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นายกฤษฎา สังขมงคล

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AN IMPROVEMENT OF PROBABILITY APPROXIMATION OF
RANDOMIZED ORTHOGONAL ARRAY SAMPLING

Mr. Kitsada Sungkamongkol

A Thesis Submitted in Partial Fulfillment of the Requirements
for the Degree of Master of Science Program in Mathematics

Department of Mathematics
Faculty of Science
Chulalongkorn University

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ให้ X เป็นเวกเตอร์สุ่มที่มีการแจกแจงแบบสม่ำเสมอบน $[0,1]^3$ และกำหนดให้ f เป็นฟังก์ชันจาก $\mathbb{R}^3$ ไปยัง $\mathbb{R}$ ซึ่งสามารถหาปริพันธ์ได้และนิยามให้

$$\mu = Ef(X) = \int_{[0,1]^3} f(x)dx$$

ตัวประมาณค่าอย่างง่ายตัวหนึ่งของ μ คือ

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

โดยที่ $X_1, X_2, ..., X_n$ เป็นเวกเตอร์สุ่มที่เป็นอิสระต่อกันและมีการแจกแจงแบบสม่ำเสมอบน $[0,1]^3$ อย่างไรก็ตามมีวิธีในการสุ่มเลือก $X_1, X_2, ..., X_n$ อยู่หลายวิธี หนึ่งในนั้นคือ การสุ่มตัวอย่างแบบแถวเชิงตั้งฉากโดย ในปี ค.ศ. 1996 ลอฮ์ เป็นบุคคลแรกที่พิจารณาการประมาณค่าการแจกแจงของตัวแปรสุ่ม $W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}$ เมื่อ $Var(\hat{\mu}) > 0$ ด้วยการแจกแจงปกติและให้ขอบเขตแบบสม่ำเสมอ ในปี ค.ศ. 2008 เนียมมณี และไหลภาภรณ์ ได้ปรับปรุงอัตราการลู่เข้าของ ลอฮ์ ให้เป็น $O(q^{-\frac{1}{2}})$ ด้วยสมมุติฐานที่ว่าโมเมนต์ที่หกของ $f \circ X$ มีค่าจำกัด

ในวิทยานิพนธ์ฉบับนี้เราจะปรับปรุงผลลัพธ์ทั้งคู่ภายใต้โมเมนต์ที่สี่ของ $f \circ X$ มีค่าจำกัด ในส่วนที่สองเราปรับปรุงอสมการความเข้มข้นแบบไม่สม่ำเสมอสำหรับการสุ่มตัวอย่างแบบแถวเชิงตั้งฉากซึ่งถูกให้โดย เนียมมณี และไหลภาภรณ์ ในปี ค.ศ.2006

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Let X be a random vector uniformly distributed on $[0,1]^3$ and let f be an integrable function from $\mathbb{R}^3$ into $\mathbb{R}$ and define

$$\mu = Ef(X) = \int_{[0,1]^3} f(x)dx.$$

A simple estimator of μ is

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f(X_i)$$

where $X_1, X_2, ..., X_n$ are independent random vectors and uniformly distributed on $[0,1]^3$. However, there are many methods to choose the points X_i's. One of those is the orthogonal array. In 1996, Loh was the first one who considered the normal approximation of $W = \dfrac{\widehat{\mu} - \mu}{\sqrt{Var(\widehat{\mu})}}$ where $Var(\widehat{\mu}) > 0$ and gave a uniform bound. In 2008, Neammanee and Laipaporn improved the rate of convergence of Loh to be $O(q^{-\frac{1}{2}})$ with the assumption that the sixth moment of $f \circ X$ is finite.

In this thesis we improve their results under the finiteness of the fourth moment of $f \circ X$. In the second part, we improve a non-uniform concentration inequality for a randomized orthogonal array which is given by Neammanee and Laipaporn in 2006.

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CONTENTS

page

ABSTRACT IN THAI iv

ABSTRACT IN ENGLISH v

ACKNOWLEDGEMENTS vi

CONTENTS vii

CHAPTER

I INTRODUCTION 1

II PRELIMINARIES 4

III AN IMPROVEMENT OF NORMAL APPROXIMATION OF RANDOMIZED ORTHOGONAL ARRAYS SAMPLING 11

IV AN IMPROVEMENT OF A NON-UNIFORM CONCENTRATION INEQUALITY FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING 17

REFERENCES 27

VITA 29

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

CHAPTER I
INTRODUCTION

In many scientific fields, we are often confronted with the problem of estimating a value of integral over a high-dimensional domain. Among numerical integration techniques, Monte Carlo methods are especially useful and competitive for high-dimensional integration.

Let $f : [0,1]^d \to \mathbb{R}$ be a measurable function. Our aim is to estimate

$$\mu = \int_{[0,1]^d} f(x)dx.$$

This is equivalent to finding the expectation of $f \circ X$ where X is a random vector uniformly distributed on a unit hypercube $[0,1]^d$. The simple Monte Carlo method is to draw $X_1, X_2, ...X_n$ independently and uniformly distributed from $[0,1]^d$ and to use

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f \circ X_i$$

as an estimator of μ. There are various alternative ways to select the point X_i's. For examples, lattic sampling(see Patterson[18]), latin hypercube sampling(see, Owen[15]), the orthogonal array sampling(see, Loh[10], Owen[16], Tang[21]), scrambled net sampling(see, Owen[13] and [14]). In this work we investigate orthogonal array sampling.

Let d, n, q and t be positive integers with $t \leq d$ and $q \geq 2$. **An orthogonal array of strength t with index λ** $(\lambda \geq 1)$ is an $n \times d$ matrix of n rows and d columns with elements taken from the set $\{0, 1, \ldots, q-1\}$ such that in any $n \times t$ submatrix, each of the q^t possible rows occurs the same number of times of course $n = \lambda q^t$. The class of all such arrays is denoted by $OA(n, d, q, t)$(see Raghavarao [19] for more details).

Loh(1996) considered the class $OA(n, 3, q, 2)$ when $n = q^2$ and constructed the sampling $X_1, X_2, ..., X_{q^2}$ on the unit cube $[0,1]^3$ as follows: Let

(a) π_1, π_2, π_3 be random permutations of $\{0, 1, ..., q-1\}$,

(b) $U_{i_1,i_2,i_3,j}$ be $[0,1]$ uniform random variables where $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$, $j \in \{1, 2, 3\}$; and

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's be all stochastically independent.
An orthogonal array-based sample of size q^2, $\{X_1, X_2, ..., X_{q^2}\}$, is defined to be

$$\{X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) : 1 \le i \le q^2\},$$

where, for each $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j \in \{1, 2, 3\}$,

$$X\big(i_1, i_2, i_3\big) = \big(X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)\big),$$
$$X_j(i_1, i_2, i_3) = \frac{i_j + U_{i_1,i_2,i_3,j}}{q},$$

and $a_{i,j}$ is the $(i,j)^{th}$ element of some arbitary but fixed $A \in OA(q^2, 3, q, 2)$.

So the estimator $\hat{\mu}$ of μ can be expressed in the form of

$$\hat{\mu} = \frac{1}{q^2} \sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})).$$

Assume that $Var(\hat{\mu}) > 0$, and define

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}.$$

Note that

$$EW = 0 \ \text{ and } \ Var(W) = EW^2 = 1.$$

From now on, we let X be a uniform random vector on $[0,1]^3$ and Φ the standard normal distribution. Loh(1996) gave a uniform bound on the normal approximation of W in Theorem 1.1.

Theorem 1.1. *Suppose that $E(f \circ X)^r < \infty$ for some even integer $r \ge 4$. Then*

$$\sup\big\{\big|P(W \le w) - \Phi(w)\big| : -\infty < w < \infty\big\} = O(q^{\frac{2-r}{2r-2}}), \quad as \ \ q \to \infty.$$

Neammanee and Laipaporn([11]) improved the rate of convergence of Theorem 1.1 to be $O(\frac{1}{q^{\frac{1}{2}}})$ in Theorem 1.2 and they also investigated a non-uniform concentration inequality of W in Theorem 1.3.

Theorem 1.2. *Suppose that $E(f \circ X)^6 < \infty$. Then*

$$\sup\big\{\big|P(W \le z) - \Phi(z)\big| : -\infty < z < \infty\big\} = O(q^{-\frac{1}{2}}), \quad as \ \ q \to \infty.$$

Theorem 1.3. *Assume that* $E(f\circ X)^4<\infty$*. Then, there exists a constant* C *such that*

$$P(z\le W\le z+\lambda)\le\frac{C\lambda}{1+z}+\frac{1}{1+z}O(\frac{1}{\sqrt{q}}),\ \text{as } q\to\infty,$$

for any nonnegative real numbers z,λ.

In this work, we relax the condition on the moment of $f\circ X$ in Theorem 1.2 to $E(f\circ X)^4<\infty$ and improve the bound of Theorem 1.3. These are our main results.

Theorem 1.4. *(A Uniform bound)*
Suppose that $E(f\circ X)^4<\infty$*, then*

$$\sup\left\{\left|P(W\le z)-\Phi(z)\right|:-\infty<z<\infty\right\}=O(q^{-\frac{1}{2}}),\quad \text{as}\ \ q\to\infty.$$

Theorem 1.5. *(A Non-uniform concentration inequality)*
Assume that $E(f\circ X)^4<\infty$*. Then, there exists a constant* C *such that*

$$P(z\le W\le z+\lambda)\le\frac{C\lambda}{(1+z)^3}+\frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}),\ \text{as } q\to\infty,$$

for any nonnegative real numbers z,λ.

We organize our thesis as follows. In chapter 2 we give some basic concepts in probability theory, background of Stein's method and some useful properties of Stein's solution. In chapter 3 we give a uniform bound in normal approximation of randomized orthogonal array sampling. In chapter 4 we give a non-uniform concentration inequality for randomized orthogonal array sampling.

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CHAPTER II
PRELIMINARIES

In this chapter, we give some basic concepts in probability which will be used in our work. The proof is omited but can be found in [1, 17].

2.1 Probability Space and Random Variables

A **measurable space** $(\Omega, \mathcal{F})$ is a set Ω with a σ-algebra $\mathcal{F}$ of subset of Ω. A **probability measure** P is a measure on $\mathcal{F}$ with $P(\Omega) = 1$. Then $(\Omega, \mathcal{F}, P)$ is called a **probability space**. The set Ω will be referred as a **sample space** and its elements are called **points** or **elementary events**. Member of $\mathcal{F}$ are called **events.** For any event A, the value $P(A)$ is called the **probability of** A.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A function $X : \Omega \to \mathbb{R}$ is said to be a **random variable** if for every Borel set B in $\mathbb{R}$,

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

A **random vector** $X = (X_1, X_2, ..., X_k)$ is a finite family of random variables $X_1, X_2, ..., X_k$ where $X_i : \Omega \to \mathbb{R}$ for all $i = 1, 2, ..., k$ and X_i is called **component** of the random vector. We shall usually use the notation $P(X \in B)$ in stead of $P(\{\omega \in \Omega | X(\omega) \in B\})$. In the case where $B = (-\infty, a]$ or $[a, b]$, $P(X \in B)$ is denoted by $P(X \leq a)$ or $P(a \leq X \leq b)$, respectively.

Let X be a random variable. A function $F : \mathbb{R} \to [0, 1]$ which is defined by

$$F(x) = P(X \leq x)$$

is called the **distribution function** of X.

Let X be a random variable with the distribution function F. X is said to be a **discrete random variable** if the image of X is countable and X is called a **continuous random variable** if F can be written in the form

$$F(x) = \int_{-\infty}^{x} f(t)dt$$

for some nonnegative integrable function f on $\mathbb{R}$. In this case, we say that f is the **probability function** of X.

Now we will give some examples of random variables.

We say that X is a **normal** random variable with parameter μ and σ^2, written as $X \sim N(\mu, \sigma^2)$, if its probability function is defined by

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(-\frac{1}{2\sigma^2}(x-\mu)^2\Big).$$

Moreover, if $X \sim N(0,1)$ then X is said to be a **standard normal** random variable.

We say that a discrete random variable X is **uniform** with parameter n if there exist $x_1, x_2, ..., x_n$ such that $P(X = x_i) = \frac{1}{n}$ for any $i = 1, 2, ..., n$.

If X is a continuous random variable with probability function

$$f(x) = \begin{cases} \frac{1}{b-a} & \text{if} \quad a \le x \le b, \\ 0 & otherwise, \end{cases}$$

we say that X is **uniform** on $[a, b]$.

We say that $X = (X_1, X_2, ..., X_n)$ is a **continuous random vector** if and only if there are measurable function $f : \mathbb{R}^n \to [0, \infty)$ and **joint distribution function** F of X satisfying

$$F(x_1, x_2, ..., x_n) = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} ... \int_{-\infty}^{x_n} f(t_1, t_2, ..., t_n) dt_1 dt_2 ... dt_n$$

for all $(x_1, x_2, ..., x_n) \in \mathbb{R}^n$. Then we call the function, **joint probability function** of X.

2.2 Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{F}_\alpha$ are sub σ-algebras of $\mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{F}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if for any subset $J = \{j_1, j_2, ..., j_k\}$ of Λ,

$$P\Big(\bigcap_{m=1}^{k} A_m\Big) = \prod_{m=1}^{k} P(A_m)$$

where $A_m \in \mathcal{F}_{j_m}$ for $m = 1, ..., k$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{E}_\alpha \subseteq \mathcal{F}$ for all $\alpha \in \Lambda$. We say that $\{\mathcal{E}_\alpha | \alpha \in \Lambda\}$ is **independent** if and only if $\{\sigma(\mathcal{E}_\alpha) | \alpha \in \Lambda\}$ is independent where $\sigma(\mathcal{E}_\alpha)$ is the smallest σ-algebra generated by $\mathcal{E}_\alpha$.

We say that the set of random variables $\{X_\alpha | \alpha \in \Lambda\}$ is **independent** if $\{\sigma(X_\alpha)| \alpha \in \Lambda\}$ is independent, where $\sigma(X) = \{X^{-1}(B) \mid B \text{ is a Borel subset of } \mathbb{R}\}$. We say that $X_1, X_2, ..., X_n$ are **independent** if $\{X_1, X_2, ..., X_n\}$ is **independent**.

Theorem 2.1. *Random variables $X_1, X_2, ..., X_n$ are* **independent** *if and only if for any Borel sets $B_1, B_2, ..., B_n$ we have*

$$P(\bigcap_{i=1}^{n}\{X_i \in B_i\}) = \prod_{i=1}^{n} P(X_i \in B_i).$$

Proposition 2.2. *If $X_{ij}\ ; i = 1, 2, ..., n,\ j = 1, 2, ..., m_i$ are independent and $f_i : \mathbb{R}^{m_i} \to \mathbb{R}$ are measurable, then $\{f_i(X_{i1}, X_{i2}, ..., X_{im_i}),\ i = 1, 2, ..., n\}$ is independent.*

2.3 Expectation, Variance and Conditional Expectation

Let X be any random variable on a probability space $(\Omega, \mathcal{F}, P)$.
If $\int_\Omega |X|dP < \infty$, then we define its **expected value** to be

$$E(X) = \int_\Omega X dP.$$

Proposition 2.3.

1. *If X is a discrete random variable and $\sum_{x \in ImX} |x|P(X = x) < \infty$, then*

$$E(X) = \sum_{x \in ImX} xP(X = x),$$

2. *If X is a continuous random variable with probability function f and $\int_{\mathbb{R}} |x|f(x)dx < \infty$, then*

$$E(X) = \int_{\mathbb{R}} xf(x)dx.$$

Proposition 2.4. *Let X and Y be random variables such that $E(|X|) < \infty$ and $E(|Y|) < \infty$ and $a, b \in \mathbb{R}$. Then we have the followings:*

1. *$E(aX + bY) = aE(X) + bE(Y)$,*

2. *If $X \leq Y$, then $E(X) \leq E(Y)$,*

3. $|E(X)| \leq E(|X|)$.

Let X be a random variable which $E(|X|^k) < \infty$. Then $E(|X|^k)$ is called the **k-th moment** of X about the origin and $E[(X - E(X))^k]$ is called the **k-th moment** of X about the mean.

We call the second moment of X about the mean, the **variance** of X and denoted by $Var(X)$. Then

$$Var(X) = E[X - E(X)]^2.$$

We note that

1. $Var(X) = E(X^2) - E^2(X)$,

2. If $X \sim N(\mu, \sigma^2)$ then $E(X) = \mu$ and $Var(X) = \sigma^2$.

Proposition 2.5. *If $X_1, ..., X_n$ are independent and $E|X_i| < \infty$ for $i = 1, 2, ..., n$, then*

1. $E(X_1 X_2 ... X_n) = E(X_1)E(X_2)...E(X_n)$,

2. $Var(a_1 X_1 + \cdots + a_n X_n) = a_1^2 Var(X_1) + \cdots + a_n^2 Var(X_n)$ *for any real number* $a_1, ..., a_n$.

The following inequalities are useful in our work.

1. **Hölder's inequality :**

$$E(|XY|) \leq E^{\frac{1}{p}}(|X|^p) E^{\frac{1}{q}}(|X|^q)$$

where $0 < p, q < 1$, $\frac{1}{p} + \frac{1}{q} = 1$ and $E(|X|^p) < \infty$,$E(|Y|^q) < \infty$.

2. **Chebyshev's inequality :**

$$P(\{|X - E(X)| \geq \varepsilon\}) \leq \frac{Var(X)}{\varepsilon^2} \quad \text{for all } \ \varepsilon > 0$$

where $E(X^2) < \infty$.

Let X be a finite expected value random variable on a probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. Define a probability measure $P_{\mathcal{D}} : \mathcal{D} \to [0, 1]$ by

$$P_{\mathcal{D}}(E) = P(E)$$

and sign-measure $\mathcal{Q}_X : \mathcal{D} \to \mathbb{R}$ by

$$\mathcal{Q}_X(E) = \int_E X dP \quad \text{for all} \;\; E \in \mathcal{D}.$$

Then, by definition of the integral implies $\mathcal{Q}_X$ absolutely continuous with respect to $P_{\mathcal{D}}$ and there exists a unique measurable function $E^{\mathcal{D}}(X)$ on $(\Omega, \mathcal{F}, \mathcal{D})$ such that

$$\int_E E^{\mathcal{D}}(X) dP_{\mathcal{D}} = \mathcal{Q}_X(E) = \int_E X dP \quad \text{for any } E \in \mathcal{D}.$$

We will say that $E^{\mathcal{D}}(X)$ is the **conditional expectation** of X with respect to $\mathcal{D}$.

Moreover, for any random variables X and Y on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, we will denote $E^{\sigma(Y)}(X)$ by $E^Y(X)$.

Theorem 2.6. *Let X be a random variable on probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|) < \infty$, then the followings hold for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$.*

1. *If X is random variable on $(\Omega, \mathcal{D}, P_{\mathcal{D}})$, then $E^{\mathcal{D}}(X) = X$ a.s.$[P_{\mathcal{D}}]$,*
2. *$E^{\mathcal{F}}(X) = X$ a.s.$[P]$,*
3. *If $\sigma(X)$ and $\mathcal{D}$ are independent, then $E^{\mathcal{D}}(X) = E(X)$ a.s.$[P_{\mathcal{D}}]$.*

Theorem 2.7. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|X|)$ and $E(|Y|)$ are finite. Then for any sub σ-algebra $\mathcal{D}$ of $\mathcal{F}$ the followings hold.*

1. *If $X \le Y$, then $E^{\mathcal{D}}(X) \le E^{\mathcal{D}}(Y)$ a.s. $[P_{\mathcal{D}}]$,*
2. *$E^{\mathcal{D}}(aX + bY) = aE^{\mathcal{D}}(X) + bE^{\mathcal{D}}(X)$ a.s. $[P_{\mathcal{D}}]$ for any $a, b \in \mathbb{R}$.*

Theorem 2.8. *Let X and Y be random variables on the same probability space $(\Omega, \mathcal{F}, P)$ such that $E(|XY|)$ and $E(|Y|)$ are finite and $\mathcal{D}_1, \mathcal{D}_2$ any sub σ-algebra of $\mathcal{F}$. If X is a random variable with respect to $\mathcal{D}_1$, then*

1. *$E^{\mathcal{D}_1}(XY) = XE^{\mathcal{D}_1}(Y)$ a.s. $[P_{\mathcal{D}_1}]$,*
2. *$E^{\mathcal{D}_2}(XY) = E^{\mathcal{D}_2}(XE^{\mathcal{D}_1}(Y))$ a.s. $[P_{\mathcal{D}_2}]$.*

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $\mathcal{D}$ a sub σ-algebra of $\mathcal{F}$. For any event A on $\mathcal{F}$, we defined the **conditional probability of** A **given** $\mathcal{D}$ by

$$P(A|\mathcal{D}) = E^{\mathcal{D}}(\mathbb{I}_A)$$

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where $\mathbb{I}_A$ is defined by

$$\mathbb{I}_A(w) = \begin{cases} 1 & \text{if } w \in A \\ 0 & \text{if } w \notin A. \end{cases}$$

2.4 Stein's Method for Normal Approximation

The Stein's method for obtaining an explicit bound for the error in the normal approximation for dependent random variables was investigated in 1972. Stein's technique is free of Fourier methods and relied instead on the elementary differential equation. Stein's method has been applied with much success in the area of normal approximation.This method was adapted and applied to the Poisson approximation by Chen(1974, 1975). There are at least three approaches to use Stein's method when the limit distribution is normal, i.e., a concentration inequality approach, an inductive approach and a coupling approach.

In this section we give basic results on the Stein's equation and its solution.

Let Z be a standard normal distributed random variable and let $\mathcal{C}_{bd}$ be the set of continuous and piecewise continuously differential functions $g : \mathbb{R} \to \mathbb{R}$ with $E|g'(Z)| <\infty$.

For $g \in \mathcal{C}_{bd}$ and any real valued function s with $E|s(Z)| < \infty$, the equation

$$g'(w) - wg(w) = s(w) - Es(Z) \tag{2.1}$$

is called **Stein's equation**.

If $s(w) = \mathbb{I}_{(-\infty,z]}(w)$, then the Stein's equation becomes

$$g'(w) - wg(w) = \mathbb{I}_{(-\infty,z]}(w) - \Phi(z). \tag{2.2}$$

Hence

$$E(g'(W) - Wg(W)) = P(W \leq z) - \Phi(z) \tag{2.3}$$

for any random variable W and the solution g_z of Stein's equation (2.2) is given by

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(w)[1-\Phi(z)] & \text{if } w \leq z \\ \sqrt{2\pi}e^{\frac{w^2}{2}}\Phi(z)[1-\Phi(w)] & \text{if } w \geq z. \end{cases} \tag{2.4}$$

The following properties of g_z are used in this work.

Proposition 2.9. ([1], [20]) *For all real numbers w, u, v, z, we have*

1. $|g_z'(w)| \leq 1$,
2. $0 < g_z(w) \leq \min(\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|})$,
3. $$g_z'(w+u) - g_z'(w+v) \leq \begin{cases} 1 & \text{if } \ w+u \leq z, w+v > z \\ \left(|w| + \frac{\sqrt{2\pi}}{4}\right)(|u|+|v|) & \text{if } \ u \geq v \\ 0 & \text{otherwise.} \end{cases}$$

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CHAPTER III
AN IMPROVMENT OF NORMAL APPROXIMATION OF RANDOMIZED ORTHOGONAL ARRAYS SAMPLING

In this chapter, we use the same notations as in the previous chapters. Loh([10]) used a random function $\rho_\pi : \{0,1,...,q-1\}^2 \rightarrow \{0,1,...,q-1\}$ defined by

$$\big(i_1, i_2, \rho_\pi(i_1,i_2)\big) = \big(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})\big)$$

for some $i \in \{1,2,..,q^2\}$ and showed that W in chapter 1 can be rewritten as the sum

$$W = \sum_{i_1=0}^{q-1}\sum_{i_2=0}^{q-1} Y\big(i_1, i_2, \rho_\pi(i_1,i_2)\big)$$

where

$$Y(i_1,i_2,i_3) = \frac{1}{q^2\sqrt{Var(\hat{\mu})}}\Big[f\circ X(i_1,i_2,i_3) - \mu - \sum_{j=1}^{3}\mu_j(i_j) - \sum_{1\le k<l\le 3}\mu_{k,l}(i_k,i_l)\Big],$$

$$\mu_j(i_j) = \frac{1}{q^2}\sum_{\substack{i_k=0\\k\neq j}}^{q-1}[\mu(i_1,i_2,i_3)-\mu],$$

$$\mu_{k,l}(i_k,i_l) = \frac{1}{q}\sum_{\substack{i_j=0\\j\neq k,l}}^{q-1}[\mu(i_1,i_2,i_3)-\mu-\mu_k(i_k)-\mu_l(i_l)],$$

$$\mu(i_1,i_2,i_3) = Ef\circ X(i_1,i_2,i_3).$$

He also gave a uniform bound of orthogonal array sampling designs in Theorem 3.1.

Theorem 3.1. *Suppose that $E(f\circ X)^r < \infty$ for some even integer $r \geq 4$. Then*

$$\sup\Big\{\big|P(W\le z) - \Phi(z)\big| : -\infty < z < \infty\Big\} = O(q^{\frac{2-r}{2r-2}}), \quad as \;\; q\rightarrow\infty.$$

Neammanee and Laipaporn ([11]) improved the rate of convergence of Theorem 3.1 to be $O(\frac{1}{q^{\frac{1}{2}}})$ in Theorem 3.2.

Theorem 3.2. *Suppose that* $E(f \circ X)^6 < \infty$. *Then*

$$\sup\left\{\left|P(W \le z) - \Phi(z)\right| : -\infty < z < \infty\right\} = O(q^{-\frac{1}{2}}), \quad as \ \ q \to \infty.$$

In this chapter, we relax the condition on the moment of $f \circ X$ from $E(f \circ X)^6 < \infty$ to $E(f \circ X)^4 < \infty$ as in Theorem 3.3. Here is our main result.

Theorem 3.3. *Suppose that* $E(f \circ X)^4 < \infty$. *Then*

$$\sup\left\{\left|P(W \le z) - \Phi(z)\right| : -\infty < z < \infty\right\} = O(q^{-\frac{1}{2}}), \quad as \ \ q \to \infty.$$

To prove Theorem 3.3, we need the following lemmas and some notations. For each i, j and $k \in \{0, 1, ..., q-1\}$, we let I and K be uniformly distributed random variables on $\{0, 1, ..., q-1\}$, (I, K) uniformly distributed on $\{(i,k)|i, k = 0, 1, ..., q-1, i \neq k\}$. Let

$$\widetilde{W} = W - S_1 - S_2 + S_3 + S_4$$

where

$$S_1 = \sum_{j=0}^{q-1} Y\big(I, j, \rho_\pi(I, j)\big), \qquad S_2 = \sum_{j=0}^{q-1} Y\big(K, j, \rho_\pi(K, j)\big),$$
$$S_3 = \sum_{j=0}^{q-1} Y\big(I, j, \rho_\pi(K, j)\big), \qquad S_4 = \sum_{j=0}^{q-1} Y\big(K, j, \rho_\pi(I, j)\big).$$

Note that $(W, \widetilde{W})$ is an exchangeable pair, i.e., for every $a, b \in \mathbb{R}$

$$P(W \le a, \widetilde{W} \le b) = P(W \le b, \widetilde{W} \le a).$$

Lemma 3.4. ([11]) *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous and piecewise continuously differentiable function. Then*

$$EWg(W) = E\int_{-\infty}^{\infty} g'(W+t)M(t)dt - \Delta g(W) \tag{3.1}$$

and

$$\left|\Delta g(W)\right| \le \frac{1}{q-1}\Big[Eg^2\big(W\big)\Big]^{\frac{1}{2}}, \tag{3.2}$$

where

$$M(t) = \frac{q}{4}(\widetilde{W} - W)\Big\{\mathbb{I}(0 \le t \le \widetilde{W} - W) - \mathbb{I}(\widetilde{W} - W \le t \le 0)\Big\},$$

$$\Delta g(W) = \frac{1}{q-1} Eg(W) \sum_{i=1}^{q-1}\sum_{j=1}^{q-1} EY(i,j,\rho_\pi(i,j))$$

and $\mathbb{I}$ *is the indicator function.*

Lemma 3.5. ([8]) *If* $E(f \circ X)^r < \infty$ *for some positive even integer* r*, then*

$E\Big(\widetilde{W} - W\Big)^r \le O(q^{-\frac{r}{2}})$*, as* $q \to \infty$.

Lemma 3.6. ([10]) *If* $E(f \circ X)^r < \infty$ *for some positive even integer* r*, then*

$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^r(i,j,k) = O(q^{3-r})$*, as* $q \to \infty$.

Lemma 3.7. *If* $E(f \circ X)^4 < \infty$*, Then* $EW^4 = O(1)$.

Proof. From Lemma 3.4, if we choose $g(w) = w^3$ then we have

$$\begin{aligned} EW^4 &= 3E\int_{-\infty}^{\infty}(W+t)^2 M(t)dt - \Delta W^3 \\ &= \frac{3q}{2}EW(\widetilde{W} - W)^3 + \frac{q}{4}E(\widetilde{W} - W)^4 + \frac{3q}{4}EW^2(\widetilde{W} - W)^2 - \Delta W^3 \qquad (3.3) \end{aligned}$$

and $\Delta W^3 = \frac{1}{q-1} EW^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1} EY(i,j,\rho_\pi(i,j)).$

For convenience, we will use the notation $\overline{Y}(i,j)$ for $Y(i,j,\rho_\pi(i,j))$ and by Lemma 3.6 we have

$$\begin{aligned} &\left(\sum_i\sum_j EY(i,j,\rho_\pi(i,j))\right)^4 \\ &\le \sum_{i_1,j_1}\sum_{i_2,j_2}\sum_{i_3,j_3}\sum_{i_4,j_4} E|\overline{Y}(i_1,j_1)|E|\overline{Y}(i_2,j_2)|E|\overline{Y}(i_3,j_3)|E|\overline{Y}(i_4,j_4)| \\ &\le \frac{q^6}{4}\left(\sum_{i_1,j_1} E|\overline{Y}(i_1,j_1)|^4 + \sum_{i_2,j_2} E|\overline{Y}(i_2,j_2)|^4 + \sum_{i_3,j_3} E|\overline{Y}(i_3,j_3)|^4 + \sum_{i_4,j_4} E|\overline{Y}(i_4,j_4)|^4\right) \\ &= q^6 \sum_{i,j} E|Y(i,j,\rho_\pi(i,j))|^4 \\ &= q^5 \sum_{i,j,k} E|Y(i,j,k)|^4 \\ &= O(q^4) \end{aligned}$$

where we have used the weighted A.M.-G.M. inequality,

$$x_1^{w_1} x_2^{w_2} \dots x_n^{w_n} \le w_1 x_1 + w_2 x_2 + \dots + w_n x_n$$

where $w_1, w_2, \dots, w_n > 0$, $w_1 + w_2 + \dots + w_n = 1$ and $x_1, x_2, \dots, x_n \ge 0$, in the second inequality.

$$\begin{aligned}
|\Delta W^3| &\le E(\frac{1}{3}W^4)^{\frac{3}{4}} \left(\frac{1}{q^4} O(1) \left| \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} EY(i,j,\rho_\pi(i,j)) \right|^4 \right)^{\frac{1}{4}} \\
&\le \frac{1}{4} EW^4 + \frac{1}{q^4} O(1) E \left| \sum_{i=0}^{q-1} \sum_{j=0}^{q-1} EY(i,j,\rho_\pi(i,j)) \right|^4 \\
&\le \frac{1}{4} EW^4 + O(1).
\end{aligned}$$

Together with inequality (3.3) and Lemma 3.5 we have

$$\begin{aligned}
E|W||\widetilde{W} - W|^3 &= E(\frac{1}{q}|W|^4)^{\frac{1}{4}} (q^{\frac{1}{3}} |\widetilde{W} - W|^4)^{\frac{3}{4}} \\
&\le \frac{1}{4q} E|W|^4 + \frac{3}{4} q^{\frac{1}{3}} E|\widetilde{W} - W|^4 \\
&= \frac{1}{4q} E|W|^4 + O(\frac{1}{q^{\frac{5}{3}}})
\end{aligned}$$

and

$$\begin{aligned}
EW^2(\widetilde{W} - W)^2 &= E(\frac{1}{2q} W^4)^{\frac{1}{2}} (2q|\widetilde{W} - W|^4)^{\frac{1}{2}} \\
&\le \frac{1}{4q} EW^4 + O(\frac{1}{q^{\frac{3}{2}}}).
\end{aligned}$$

Thus
$$EW^4 \le \frac{9}{16} EW^4 + O\left(\frac{1}{\sqrt{q}}\right) + O(1).$$

Hence $EW^4 = O(1)$ and the proof is complete. □

PROOF OF THEOREM 3.3

Proof. We will prove our main result by using Stein's method. Let z be any real number. We replace a function g in equation(2.3) with the function g_z from(2.4) which implies that

$$P(W \le z) - \Phi(z) = Eg_z'(W) - EWg_z(W).$$

Thus, it suffices to bound

$$|Eg_z'(W) - EWg_z(W)|$$

instead of

$$|P(W \leq z) - \Phi(z)|.$$

By (3.1) we have,

$$EWg_z(W) = E\int_{-\infty}^{\infty} g_z'(W+t)M(t)dt - \Delta g_z(W) \tag{3.4}$$

and

$$\begin{aligned} \big|\Delta g_z(W)\big| &\leq \frac{1}{q-1}\big\{Eg_z^2(W)\big\}^{\frac{1}{2}} \\ &= O(\frac{1}{q}) \end{aligned} \tag{3.5}$$

where we have used Proposition 2.9(2) in the last equality.
Thus, from (3.4), (3.5) and Propositon 2.9(1),

$$\begin{aligned} \big|P(W \leq z) - \Phi(z)\big| = \big|Eg_z'(W) - EWg_z(W)\big| & \\ \leq \big|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\big| & \\ + \big|Eg_z'(W)E\int_{-\infty}^{\infty}M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty}M(t)dt\big| & \\ + \big|Eg_z'(W)\big|\Big|1 - E\int_{-\infty}^{\infty}M(t)dt\Big| + \big|\Delta g_z(W)\big| & \\ \leq \big|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\big| & \\ + \big|Eg_z'(W)E\int_{-\infty}^{\infty}M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty}M(t)dt\big| & \\ + \Big|1 - E\int_{-\infty}^{\infty}M(t)dt\Big| + O(\frac{1}{q}). & \end{aligned} \tag{3.6}$$

From [10], p.1217 and p.1221, we know that

$$\Big|1 - E\int_{-\infty}^{\infty}M(t)dt\Big| = O(\frac{1}{q}), \tag{3.7}$$

and

$$\big|Eg_z'(W)E\int_{-\infty}^{\infty}M(t)dt - Eg_z'(W)\int_{-\infty}^{\infty}M(t)dt\big| = O(\frac{1}{\sqrt{q}}). \tag{3.8}$$

Hence, by (3.6)-(3.8)

$$\big|P(W \le z) - \Phi(z)\big| \le O(\frac{1}{\sqrt{q}}) + \big|E\int_{-\infty}^{\infty}\{g_z'(W) - g_z'(W+t)\}M(t)dt\big|. \qquad (3.9)$$

From Lemma 3.5, we note that

$$E\int_{-\infty}^{\infty}|t|M(t)dt = \frac{q}{8}E|\widetilde{W} - W|^3 \le \frac{q}{8}\{E|\widetilde{W} - W|^4\}^{\frac{3}{4}} = O(\frac{1}{\sqrt{q}}). \qquad (3.10)$$

From [8] p.59, we know that

$$E\int_{-\infty}^{\infty}\mathbb{I}(z - t \le W \le z)M(t)dt \le O(\frac{1}{\sqrt{q}}) \qquad (3.11)$$

then, by Proposition 2.9(3), Lemma 3.7, (3.10) and (3.11),

$$\begin{aligned}
&\Big|E\int_{\mathbb{R}}\{g_z'(W) - g_z'(W+t)\}M(t)dt\Big| \\
&\le E\int_{\substack{W\le z\\ W+t>z}} M(t)dt + E\int_{t\le 0}(|W| + \frac{\sqrt{2\pi}}{4})(0 + |t|)M(t)dt \\
&\le E\int_{t>0}\mathbb{I}(z - t \le W \le z)M(t)dt + E\int_{t\le 0}|W||t|M(t)dt + \frac{\sqrt{2\pi}}{4}E\int_{t\le 0}|t|M(t)dt \\
&\le O(\frac{1}{\sqrt{q}}) + \frac{q}{4}E|W|\frac{|\widetilde{W} - W|^3}{2} + O(\frac{1}{\sqrt{q}}) \\
&= O(\frac{1}{\sqrt{q}}) + \frac{q}{8}(EW^4)^{\frac{1}{4}}(E|\widetilde{W} - W|^4)^{\frac{3}{4}} + O(\frac{1}{\sqrt{q}}) \\
&\le O(\frac{1}{\sqrt{q}}) + \frac{q}{8}O(\frac{1}{q^{\frac{3}{2}}}) \\
&= O(\frac{1}{\sqrt{q}}). \qquad (3.12)
\end{aligned}$$

Hence, by (3.9) and (3.12), we have

$$\big|P(W \le z) - \Phi(z)\big| \le O(\frac{1}{\sqrt{q}}).$$

□

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CHAPTER IV
AN IMPROVEMENT OF A NON-UNIFORM CONCENTRATION INEQUALITY FOR RANDOMIZED ORTHOGONAL ARRAY SAMPLING

In this chapter we give a non-uniform concentration inequality of W defined in chapter 3 and write C instead of a positive value with possibly different values in different places. First of all we give the definitions of uniform and non-uniform concentration inequalities as follows:

Let X be a random variable. The function $Q_X : [0, \infty) \to \mathbb{R}$ which defined by

$$Q_X(\lambda) = \sup_x P(x \leq X \leq x + \lambda)$$

is called a **uniform (Lévy) concentration function** of X and the function
$Q_X : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ which defined by

$$Q_X(x; \lambda) = P(x \leq X \leq x + \lambda)$$

is called a **non-uniform (Lévy) concentration function** of X.

The upper bounds of uniform and non-uniform concentration functions are called **uniform** and **non-uniform concentration inequalities** respectively.

Neammanee and Laipaporn([9]) gave a non-uniform concentration inequality for W in Theorem 4.1.

Theorem 4.1. *Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C, such that*

$$P(z \leq W \leq z + \lambda) \leq \frac{C\lambda}{1+z} + \frac{1}{1+z} O(\frac{1}{\sqrt{q}}), \text{ as } q \to \infty,$$

for any nonnegative real numbers z, λ.

In this chapter, we improve the bound in Theorem 4.1. Here is our main result.

Theorem 4.2. *Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C such that*

$$P(z \leq W \leq z + \lambda) \leq \frac{C\lambda}{(1+z)^3} + \frac{1}{(1+z)^2} O(\frac{1}{\sqrt{q}}), \text{as } q \to \infty,$$

for any nonnegative real numbers z, λ.

To prove Theorem 4.2, we need the following lemmas and some notations. For each i, j and $k \in \{0, 1, ..., q-1\}$, and $z > 0$, we let I and K be uniformly distributed random variables on $\{0, 1, ..., q-1\}$, (I, K) uniformly distributed on $\big\{(i, k) | i, k = 0, 1, ..., q-1, i \neq$ k$\big\}$. Let

$$Y_z(i,j,k) = Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big| > 1+z\big),$$
$$\widehat{Y}_z(i,j,k) = Y(i,j,k)\mathbb{I}\big(\big|Y(i,j,k)\big| \leq 1+z\big),$$
$$\widehat{Y} = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))$$

and
$$\widetilde{Y} = \widehat{Y} - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where
$$\widehat{S}_{1,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(I,j)), \quad \widehat{S}_{2,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(K,j)),$$
$$\widehat{S}_{3,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I,j,\rho_\pi(K,j)), \quad \widehat{S}_{4,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K,j,\rho_\pi(I,j)).$$

Lemma 4.3. ([9]) *Assume that* $E(f \circ X)^r < \infty$ *for some positive even integer* r*. Then*

1. $E|\widetilde{Y} - \widehat{Y}|^r \leq O(\frac{1}{q^{\frac{r}{2}}})$, *as* $q \to \infty$.
2. *For any positive integer* n *and* t *such that* $n+t$ *is an even number and* $n + t \leq r$,
$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}E|Y_z^n(i,j,k)| = \frac{O(q^{3-n-t})}{(1+z)^t}, \text{ as } q \to \infty.$$

Lemma 4.4. ([8]) *Let* $g : \mathbb{R} \to \mathbb{R}$ *be a continuous and piecewise continuously differrentiable function, then*

$$E\widehat{Y}g\big(\widehat{Y}\big) = E\int_{-\infty}^{\infty} g'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}g\big(\widehat{Y}\big) \tag{4.1}$$

where

$$K(t) = \frac{q-1}{4}\Big(\widetilde{Y} - Y\Big)\Big(\mathbb{I}\big(0 \leq t \leq \widetilde{Y} - Y\big) - \mathbb{I}\big(\widetilde{Y} - Y \leq t < 0\big)\Big)$$

and

$$\widetilde{\Delta}g\big(\widehat{Y}\big) = \frac{1}{q}Eg\big(\widehat{Y}\big)\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k).$$

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Lemma 4.5. ([8]) *If* $E(f \circ X)^2 < \infty$, *then* $E(\widehat{Y} - \widetilde{Y})^2 = \frac{4}{q} + O(\frac{1}{q^2})$, *as* $q \to \infty$.

Lemma 4.6. ([7]) *If* $E(f \circ X)^4 < \infty$, *then*

1. $E\left(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k)\right)^4 \leq O(q^2)$, *as* $q \to \infty$.

2. $E\left(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\right)^4 \leq \frac{1}{(1+z)^2} O(\frac{1}{q^2})$, *as* $q \to \infty$.

Lemma 4.7. *Assume that* $E(f \circ X)^4 < \infty$. *Let* $\gamma = \max\left(\frac{q(q-1)}{4(q-4)} E|\widehat{Y} - \widetilde{Y}|^3, \frac{1}{\sqrt{q}}\right)$ *and*

$$U_\gamma = \sum_{i \neq k} \left| \sum_{j=0}^{q-1} \{\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \widehat{Y}_z(k,j,\rho_\pi(k,j)) - \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \widehat{Y}_z(k,j,\rho_\pi(i,j))\} \right|$$

$$\times \min\left(\gamma, \sum_{i \neq k} \left| \sum_{j=0}^{q-1} \{\widehat{Y}_z(i,j,\rho_\pi(i,j)) + \widehat{Y}_z(k,j,\rho_\pi(k,j)) - \widehat{Y}_z(i,j,\rho_\pi(k,j)) - \widehat{Y}_z(k,j,\rho_\pi(i,j))\} \right|\right).$$

Then

1. $EU_\gamma \geq 3q + O(1)$, *as* $q \to \infty$.

2. $Var(U_\gamma) \leq \gamma^2 O(q^2)$, *as* $q \to \infty$.

Proof. (1.) By the fact that $\min(a,b) \geq b - \frac{b^2}{4a}$ for any $a, b > 0$,

$$\begin{aligned} E(U_\gamma) &= q(q-1)E(\widehat{Y} - \widetilde{Y})\min(\gamma, |\widehat{Y} - \widetilde{Y}|) \\ &\geq q(q-1)E(\widehat{Y} - \widetilde{Y})^2 - \frac{q(q-1)}{4\gamma} E|\widehat{Y} - \widetilde{Y}|^3 \\ &\geq q(q-1)E(\widehat{Y} - \widetilde{Y})^2 - \frac{q(q-1)}{4\left(\frac{q(q-1)}{4(q-4)} E|\widehat{Y} - \widetilde{Y}|^3\right)} E|\widehat{Y} - \widetilde{Y}|^3. \end{aligned}$$

By Lemma 4.5 we have

$$\begin{aligned} E(U_\gamma) &\geq q(q-1)\left(\frac{4}{q} + O(\frac{1}{q^2})\right) - (q-4) \\ &= 3q + O(1). \end{aligned}$$

(2.) For each $i,k \in \{0,1,...,q-1\}$ and random permutations β,α on $\{0,1,...,q-1\}$ we let

$$s_\gamma[(i,k),(\beta,\alpha)] = \Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\}\Big|$$

$$\times \min\Big(\gamma, \Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i,j,\beta(j)) + \widehat{Y}_z(k,j,\alpha(j)) - \widehat{Y}_z(i,j,\alpha(j)) - \widehat{Y}_z(k,j,\beta(j))\}\Big|\Big)$$

$$\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] = s[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))] - Es[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]$$

$$T_\gamma = \sum_{i\neq k}\hat{s}[(i,k),(\rho_\pi(i,\cdot),\rho_\pi(k,\cdot))]$$

and

$$\begin{aligned}\widetilde{T}_\gamma = T_\gamma &- \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))]\\ &- \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\\ &+ \hat{s}_\gamma[(\bar{I},\bar{K}),(\rho_\pi(\bar{L},\cdot),\rho_\pi(\bar{M},\cdot))]\\ &+ \hat{s}_\gamma[(\bar{L},\bar{M}),(\rho_\pi(\bar{I},\cdot),\rho_\pi(\bar{K},\cdot))].\end{aligned}$$

We note that $Var(U_\gamma) = ET_\gamma^2$.
By equations (2.24),(2.25) and (2.26) p.26 and p.27 of [9] we have

$$ET_\gamma^2 = \frac{q(q-1)}{4}E(\widetilde{T}_\gamma - T_\gamma)^2 + \frac{1}{q(q-1)}\sum_{i\neq k}\sum_{l\neq m}E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot),\rho_\pi(m,\cdot))]T_\gamma \quad (4.2)$$

$$E(\widetilde{T}_\gamma - T_\gamma)^2 \leq \left\{\gamma^2 O(\frac{1}{q}) + \frac{\gamma^2}{(1+z)^2}O(\frac{1}{q^2})\right\} \quad (4.3)$$

$$\text{and} \quad \sum_{i\neq k}\sum_{l\neq m}E\hat{s}_\gamma[(i,k),(\rho_\pi(l,\cdot)),\rho_\pi(m,\cdot)]T_\gamma \leq \gamma^2 O(q^4). \quad (4.4)$$

Then by (4.2),(4.3) and (4.4),

$$\begin{aligned}Var(U_\gamma) &= ET_\gamma^2\\ &\leq \frac{q(q-1)}{4}\left\{\gamma^2 O(\frac{1}{q}) + \frac{\gamma^2}{(1+z)^2}O(\frac{1}{q^2})\right\} + \frac{1}{q(q-1)}\left\{\gamma^2 O(q^4)\right\}\\ &\leq \gamma^2 O(q^2).\end{aligned}$$

□

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

Lemma 4.8. *If $E(f \circ X)^4 < \infty$, then $E\widehat{Y}^4 = O(1)$.*

Proof. First we note that

$$\begin{aligned} E\widehat{Y}^4 &= E\left(\sum_{i,j} \widehat{Y}_z(i,j,\rho_\pi(i,j))\right)^4 \\ &= E\left(\sum_{i,j} Y(i,j,\rho_\pi(i,j)) - \sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4 \\ &\leq CEW^4 + CE\left(\sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4. \end{aligned}$$

From Lemma 3.7, we proved that $EW^4 = O(1)$, hence it remains to bound $E\left(\sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4$.

Note that $\quad E\left(\sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4 = B_1 + B_2 + B_3 + B_4 + B_5$

where

$$B_1 = \sum_{i,j} EY_z^4(i,j,\rho_\pi(i,j)),$$

$$B_2 = \sum_{i_1,j_1} \sum_{\substack{i_2,j_2 \\ (i_2,j_2)\neq(i_1,j_1)}} EY_z^3(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2)),$$

$$B_3 = \sum_{i_1,j_1} \sum_{\substack{i_2,j_2 \\ (i_2,j_2)\neq(i_1,j_1)}} EY_z^2(i_1,j_1,\rho_\pi(i_1,j_1))Y_z^2(i_2,j_2,\rho_\pi(i_2,j_2)),$$

$$B_4 = \sum_{i_1,j_1} \sum_{\substack{i_2,j_2 \\ (i_2,j_2)\neq(i_1,j_1)}} \sum_{\substack{i_3,j_3 \\ (i_3,j_3)\neq(i_1,j_1) \\ (i_3,j_3)\neq(i_2,j_2)}} EY_z^2(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2))Y_z(i_3,j_3,\rho_\pi(i_3,j_3)),$$

$$B_5 = \sum_{i_1,j_1} \sum_{\substack{i_2,j_2 \\ (i_2,j_2)\neq(i_1,j_1)}} \sum_{\substack{i_3,j_3 \\ (i_3,j_3)\neq(i_1,j_1) \\ (i_3,j_3)\neq(i_2,j_2)}} \sum_{\substack{i_4,j_4 \\ (i_4,j_4)\neq(i_1,j_1) \\ (i_4,j_4)\neq(i_2,j_2) \\ (i_4,j_4)\neq(i_3,j_3)}}$$

$$EY_z(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2))Y_z(i_3,j_3,\rho_\pi(i_3,j_3))Y_z(i_4,j_4,\rho_\pi(i_4,j_4)).$$

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

By Lemma 4.3(2) we have

$$
\begin{aligned}
|B_1| &\le \frac{1}{q}\sum_{i,j,k} E|Y_z^4(i,j,k)| = \frac{1}{q}O(q^{3-4}) = O(\frac{1}{q^2}),\\
|B_2| &\le \frac{1}{q^2}\left[\sum_{i_1,j_1,k_1} E|Y_z^3(i_1,j_1,k_1)|\right]\left[\sum_{i_2,j_2,k_2} E|Y_z(i_2,j_2,k_2)|\right]\\
&= \frac{1}{q^2(1+z)}O(q^{3-3-1})(\frac{1}{1+z})O(q^{3-1-1})\\
&= \frac{1}{(1+z)^2}O(\frac{1}{q^2}),\\
|B_3| &\le \frac{1}{q^2}\left[\sum_{i,j,k} E|Y_z^2(i,j,k)|\right]^2\\
&= \frac{1}{q^2}\left(\frac{1}{(1+z)^2}O(q^{3-2-2})\right)^2\\
&= \frac{1}{(1+z)^4}O(\frac{1}{q^4}),\\
|B_4| &\le \frac{1}{q^3}\left[\sum_{i_1,j_1,k_1} E|Y_z^2(i_1,j_1,k_1)|\right]\left[\sum_{i_2,j_2,k_2} E|Y_z(i_2,j_2,k_2)|\right]^2\\
&= \frac{1}{q^3(1+z)^2}O(q^{3-2-2})\left(\frac{1}{1+z}O(q^{3-1-1})\right)^2\\
&= \frac{1}{(1+z)^4}O(\frac{1}{q^2})\\
\text{and}\quad |B_5| &\le \frac{1}{q^4}\left[\sum_{i,j,k} E|Y_z(i,j,k)|\right]^4\\
&= (\frac{1}{q^4})\left((\frac{1}{(1+z)})O(q^{3-1-1})\right)^4\\
&= \frac{1}{(1+z)^4}O(1).
\end{aligned}
$$

Hence $E\left(\sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4 < C$. Therefore $E\widehat{Y}^4 = O(1)$. □

PROOF OF THEOREM 4.2

Proof. Note that $P(z \le W \le z+\lambda) \le P(W \ne \widehat{Y}) + P(z \le \widehat{Y} \le z+\lambda)$

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

and by Lemma 3.6,

$$
\begin{aligned}
P(W \neq \widehat{Y}) &= P\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z) \geq 1\Big) \\
&\leq E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z)\Big) \\
&= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}E\mathbb{I}(|Y(i,j,k)| > 1+z) \\
&\leq \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\frac{E|Y(i,j,k)|^4}{(1+z)^4} \\
&\leq \frac{1}{q}\frac{1}{(1+z)^4}O(q^{3-4}) \\
&= \frac{1}{(1+z)^4}O(\frac{1}{q^2}).
\end{aligned}
$$

If we can show that $P(z \leq \widehat{Y} \leq z+\lambda) \leq \frac{C\lambda}{(1+z)^3} + \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}})$. Then we have Theorem 4.2.

Let γ be defined as in Lemma 4.7.

Case 1 $(1+z)^2\gamma \geq 1$.

By Lemma 4.8 and the fact that $\gamma \geq \frac{1}{(1+z)^2}$,

$$
\begin{aligned}
P(z \leq \widehat{Y} \leq z+\lambda) &\leq P(z \leq \widehat{Y}) \\
&= P(1+z \leq 1+\widehat{Y}) \\
&\leq \frac{E|\widehat{Y}+1|^4}{(1+z)^4} \\
&\leq C\frac{E|\widehat{Y}|^4}{(1+z)^4} + \frac{C}{(1+z)^4} \\
&\leq \frac{C}{(1+z)^4} \\
&\leq \frac{C\gamma}{(1+z)^2}.
\end{aligned}
$$

By H**ö**lder's inequality and Lemma 4.3(1), $E|\widetilde{Y} - \widehat{Y}|^3 \leq O(\frac{1}{q\sqrt{q}})$. Then $\gamma = O(\frac{1}{\sqrt{q}})$.

Hence $P(z \leq W \leq z+\lambda) \leq \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}})$.

ศูนย์วิทยทรัพยากร
จุฬาลงกรณ์มหาวิทยาลัย

Case 2 $(1+z)^2\gamma < 1$.

Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(t)=\begin{cases}0 & \text{if} \quad t<z-\gamma,\\ (1+t+\gamma)^3(t-z+\gamma) & \text{if} \quad z-\gamma\le t\le z+\lambda+\gamma,\\ (1+t+\gamma)^3(\lambda+2\gamma) & \text{if} \quad t>z+\lambda+\gamma.\end{cases}$$

Then f is a non decreasing function satisfying

$$f'(t)\ge\begin{cases}(1+z)^3 & \text{for} \quad z-\gamma<t<z+\lambda+\gamma,\\ 0 & \text{otherwise}\ .\end{cases}$$

Since f is a continuous and piecewise continuously differrentiable function, by Lemma 4.4 we have

$$E\widehat{Y}f(\widehat{Y})=E\int_{-\infty}^{\infty}f'(\widehat{Y}+t)K(t)dt+\widetilde{\Delta}f(\widehat{Y}).$$

Let U_γ be defined as in Lemma 4.7, we observe that

$$\begin{aligned}E\widehat{Y}f(\widehat{Y})-\widetilde{\Delta}f(\widehat{Y})&=E\int_{-\infty}^{\infty}f'(\widehat{Y}+t)K(t)dt\\ &\ge(1+z)^3E\mathbb{I}(z\le\widehat{Y}\le z+\lambda)\int_{|t|\le\gamma}K(t)dt\\ &=\frac{(q-1)(1+z)^3}{4}E\mathbb{I}(z\le\widehat{Y}\le z+\lambda)|\widehat{Y}-\widetilde{Y}|\min(\gamma,|\widehat{Y}-\widetilde{Y}|)\\ &=\frac{(q-1)(1+z)^3}{4q(q-1)}E\mathbb{I}(z\le\widehat{Y}\le z+\lambda)U_\gamma\\ &\ge\frac{(1+z)^3}{4q}E\mathbb{I}(z\le\widehat{Y}\le z+\lambda)U_\gamma\mathbb{I}(U_\gamma\ge q)\\ &\ge\frac{(1+z)^3}{4}E\mathbb{I}(z\le\widehat{Y}\le z+\lambda)\mathbb{I}(U_\gamma\ge q)\\ &=\frac{(1+z)^3}{4}E\big\{\mathbb{I}(z\le\widehat{Y}\le z+\lambda)-\mathbb{I}(z\le\widehat{Y}\le z+\lambda,U_\gamma\le q)\big\}\\ &\ge\frac{(1+z)^3}{4}\big\{P(z\le\widehat{Y}\le z+\lambda)-P(U_\gamma\le q)\big\}.\end{aligned}\tag{4.5}$$

By this fact, (4.5), Lemma 4.7(1,2) and Lemma 4.8,

$$\begin{aligned}P(z\le\widehat{Y}\le w)&\le\frac{4}{(1+z)^3}E\widehat{Y}f(\widehat{Y})-\frac{4}{(1+z)^3}\widetilde{\Delta}f(\widehat{Y})+P(U_\gamma\le q)\\ &\le\frac{4}{(1+z)^3}(\lambda+2\gamma)E|\widehat{Y}||1+\gamma+\widehat{Y}|^3\\ &\quad+\frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})|+P(EU_\gamma-U_\gamma\ge3q+O(1)-q)\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C}{(1+z)^3}(\lambda+2\gamma)E|\widehat{Y}||(1+\gamma)^3+\widehat{Y}^3| \\
&\quad + \frac{4}{(1+z)^3}|\widetilde{\Delta} f(\widehat{Y})| + P(EU_\gamma - U_\gamma \ge q) \\
&\le \frac{C}{(1+z)^3}(\lambda+2\gamma)\Big\{E|\widehat{Y}| + E|\widehat{Y}|^4\Big\} \\
&\quad + \frac{4}{(1+z)^3}|\widetilde{\Delta} f(\widehat{Y})| + \frac{1}{q^2}Var(U_\gamma) \\
&\le \frac{C\lambda}{(1+z)^3} + \frac{C\gamma}{(1+z)^3} + \frac{4}{(1+z)^3}|\widetilde{\Delta} f(\widehat{Y})| + \frac{1}{q^2}\gamma^2 O(q^2). \qquad (4.6)
\end{aligned}
$$

By (4.6), $(1+z)^2\gamma < 1$ and the fact that $\gamma = O(\frac{1}{\sqrt{q}})$,

$$
P(z \le \widehat{Y} \le z+\lambda) \le \frac{C\lambda}{(1+z)^3} + \frac{4}{(1+z)^3}|\widetilde{\Delta} f(\widehat{Y})| + \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}). \qquad (4.7)
$$

From, Lemma 4.6(1,2), Lemma 4.8 and the fact that $\gamma = O(\frac{1}{\sqrt{q}})$,

$$
\begin{aligned}
|\widetilde{\Delta} f(\widehat{Y})| &= \Big|\frac{1}{q}Ef(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\Big| \\
&\le \frac{1}{q}(\lambda+2\gamma)E\big|(1+\gamma+\widehat{Y})^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\big| \\
&\le \frac{C}{q}(\lambda+2\gamma)E\big|(1+\gamma)^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\big| \\
&\quad + \frac{C}{q}(\lambda+2\gamma)E\big|\widehat{Y}^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i,j,k)\big| \\
&\le C(\lambda+2\gamma)(1+\gamma)^3\frac{1}{q}\sum_{k=0}^{q-1}E\big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,k)\big| \\
&\quad + \frac{C}{q}(\lambda+2\gamma)E\big|\widehat{Y}^3\big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k) - \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y_z(i,j,k)\big)\big| \\
&\le C(\lambda+2\gamma)(1+\gamma)^3E\big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i,j,\rho_\pi(i,j))\big| \\
&\quad + \frac{C}{q}(\lambda+2\gamma)E\big|\widehat{Y}^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k)\big|
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{C}{q}(\lambda+2\gamma)E\big|\widehat{Y}^3\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k)\big| \\
&\le C(\lambda+2\gamma)(1+\gamma)^3E|\widehat{Y}| \\
&\quad+\frac{C}{q}(\lambda+2\gamma)\{E\widehat{Y}^4\}^{\frac{3}{4}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y(i,j,k))^4\}^{\frac{1}{4}} \\
&\quad+\frac{C}{q}(\lambda+2\gamma)\{E\widehat{Y}^4\}^{\frac{3}{4}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}Y_z(i,j,k))^4\}^{\frac{1}{4}} \\
&\le C(\lambda+2\gamma)(1+\gamma)^3+\frac{C}{q}(\lambda+2\gamma)O(q^{\frac{1}{2}}) \\
&\quad+\frac{C}{q}(\lambda+2\gamma)\frac{1}{(1+z)^{\frac{1}{2}}}O(q^{-\frac{1}{2}}) \\
&\le C(\lambda+2\gamma)\left(1+O(\frac{1}{\sqrt{q}})\right) \\
&\le C\lambda+O(\frac{1}{\sqrt{q}}).
\end{aligned}
$$

From this fact and (4.7), we can conclude that

$$
P(z\le\widehat{Y}\le z+\lambda)\le\frac{C\lambda}{(1+z)^3}+\frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}).
$$

□

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