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Original Article

Estimating conditional heteroscedastic nonlinear autoregressive model by using smoothing spline and penalized spline methods

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Abstract

We propose smoothing spline (SS) and penalized spline (PS) methods in a class of nonparametric regression methods for estimating the unknown functions in a conditional heteroscedastic nonlinear autoregressive (CHNLAR) model. The CHNLAR model consists of a trend and heteroscedastic functions in terms of past data at lag 1. The SS and PS methods were tested in estimating the unknown functions used to transform data so that it fits the trend and the heteroscedastic functions. In a simulation study, time series data were generated and hypothesis testing of the bias was used to check the accuracy. The SS and PS methods exhibit a good power estimation in most cases of generated data. As real data, gold price was modeled by using SS and PS methods in the CHNLAR model. The results show that the SS method performed better than the PS method.

Keywords: conditional heteroscedastic nonlinear autoregressive model, smoothing spline method, penalized spline method

1. Introduction

Currently the economic growth is of interest to developing countries. These data are mostly stored in the form of time series data, whether it is daily, monthly, quarterly, or yearly; typical examples are unemployment rate, economic growth rate, gold price, and currency exchange rates. These indicators are sensitive with rapid fluctuations caused by external factors, such as natural disasters, wars, and epidemics, that are not controllable or predictable. Because of such perturbations, it is difficult to make accurate economic forecasts.

Heteroscedasticity or volatility means that an economic time series data displays quick changes in its time-trace. A heteroscedastic model is useful to study for estimating or forecasting time series data, and this is the right approach to take when the time series has clear evidence of changing mean and variance.

There are several methods to model heteroscedastic time series, such as the autoregressive conditional hetero-scedastic model (ARCH) by Engle (1982), who was the first to introduce the ARCH model. The mean-corrected asset return is serially uncorrelated, but heteroscedastically changes over time. Ghosh, Pual and Prajneshu (2010) applied zero conditional mean residual series to identify time varying volatility in a data set, by using a mixture periodic ARCH model. Bollerslev (1986) extended the model type to Generalized ARCH (GARCH), and assumed that the mean equation can be adequatedly described by an ARMA model. Pual, Ghosh and Prajneshu (2009) carried used autoregressive intergrated moving average (ARIMA) and GARCH model for modeling and forecasting. Peng and Yao (2003) showed that the conditional maximum quasilikelihood estimator suffers from complex limit distributions and slow convergence rates in an ARCH and GARCH model with heavy-tailed errors. The nonlinear autoregressive (NLAR) model developed from the nonlinear regressive model was introduced by Jones (1978). Gouri´eroux Monfort (1992); Masry and Tjøstheim (1995) have proposed conditional heteroscedastic nonlinear auto-regressive (CHNLAR) models for financial time series. For simplicity, a single timestep lag in the CHNLAR model was

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studied to model the foreign exchange rates (Bossaerts, H¨ardle, & Hafner, 1996).

The parametric and nonparametric methods are the alternatives when estimating the regression between two sets of variables that consist of a vector of predictors and a response variable. A parametric regression model requires the user giving the form of the underlying regression function. The selection of a parametric model depends much on the problem and may be too restrictive in some applications. To overcome these difficulties, one may remove the restriction that the regression function belongs to a parametric family. This approach leads to so-called nonparametric regression.

Typically, the nonparametric regression methods are based on a smoothing technique. A smoother is an operator that summarizes the trend of a response variable as a function of one or more predictor variables. The single predictor case is called scatterplot smoothing, and can be used to enhance the visual appearance of the scatterplot of response versus predictor variable. There are many smoothing techniques, e.g., smoothing splines (Green & Silverman, 1994; Wahba, 1990), and penalized splines (Ruppert, Wand, & Carroll, 2003). These smoothing techniques are generally based on the assumption of homoscedastic variance, which may not be suitable when the data involve high volatility.

For these various reasons, we are interested in extending the NLAR model to a CHNLAR model, for approximating heteroscedastic values by adjusting the past value. The smoothing spline and penalized spline methods are applied to estimate the trend and the heteroscedastic values in both simulated and real data.

2. CHNLAR Model

Some nonlinear time series models focus on various volatility forms, such as ARCH model, GARCH model, threshold autoregressive model, and nonparametric autoregressive model. The nonparametric autoregressive conditional heteroscedastic (NARCH) model (Fan & Yao, 2003) adopted the nonparametric and nonlinear time series model and is called a conditional heteroscedastic nonlinear autoregressive (CHNLAR) model. It can be written as

$$y_t = \mu\left(y_{t-1},...,y_{t-p}\right) + \sigma\left(y_{t-1},...,y_{t-p}\right)\varepsilon_t,$$

and $\sigma(\bullet)$ is a function called the nonparametric autoregressive (NAR) model or the nonlinear autoregressive (NLAR) model.

In this current study, we employ the first-order conditional heteroscedastic nonlinear autoregressive (CHN LAR) model

$$y_t = \mu\left(y_{t-1}\right) + \sigma\left(y_{t-1}\right)\varepsilon_t,\ \ t = 2,3,\ldots,n, \qquad (1)$$

where $y_t, t = 2,3,\ldots,n$ are observed and depend on $y_{t-1}, t = 2,3,\ldots,n$ with lag 1, $\mu\left(y_{t-1}\right)$ is the trend function of CHNLAR model, $\sigma\left(y_{t-1}\right)$ is the heteroscedastic function of CHNLAR model, and ε_t, $t = 2,3,\ldots,n,$ denotes a random variable in the error term, with mean zero and variance one.

3. Nonparametric Regression Method

The popular nonparametric regression methods include smoothing splines and penalized splines. The concept of these methods is to interpolate the data in most suitable form of the fitting function with a smoothing parameter.

In this section we study the following nonlinear autoregressive (NLAR) model.

The NLAR model is written as

$$y_t = \mu\left(y_{t-1}\right) + \varepsilon_t,\ t = 2,3,\ldots,n, \qquad (2)$$

where $y_t, t = 2,3,\ldots,n$ are observed dependent variables, $y_{t-1}, t = 2,3,\ldots,n$ are the past values with lag 1 , $\mu\left(y_{t-1}\right)$ is the trend function of nonlinear autoregressive model, and ε_t, $t = 2,3,\ldots,n$ denote the random values with mean zero and variance one in the error terms.

3.1 Smoothing spline (SS) method

The smoothing spline was studied by Wahba (1990) and the smoothing spline is a natural polynomial spline $\left(S_\lambda^{(K)}(\mu)\right)$ that depends on the smoothing parameter $\left(\lambda\right)$:

$$S_\lambda^{(K)}(\mu) = \sum_{t=1}^{n}\left\{y_t - \mu\left(x_t\right)\right\}^2 + \lambda\int_a^b\left\{\mu^{(m)}\left(x_t\right)\right\}^2 dx_t, \qquad (3)$$

where K is the number of knots in the trend function with domain $\left[a,b\right]$, superscript (m) indicates the m th derivative of $\mu\left(x_t\right)$, y_t is the dependent variable, and $\mu\left(x_t\right)$ is the trend function that is a nonparametric regression function of independent variables.

Green and Silverman (1994) emphasized $m = 2$ case as so-called natural cubic spline to fit the nonparametric regression function by minimizing

$$S_\lambda^{(K)}(\mu) = \sum_{t=1}^{n}\left\{y_t - \mu\left(x_t\right)\right\}^2 + \lambda\int_a^b\left\{\mu''\left(x_t\right)\right\}^2 dx_t. \qquad (4)$$

In this case, we propose the NLAR model via smoothing spline method, and the natural cubic spline can be written as

$$S_\lambda^{(K)}(\mu) = \sum_{t=2}^{n}\left\{y_t - \mu\left(y_{t-1}\right)\right\}^2 + \lambda\int_a^b\left\{\mu''\left(y_{t-1}\right)\right\}^2 dy_t. \qquad (5)$$

The natural cubic spline has given value and second derivative at each knot y_t namely these are

$$\boldsymbol{\mu} = \mu\left(y_{t-1}\right),$$

$$\boldsymbol{\gamma} = \mu''\left(y_{t-1}\right),\ t = 2,3,\ldots,n,$$

Let $\boldsymbol{\mu}$ be the vector $(\mu_1,\ldots,\mu_{n-1})^{\mathrm{T}}$ and let $\boldsymbol{\gamma}$ be the vector $(\gamma_1,\ldots,\gamma_{n-1})^{\mathrm{T}}$.

The natural cubic spline then depends on the two matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ with

$$\boldsymbol{Q}=\begin{pmatrix} h_1^{-1} & 0 & \cdots & 0 \\ -h_1^{-1}-h_2^{-1} & h_2^{-1} & \cdots & 0 \\ h_2^{-1} & -h_2^{-1}-h_3^{-1} & \cdots & 0 \\ 0 & h_3^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & h_{n-1}^{-1} \end{pmatrix}_{(n-1)\times(n-3)},$$

where $h_t = y_t - y_{t-1}$, for $t=2,\ldots,n-1$, and $\boldsymbol{Q}$ is an $(n-1)\times(n-3)$ matrix.

$\boldsymbol{R}$ is a symmetric $(n-3)\times(n-3)$ matrix

$$\boldsymbol{R}=\begin{pmatrix} \frac{1}{3}(h_2+h_3) & \frac{1}{6}h_3 & \cdots & 0 \\ \frac{1}{6}h_3 & \frac{1}{3}(h_3+h_4) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{3}(h_{n-2}+h_{n-1}) \end{pmatrix}_{(n-3)\times(n-3)}.$$

The matrix $\boldsymbol{K}$ is defined by

$$\boldsymbol{K}=\boldsymbol{Q}\boldsymbol{R}^{-1}\boldsymbol{Q}^{\mathrm{T}}. \tag{6}$$

The vectors $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ specify a natural cubic spline $\mu(y_t)$ if and only if the condition

$$\boldsymbol{Q}^{\mathrm{T}}\boldsymbol{\mu}=\boldsymbol{R}\,\boldsymbol{\gamma} \tag{7}$$

is satisfied. If (7) is satisfied the roughness penalty will satisfy

$$\int_a^b \left\{\mu''(y_{t-1})\right\}^2 dy_t = \boldsymbol{\gamma}^{\mathrm{T}}\boldsymbol{R}\,\boldsymbol{\gamma}=\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{K}\,\boldsymbol{\mu}. \tag{8}$$

To illustrate, in the matrix form introduced by Green & Silverman (1994)

$$RSS=\sum_{t=2}^{n}\left\{y_t-\mu(y_{t-1})\right\}^2=(\boldsymbol{y}-\boldsymbol{\mu})^{\mathrm{T}}(\boldsymbol{y}-\boldsymbol{\mu}), \tag{9}$$

where $\boldsymbol{y}=(y_2,\ldots,y_n)^{\mathrm{T}}$ with y_t corresponding value to y_{t-1} and

$$\boldsymbol{\mu}=\left(\mu(y_1),\ldots,\mu(y_{n-1})\right)^{\mathrm{T}}. \tag{10}$$

The roughness penalty term $\int \mu''^2$ equals $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\mu}$ in (8) to obtain

$$\begin{aligned} S_\lambda(\mu) &= (\boldsymbol{y}-\boldsymbol{\mu})^{\mathrm{T}}(\boldsymbol{y}-\boldsymbol{\mu})+\lambda\,\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{\mu} \\ &= \boldsymbol{\mu}^{\mathrm{T}}(\boldsymbol{I}+\lambda\boldsymbol{K})\boldsymbol{\mu}-2\boldsymbol{y}^{\mathrm{T}}\boldsymbol{\mu}+\boldsymbol{y}^{\mathrm{T}}\boldsymbol{y}, \end{aligned} \tag{11}$$

Since $\lambda\boldsymbol{K}$ is non-negative definite, the matrix $\boldsymbol{I}+\lambda\boldsymbol{K}$ is strictly positive definite. It therefore follows that (11) has a unique minimum, and the smoothing spline estimator is

$$\hat{\boldsymbol{\mu}}_\lambda=(\boldsymbol{I}+\lambda\boldsymbol{K})^{-1}\boldsymbol{y}, \tag{12}$$

where $\boldsymbol{I}$ denotes the n-dimensional identity matrix.

3.2 Penalized spline (PS) method

Penalized spline smoother is estimated using the truncated power function (Ruppert & Carroll, 2000), and the penalized spline model is written as

$$\mu(x_t)=\sum_{j=0}^{m-1}\alpha_j x_t^j+\sum_{k=1}^{K}\beta_k\,(x_t-\tau_k)^{2m-1},\ t=1,2,\ldots,n, \tag{13}$$

where α_j and β_k denote regression coefficients in the truncated power function.

The natural cubic spline with $m=2$ is called the low-rank thin-plate spline and it tends to have very good numerical properties. The low-rank thin-plate spline representation of $\mu(.)$ is

$$\mu(x_t,\boldsymbol{\theta})=\alpha_0+\alpha_1 x_t+\sum_{k=1}^{K}\beta_k\left|x_t-\tau_k\right|^3,\ t=1,2,\ldots,n, \tag{14}$$

where $\boldsymbol{\theta}=(\alpha_0,\alpha_1,\beta_1,\ldots,\beta_K)^{\mathrm{T}}$ is the vector of regression coefficients, and $\tau_1<\tau_2<\ldots<\tau_K$ are fixed knots.

In this case, we focus on the NLAR model based on penalized spline method, then the natural cubic spline can be written as

$$\mu(y_{t-1},\boldsymbol{\theta})=\alpha_0+\alpha_1 y_{t-1}+\sum_{k=1}^{K}\beta_k\left|y_{t-1}-\tau_k\right|^3,\ t=2,3,\ldots,n. \tag{15}$$

To avoid overfitting, we minimize

$$\sum_{t=1}^{n}\left\{y_t-\mu(y_{t-1},\boldsymbol{\theta})\right\}^2+\frac{1}{\lambda}\boldsymbol{\theta}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\theta}, \tag{16}$$

where λ is the smoothing parameter and $\boldsymbol{D}$ is a known positive semi-definite penalty matrix. The thin-plate spline penalty matrix is

$$\boldsymbol{D}=\begin{bmatrix}\mathbf{0}_{2\times 2} & \mathbf{0}_{2\times K}\\ \mathbf{0}_{K\times 2} & \boldsymbol{\Omega}_K\end{bmatrix}, \tag{17}$$

where the (l,k) th entry of $\boldsymbol{\Omega}$ is $|\tau_l-\tau_k|^3$ and penalizes only the coefficient of $|y_{t-1}-\tau_k|^3$.

Just as with the linear model, we can generalize penalized spline in general linear mixed model (Brumback, Ruppert, & Wand, 1999) as

$$\boldsymbol{y}=\boldsymbol{Y\alpha}+\boldsymbol{Z}_K\boldsymbol{\beta}+\boldsymbol{\varepsilon}, \tag{18}$$

where $\boldsymbol{y}=(y_2,\ldots,y_n)^{\mathrm{T}}$, $\boldsymbol{Y}$ be the matrix with the $t-1$ th row $\boldsymbol{Y}_t=(1,y_{t-1})$, $\boldsymbol{Z}_K$ is the matrix with the t th row $\boldsymbol{Z}_{Kt}=\left\{|y_{t-1}-\tau_1|^3,\ldots,|y_{t-1}-\tau_K|^3\right\}$, $\boldsymbol{\alpha}=(\alpha_1,\alpha_2)^{\mathrm{T}}$, $\boldsymbol{\beta}=(\beta_1,\ldots,\beta_K)^{\mathrm{T}}$, and $\boldsymbol{\varepsilon}$ is $N(0,\sigma_\varepsilon^2 I)$. Consider the vector $\boldsymbol{\alpha}$ as fixed parameters and the vector $\boldsymbol{\beta}$ as a set of random parameters with $E(\boldsymbol{\beta})=0$ and $\mathrm{cov}(\boldsymbol{\beta})=\sigma_\beta^2$. This class of penalized spline smoothers $(\hat{\mu}(.))$ may also be expressed as

$$\hat{\boldsymbol{\mu}}=\boldsymbol{C}\left(\boldsymbol{C}^{\mathrm{T}}\boldsymbol{C}+\lambda^3\boldsymbol{D}\right)^{-1}\boldsymbol{C}^{\mathrm{T}}\boldsymbol{y}, \tag{19}$$

where

$$\boldsymbol{C}=\left[1\quad y_{t-1}\quad |y_{t-1}-\tau_k|^3_{1\le k\le K}\right]_{1\le t\le n},\ \boldsymbol{D}=\begin{bmatrix}\mathbf{0}_{2\times 2} & \mathbf{0}_{2\times K}\\ \mathbf{0}_{K\times 2} & \left(\boldsymbol{\Omega}_K^{1/2}\right)^{\top}\boldsymbol{\Omega}_K^{1/2}\end{bmatrix}, \tag{20}$$

and $\lambda=\sigma_\beta^2/\sigma_\varepsilon^2$ is a smoothing parameter.

4. Proposed Trend and Heteroscedastic Estimators

The trend $\mu(y_{t-1})$ and heteroscedasticity $\sigma^2(y_{t-1})$ can also be considered in CHNLAR model. As an initial step, we start by estimating the trend $\mu(y_{t-1})$ using the concept of NLAR model written as

$$y_t=\mu(y_{t-1})+\delta_t,\ t=2,3,\ldots,n, \tag{21}$$

where $\delta_t=\sigma(y_{t-1})\ \varepsilon_t$. Next, we obtain $\hat{\mu}(y_{t-1})$ from smoothing spline (SS) and penalized spline (PS) where the residuals can be estimated as

$$\hat{\delta}_t = y_t-\hat{\mu}(y_{t-1}) \tag{22}$$

$$\left(\hat{\delta}_t\right)^2=\left(\sigma(y_{t-1})\ \hat{\varepsilon}_t\right)^2. \tag{23}$$

We transform $\sigma(y_{t-1})=\exp\left\{\frac{h(y_{t-1})}{2}\right\}$, and take log with residuals in (23)

$$\log\hat{\delta}_t^2 = h(y_{t-1})+\log\hat{\varepsilon}_t^2, \tag{24}$$

$$\log\hat{\delta}_t^2-E\left[\log\hat{\varepsilon}_t^2\right] = h(y_{t-1})+\log\hat{\varepsilon}_t^2-E\left[\log\hat{\varepsilon}_t^2\right]. \tag{25}$$

If we require $\hat{\varepsilon}_t$ to be normally distributed with mean zero and variance one, then $E\left[\log\hat{\varepsilon}_t^2\right]=-1.2704$ in (25) and hence we can apply it in SS and PS to obtain

$$\log\hat{\delta}_t^2+1.2704 = h(y_{t-1})+\log\hat{\varepsilon}_t^2+1.2704 \tag{26}$$

$$\tilde{y}_t = h(y_{t-1})+\tilde{\varepsilon}_t \tag{27}$$

where $\tilde{y}_t=\log\hat{\delta}_t^2+1.2704$ and $\tilde{\varepsilon}_t=\log\hat{\varepsilon}_t^2+1.2704$. Next, we get a smooth estimate $\hat{h}(y_{t-1})$ from SS and PS by using (27) and update the heteroscedastic estimate to

$$\hat{\sigma}(y_{t-1}) = \exp\left\{\frac{\hat{h}(y_{t-1})}{2}\right\}. \tag{28}$$

At the second stage of estimation we update the trend estimate by using the following model

$$y_t = \mu(y_{t-1})+\exp\left\{\frac{\hat{h}(y_{t-1})}{2}\right\}\hat{\varepsilon}_t \tag{29}$$

$$\exp\left\{-\frac{\hat{h}(y_{t-1})}{2}\right\}y_t = \exp\left\{-\frac{\hat{h}(y_{t-1})}{2}\right\}\mu(y_{t-1})+\hat{\varepsilon}_t \tag{30}$$

$$\breve{y}_t = g(y_{t-1})+\hat{\varepsilon}_t \tag{31}$$

where

$$\breve{y}_t=\exp\left\{-\frac{\hat{h}(y_{t-1})}{2}\right\}y_t,$$

and

$g(y_{t-1})=\exp\left\{-\frac{\hat{h}(y_{t-1})}{2}\right\}\mu(y_{t-1})$ on (30). If $\hat{g}(y_{t-1})$ is obtained by SS and PS, the second stage estimate of $\mu(y_{t-1})$ is given by

$$\hat{\mu}(y_{t-1}) = \exp\left\{\frac{\hat{h}(y_{t-1})}{2}\right\}\hat{g}(y_{t-1}). \quad (32)$$

Finally, the estimates of $\mu(y_{t-1})$ and $\sigma(y_{t-1})$ converge to $\hat{\mu}(y_{t-1})$ and $\hat{\sigma}(y_{t-1})$.

5. Simulation Study

The simulation study to assess the performances of smoothing spline (SS) method and penalized spline (PS) was divided into two parts. The first part is to study in CHNLAR model

$$y_t = \mu(y_{t-1}) + \sigma(y_{t-1})\ \varepsilon_t,\ t = 2,3,\ldots,n, \quad (33)$$

where $\mu(y_{t-1})$ and $\sigma(y_{t-1})$ are generated following

$$\mu(y_{t-1}) = 0.1(y_{t-1}),$$

$$\sigma(y_{t-1}) = \exp\{0.05 \times y_{t-1}\},$$

where $y_1 \sim$Normal(0,1). In Figure 1, we present y_t in CHNLAR model at sample sizes n = 50, 100, 200, and 300. The error process $\varepsilon_t, t = 2,3,\ldots,n$ in (33) is assumed to follow the normal distribution with mean zero and variance one.

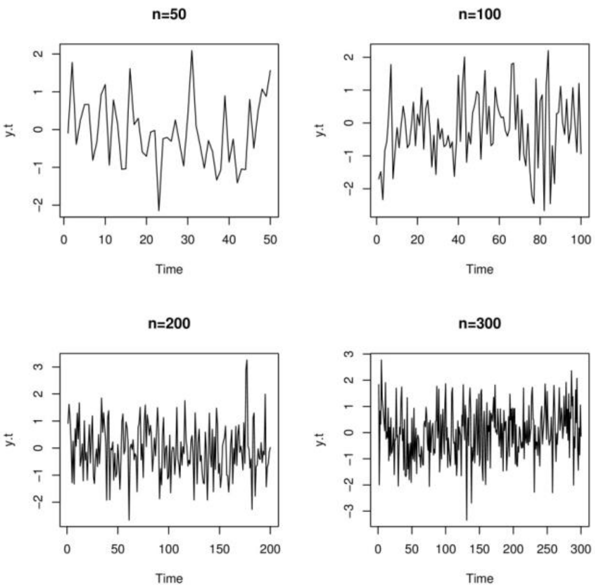

Figure 1. The time series data in CHNLAR model.

In the second part, to estimate $\hat{\mu}(y_{t-1})$ and $\hat{\sigma}(y_{t-1})$, we compute the bias and the Mean Square Error (MSE) of $\mu(.)$ and $\sigma(.)$ by

$$\mu_{bias} = \frac{1}{n}\sum_{t=2}^{n}\frac{\hat{\mu}(y_{t-1}) - \mu(y_{t-1})}{\mu(y_{t-1})},$$

$$\sigma_{bias} = \frac{1}{n}\sum_{t=2}^{n}\frac{\hat{\sigma}(y_{t-1}) - \sigma(y_{t-1})}{\sigma(y_{t-1})},$$

$$MSE(\mu) = \frac{1}{n}\sum_{t=2}^{n}\left(\hat{\mu}(y_{t-1}) - \mu(y_{t-1})\right)^2,$$

$$MSE(\sigma) = \frac{1}{n}\sum_{t=2}^{n}\left(\hat{\sigma}(y_{t-1}) - \sigma(y_{t-1})\right)^2.$$

We simulated data with the sample sizes n = 50, 100, 200, and 300, and repeated the data generation and model fitting 500 times.

Table 1 presents the average MSE of SS and PS methods for all sample sizes. The average MSE of $\mu(.)$ and $\sigma(.)$ decreased with sample size. For $\mu(.)$, the average MSE of PS is less than with SS, but the average MSE of PS was larger than with SS for $\sigma(.)$.

Tables 2 and 3 show various Monte Carlo (MC) summary statistics of the estimates obtained by the SS and PS methods. The third and the fourth columns of these tables represent the MC sample mean and standard deviation of biases. The sample means of the lower and upper bounds of 95% confidence interval are given in the next two columns. The last two columns of these tables list the t-statistic, and p-value for hypothesis testing (H_0: bias = 0 versus H_1: bias ≠ 0). If the p-value is less than 0.05, we reject the null hypothesis (H_0) that there is difference between the observed values and the fitted values. If the p-value is larger than 0.05, we conclude that we have an unbiased estimator. Based on the p-values, we can claim the following. From Tables 2 and 3, the SS and PS methods provide asymptotically unbiased estimates of $\mu(.)$ and $\sigma(.)$. However $\sigma(.)$ did not get an asymptotically unbiased estimate when the sample sizes was 200 $(n = 200)$ with the SS method.

Histograms of the biases of all parameter estimates are presented in Figures 2-5. It is apparent that for the distribution of $\sigma(.)$ the biases appear to be normally distri-buted for all sample sizes.

Table 1. The average MSE of SS and PS methods with different sample sizes (500 replications).

	SS method		PS method	
sample size	$\mu(.)$	$\sigma(.)$	$\mu(.)$	$\sigma(.)$
n = 50	0.300	0.0251	0.0249	0.1427
n = 100	0.0105	0.0114	0.0095	0.0622
n = 200	0.0054	0.0061	0.0033	0.0193
n = 300	0.00372	0.00413	0.0020	0.0100

Table 2. The simulation of smoothing spline method for different sample sizes (500 replications).

bias	sample size	mean	s.d.	lci	uci	t-stat	p-value
μ_{bias}	n = 50	6.507	157.106	-7.296	20.3116	0.9261	0.3548
	n = 100	-1.172	23.367	-3.225	0.881	-1.1216	0.2626
	n = 200	-0.2977	10.2154	-1.1953	0.5998	-0.6517	0.5149
	n = 300	3.5053	49.927	-0.8815	7.8921	1.5699	0.1171
σ_{bias}	n = 50	-0.0114	0.1577	-0.025	0.0024	-1.6176	0.1064
	n = 100	0.0020	0.1072	-0.0073	0.0114	0.4283	0.6686
	n = 200	-0.0074	0.0776	-0.0142	-0.0005	-2.1335	0.0333*
	n = 300	0.00184	0.0644	-0.0038	0.0075	0.6412	0.5217

* indicates significance at 5% level

Table 3. The simulation of penalized spline method for different sample sizes (500 replications).

bias	sample size	mean	s.d.	lci	uci	t-stat	p-value
μ_{bias}	n = 50	-4.734	79.833	-11.749	2.279	-1.326	0.1854
	n = 100	1.485	47.561	-2.693	5.664	0.698	0.485
	n = 200	0.0454	32.9606	-2.8506	2.9415	0.0308	0.9754
	n = 300	-0.5388	10.4931	-1.4608	0.3831	-1.1483	0.2514
σ_{bias}	n = 50	0.00429	0.172	-0.010	0.019	0.556	0.578
	n = 100	-0.0001	0.1120	-0.009	0.009	-0.026	0.9792
	n = 200	-0.0018	0.0790	-0.0087	0.0051	-0.5170	0.6054
	n = 300	0.0035	0.0617	-0.0018	0.0090	1.3005	0.194

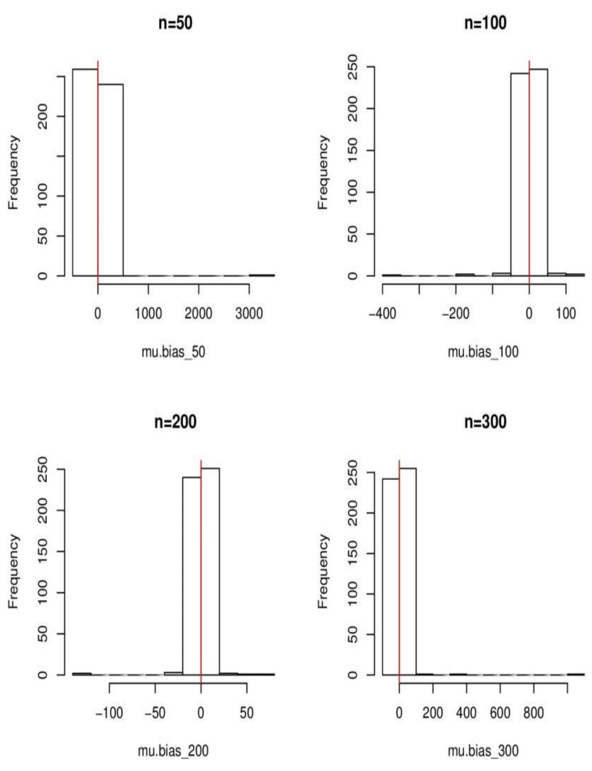

Figure 2. Histogram of bias in $\mu(.)$ with smoothing spline method.

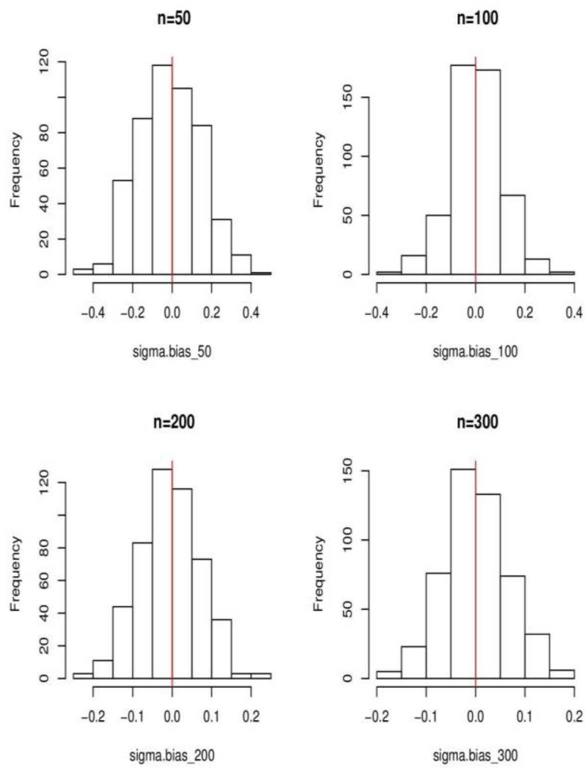

Figure 3. Histogram of bias in $\sigma(.)$ with smoothing spline method.

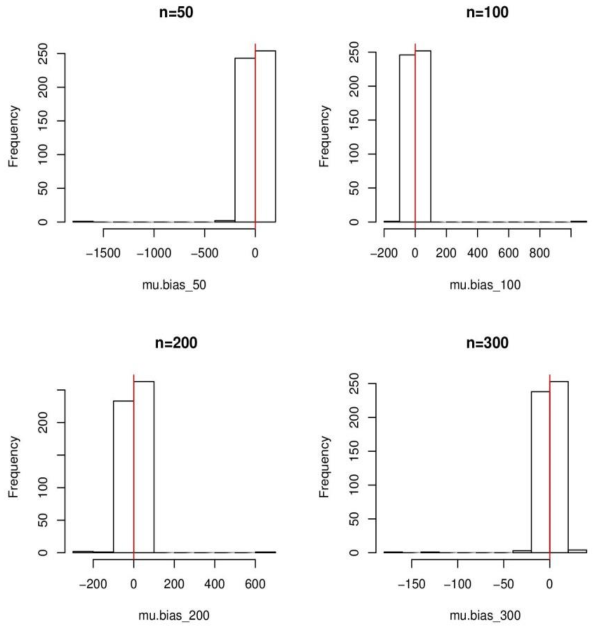

Figure 4. Histogram of bias in $\mu(.)$ with penalized spline method.

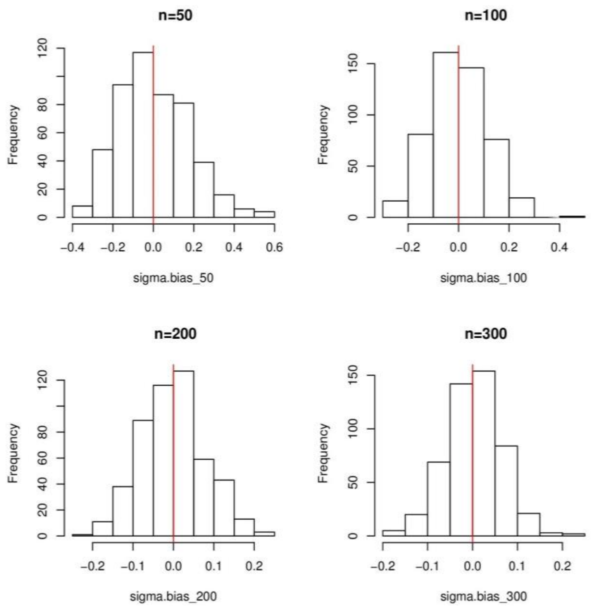

Figure 5. Histogram of bias in $\sigma(.)$ with penalized spline method.

6. Applications for Real Data

In this section, we will consider the application of CHNLAR model using the smoothing spline (SS) and penalized spline (PS) methods that we developed in the previous chapter. As a real data set, we use the monthly gold price (US Dollars per Troy Ounce) from January, 1984 to December 2013, which consists of 360 records and is shown in Figure 6. These data were obtained from http://www.index mundi.com.

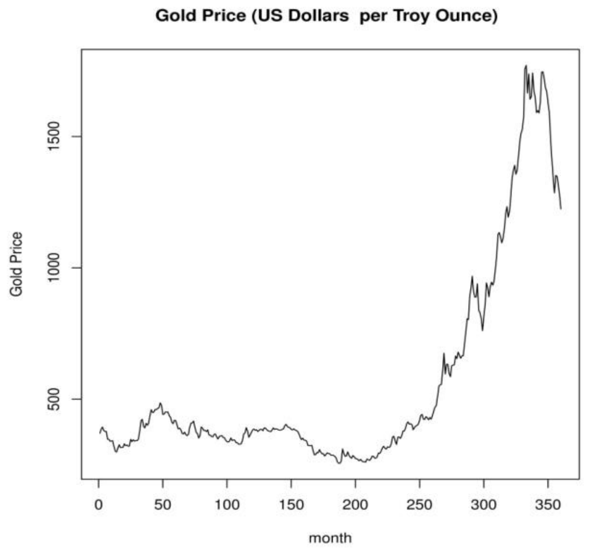

Figure 6. The monthly gold prices from January, 1984 to December, 2013.

The modeling steps were as follows.
At first, we considered the CHNLAR model following

$$y_t = \mu(y_{t-1}) + \sigma(y_{t-1})\ \varepsilon_t,\ t = 2,3,\ldots,360, \qquad (34)$$

where ε_t's are independently and identically distributed with mean zero and variance one. In this case, we let y_t denote the gold price of month t, where $t=1$ represents January of 1984 and $t=360$ represents December of 2013.

Then we fitted the CHNLAR model to obtain the trend function, $\mu(.)$ and the heteroscedastic function, $\sigma(.)$. We got $\hat{\mu}(y_{t-1}), \hat{\sigma}(y_{t-1}), t = 2,3,\ldots,360$ using SS and PS methods.

Let $\hat{\mu}(y_{t-1})$ and $\hat{\sigma}(y_{t-1})$ denote the converged estimates of $\mu(.)$ and $\sigma(.)$,and let

$$\hat{\varepsilon}_t = \frac{y_t - \hat{\mu}(y_{t-1})}{\hat{\sigma}(y_{t-1})},\ t = 2,3,\ldots,360, \qquad (35)$$

denote the standardized residuals based on the converged values of $\hat{\mu}(y_{t-1})$ and $\hat{\sigma}(y_{t-1})$.

Finally, we obtain estimated values of $\hat{y}_2,\ldots,\hat{y}_n$ using the estimated trend and heteroscedastic based on the CHNLAR model:

$$\hat{y}_t = \hat{\mu}(y_{t-1}) + \hat{\sigma}(y_{t-1})\ \hat{\varepsilon}_t,\ t = 2,3,\ldots,360, \qquad (36)$$

where the forecast trend $\hat{\mu}(y_{t-1})$ and the forecast heteroscedastic $\hat{\sigma}(y_{t-1})$ from SS and PS methods are presented on Figure 7 and 8.

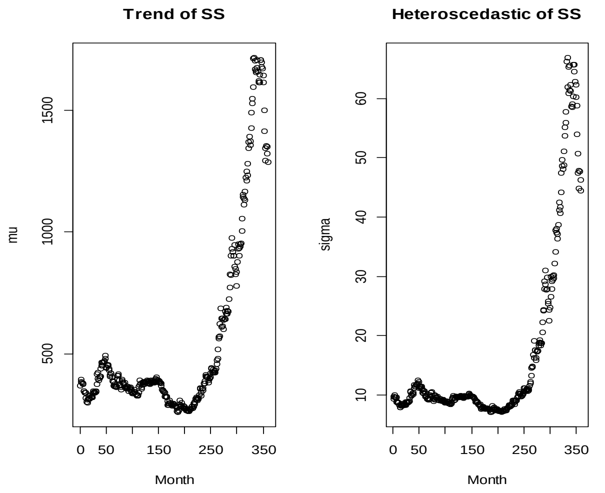

Figure 7. Forecasting trend and heterocsedasticity by smoothing spline (SS) method.

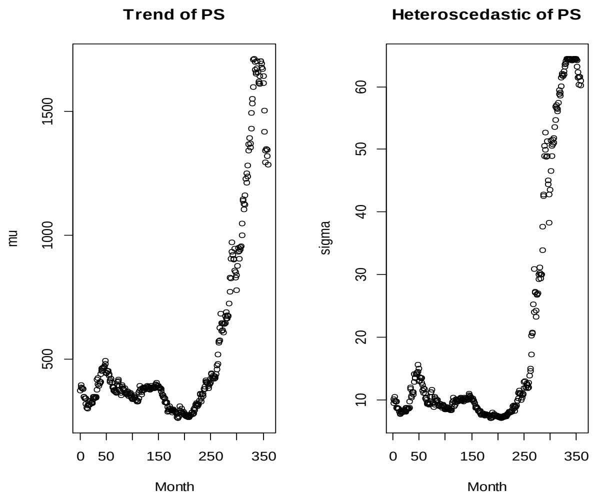

Figure 8. Forecasting trend and heterocsedasticity by penalized spline (PS) method.

The fitted values from February, 1984 to December, 2013 of the SS and PS methods are shown in Figure 9 and also MSE, mean, and standard deviation of $\mu(.)$ and $\sigma(.)$ are shown in Table 4. From Table 4, the MSE of PS is larger than that of SS, but the $\mu(.)$ show the slightly different values.

Table 4. The mean (standard deviation) and Mean Square Error (MSE) of smoothing spline (SS) and penalized spline (PS) methods.

Estimator	SS method	PS method
$\mu(.)$	564.4387 (396.7862)	564.4463 (396.755)
$\sigma(.)$	16.92417 (15.54389	20.4015 (19.66588)
MSE	5.653	8.372

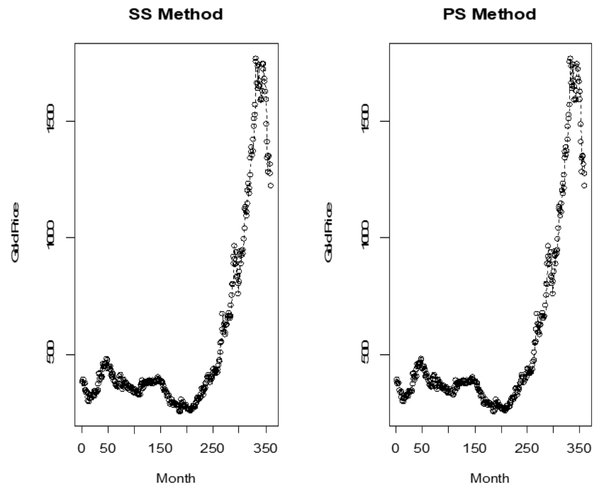

Figure 9. The gold prices with line plot of fitted values by smoothing spline (SS) and penalized spline (PS) methods.

7. Conclusions

In this study, we used nonparametric regression methods such as the smoothing spline method and the penalized spline method to estimate a smooth unknown trend and heteroscedasticity in an CHNLAR model. Through a Monte Carlo simulation study, we evaluated the performance of the smoothing spline methods and showed that the trend estimator ($\mu(.)$) and heteroscedasticity estimator ($\sigma(.)$)work reasonably well for most data of all sample sizes, except in one case ($n = 200$) where the heteroscedasticity estimator had a bias. The point volatility estimators approach their corresponding true values as the sample size increases.

In a Monte Carlo study, we showed that the trend estimator of penalized spline works well for all small sample sizes, when the smoothing parameter is high enough, indicating that small sample sizes allow reliable interpolation by these models.

In an application to actual data, we were also interested in comparing the power of estimating values, assessed by considering the Mean Square Error (MSE). The MSEs with smoothing spline were smaller than with penalized spline. However, we consider the mean of the trend and heteroscedastic estimator, and we can see that the means of trend with smoothing spline method were slightly different from the penalized spline method but the variance and MSE of smoothing spline method is smaller than with penalized spline method. It can be concluded that the forecasting performance of CHNLAR depends on heteroscedasticity.

In future work, we intend to study higher order lags in trend and heteroscedasticity of the CHNALR model.

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