psi$ and $\phi \in \Phi$, then the following contractivity conditions are equivalent:*

(*i*) $\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X$ *with* $(x, y) \in \Re$,

(*ii*) $\psi(d(Tx, Ty)) \leq \phi(d(x, y)), \quad \forall x, y \in X$ *with* $[x, y] \in \Re$.

Proposition 3.3.5. *Let (X,d) be a metric space endowed with a binary relation $\Re$ and T be a self-mapping on X. Suppose that the following conditions hold:*

(i) *T is a $(\psi,\phi,\Re)$-contraction with $\psi(t) > \phi(t)$ for all $t > 0$;*

(ii) *$\Re$ is T-orbitally transitive;*

(iii) *X is $(\Re,d)$-increasing regular.*

Then T is orbitally $\Re^{\#}$-continuous.

Proof. Let $x,u \in X$ and $\{n_j\}$ be a increasing sequence of positive integers. Suppose that $T^{n_j}x \to u$ and $T^{n_j}x\Re^{\#}T^{n_{j+1}}x$ for all $j \in \mathbb{N}$. Since $\Re$ is T-orbitally transitive, we get

$$TT^{n_j}x\Re^{\#}Tu$$

for all $j \in \mathbb{N}$. Using the condition (i), we have

$$\psi(d(TT^{n_j}x,Tu)) \leq \phi(d(T^{n_j}x,u)) < \psi(d(T^{n_j}x,u)) \tag{3.3.2}$$

for all $j \in \mathbb{N}$. By the properties of ψ, we have

$$d(TT^{n_j}x,Tu) \leq d(T^{n_j}x,u))$$

for all $j \in \mathbb{N}$. So the sequence $\{d(TT^{n_j}x,Tu)\}$ is decreasing and bounded below. Then there exists $c \geq 0$ such that $d(TT^{n_j}x,Tu) \to c$ as $j \to \infty$. By (3.3.2), property of ψ and ϕ, taking $j \to \infty$ we get

$$\begin{aligned}\psi(c) &\leq \liminf_{j\to\infty} \psi(d(TT^{n_j}x,Tu)) \\ &\leq \limsup_{j\to\infty} \psi(d(TT^{n_j}x,Tu)) \\ &\leq \limsup_{j\to\infty} \phi(d(TT^{n_j}x,Tu)) \\ &\leq \phi(c).\end{aligned}$$

Since $\psi(t) > \phi(t)$ for all $t > 0$, we have $c = 0$. That is, $\lim_{j\to\infty} d(TT^{n_j}x,Tu) = 0$. Therefore, T is orbitally $\Re^{\#}$-continuous. □

Proposition 3.3.6. *Let (X,d) be a metric space endowed with a binary relation $\Re$ and T be a self-mapping on X. If the following conditions hold:*

(i) *T is a $(\psi,\phi,\Re)$-contraction with $\psi(t) > \phi(t)$ for all $t > 0$;*

(ii) *$\emptyset \neq Fix(T)$ is $\Re^s$-connected.*

Then T has a unique fixed point.

Proof. Suppose by contradiction that there exist $x,y \in Fix(T)$ such that $x \neq y$. Then there exists a path in $\Re^s$ of some finite length m from x to y such that

$$u_0 = x, u_m = y, u_i \neq u_{i+1}$$

and $[u_i,u_{i+1}] \in \Re$ for all $i \in \{0,1,\ldots,m-1\}$. Since $u_i \in Fix(T)$, we have $Tu_i = u_i$ for all $i \in \{0,1,\ldots,m\}$. By the assumption (i), we have

$$\psi(d(u_i,u_{i+1})) = \psi(d(Tu_i,Tu_{i+1})) \leq \phi(d(u_i,u_{i+1})) < \psi(d(u_i,u_{i+1}))$$

for all $i \in \{0,1,\ldots,m-1\}$, which is a contradiction. Hence, T has a unique fixed point. □

Theorem 3.3.7. *Let (X,d) be a metric space endowed with a binary relation $\Re$ and T be a self-mapping on X. Suppose that the following conditions hold:*

(i) *$\Re$ is T-orbitally transitive;*

(ii) *there exists a $(T,\Re)$-Picard sequence;*

(iii) *TX is $(\Re,d)$-increasing precomplete;*

(iv) *T is a $(\psi,\phi,\Re)$-contraction with $\psi(t) > \phi(t)$ for all $t > 0$;*

(v) *T is orbitally $\Re^{\shortparallel}$-continuous.*

Then T has a fixed point. Moreover, if $\{x_n\}$ is any $(T,\Re)$- Picard sequence, then either $\{x_n\}$ contains a fixed point of T or $\{x_n\}$ converges to a fixed point of T.

Proof. It follows from (i) that there exists a sequence $\{x_n\} \subseteq X$ such that $x_{n+1} = Tx_n$ and $x_n \Re x_{n+1}$ for all $n \in \mathbb{N}$. If there exists $n^* \in \mathbb{N}_0$ such that $x_{n^*} = Tx_{n^*}$, then x_{n^*} is a fixed point of T. So we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ and then $d(x_n, x_{n+1}) > 0$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is $\Re$-increasing sequence. From the contractive condition, we have

$$\psi(d(x_n, x_{n+1})) = \psi(d(Tx_{n-1}, Tx_n)) \leq \phi(d(x_{n-1}, x_n)) < \psi(d(x_{n-1}, x_n)) \quad (3.3.3)$$

for all $n \in \mathbb{N}$. Since ψ is a nondecreasing function, we have

$$d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$$

for all $n \in \mathbb{N}$. So, the sequence $\{d(x_n, x_{n+1})\}$ is decreasing and bounded below. Then there exists $p \geq 0$ such that $d(x_n, x_{n+1}) \to p$ as $n \to \infty$. By (3.3.3), property of ψ and ϕ, taking $n \to \infty$ we have

$$\begin{aligned} \psi(p) &\leq \liminf_{n\to\infty} \psi(d(x_n, x_{n+1})) \\ &\leq \limsup_{n\to\infty} \psi(d(x_n, x_{n+1})) \\ &\leq \limsup_{n\to\infty} \phi(d(x_{n-1}, x_n)) \\ &\leq \phi(p). \end{aligned}$$

Since $\psi(t) > \phi(t)$ for all $t > 0$, we have $p = 0$. That is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \quad (3.3.4)$$

Now, we will show that $\{x_n\}$ is a Cauchy sequence. Assume that $\{x_n\}$ is not a Cauchy sequence. By 3.3.4 and Lemma 2.5.22, there exist $\epsilon_0 > 0$ and two sequences $\{x_{n(k)}\}$ and $\{x_{m(k)}\}$ of $\{x_n\}$ such that $k \leq n(k) \leq m(k)$,

$$d(x_{n(k)}, x_{m(k)-1}) \leq \epsilon_0 < d(x_{n(k)}, x_{m(k)}) \quad \forall k \in \mathbb{N}_0$$

and

$$\lim_{n\to\infty} d(x_{n(k)}, x_{m(k)}) = \lim_{n\to\infty} d(x_{n(k)-1}, x_{m(k)-1}) = \epsilon_0.$$

Since $\Re$ is T-orbitally transitive, we have

$$x_{n(k)-1} \Re^{\shortparallel} x_{m(k)-1} \text{ and } Tx_{n(k)-1} \Re^{\shortparallel} Tx_{m(k)-1}.$$

It follows form (3.3.1), we get

$$\psi(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \leq \phi(d(x_{n(k)-1}, x_{m(k)-1})).$$

Tanking $k \to \infty$ and using the property of ψ and ϕ we get

$$\begin{aligned} \psi(\epsilon_0) &\leq \liminf_{k\to\infty} \psi(d(Tx_{n(k)-1}, Tx_{m(k)-1})) \\ &\leq \limsup_{k\to\infty} \psi(d(x_{n(k)-1}, x_{m(k)-1})) \\ &\leq \limsup_{k\to\infty} \phi(d(x_{n(k)-1}, x_{m(k)-1})) \\ &\leq \phi(\epsilon_0). \end{aligned}$$

It follows that $\epsilon_0 = 0$, which is a contradiction. So $\{x_n\}$ is a Cauchy sequence and it is also an $\Re$-increasing. By the $\Re$-increasingly precompleteness of TX, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Since $T^n x_0 \Re^{\natural} T^{n+1} x_0$, it follows from orbitally $\Re^{\natural}$-continuity of T we get $x_{n+1} \to Tz$ as $n \to \infty$. Therefore, $Tz = z$, that is z is a fixed point of T. This completes the proof. □

In the following result, we avoid the orbitally $\Re^{\natural}$-continuity of T.

Theorem 3.3.8. *Theorem 3.3.7 also holds if we replace hypothesis* (v) *by the following one*

(iv) *X is $(\Re, d)$-increasing regular.*

Proof. It follows from Proposition 3.3.5 and Theorem 3.3.7 that T has a fixed point. □

The following theorem guarantees the uniqueness of the fixed point in Theorem 3.3.7 (respectively, Theorem 3.3.8).

Theorem 3.3.9. *In addition to the hypothesis of Theorem 3.3.7 (respectively, Theorem 3.3.8), assume that $Fix(T)$ is $\Re^s$-connected, then the fixed point of T is unique.*

Proof. From Theorem 3.3.7 (respectively, Theorem 3.3.8), we get $Fix(T) \neq \emptyset$. So we can conclude from 3.3.6 that T has a unique fixed point. $\square$

Now, we give an example to illustrate the utility of Theorem 3.3.9.

Example 3.3.10. Let $X = (0,\infty)$ and $d : X \times X \to [0,\infty)$ be the Euclidean metric defined by $d(x,y) = |x-y|$ for all $x,y \in X$. Consider the sequence $\{\alpha_n\}$ defined as

$$\alpha_n = \frac{n(n+1)}{2} \qquad \text{for all } n \in \mathbb{N}.$$

Define a binary relation $\Re$ on X by

$$\Re := \{(\alpha_1,\alpha_1)\} \cup \{(x,y) : 0 < x < y \leq 2\} \cup \{(\alpha_i,\alpha_{i+1}) : i \in \mathbb{N} \setminus \{1\}\}.$$

We now define a mapping $T : X \to X$ by

$$Tx = \begin{cases} \alpha_1 & \text{if } 0 < x \leq 2, \\ \alpha_1 + \left(\dfrac{\alpha_2(x-2)}{\alpha_3}\right) & \text{if } 2 \leq x \leq \alpha_2, \\ \alpha_1 + \dfrac{\alpha_{i+1}}{\alpha_{i+2}} + \left(\dfrac{\frac{\alpha_{i+2}}{\alpha_{i+3}} - \frac{\alpha_{i+1}}{\alpha_{i+2}}}{\alpha_{i+2} - \alpha_{i+1}}\right)(x - \alpha_{i+1}) & \text{if } \alpha_{i+1} \leq x \leq \alpha_{i+2}, i = 1,2,3,\ldots. \end{cases}$$

It is easy to see that T is orbitally $\Re^{\#}$-continuous and TX is $\Re$-increasingly precomplete. Moreover, there exists $(T,\Re)$-Picard sequence since $x_0 = \alpha_1 \in X$ such that $x_0 \Re T x_0$ and $\Re$ is T-closed. Next, we will show that T is a $(\psi,\phi,\Re)$-contraction with $\psi(s) > \phi(s)$ for all $s > 0$ and ψ, ϕ which are defined by

$$\psi(s) = \begin{cases} \ln(1+s) + 1 & \text{if } s \leq 1 \\ s + 1 & \text{if } s > 1 \end{cases},$$

and

$$\phi(s) = \begin{cases} \ln(1+s) & \text{if } s < 1 \\ s & \text{if } s \geq 1 \end{cases},$$

We see that $\psi \in \Psi$ and $\phi \in \Phi$ with $\psi(s) > \phi(s)$ for all $s > 0$. Next, we will show that T is an $(\psi,\phi,\Re)$-contraction. Let $x \Re^{\#} y$ and $Tx \Re^{\#} Ty$. This implies that $x = \alpha_i$ and

$y = \alpha_{i+1}$ for some $i \in \mathbb{N} \backslash \{1\}$. Then $d(x,y) \geq d(\alpha_2, \alpha_3) = 3$ and $0 < d(Tx,Ty) < 1$. It follows that

$$\ln(d(Tx,Ty)+1)+1 \leq d(x,y) \tag{3.3.5}$$

That is,

$$\psi(d(Tx,Ty)) \leq \phi(d(x,y)). \tag{3.3.6}$$

This yields that T is an $(\psi, \phi, \Re)$-contraction. Furthermore, it easy to show that $Fix(T)$ is $\Re^s$-connected. Therefore, all the conditions of Theorem 3.3.9 are satisfied and so T has a unique fixed point, namely $x = 1$ (see Figure 3.4).

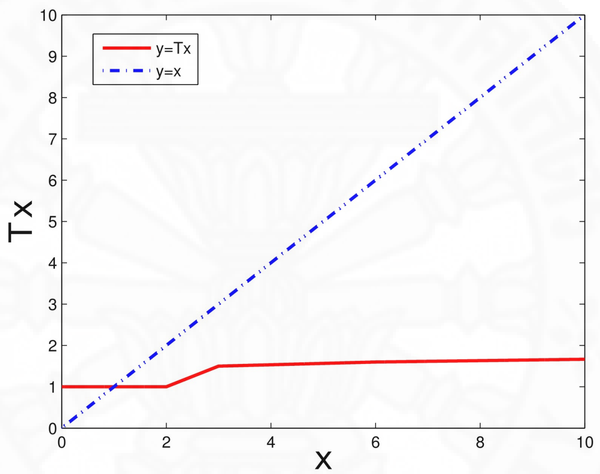


Figure 3.4: Graphs of $y = x$ and $y = Tx$ in Example 3.3.10.

Remark 3.3.11. From the Example 3.3.10, ψ is not an altering distance function and ϕ is not continuous. So fixed point result of Yan *et al.* in [44] are not applicable. Moreover, a binary relation is not a transitive, so fixed point result of Sawangsup *et al.* [19] are not applicable. It also can be pointed that the fixed point results in [44, 50, 51, 52] are not applicable.

CHAPTER 4

APPLICATIONS TO NONLINEAR MATRIX EQUATIONS AND NUMERICAL EXPERIMENTS

In this chapter, we will show that how to use our fixed point results in Chapter 3 for solving the existence and uniqueness of a solution of nonlinear matrix equations.

4.1 Solutions of a nonlinear matrix equation arising from $\mathcal{Z}^b_{\Re}$-contraction mappings via Ky Fan norms

On the basis of the fixed point theorems in Section 3.1, we study the nonlinear matrix equation

$$X = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i, \tag{4.1.1}$$

where $A_1, A_2, \ldots, A_m$ are arbitrary $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is a continuous order preserving maps from the set of all $n \times n$ Hermitian matrices $H(n)$ into the set of all $n \times n$ positive definite matrices $P(n)$ such that $\mathcal{G}(0) = 0$. In this process, we consider the nonlinear matrix equation (4.1.1) in a complete b-metric space $(H(n), D_{tr})$ with the coefficient $b = 2^{p-1}$, where $p \geq 1$ is a real number, such that the b-metric D_{tr} induced by the Ky Fan norm is defined by

$$D_{tr}(X,Y) = (\|X - Y\|_{tr})^p$$

for all $x, y \in H(n)$.

Studying the existence and uniqueness of a solution of the nonlinear matrix equation (4.1.1), we define a self-mapping $\mathcal{K}$ on $H(n)$ by

$$\mathcal{K}(X) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i \tag{4.1.2}$$

for all $X \in H(n)$, where $A_1, A_2, \ldots, A_m$ are arbitrary $n \times n$ matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}$ is a continuous order preserving maps from the set of all $n \times n$ Hermitian matrices $H(n)$ into the set of all $n \times n$ positive definite matrices $P(n)$ such that $\mathcal{G}(0) = 0$.

Applying fixed point theorems for $\mathcal{Z}^b_{\Re}$-contraction mappings, we will show that $\mathcal{K}$ has a unique fixed point and then the nonlinear matrix equation (4.1.1) has a unique solution. The following theorems guarantee the existence and uniqueness of solution of the nonlinear matrix equation (4.1.1).

Theorem 4.1.1. *Consider the matrix equation (4.1.1). Suppose that there exist positive real numbers M and $p \geq 1$ such that*

(*i*) *for every $X, Y \in H(n)$ such that $(X,Y) \in \preceq$, the following inequality hold:*

$$|tr(\mathcal{G}(Y) - \mathcal{G}(X))|^p \leq \frac{1}{b^{\frac{1}{p}} M} \left[\psi(|tr(Y - X)|^p)\right], \tag{4.1.3}$$

where $b = 2^{p-1}$ and $\psi : [0,\infty) \to [0,\infty)$ is a function such that $\psi(t) < t$ for all $t > 0$;

(*ii*) $\sum_{i=1}^{m} A_i A_i^* \prec M I_n$ *and* $\sum_{i=1}^{m} A_i^* \mathcal{G}(Q) A_i \succ 0$.

Then the matrix equation (4.1.1) has a solution. Moreover, the iteration

$$X_j = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X_{j-1}) A_i \tag{4.1.4}$$

for all $j \in \mathbb{N}$, where $X_0 \in H(n)$ such that $X_0 \preceq Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X_0) A_i$, converges in the sense of b-metric D_{tr} to the solution of matrix equation (4.1.1).

Proof. Let a mapping $\mathcal{K} : H(n) \to H(n)$ be defined by

$$\mathcal{K}(X) = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}(X) A_i \quad \text{for all} \quad X \in H(n). \tag{4.1.5}$$

Then $\mathcal{K}$ is well defined and $\preceq$ on $H(n)$ is $\mathcal{K}$-closed. Clearly, a fixed point of $\mathcal{K}$ is a solution of the equation (4.1.1). Now, we will show that there is a b-simulation function ξ so that $\mathcal{K}$ is $\mathcal{Z}^b_{\preceq}$-contraction mapping with respect to ξ.

Let $X,Y \in H(n)$ such that $(X,Y) \in \preceq$. This means that $X \preceq Y$ and thus $\mathcal{G}(X) \preceq \mathcal{G}(Y)$. Therefore,

$$\begin{aligned}
b(\|\mathcal{K}(Y)-\mathcal{K}(X)\|_{tr})^p &= b[tr(\mathcal{K}(Y)-\mathcal{K}(X))]^p \\
&= b\left[tr(\sum_{i=1}^{m} A_i^*(\mathcal{G}(Y)-\mathcal{G}(X))A_i)\right]^p \\
&= b\left[\sum_{i=1}^{m} tr(A_i^*(\mathcal{G}(Y)-\mathcal{G}(X))A_i)\right]^p \\
&= b\left[\sum_{i=1}^{m} tr(A_iA_i^*(\mathcal{G}(Y)-\mathcal{G}(X)))\right]^p \\
&= b\left[tr\Big((\sum_{i=1}^{m} A_iA_i^*)(\mathcal{G}(Y)-\mathcal{G}(X))\Big)\right]^p \\
&\leq b\left[\|\sum_{i=1}^{m} A_iA_i^*\|\right]^p (\|\mathcal{G}(Y)-\mathcal{G}(X)\|_{tr})^p \\
&\leq b\left[\frac{\|\sum_{i=1}^{m} A_iA_i^*\|}{b^{\frac{1}{p}}M}\right]^p \left[\psi\Big((\|(Y-X)\|_{tr})^p\Big)\right] \\
&< \psi\Big((\|(Y-X)\|_{tr})^p\Big)
\end{aligned}$$

and then

$$0 < \psi\Big((\|(Y-X)\|_{tr})^p\Big) - b(\|\mathcal{K}(Y)-\mathcal{K}(X)\|_{tr})^p. \tag{4.1.6}$$

Putting $\xi(t,s) = \psi(s) - t$ for all $s,t > 0$, obviously ξ is a b-simulation function. From the inequality (4.1.6), we have

$$0 \leq \zeta\Big(b(\|\mathcal{K}(Y)-\mathcal{K}(X)\|_{tr})^p, (\|Y-X\|_{tr})^p\Big). \tag{4.1.7}$$

Therefore, $\mathcal{K}$ is a $\mathcal{Z}^b_{\preceq}$-contraction mapping.

From $\sum_{i=1}^{m} A_i^*\mathcal{G}(Q)A_i \succ 0$, we have $Q \preceq \mathcal{K}(Q)$ and hence $H(n)(\mathcal{K};\preceq) \neq \emptyset$. This means that $Q \in H(n)(\mathcal{K};\preceq)$. Now from Theorem 3.1.5, there exists $Z \in H(n)$ such that $\mathcal{K}(Z) = Z$, that is, the matrix equation (4.1.1) has a solution. □

Theorem 4.1.2. *Under the assumptions of Theorem 4.1.1, the equation (4.1.1) has a unique solution $Z \in H(n)$.*

Proof. Since for every $X,Y \in H(n)$ there is a greatest lower bound and a least upper bound, we obtain that $\Upsilon(x,y,\Re)$ is nonempty for each $x,y \in H(n)$. Thus, it follows from Theorem 3.1.8 that $\mathcal{K}$ has a unique fixed point in $H(n)$. This implies that Equation (4.1.1) has a unique solution in $H(n)$. □

Next, we give a numerical example to show the correctness of Theorem 4.1.1.

Example 4.1.3. Let

$$Q = \begin{pmatrix} 7 & 3 & 0 & 0 \\ 3 & 7 & 3 & 0 \\ 0 & 3 & 7 & 3 \\ 0 & 0 & 3 & 7 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -0.0558 & 0.0064 & 0 & -0.2069 \\ -0.0006 & -0.0021 & 0 & 0 \\ 0.0201 & 0 & 0 & 0.1201 \\ 0.1212 & 0.3131 & 0.0003 & 0.0424 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0.0121 & 0.0666 & 0.1301 \\ 0.0341 & 0.0141 & 0.1201 & 0.2004 \\ 0.0011 & 0 & 0 & 0.0011 \\ 0.0114 & 0.0541 & 0.1111 & 0.0511 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0.0014 & 0.0021 & -0.0421 & -0.1158 \\ 0 & 0.1471 & -0.0451 & 0.0112 \\ 0.0125 & 0.1214 & 0.01142 & 0.2999 \\ 0.1254 & -0.1010 & 0.1241 & 0 \end{pmatrix}.$$

Define $\psi : [0,\infty) \to [0,\infty)$ by $\psi(t) = \frac{t}{2}$. We consider Equation (4.1.1) with $\mathcal{G}(X) = X$ that is

$$X = Q + A_1^*(X)A_1 + A_2^*(X)A_2 + A_3^*(X)A_3. \tag{4.1.8}$$

All the hypotheses of Theorem 4.1.2 are satisfied with $M = \frac{1}{4\sqrt{2}}$ and $p = 2$. We will consider the iteration

$$X_j = Q + A_1^* X_{j-1} A_1 + A_2^* X_{j-1} A_2 + A_3^* X_{j-1} A_3 \tag{4.1.9}$$

for all $j \in \mathbb{N}$, where $X_0 = Q$, and the error $E_j := (\|X_j - X_{j-1}\|_{tr})^p$ for all $j \in \mathbb{N}$. After 6 iterations, we can approximate a solution $\hat{X}$ of Equation (4.1.8) by

$$\hat{X} \approx X_6 = \begin{pmatrix} 7.3321 & 3.3284 & 0.1998 & 0.4277 \\ 3.3284 & 8.2894 & 2.9635 & 0.5570 \\ 0.1998 & 2.9635 & 7.5132 & 3.5935 \\ 0.4277 & 0.5570 & 3.5935 & 8.9026 \end{pmatrix}$$

with $E_6 = 9.3708e - 005$.

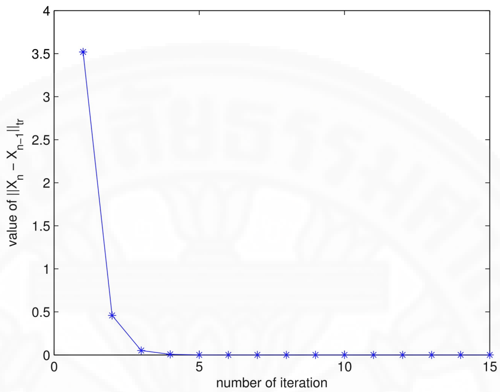


Figure 4.1: The error of the iteration process (4.1.9) for the Equation (4.1.8) given in Example 4.1.3.

4.2 Solutions of a nonlinear matrix equation arising from $(F,\gamma)_{\Re}$-contraction mappings via Thompson metrics

In this section, we apply fixed point results for $(F,\gamma)_{\Re}$-contraction mappings via Thompson metrics to solve the nonlinear matrix equation

$$X^r = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i, \tag{4.2.1}$$

where $r \geq 1$, $A_1, A_2, \ldots, A_m$ are $n \times n$ nonsingular matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ are continuous order preserving self-mappings on $P(n)$.

Studying the existence and uniqueness of a solution of the nonlinear matrix equation (4.2.1), we define a self-mapping $\mathcal{K}$ on $P(n)$ by

$$\mathcal{K}(X) = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i\right)^{\frac{1}{r}} \tag{4.2.2}$$

for all $X \in P(n)$, where $r \geq 1$, $A_1, A_2, \ldots, A_m$ are $n \times n$ nonsingular matrices, Q is a Hermitian positive definite matrix, and $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ are continuous order preserving self-mappings on $P(n)$.

Applying fixed point theorems for $(F,\gamma)_{\Re}$-contraction mappings, we will show that $\mathcal{K}$ has a unique fixed point and then the nonlinear matrix equation (4.2.1) has a unique positive definite solution. The following theorems guarantee the existence and uniqueness of positive definite solution of the nonlinear matrix equation (4.2.1).

Theorem 4.2.1. *Consider the matrix equation* (4.2.1). *Let $Q \in P(n)$ and for each $i = 1,2,\ldots,m$ $\mathcal{G}_i : P(n) \to P(n)$ be a continuous order-preserving mapping. Suppose that there are positive number τ and $r \geq 1$ such that for every $X, Y \in P(n)$ such that $(X,Y) \in \preceq$, we have*

$$d_T(\mathcal{G}_i(X), \mathcal{G}_i(Y)) \leq r d_T(X,Y) e^{-\frac{1}{[1+d_T(X,Y)]} - \tau} \tag{4.2.3}$$

for all $i = 1,2,\ldots,m$. Then the matrix equation (4.2.1) *has a unique positive solution. Moreover, the iteration*

$$X_j = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1}) A_i\right)^{\frac{1}{r}} \tag{4.2.4}$$

for all $j \in \mathbb{N}$, where $X_0 \in P(n)$ satisfies $X_0 \preceq \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0) A_i\right)^{\frac{1}{r}}$, converges in the sense of the Thompson metric d_T to a unique solution of the matrix equation (4.2.1).

Proof. From a self-mapping $\mathcal{K}$ on $P(n)$ defined as the equation (4.2.2), it follows that $\mathcal{K}$ is well-defined, $\preceq$ on $P(n)$ is $\mathcal{K}$-closed, and a fixed point of $\mathcal{K}$ is a positive solution of the matrix equation (4.2.1). Now we show that $\mathcal{K}$ is an $(F,\gamma)_{\Re}$-contraction mapping with $\tau > 0$ and the mapping $F, \gamma : \mathbb{R}^+ \to \mathbb{R}$ by

$$F(\alpha) = \ln\alpha \quad \text{and} \quad \gamma(\alpha) = \ln\alpha - \tfrac{1}{\lfloor 1+\alpha \rfloor}.$$

Let $(X,Y) \in \mathcal{A} = \{(X,Y) \in \preceq : G_i(X) \neq G_i(Y)\ \forall i = 1,2,\ldots,m\}$. Then $X \neq Y$ and $(G_i(X), G_i(Y)) \in \preceq$ since G_i is an order preserving mapping. Then

$$\begin{aligned}
d_T(\mathcal{K}(X),\mathcal{K}(Y))) &= d_T\left(\left(Q+\sum_{i=1}^{m}A_i^*\mathcal{G}_i(X)A_i\right)^{\frac{1}{r}}, \left(Q+\sum_{i=1}^{m}A_i^*\mathcal{G}_i(Y)A_i\right)^{\frac{1}{r}}\right)\\
&\leq \frac{1}{r}d_T\left(Q+\sum_{i=1}^{m}A_i^*\mathcal{G}_i(X)A_i, Q+\sum_{i=1}^{m}A_i^*\mathcal{G}_i(Y)A_i\right)\\
&\leq \frac{1}{r}d_T\left(\sum_{i=1}^{m}A_i^*\mathcal{G}_i(X)A_i, \sum_{i=1}^{m}A_i^*\mathcal{G}_i(Y)A_i\right)\\
&\leq \frac{1}{r}\max_{i\in\{1,2,\ldots,m\}} d_T(\mathcal{G}_i(X),\mathcal{G}_i(Y))\\
&\leq d_T(X,Y)\, e^{-\frac{1}{\lfloor 1+d_T(X,Y)\rfloor}-\tau}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\ln(d_T(\mathcal{K}(X),\mathcal{K}(Y)))) &\leq \ln\left(d_T(X,Y)\, e^{-\frac{1}{\lfloor 1+d_T(X,Y)\rfloor}-\tau}\right)\\
&= \ln(d_T(X,Y)) - \frac{1}{\lfloor 1+d_T(X,Y)\rfloor} - \tau
\end{aligned}$$

and so

$$\tau + \ln(d_T(\mathcal{K}(X),\mathcal{K}(Y)))) \leq \ln(d_T(X,Y)) - \frac{1}{\lfloor 1+d_T(X,Y)\rfloor}.$$

Therefore, $\mathcal{K}$ is an $(F,\gamma)_{\Re}$-contraction mapping with $\tau > 0$. Moreover, there exists $Q^{\frac{1}{r}} \in P(n)$ such that

$$Q^{\frac{1}{r}} \preceq \left(Q+\sum_{i=1}^{m}A_i^*\mathcal{G}_i(Q^{\frac{1}{r}})A_i\right)^{\frac{1}{r}} = \mathcal{K}\left(Q^{\frac{1}{r}}\right).$$

This implies that $Q^{\frac{1}{r}} \in P(n)(\mathcal{K},\preceq)$. Using Theorem 3.2.6, we conclude that there exists $X^* \in P(n)$ such that $\mathcal{K}(X^*) = X^*$. That is, X^* is a positive definite solution of the Equation (4.2.1).

Finally, since for every $X,Y \in P(n)$ there is a greatest lower bound and a least upper bound, we have $\Upsilon(X,Y,\Re)$ is nonempty for each $X,Y \in P(n)$. Thus, it follows from Theorem 3.2.10 that $\mathcal{K}$ has a unique fixed point in $P(n)$. This implies that Equation (4.2.1) has a unique solution in $P(n)$. □

The method in Theorem 4.2.1 for finding the solution of Equation (4.2.1) is detailed in the following steps.

Step 1. Check that (4.2.3) holds for all $i = 1, 2, \ldots, m$.

Step 2. Initialize the starting point $X_0 \in P(n)$ satisfying $X_0 \preceq \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0) A_i \right)^{\frac{1}{r}}$.

Step 3. Set up E and e as the tolerances for the stopping criteria in the algorithm.

Step 4. Calculate a unique positive definite solution $\hat{X}$ of the matrix equation (4.2.1) from the iteration (4.2.4).

Based on the various techniques for approximating the root of matrices, we have many choices for constructing the method for finding the solution of Equation (4.2.1) by using the above steps. For instance, the algorithm for finding the solution of Equation (4.2.1) by using our step with the Newton's method [53] is as follows.

Algorithm for finding the solution of Equation (4.2.1)

1. Check that (4.2.3) holds for all $i = 1,2,\dots,m$.

Initialize:

2. Set the starting point $X_0 \in P(n)$ satisfying $X_0 \preceq \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0) A_i\right)^{\frac{1}{r}}$.

3. Set the $E > 0$ and $e > 0$ as the tolerances for the stopping criteria.

4. Set the iteration step $j := 1$.

do

5. Calculate $B_j = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1}) A_i$.

6. Set $Y_1 := B_j$.

7. Set the iteration step $k := 1$.

    do

    8. Calculate $Y_{k+1} := \frac{1}{r}\left[(r-1)Y_k + B_j Y_k^{1-r}\right]$.

    9. Update $k := k+1$.

    while $d_T(Y_k, Y_{k-1}) \geq E$

    end while

10. $X_j := Y_k$

11. Update $j := j+1$.

while $d_T(X_{j-1}, X_{j-2}) \geq e$

end while

12. Obtain the solution $\hat{X} := X_{j-1}$.

We can summarize the suggested algorithm as in the flowchart in Fig. 4.4.

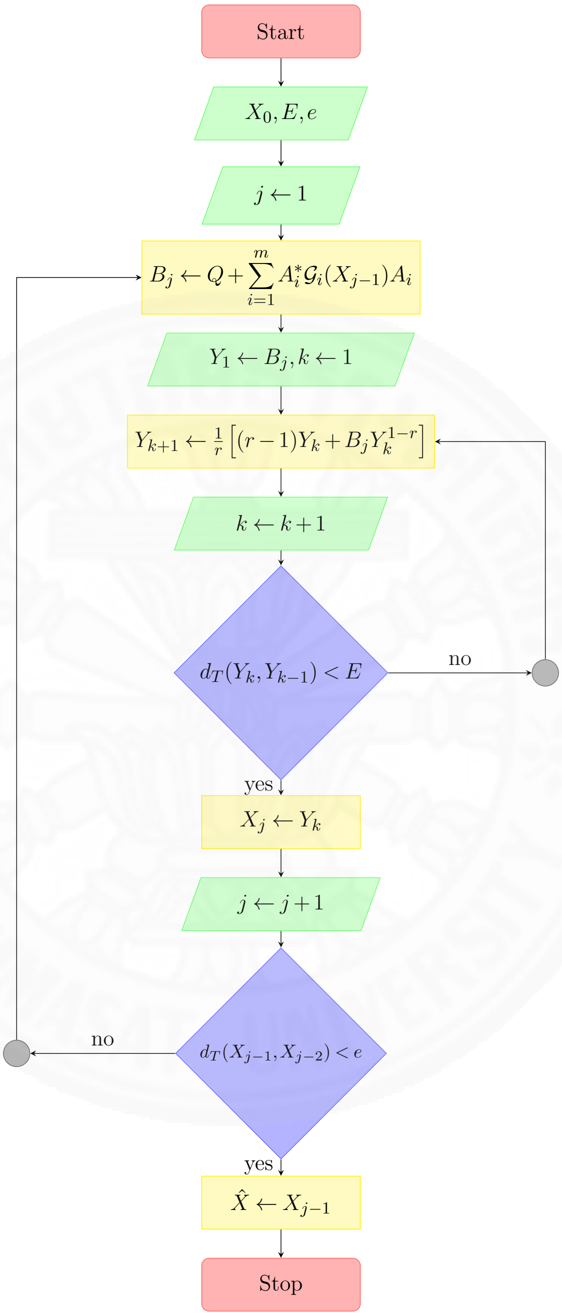


Figure 4.2: The flowchart of the algorithm for finding the solution of Equation (4.2.1).

4.3 Solutions of a nonlinear matrix equation arising from $(\psi,\phi,\Re)$-contraction mappings via Thompson metrics

In this section, we apply fixed point results for $(\psi,\phi,\Re)$-contraction mappings via Thompson metrics to solve the nonlinear matrix equation

$$X^r = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i, \tag{4.3.1}$$

where $r \geq 1$, $A_1, A_2, \ldots, A_m$ are $n \times n$ nonsingular matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ are continuous order preserving self-mappings on $P(n)$.

Studying the existence and uniqueness of a solution of the nonlinear matrix equation (4.3.1), we define a self-mapping $\mathcal{K}$ on $P(n)$ by

$$\mathcal{K}(X) = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i\right)^{\frac{1}{r}} \tag{4.3.2}$$

for all $X \in P(n)$, where $r \geq 1$, $A_1, A_2, \ldots, A_m$ are $n \times n$ nonsingular matrices, Q is a Hermitian positive definite matrix and $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ are continuous order preserving self-mappings on $P(n)$.

Applying fixed point theorems for $(\psi,\phi,\Re)$-contraction mappings, we will show that $\mathcal{K}$ defined as (4.3.2) has a unique fixed point and then the nonlinear matrix equation (4.3.1) has a unique positive definite solution. The following theorems guarantee the existence and uniqueness of positive definite solution of the nonlinear matrix equation (4.3.1).

Theorem 4.3.1. *Consider the matrix equation* (4.3.1). *Let $Q \in P(n)$ and for each $i = 1, 2, \ldots, m$, $\mathcal{G}_i : P(n) \to P(n)$ be continuous order preserving mappings. If for every $X, Y \in P(n)$ such that $X \preceq Y$ with $\mathcal{G}_i(X) \neq \mathcal{G}_i(Y)$ there exists $\psi \in \Psi$ and $\phi \in \Phi$ with $\psi(t) > \phi(t)$ for all $t > 0$ such that*

$$\psi\left(\frac{1}{r} d_T(\mathcal{G}_i(X), \mathcal{G}_i(Y))\right) \leq \phi\left(d_T(X,Y)\right) \tag{4.3.3}$$

for all $i = 1, 2, \ldots, n$, where $r \in [1, \infty)$. Then the matrix equation (4.3.1) *has a* *solution. Moreover, the iteration*

$$X_j = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1}) A_i\right)^{\frac{1}{r}} \tag{4.3.4}$$

for all $j \in \mathbb{N}$*, where* $X_0 \in P(n)$ *satisfies* $X_0 \preceq \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0) A_i\right)^{\frac{1}{r}}$*, converges in the sense of the Thompson metric* d_T *to a unique solution of the matrix equation* (4.3.1)*.*

Proof. We define the mapping $\mathcal{K} : P(n) \to P(n)$ by

$$\mathcal{K}(X) = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i\right)^{\frac{1}{r}} \quad \text{for all} \quad X \in P(n). \tag{4.3.5}$$

Then $\mathcal{K}$ is well defined, $\preceq$ on $P(n)$ is $\mathcal{K}$-closed and a fixed point of $\mathcal{K}$ is a solution of equation (4.3.1). Now we show that $\mathcal{K}$ is a $(\psi, \phi, \Re)$-contraction. Let $X, Y \in P(n)$ such that $X \preceq Y$ with $\mathcal{G}_i(X) \neq \mathcal{G}_i(Y)$. This implies that $X \prec Y$. Since $\mathcal{G}_i$ are order preserving mapping, we obtain that $\mathcal{G}_i(X) \prec \mathcal{G}_i(Y)$. Thus,

$$\begin{aligned}
\psi\left(d_T(\mathcal{K}(X), \mathcal{K}(Y))\right) &= \psi\left(d_T\left(\left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i\right)^{\frac{1}{r}}, \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(Y) A_i\right)^{\frac{1}{r}}\right)\right) \\
&\leq \psi\left(\frac{1}{r} d_T\left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i, Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(Y) A_i\right)\right) \\
&\leq \psi\left(\frac{1}{r} d_T\left(\sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i, \sum_{i=1}^{m} A_i^* \mathcal{G}_i(Y) A_i\right)\right) \\
&\leq \psi\left(\frac{1}{r} \max_i d_T\left(\mathcal{G}_i(X), \mathcal{G}_i(Y)\right)\right) \\
&\leq \phi\left(d_T(X, Y)\right).
\end{aligned}$$

So $\mathcal{K}$ is a $(\psi, \phi, \Re)$-contraction. Moreover, there exists $Q^{\frac{1}{r}} \in P(n)$ such that $Q^{\frac{1}{r}} \preceq \mathcal{K}\left(Q^{\frac{1}{r}}\right)$. By Proposition 2.7.25, there exists $(\mathcal{K} \preceq)$-Picard sequence in $P(n)$. From Theorem 3.3.7, $\mathcal{K}^n\left(Q^{\frac{1}{r}}\right)$ converges to a solution Z of the matrix equation (4.3.1) in $P(n)$. □

Similarly, we obtain the following theorems.

Theorem 4.3.2. *Consider the matrix equation* (4.3.1)*. Let* $Q \in P(n)$ *and for each* $i = 1, 2, \ldots, m$*,* $\mathcal{G}_i : P(n) \to P(n)$ *be continuous order preserving mappings. If* *for every* $X,Y \in P(n)$ *such that* $X \succeq Y$ *with* $\mathcal{G}_i(X) \neq \mathcal{G}_i(Y)$ *there exists* $\psi \in \Psi$ *and* $\phi \in \Phi$ *with* $\psi(t) > \phi(t)$ *for all* $t > 0$ *such that*

$$\psi\left(\frac{1}{r} d_T(G_i(X), G_i(Y))\right) \leq \phi\left(d_T(X,Y)\right) \tag{4.3.6}$$

for all $i = 1,2,\ldots,n$, *where* $r \in [1,\infty)$ *and there exists* $X_0 \succeq \mathcal{K}(X_0)$ *and* $\mathcal{K}^(Q^{\frac{1}{r}})$ *is convergent in* $(P(n), d_T)$, *where the mapping* $\mathcal{K}$ *is denoted by* $X \to \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X) A_i\right)^{\frac{1}{r}}$. *Then the matrix equation* (4.3.1) *has a solution. Moreover, the iteration*

$$X_j = \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1}) A_i\right)^{\frac{1}{r}} \tag{4.3.7}$$

for all $j \in \mathbb{N}$, *converges in the sense of the Thompson metric* d_T *to a unique solution of the matrix equation* (4.3.1).

Theorem 4.3.3. *In addition to the hypothesis of Theorem 4.3.1, the equation* (4.3.1) *has a unique solution* $Z \in P(n)$.

Proof. From Theorem 4.3.1, we have $Fix(\mathcal{K}) \neq \emptyset$. Since for every $X,Y \in P(n)$ there is a greatest lower bound and a least upper bound, we have $Fix(\mathcal{K})$ is $\preceq^s$-connected. It follows from Theorem 3.3.9 that $\mathcal{K}$ has a unique fixed point in $P(n)$. This implies that Equation (4.3.1) has a unique solution in $P(n)$. □

Next, we give a numerical example to show the correctness of Theorem 4.3.3.

Example 4.3.4. Let

$$Q = \begin{pmatrix} 0.0450 & 0.0225 & 0 & 0 \\ 0.0225 & 0.0450 & 0.0225 & 0 \\ 0 & 0.0225 & 0.0450 & 0.0225 \\ 0 & 0 & 0.0225 & 0.0450 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0.0254 & 0.0214 & 0.1026 & 0.0146 \\ 0.0189 & 0.9141 & 0.3231 & 0.1069 \\ 0.0129 & 0.2254 & 0.3125 & 0.4412 \\ 0.9073 & 0.0264 & 0.0114 & 0.2034 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0.0321 & 0.5002 & 0.1407 & 0.7034 \\ 0.5011 & 0.9402 & 0.3102 & 0.1471 \\ 0.1761 & 0.2543 & 0.5001 & 0.7441 \\ 0.3147 & 0.2241 & 0.6105 & 0.1646 \end{pmatrix}.$$

Define $\mathcal{G} : P(n) \to P(n)$ by $\mathcal{G}_1(X) = \mathcal{G}_2(X) = X^{\frac{1}{2}}$. Consider Equation (4.3.1) with $\mathcal{G}_1(X) = \mathcal{G}_2(X) = X^{\frac{1}{2}}$ and $r = 1$ that is

$$X = Q + A_1^* X^{\frac{1}{2}} A_1 + A_2^* X^{\frac{1}{2}} A_2. \tag{4.3.8}$$

Define functions $\psi, \phi : [0,\infty) \to [0,\infty)$ by $\psi(t) = 2t$ and $\phi(t) = t$ for all $[0,\infty)$. Let $X, Y \in P(n)$ such that $X \succeq Y$ with $\mathcal{G}_i(X) \neq \mathcal{G}_i(Y)$. We see that $\mathcal{G}_i$ are continuous order preserving for all $i = 1,2$. Then $X \succ Y$ and $\mathcal{G}_i(X) \succ \mathcal{G}_i(Y)$ for all $i = 1,2$ and

$$d_T(G_i(X), G_i(Y)) = d_T(X^{\frac{1}{2}}, Y^{\frac{1}{2}}) \leq \frac{1}{2} d_T(X,Y). \tag{4.3.9}$$

It follows that

$$\psi\left(d_T(G_i(X), G_i(Y))\right) \leq \phi\left(d_T(X,Y)\right) \tag{4.3.10}$$

for all $i = 1,2$. Moreover, there exists $X_0 = 16I_4$ such that $16I_4 \succeq \mathcal{K}(16I_4)$ and $(\mathcal{K}^n(Q))$ is convergent. All the hypotheses of Theorem 4.3.3, we can conclude that the Equation 4.3.8 has a unique solution $\hat{X}$ in $P(n)$. We will consider the iteration

$$X_j = Q + A_1^* X_{j-1}^{\frac{1}{2}} A_1 + A_2^* X_{j-1}^{\frac{1}{2}} A_2, \tag{4.3.11}$$

for all $j \in \mathbb{N}$, where $X_0 = 16I_4$, and the error $E_j := d_T(X_j, X_{j-1})$ for all $j \in \mathbb{N}$. After 18 iterations, we can approximate a solution $\hat{X}$ of Equation (4.3.8) by

$$\hat{X} \approx X_8 = \begin{pmatrix} 2.4090 & 3.1286 & 2.1331 & 2.1313 \\ 3.1286 & 6.3328 & 3.9076 & 3.7938 \\ 2.1331 & 3.9076 & 2.8996 & 2.7232 \\ 2.1313 & 3.7938 & 2.7232 & 3.0994 \end{pmatrix}$$

with $E_{18} = 5.0751e - 06$.

In addition, we can use another method involving Theorem 4.3.3 to find the solution of Equation (4.3.1) by setting tolerances E and e as the following steps.

Step 1. Check that (4.3.3) (or (4.3.6)) holds for all $i = 1, 2, \ldots, m$.

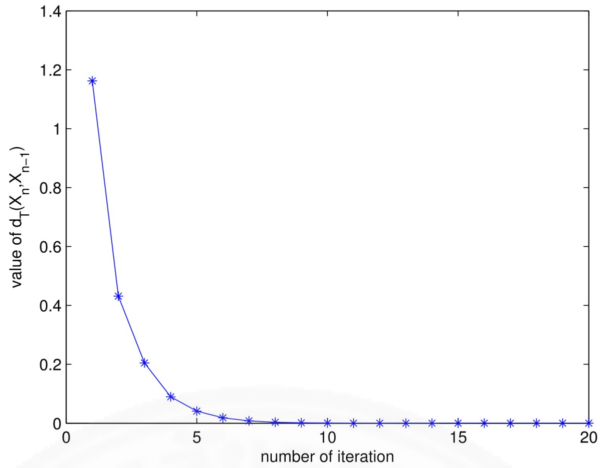


Figure 4.3: The error of the iteration process (4.3.11) for the Equation (4.3.8) given in Example 4.3.4.

Step 2. Initialize the starting point $X_0 \in P(n)$ satisfying $X_0 \succeq \left(Q+\sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0)A_i\right)^{\frac{1}{r}}$ (or set the starting point $X_0 = Q^{\frac{1}{2}} \in P(n)$).

Step 3. Set up E and e as the tolerances for the stopping criteria in the algorithm.

Step 4. Calculate a unique positive definite solution $\hat{X}$ of the matrix equation (4.3.1) from the iteration $X_j = \left(Q+\sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1})A_i\right)^{\frac{1}{r}}$.

Based on the various techniques for approximating the root of matrices, we have many choices for constructing the method for finding the solution of Equation (4.3.1) by using the above steps. For instance, the algorithm for finding the solution of Equation (4.3.1) by using our step with the Newton's method [53] is as follows.

Algorithm for finding the solution of Equation (4.3.1)

1. Check that (4.3.3) (or (4.3.6)) holds for all $i = 1, 2, \dots, m$.

Initialize:

2. Set the starting point $X_0 \in P(n)$ satisfying $X_0 \preceq \left(Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_0) A_i \right)^{\frac{1}{r}}$ (or set the starting point $X_0 = Q^{\frac{1}{2}} \in P(n)$).

3. Set the $E > 0$ and $e > 0$ as the tolerances for the stopping criteria.

4. Set the iteration step $j := 1$.

do

5. Calculate $B_j = Q + \sum_{i=1}^{m} A_i^* \mathcal{G}_i(X_{j-1}) A_i$.

6. Set $Y_1 := B_j$.

7. Set the iteration step $k := 1$.

 do

 8. Calculate $Y_{k+1} := \frac{1}{r}\left[(r-1)Y_k + B_j Y_k^{1-r}\right]$.

 9. Update $k := k+1$.

 while $d_T(Y_k, Y_{k-1}) \geq E$

 end while

10. $X_j := Y_k$

11. Update $j := j+1$.

while $d_T(X_{j-1}, X_{j-2}) \geq e$

end while

12. Obtain the solution $\hat{X} := X_{j-1}$.

We can summarize the suggested algorithm as in the flowchart in Fig. 4.4.

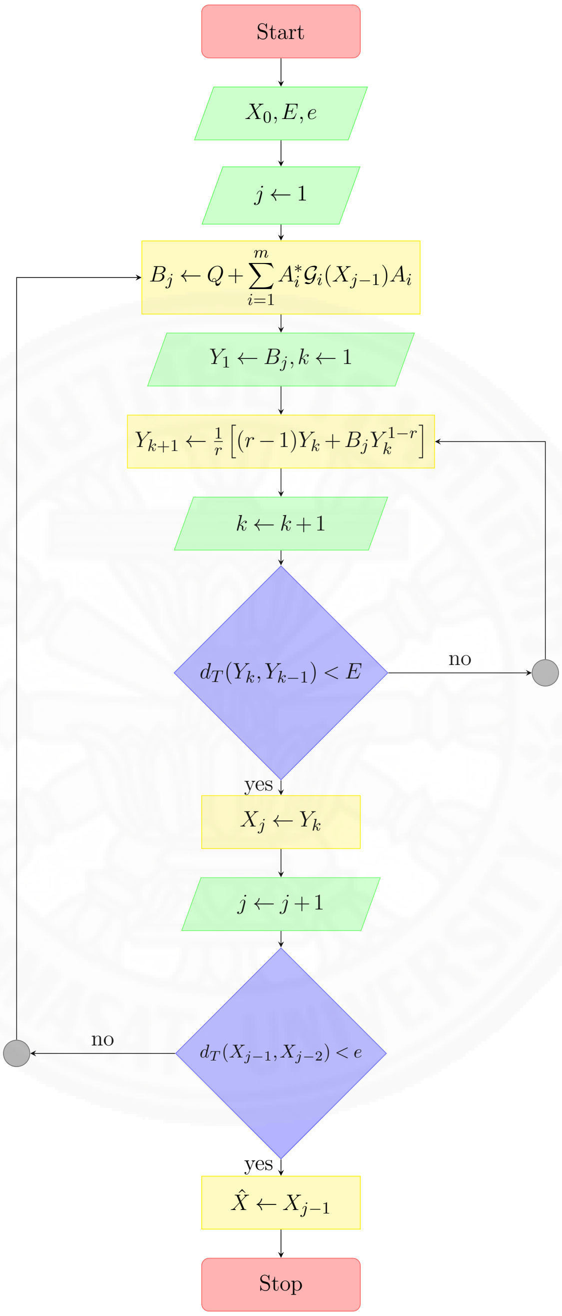


Figure 4.4: The flowchart of the algorithm for finding the solution of Equation (4.3.1).

CHAPTER 5

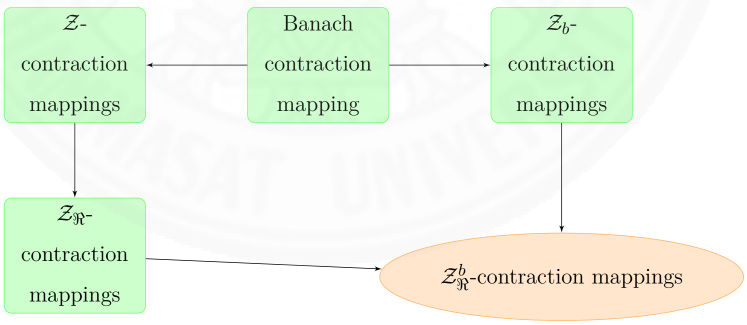
CONCLUSIONS AND RECOMMENDATIONS

5.1 Conclusions

In this dissertation, we improved and generalized fixed point results in the past and applied our obtained fixed point results to guarantee the existence and uniqueness of a solution of nonlinear matrix equations. So we consider our fixed point results in Chapter 3 and investigate our applications in Chapter 4, respectively.

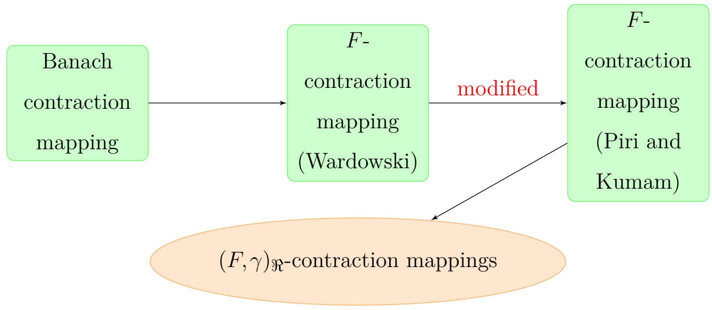
In Chapter 3, we improved several control functions and defined new contraction mappings in terms of some control functions in the setting of metric and b-metric spaces endowed with a binary relation.

In Section 3.1, we introduced the concept of $\mathcal{Z}_{\Re}^{b}$-contraction mappings, which are the extension from concepts of other previous known mappings as follows:



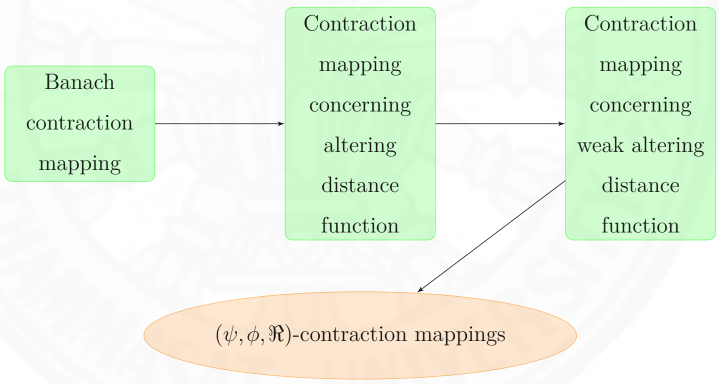
We also established fixed point results for $\mathcal{Z}_{\Re}^{b}$-contraction mappings in complete b-metric spaces endowed with a binary relation, which is more general than other fixed point results in the literature.

In Section 3.2, we introduced the concept of $(F,\gamma)_{\Re}$-contraction mappings, which are the extension from concepts of other previous known mappings as follows:



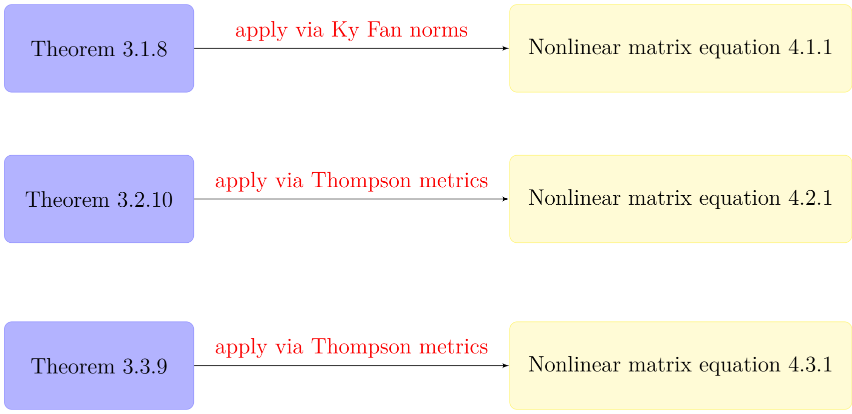
Moreover, we established fixed point results for $(F,\gamma)_{\Re}$-contraction mappings in complete metric spaces endowed with a binary relation and also gave an example to illustrate the utility of our fixed point results such that other previous fixed point results are not applicable.

In Section 3.3, we introduced the concept of $(\psi,\phi,\Re)$-contraction mappings, which are the extension from concepts of other previous mappings as follows:



Furthermore, we established fixed point results for $(\psi,\phi,\Re)$-contraction mappings in complete metric spaces endowed with with a T-orbital transitivity and also give an example to show the benefit of our fixed point results such that other previous fixed point results are not applicable.

In Chapter 4, we applied our fixed point results to consider the existence and uniqueness of solution of nonlinear matrix equations as follows:



5.2 Recommendations

The advantages of our fixed point results in this dissertation are to solve problems that some fixed point results in the past cannot be applied (see Example 3.2.7 and Example 3.3.10). Indeed, the new contractive conditions in our main results of this dissertation hold for each two elements which are related under the weak condition with many conditions in this part.

However, our fixed point results in this dissertation cannot solve every problems in the world. The development of several fixed point results are necessary. So we pose the following open problems for further investigations.

- Can some conditions of b-simulation functions be reduced to weaker conditions?
- Can the fixed point results for $(F,\gamma)_{\Re}$-contraction mappings and $(\psi,\phi,\Re)$-contraction mappings be extended to b-metric spaces endowed with a binary relation?
- Can the fixed point results for other contractions be applied to a nonlinear matrix equation (4.1.1) and a nonlinear matrix equation (4.2.1)?

REFERENCES

[1] Banach S. Sur les opérations dans les ensembles abstraits et leurs applications. Fund. Math. 1922;3:133-81.

[2] Dhutta P, Choudhury B. A generalization of contractions in partially ordered metric spaces. Appl. Anal. 2008;87:109-16.

[3] Sastry KPR, Babu GVR. Some fixed point theorems by altering distance between the points. Indian J. Pure Appl. Math. 1999;30:641-7.

[4] Amini-Harandi A, Emami H. A fixed point theorem for contraction type maps in partially ordered metric spaces and application to ordinary differential equations. Nonlinear Anal. 2010;72:2238-42.

[5] Harjani J, Sadarangni, K. Fixed point theorems for weakly contraction mappings in partially ordered sets. Nonlinear Anal. 2009;71:3403-10.

[6] Burgic D, Kalabušić S, Kulenović M. Global attractivity results for mixed monotone mappings in partially ordered complete metric spaces. Fixed Point Theory Appl. 2009, Article ID 762478.

[7] Cirić L, Cakić N, Rajović M, Uma J. Monotone generalized nonlinear contractions in partially ordered metric spaces. Fixed Point Theory Appl. 2008, Article ID 131294.

[8] Bhaskar TG, Lakshmikantham V. Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. 2006;65:1379-93.

[9] Lakshmikantham V, Cirić L. Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. 2009;70:4341-9.

[10] Nieto JJ, Rodriguez-López R. Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order. 2005;22:223-39.

[11] Nieto JJ, Rodriguez-López R. Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary differential equations. Acta Math. Sin. 2007;23:2205-12.

[12] O'Regan D, Petrusel, A. Fixed point theorems for generalized contractions in ordered metric spaces. J. Math. Anal. Appl. 2008;341:1241-52.

[13] Shatanawi W, Postolache M. Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces. Fixed Point Theory Appl. 2013, Article ID 271.

[14] Shatanawi W, Postolache M. Common fixed point results of mappings for nonlinear contraction of cyclic form in ordered metric spaces. Fixed Point Theory Appl. 2013, Article ID 60.

[15] Karapinar E, Shatanawi W, Mustafa Z. Quadruple fixed point theorems under nonlinear contractive conditions in partially ordered metric spaces. J. Appl. Math. 2012, Article ID 951912.

[16] Shatanawi W. Some fixed point results for a generalized-weak contraction mappings in orbitally metric spaces. Chaos Solitons Fractals. 2012;45:520-6.

[17] Ran ACM, Reurings MCB. A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Amer. Math. Soc. 2003;132:1435-43.

[18] Sawangsup K, Sintunavarat W, de Hierro AFRL. Fixed point theorems for $F_{\mathfrak{R}}$-contractions with applications to solution of nonlinear matrix equations. J. Fixed Point Theory Appl. 2017;19(3):1711-25.

[19] Sawangsup K, Sintunavarat W. Fixed point and multidimensional fixed point theorems with applications to nonlinear matrix equations in terms of weak altering distance functions. Open Mathematics. 2017;15(1):111-25.

[20] Sawangsup K, Sintunavarat W. On modified $\mathcal{Z}$-contractions and an iterative scheme for solving nonlinear matrix equations. J. Fixed Point Theory Appl. 2018;20(2): 80.

[21] Lim Y. Solving the nonlinear matrix equation $X = Q + \sum_{i=1}^{m} M_i X^{\delta_i} M_i^*$ via a contraction principle. Linear Algebra Appl. 2009;430:1380-3.

[22] Khojasteha F, Shuklab S, Radenović S. A new approach to the study of fixed point theory for simulation function. Filomat. 2015;29:6:1189-94.

[23] Argoubi H, Samet B, Vetro C. Nonlinear contractions involving simulation functions in a metric space with a partial order. J. Nonlinear Sci. Appl. 2015;8(2015):1082-94.

[24] Demma M, Saadati R, Vetro P. Fixed Point Results on b-Metric Space via Picard Sequences and b-Simulation Functions. Iranian Journal of Mathematical Sciences and Informatics. 2016;11(1);123-36.

[25] Khan MS, Swaleh M, Sessa S. Fixed point Theorems by altering distances between the points. Bull. Aust. Math. Soc. 1984;30 (1):1-9.

[26] Wardowski D. Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl. 2012;94: 6 pages.

[27] Radenovic S, Chandok S. Simulation type functions and coincidence point results. Fixed Point Theory and Appl. 2014, Article ID 190.

[28] Bakhtin IA. The contraction mapping principle in almost metric spaces. Funct. Anal. 30. Unianowsk, Gos. Ped. Inst. 1989;26-37.

[29] Czerwik S. Nonlinear set-valued contraction mappings in B-metric spaces. Atti Sem. Mat. Univ. Modena. 1998;46:263-76.

[30] Boriceanu M. Fixed point theory for multivalued generalized contraction on a set with two b-metric. studia, univ Babes, Bolya: Math, Liv(3)(2009): 1-14.

[31] Aghajani A, Abbas M, Roshan JR. Common fixed point of generalized weak contractive mappings in partially ordered *b*-metric spaces. Math. Slovaca. 2014;64(4):941-60.

[32] Boriceanu M, Bota M, Petrusel A. Multivalued fractals in *b*-metric spaces. Cent. Eur. J. Math. 2010;8(2):367-77.

[33] Alam A, Imdad M. Relation-theoretic contraction principle. Fixed point Theory Appl. 2015;17(4):693-702.

[34] Samet B, Turinici M. Fixed point theorems on a metric space endowed with an arbitrary binary relation and applications. Commun. Math. Anal. 2012;13(2012):82-97.

[35] Lipschutz S. SchaumŠs Outlines of Theory and Problems of Set Theory and Related Topics. McGrawŰHill, New York. 1964.

[36] Kolman B, Busby RC, Ross S. Discrete Mathematical Structures. 3rd.

[37] Alam A, Imdad M. Relation-theoretic metrical coincidence theorems. under proses in Filomat. 2017. arXiv:1603.09159 [math.FA].

[38] Shahzad N, de Hierro AFRL, Khojasteh F. Some new fixed point theorems under $(\mathcal{A},\mathcal{S})-$contractivity conditions. Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas. 2016;pages 1-18.

[39] Maddux RD. Relation Algebras. Stud. Logic Found. Math. 150, Elsevier B. V., Amsterdam. 2006.

[40] Imdad M, Khan Q, Alfaqih WM, Gubran R. A relation theoretic $(F,\Re)-$contraction principle with applications to matrix equations. To appear in Bulletin of Mathematical Analysis and Applications. 2018.

[41] Ciric LB. On contraction type mappings. Math. Balkanica. 1971;1971(1):52-7.

[42] de Hierro AFRL, Shahzad N. Common fixed point theorems under $(\mathcal{R}, \mathcal{S})-$contractivity conditions. Fixed Point Theory and Applications. 2016;2016(1):1-25.

[43] Piri H, Kumam P. Some fixed point theorems concerning F-contraction in complete metric spaces. Fixed Point Theory and Applications. 2014;2014:210.

[44] Yan F, Su Y, Feng Q. A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differenctial equations. Fixed point Theory Appl. 2012;Article ID 152.

[45] Asgari MS, Mousavi B. Coupled fixed point theorems with respect to binary relations in metric spaces. J. Nonlinear Sci. Appl. 2015;8(2015):153-62.

[46] Long JH, Hu XY, Zhang L. On the Hermitian positive definite solution of the nonlinear matrix equation $X + A^*x^{-1}A + B^*X^{-1}B = I$. Bull. Braz. Math. Soc. 2015;39(3):317-86.

[47] Hussain N, Ahmad J, Azam A. On Suzuki-Wardowski type fixed point theorems. J. Nonlinear Sci. Appl. 2015;8(6):1095-111.

[48] Parodi M. La localisation des valeurs caracteristique des matrics et ses applications. Gauthiervillars. Paris. 1959.

[49] Imdad M, Khan QH, Alfaqih WM, Gubran R. A relation-theoretic $(F, \mathcal{R})$-contraction principle with applications to matrix equations, Bulletin of Mathematical Analysis & Applications. 2018;10(1):1-12.

[50] Rus MD. The fixed point problem for systems of coordinate-wise uniformly monotone operators and applications. Mediterr. J. Math. 2014;11:109-22.

[51] Soleimani Rad G, Shukla S, Rahimi H. Some relations between n-tuple fixed point and fixed point results, Revista de la Real Academia de Ciencias Exactas. Fisicas y Naturales. Serie A. Matematicas. 2015 Sep;109(2):471-81.

[52] Su Y, Petrusel A, Yao JC. Multivariate fixed point theorems for contractions and nonexpansive mappings with applications. Fixed Point Theory Appl. 2016:9, 19 pp.

[53] Hoskins WD, Walton DJ. A faster, more stable method for computing the pth roots of positive definite matrices. Linear Algebra Appl. 1979;26(1979):139-63.

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Publications

1. Sawangsup K, Sintunavarat W. On solving nonlinear matrix equations in terms of b-simulation functions in b-metric spaces with numerical solutions. Comput. Appl. Math. 2018;37(5):5829-43.

2. Sawangsup K, Sintunavarat W. On modified $\mathcal{Z}$-contractions and an iterative scheme for solving nonlinear matrix equations. J. Fixed Point Theory Appl. 2018;20(2):80.

3. Sawangsup K, Sintunavarat W. Fixed point and multidimensional fixed point theorems with applications to nonlinear matrix equations in terms of weak altering distance functions. Open Mathematics. 2017;15(1):111-25.

4. Sawangsup K, Sintunavarat W, Roldán López de Hierro, A, F. Fixed point theorems for $F_{\mathfrak{R}}$-contractions with applications to solution of nonlinear matrix equations. J. Fixed Point Theory Appl. 2017;19(3):1711-25.

International Oral Presentations

1. Sawangsup K, Sintunavarat W. Fixed point results for orthogonal $\mathcal{Z}$-contraction mappings in O-complete metric spaces, 2019 the 8^{th} International Conference on Pure and Applied Mathematics (ICPAM2019), 22-25 July 2019, Novotel Brussels City Centre, Brussels, Belgium.

2. Sawangsup K, Sintunavarat W. Coincidence point results for mappings and relations via simulation functions with an application, The 10^{th} Asian Conference on Fixed Point Theory and Optimization, 16-18 July 2018, The Empress Hotel, Chiang Mai, Thailand.

3. Sawangsup K, Sintunavarat W. On a modified $\mathcal{Z}$-contractions with the iteration scheme for numerical reckoning solutions of nonlinear matrix equations, International Conference on Applied Statistics 2016 (ICAS 2016), 13-15 July, 2016, Phuket Graceland Resort and Spa, Phuket, Thailand.

4. Sawangsup, K, Sintunavarat, W, Roldán López de Hierro, A.F. Fixed point theorems for $F_{\mathfrak{R}}$-contractions with applications to resolution of nonlinear matrix equations, The 9^{th} Asian Conference on Fixed Point Thoery and Optimization, 18-20 May 2016, the Faculty of Science, King Mongkut's University of Technology, Thailand.

National Oral Presentations

1. Sawangsup, K., Sintunavarat, W. Some weaker conditions on relation-theoretic F-contraction mappings with application to matrix equation, The 24^{th} Annual Meeting in Mathematics (AMM 2019), 15-17 May 2019, at Central Laboratory, Faculty of Science, Burapha University, Chonburi, Thailand.

2. Sawangsup K, Sintunavarat W. The relation-theoretic $(\psi, \phi, \Re)$-contraction with applications to nonlinear matrix equations, Phayao Research Conference 2019, 24-25 January 2019, Phayao, Thailand.

3. Sawangsup K, Sintunavarat W. On the solving coupled systems of functional equations in terms of weak altering distance functions, The 10^{th} National Science Research Conference "Science Leading for Thailand Innovation 4.0", 24-25 May 2018, Mahasarakham University, Thailand.

4. Sawangsup, K., Sintunavarat, W. & Cho, Y.J. Fixed point theorems for orthogonal F-contraction mappings on O-complete metric spaces, The 23^{rd} Annual Meeting in Mathematics (AMM 2018), 3-5 May 2018, Mandarin Oriental, Bangkok, Thailand.

5. Sawangsup K, Sintunavarat W. On the solving a nonlinear matrix equation by fixed-point iteration in terms of b-simulation functions, The 22^{nd} Annual Meeting in Mathematics (AMM 2017), 2-4 June 2017, Lotus Hotel Pang Suan Kaew, Chiang Mai, Thailand.

6. Sawangsup K, Sintunavarat W. Discussion on relation-theoretic for JS-quasi-contractions of uni/multi-dimensional mappings with transitivity, The 42^{nd} Congress on Science and Technology of Thailand (STT 42), 1 December 2016, Centara Grand at Central Plaza Ladprao, Bangkok, Thailand.

Poster Presentation

1. Sawangsup K, Sintunavarat W. Fixed point theorems with applications to nonlinear matrix equations in terms of simulation functions, The 9^{th} National Science Research Conference, 25-26 May 2017, Faculty of Science, Chonburi, Thailand.

2. Sawangsup K, Sintunavarat W. Fixed point theorems with weak altering distance functions under a transitive relation, Science Research Conference 8^{th}, 30-31 May 2016, School of Science, University of Phayao, Thailand.

Awards

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2. Honorable mention on poster presentation, Sawangsup K, Sintunavarat W. Fixed point theorems with weak altering distance functions under a transitive relation, Science Research Conference 8^{th}, 30-31 May 2016, School of Science, University of Phayao, Thailand.