

*Original Article*

# Differential transformation method for vibration of membranes

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**Abstract**

The purpose of this research is to apply the three-dimensional differential transform method to estimate solutions to the equation of motion for vibration in a membrane with certain types of boundary conditions. The analytical solutions without external force or damping under specific initial and boundary conditions are presented. We found by comparison that the analytical solution and the estimated solution are in good agreement, in case there is no damping or external forces. Furthermore, the differential transform method can be used to find approximate solutions taking into account both external forces and damping. This cannot be achieved via an analytical solution.

**Keywords:** vibration of membrane, differential transformation method, analytical solution, approximate solution

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**1. Introduction**

Many researchers have attempted to understand various phenomena occurring in nature by applying knowledge from different fields, such as mechanical engineering, electrical engineering, industrial engineering, energy, and medicine. Most of these problems have been studied by employing some form of mathematical modeling, often with ordinary differential equations (ODE) or partial differential equations (PDE). These problems require sufficiently accurate solutions, either analytical or approximate. The differential transform method (DTM) is among the most effective mathematical methods for finding solutions to these differential equations (Hatami, Ganji, & Sheikholeslami, 2017).

The differential transform method (DTM) is based on high-order Taylor series expansions. This method is a powerful tool for solving linear and non-linear ordinary differential equations (Arikoglu, & Ozkol, 2006; Ayaz, 2004; Catal, 2008) and for solving two- and three-dimensional partial differential equations in both linear and non-linear problems. DTM can be used to solve differential equations subject to

initial and boundary conditions, having both linear and non-linear terms, and within an acceptable error range.

The two-dimensional differential transform method (2D-DTM) has been used to find the solutions of both linear PDEs (Ayaz, 2003; Chen & Ho, 1999; Othman, & Mahdy, 2010; Yang, Liu, & Bai, 2006) and nonlinear PDEs (Biazar & Eslami, 2010; Biazar, Eslami, & Islam, 2012; Bildik, Konuralp, Bek, & Kucukarslan, 2006; Kangalgil & Ayaz, 2009).

Additionally, the three-dimensional differential transform method (3D-DTM) has been applied to find the solutions of linear and non-linear PDEs (Bagheri & Manafianheris, 2012; Saravanan & Magesh, 2013). It is noted that the differential transform method can be used to solve multidimensional PDEs, such as the Westervelt equation (Jafari, Sadeghi, & Biswas, 2012), heat-like and wave-like equations (Tabaei, Celuk, & Tabaei, 2012), and fuzzy partial differential equations (Mirzaee & Yari, 2015), as well as linear and nonlinear systems of PDEs (Ayaz, 2004; Zedan & AliAlghamdi, 2012).

Many researchers sought to use non-linear PDEs with various transform methods, as follows. The Fitzhugh Nangumo (FN) equation is a mathematical model for solving scientific and engineering problems by using q-HATM and the fractional reduced differential transform method (FRDTM), which is based on DTM (Kumar, Singh, & Baleanu, 2017).

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The numerical solutions to non-linear fractional dynamical model of interpersonal and romantic relationships were found by applying q-homotopy analysis via the Sumudu transform method (q-HASTM) (Singh, Kumar, Qurashi, & Baleanu, 2017).

Jeffery-Hamel flow with non-parallel walls, which is represented by a non-linear PDE and occurs in fluid dynamics and other scientific applications, is solved by using an efficient hybrid computational technique, the homotopy analysis transform method (HATM) (Singh, Rashidi, Sushila, & Kumar, 2017).

The homotopy perturbation Sumudu transform method (HPSTM) and homotopy analysis Sumuda transform method (HASTM) are more convenient than the homotopy perturbation method (HPM) and the homotopy analysis method (HAM), since they produce a comparative analytical study for a system of time fractional non-linear differential equations (Choi, Kumar, Singh, & Swroop, 2016).

Further examples of the application of PDEs by Laplace transform method to various problem include Case I-application of drum head vibration solution by separation of variables, and Case II and III-applications to signal transmission and to chemical communication in insects. All three cases have been implemented as simulations in MATLAB software (Ojwando, 2016).

A modified He-Laplace method (MHLM) is applied to solve space and time nonlinear fractional differential-difference equations (NFDDEs) (Prakash, Kothandapan, & Bharathi, 2016).

In our previous work, we studied the suspended vibrating string equation using 2D –DTM. It was found that DTM can be applied to various problems of the suspended string equation.

In this current report, we study further the vibration equation in three dimensions of the motion of a membrane using the 3D-DTM. The oscillation of a membrane-like plate is determined by the tension when there is insignificant resistance to bending. The differential transform method was applied to find solutions of the motion equation of a membrane with an external force and a damping term. We compare the results with an analytical solution. We show in detail the derivation of the transformed formula in DTM, which is of the  $n^{th}$  power form (to be shown in theorem 3.6). The obtained formula helps simplify the use of DTM in solving non-linear PDEs. It is noted that the formula requires heavy computation to find complete solutions, mainly due to the fact that the formula is in a recursive form.

**1.1 The equation of motion of a membrane**

The equation of motion for the forced transverse vibration of a membrane, after (Rao, 2011) is as follows:

$$P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f(x, y, t) = \rho(x, y) \frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where  $f(x, y, t)$  is the pressure acting in the z direction (external force),  $P$  is the intensity of tension at a point, equal to product of tensile stress and thickness of the membrane, and  $\rho(x, y)$  is the mass per unit area. We assume that

$f(x, y, t) = 0$ ,  $P = 1$ , and  $\rho(x, y) = 1$ . Then Equation (1) leads to:

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial t^2}. \tag{2}$$

The initial conditions of (2) after (Rao, 2011) are:

$$w(x, y, 0) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,$$

$$\frac{\partial w}{\partial t}(x, y, 0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b,$$

and we set  $a = 1$  and  $b = 1$ . Therefore, the boundary conditions of the equation of motion of a membrane are given by:

$$w(x, 0, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(0, y, t) = 0, \quad 0 \leq y \leq 1,$$

$$w(x, 1, t) = 0, \quad 0 \leq x \leq 1,$$

$$w(1, y, t) = 0, \quad 0 \leq y \leq 1, t \in \mathbb{R}. \tag{3}$$

**1.2 An analytical solution**

The following is the derivation of the analytical solution to the problem (2). Considering the initial conditions  $w(x, y, 0) = \sin \pi x \sin \pi y$  and  $\frac{\partial w}{\partial t}(x, y, 0) = 0$ , the boundary conditions are shown in (3). The general solution can be derived by separation of variables:

$$w(x, y, t) = X(x)Y(y)T(t), \tag{4}$$

Subject to the eigenvalue,  $-\omega^2, \alpha^2$  and  $\beta^2$ , (2) implies

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = \frac{T''(t)}{T(t)} = -\omega^2,$$

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} = -\omega^2,$$

and  $-\frac{X''(x)}{X(x)} = \frac{Y''(y)}{Y(y)} + \omega^2 = \alpha^2,$

$$-\frac{X''(x)}{X(x)} = \alpha^2,$$

then  $X''(x) + \alpha^2 X(x) = 0,$

$$\frac{Y''(y)}{Y(y)} = \alpha^2 - \omega^2,$$

assume  $-\alpha^2 + \omega^2 = \beta^2, Y''(y) + (-\alpha^2 + \omega^2)Y(y) = 0,$

then  $Y''(y) + \beta^2 Y(y) = 0,$

and  $\frac{T''(t)}{T(t)} = \omega^2,$

then  $T''(t) + \omega^2 T(t) = 0,$

$$X''(x) + \alpha^2 X(x) = 0, \tag{5}$$

$$Y''(y) + \beta^2 Y(y) = 0, \tag{6}$$

$$T''(t) + \omega^2 T(t) = 0. \tag{7}$$

We obtain a solution of (5) as  $X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x,$  where  $C_1$  and  $C_2$  are arbitrary constants. Subject to the boundary conditions  $X(0) = 0,$  we have  $C_1 = 0.$  Then  $X(x) = C_2 \sin \alpha x$  full step the boundary condition  $X(1) = 0.$  Let  $C_2 \neq 0$  give  $\alpha = m\pi; m \in I,$  we have that

$$X_m(x) = C_m \sin m\pi x. \tag{8}$$

We obtain a solution of (6) as  $Y(y) = C_3 \cos \beta y + C_4 \sin \beta y.$  According to boundary conditions  $Y(0) = 0,$  we have  $C_3 = 0.$  Then the boundary conditions  $Y(1) = 0.$

Letting  $C_4 \neq 0$  gives  $\beta = n\pi; n \in I,$  then we see that:

$$Y_n(y) = C_n \sin n\pi y. \tag{9}$$

From  $\omega^2 = \beta^2 + \alpha^2,$   
 $\omega = \pi \sqrt{m^2 + n^2}.$

We obtain a solution of (7) as

$$T_{mn}(t) = A_{mn} \cos \omega t + B_{mn} \sin \omega t.$$

Then  $T_{mn}(t) = A_{mn} \cos \pi \sqrt{m^2 + n^2} t + B_{mn} \sin \pi \sqrt{m^2 + n^2} t. \tag{10}$

According to (8), (9) and (10)

$$W_{mn}(x, y, t) = X(x)Y(y)T(t),$$

$$= \left( A_{mn} C_m C_n \sin m\pi x \sin n\pi y \cos \pi \sqrt{m^2 + n^2} t \right) \tag{11}$$

$$+ \left( B_{mn} C_m C_n \sin m\pi x \sin n\pi y \sin \pi \sqrt{m^2 + n^2} t \right),$$

where  $F_{mn} = A_{mn} C_m C_n, H_{mn} = B_{mn} C_m C_n$  and for all  $m, n \in I$

By using superposition of solutions in (11)

$$w(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x, y, t),$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left( F_{mn} \sin m\pi x \sin n\pi y \cos \pi \sqrt{m^2 + n^2} t \right), \tag{12}$$

$$+ \left( H_{mn} \sin m\pi x \sin n\pi y \sin \pi \sqrt{m^2 + n^2} t \right).$$

According to the initial condition  $w(x, y, 0) = \sin \pi x \sin \pi y$  and  $\frac{\partial w}{\partial t}(x, y, 0) = 0.$  By using Fourier series, we have

$$F_{mn} = 1 ; m=1 \text{ and } n=1,$$

$$H_{mn} = 0; m, n \in I$$

$$\omega = \sqrt{2}\pi.$$

Substituting  $F_{11} = 1, H_{mn} = 0$  and  $\omega = \sqrt{2}\pi$  in (12), we obtain the analytical solution:

$$w(x, y, t) = \sin \pi x \sin \pi y \cos \sqrt{2}\pi t. \tag{13}$$

In the case of problem (2) with a damping term

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t}.$$

By similar calculations, we obtain the solution of  $T_{m,n}(t)$  as

$$T_{m,n}(t) = e^{-\frac{t}{2}} \left( A_{m,n} \cos \frac{\sqrt{1-4\omega^2}}{2} t + B_{m,n} \sin \frac{\sqrt{1-4\omega^2}}{2} t \right). \tag{14}$$

We can see that on substituting  $\omega = \sqrt{2}\pi$  to (14),  $\sqrt{1-4\omega^2}$  becomes a complex number. Therefore  $T(t)$  are not solvable.

### 2. Three-Dimensional Differential Transform Method

The basic definitions and fundamental operations of differential transform are defined below.

**Definition 2.1** The three-dimensional differential transform of function  $w(x, y, t)$  is defined as:

$$W(k, h, m) = \frac{1}{k!h!m!} \frac{\partial^{k+h+m} w(x, y, t)}{\partial x^k \partial y^h \partial t^m} \Big|_{(0,0,0)}$$

$$k \geq 0, h \geq 0 \text{ and } m \geq 0.$$

**Definition 2.2** The inverse three-dimensional differential transform of sequence  $\{W(k, h, m)\}_{k,h,m=0}^{\infty}$  is defined as:

$$w(x, y, t) = \sum_{k=0}^{\infty} \sum_{h=0}^{\infty} \sum_{m=0}^{\infty} W(k, h, m) x^k y^h t^m. \tag{14}$$

**3. The Fundamental Operations of Three-Dimensional Differential Transform Method**

From Table 1 theorem 3.1-3.5 shown in (Yang, Liu, & Bai, 2006). The following is the derivation of theorem 3.6: If  $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t)$  then,

$$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_3=0}^{h_2} \sum_{h_2=0}^{h_3} \sum_{m_3=0}^{m_2} \sum_{m_2=0}^{m_3} W_1(k_1, h_1, m_1)W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2})W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$$

For  $V : I^+ \rightarrow \mathbb{R}$ ,  $W_n : I^+ \rightarrow \mathbb{R}; n \in I^+$ ,  $v : \mathbb{R} \rightarrow \mathbb{R}$ ,  $w_n : \mathbb{R} \rightarrow \mathbb{R}; n \in I^+, x \in \mathbb{R}$  and  $k, h, m = 0, 1, 2, 3, \dots$

By definition of the three-dimensional differential transform:

$$V(k, h, m) = \frac{1}{k!h!m!} \left[ \frac{\partial^{k+h+m} v(x, y, t)}{\partial x^k \partial y^h \partial t^m} \right]_{(0,0,0)},$$

we obtain  $v(x, y, t) = w_1(x, y, t)w_2(x, y, t)...w_{n-1}(x, y, t)w_n(x, y, t)$

Table 1. The fundamental operations of DTM.

The fundamental operations of three-dimensional differential transform method	
Original function	Transformed form
3.1 $v(x, y, t) = a(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial x^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k-i+2)(k-i+1) A(i, j, r)W(k-i+2, h-j, m-r).$
3.2 $v(x, y, t) = b(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial y^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+2)(h-i+1) B(i, j, r)U(k-i, h-j+2, m-r).$
3.3 $v(x, y, t) = c(x, y, t) \frac{\partial^2 w(x, y, t)}{\partial t^2}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (m-i+2)(m-i+1) C(i, j, r)W(k-i, h-j, m-r+2).$
3.4 $v(x, y, t) = d(x, y, t) \frac{\partial w(x, y, t)}{\partial t}$	$V(k, h, m) = \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h-i+1)D(i, j, r) D(i, j, r)W(k-i, h-j+1, m-r).$
3.5 $v(x, y, t) = x^n y^l t^s$	$V(k, h, m) = \delta(k-n, h-l, m-s) = \delta((k-n)\delta(h-l)\delta(m-s))$ , where $\delta(k-n) = \begin{cases} 1 & k=n, \\ 0 & k \neq n, \end{cases} \delta(h-l) = \begin{cases} 1 & h=l, \\ 0 & h \neq l, \end{cases} \delta(m-p) = \begin{cases} 1 & m=s, \\ 0 & m \neq s, \end{cases}$
3.6 $v(x, y, t) = p(x, y, t)w^n(x, y, t)$	$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \sum_{h_3=0}^{h_2} \sum_{h_2=0}^{h_3} \sum_{m_3=0}^{m_2} \sum_{m_2=0}^{m_3} W_1(k_1, h_1, m_1)W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2})W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$
3.7 $v(x, y, t) = q(x, y, t)w^3(x, y, t)$	$V(k, h, m) = \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1)W_3(k - k_2, h - h_2, m - m_2).$

Then by using definition (13), we have:

$$\begin{aligned}
 V(k, h, m) &= \frac{1}{k!h!m!} \left[ \frac{\partial^{k+h+m}}{\partial x^k \partial y^h \partial t^m} w_1(x, y, t) w_2(x, y, t) \dots w_{n-1}(x, y, t) w_n(x, y, t) \right]_{(0,0,0)} \\
 &= \frac{1}{k!h!m!} \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{m_{n-1}=0}^m \frac{k!}{(k_{n-1})!(k-k_{n-1})!} \frac{h!}{(h_{n-1})!(h-h_{n-1})!} \frac{m!}{(m_{n-1})!(m-m_{n-1})!} \\
 &\quad \left[ \frac{\partial^{k_{n-1}+h_{n-1}+m_{n-1}}}{\partial x^{k_{n-1}} \partial y^{h_{n-1}} \partial t^{m_{n-1}}} w_1(x, y, t) w_2(x, y, t) \dots \right. \\
 &\quad \left. w_{n-1}(x, y, t) \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
 &= \sum_{k_{n-1}=0}^k \sum_{h_{n-1}=0}^h \sum_{m_{n-1}=0}^m \\
 &\quad \left[ \frac{1}{(k_{n-1})!(h_{n-1})!(m_{n-1})!} \frac{\partial^{k_{n-1}+h_{n-1}+m_{n-1}}}{\partial x^{k_{n-1}} \partial y^{h_{n-1}} \partial t^{m_{n-1}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(k-k_{n-1})!(h-h_{n-1})!(m-m_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
 &= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \\
 &\quad \left[ \frac{1}{(k_{n-2})!(h_{n-2})!(m_{n-2})!} \frac{\partial^{k_{n-2}+h_{n-2}+m_{n-2}}}{\partial x^{k_{n-2}} \partial y^{h_{n-2}} \partial t^{m_{n-2}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
 &\quad \left. \times \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(k-k_{n-1})!(h-h_{n-1})!(m-m_{n-1})!} \frac{\partial^{k-k_{n-1}+h-h_{n-1}+m-m_{n-1}}}{\partial x^{k-k_{n-1}} \partial y^{h-h_{n-1}} \partial t^{m-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)} \\
 &= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \\
 &\quad \left[ \frac{1}{(k_{n-3})!(h_{n-3})!(m_{n-3})!} \frac{\partial^{k_{n-3}+h_{n-3}+m_{n-3}}}{\partial x^{k_{n-3}} \partial y^{h_{n-3}} \partial t^{m_{n-3}}} w_1(x, y, t) w_2(x, y, t) \dots w_{n-3}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(k_{n-2}-k_{n-3})!(h_{n-2}-h_{n-3})!(m_{n-2}-m_{n-3})!} \right. \\
 &\quad \left. \times \frac{\partial^{k_{n-2}-k_{n-3}+h_{n-2}-h_{n-3}+m_{n-2}-m_{n-3}}}{\partial x^{k_{n-2}-k_{n-3}} \partial y^{h_{n-2}-h_{n-3}} \partial t^{m_{n-2}-m_{n-3}}} w_{n-2}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(k_{n-1}-k_{n-2})!(h_{n-1}-h_{n-2})!(m_{n-1}-m_{n-2})!} \right. \\
 &\quad \left. \times \frac{\partial^{k_{n-1}-k_{n-2}+h_{n-1}-h_{n-2}+m_{n-1}-m_{n-2}}}{\partial x^{k_{n-1}-k_{n-2}} \partial y^{h_{n-1}-h_{n-2}} \partial t^{m_{n-1}-m_{n-2}}} w_{n-1}(x, y, t) \right]_{(0,0,0)} \\
 &\quad \left[ \frac{1}{(1-k_{n-1})!(1-h_{n-1})!(1-m_{n-1})!} \frac{\partial^{1-k_{n-1}+1-h_{n-1}+1-m_{n-1}}}{\partial x^{1-k_{n-1}} \partial y^{1-h_{n-1}} \partial t^{1-m_{n-1}}} w_n(x, y, t) \right]_{(0,0,0)},
 \end{aligned}$$

$$\begin{aligned}
 V(k, h, m) &= \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{k_{n-3}=0}^{k_{n-2}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{h_{n-3}=0}^{h_{n-2}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \sum_{m_{n-3}=0}^{m_{n-2}} \dots \\
 &\quad \sum_{k_3=0}^{k_2} \sum_{k_2=0}^{k_1} \sum_{h_3=0}^{h_2} \sum_{h_2=0}^{h_1} \sum_{m_3=0}^{m_2} \sum_{m_2=0}^{m_1} \\
 &\quad W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots \\
 &\quad W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2}) W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).
 \end{aligned}$$

since  $v(x, y, t) = w_1(x, y, t)w_2(x, y, t) \dots w_{n-1}(x, y, t)w_n(x, y, t)$  then,

$$V(k, h, m) = \sum_{k_{n-1}=0}^k \sum_{k_{n-2}=0}^{k_{n-1}} \sum_{h_{n-1}=0}^h \sum_{h_{n-2}=0}^{h_{n-1}} \sum_{m_{n-1}=0}^m \sum_{m_{n-2}=0}^{m_{n-1}} \dots \sum_{k_3=0}^{k_2} \sum_{k_1=0}^{k_3} \sum_{h_3=0}^{h_2} \sum_{h_1=0}^{h_3} \sum_{m_3=0}^{m_2} \sum_{m_1=0}^{m_3} W_1(k_1, h_1, m_1)W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \dots W_{n-1}(k_{n-1} - k_{n-2}, h_{n-1} - h_{n-2}, m_{n-1} - m_{n-2})W_n(k - k_{n-1}, h - h_{n-1}, m - m_{n-1}).$$

**4. Application**

In this section, we apply the three-dimensional differential transform method to the vibration of a membrane. We demonstrate four examples of the problem under different conditions. The conditions are (i) without damping term, (ii) with damping term, (iii) with external force, and (iv) with damping term and external force. The initial and boundary conditions are defined as follows.

**Example 4.1 Consider the equation of the motion for the vibration of a membrane**

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}, \tag{15}$$

with the initial and boundary conditions,

$$\begin{aligned} w(x, y, 0) &= \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, & w(x, 0, t) &= 0, \quad 0 \leq x \leq 1, \\ w(0, y, t) &= 0, \quad 0 \leq y \leq 1, & w(x, 1, t) &= 0, \quad 0 \leq x \leq 1, \\ \frac{\partial w}{\partial t}(x, y, 0) &= 0, \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1, & w(1, y, t) &= 0, \quad 0 \leq y \leq 1, t \in \mathbb{R}. \end{aligned} \tag{16}$$

Comparing (15) to the general terms of PDEs in Table 1, we have:

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1.$$

Then for all  $i \geq 0, j \geq 0$  and  $r \geq 0$ , we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r).$$

The following Kronecker symbols can be used:

$$\delta(x, y, t) = \delta(x)\delta(y)\delta(t) = \begin{cases} 1; & x = y = t = 0 \\ 0; & \text{otherwise,} \end{cases} \quad \delta(x, y) = \delta(x)\delta(y).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have

$$\begin{aligned} W(k, h, m + 2) &= \frac{1}{(m + 2)(m + 1)\delta(0, 0, 0)} \\ &\times \left[ \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (k - i + 2)(k - i + 1)A(i, j, r)W(k - i + 2, h - j, m - r) \right. \\ &\left. + \sum_{i=0}^k \sum_{j=0}^h \sum_{r=0}^m (h - i + 2)(h - i + 1)B(i, j, r)U(k - i, h - j + 2, m - r) \right], \\ &= \frac{1}{(m + 2)(m + 1)}(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m), \end{aligned}$$

that is

$$W(k, h, m + 2) = \frac{[(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m)]}{(m + 2)(m + 1)(m + 2)(m + 1)}. \tag{17}$$

Comparing the coefficients of Taylor’s series for sine with the series in definition (14)

$$\sin(\pi x)\sin(\pi y) \cong \frac{2\pi^2}{2!}xt - \frac{4\pi^4}{4!}x^3t + \frac{6\pi^6}{6!}x^5t - \frac{8\pi^8}{8!}x^7t + \frac{10\pi^{10}}{10!}x^9t - \frac{12\pi^{12}}{12!}x^{11}t + \dots$$

$$\sum_{k=0}^{\infty} \sum_{h=0}^{\infty} W(k, h, 0)x^k y^h = W(0, 0, 0)x^0 y^0 + W(1, 1, 0)xy + W(2, 1, 0)x^2 y$$

$$+ W(3, 1, 0)x^3 y + \dots + W(0, 1, 0)xy + W(1, 2, 0)xy^2$$

$$+ W(2, 2, 0)x^2 y^2 + \dots \tag{18}$$

Then, we have

$$U(k, h, 0) = \begin{cases} 0 & \text{if } k = 0, 1, 2, \dots \text{ and } h = 0, 2, 4, 6, \dots \\ \frac{(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 1, 5, 9, \dots \text{ and } h = 1, 3, 5, 7, \dots \\ -\frac{(k+h)\pi^{k+h}}{(k+h)!} & \text{if } k = 3, 7, 11, \dots \text{ and } h = 1, 3, 5, 7, \dots \end{cases}, \tag{19}$$

and from the initial condition (16)

$$W(k, h, 1) = 0, \quad k, h = 0, 1, 2, 3, \dots, \tag{20}$$

and from the boundary condition (16), we have

$$\begin{aligned} W(k, 0, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots, \\ W(0, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots, \\ W(k, 1, m) &= 0, \quad k, m = 0, 1, 2, 3, \dots, \\ W(1, h, m) &= 0, \quad h, m = 0, 1, 2, 3, \dots, \end{aligned} \tag{21}$$

for each  $k, h, m$  substituting(19)-(21), and by recursion relation in (17), we obtain the coefficients  $W(k, h, m)$  for the series solution.

That is  $u(x, y, t) = \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6}\pi^6 t^4 xy - \frac{1}{90}\pi^8 t^6 xy - \frac{1}{6}\pi^4 x^3 y + \frac{1}{6}\pi^6 t^2 x^3 y + \dots$

Comparing to (14), we cannot apply separation of variables to find the analytical solution to the problem with damping in equation (22). In the following example, we apply the DTM to find an approximate solution to the problem.

**Example 4.2 Consider the equation of motion for the vibration of a membrane with damping term.**

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t}, \tag{22}$$

comparing (22) to general terms of PDEs in table 1, we have

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, d(x, y, t) = -1.$$

Then for all  $i \geq 0, j \geq 0$  and  $r \geq 0$ , we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), D(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have:

$$W(k, h, m + 2) = [(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m) - (h + 1)W(k, h + 1, m)] / (m + 2)(m + 1). \tag{23}$$

For each  $k, h, m$  substituting(19)-(21) and by recursion in (23), we obtain the coefficients  $W(k, h, m)$  for the series solution.

That is  $w(x, y, t) = -\frac{5}{2}\pi^2 t^2 x + \frac{5}{6}\pi^6 t^4 x + \frac{5}{12}\pi^4 t^2 x^3 - \frac{5}{36}\pi^6 t^4 x^3 - \frac{1}{48}\pi^6 t^2 x^5 + \frac{1}{144}\pi^8 t^4 x^5 + \dots$

**Example 4.3 Consider the equation of motion for the vibration of a membrane with external force.**

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - w^3. \tag{24}$$

Comparing (24) to general terms of PDEs in table 1, we have:

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, q(x, y, t) = -1.$$

Then for all  $i \geq 0, j \geq 0$  and  $r \geq 0$ , we have:

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), Q(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have

$$W(k, h, m + 2) = [(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m) - \sum_{k_2=0}^k \sum_{h_2=0}^h \sum_{m_2=0}^m W_1(k_1, h_1, m_1) W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) W_3(k - k_2, h - h_2, m - m_2)] / (m + 2)(m + 1). \tag{25}$$

For each  $k, h, m$  substituting (19)-(21) and by recursion relation in (25), we obtain the coefficients  $W(k, h, m)$  for the series solution.

That is:  $w(x, y, t) = \pi^2 xy - \pi^4 t^2 xy + \frac{1}{6}\pi^6 t^4 xy + \frac{1}{30}(-\frac{\pi^8}{6} + 3(-\frac{\pi^8}{18} - \frac{\pi^{10}}{36}))t^6 xy - \frac{1}{6}\pi^4 x^3 y + \frac{1}{6}\pi^6 t^2 x^3 y + \dots$

**Example 4.4 Consider the equation of the motion for the vibration of a membrane with external force and damping term.**

$$\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{\partial w}{\partial t} - w^3. \tag{26}$$

Comparing (26) to general terms of PDEs in table 1, we have

$$a(x, y, t) = b(x, y, t) = c(x, y, t) = 1, d(x, y, t) = q(x, y, t) = -1.$$

Then for all  $i \geq 0, j \geq 0$  and  $r \geq 0$ , we have

$$A(i, j, r) = B(i, j, r) = C(i, j, r) = \delta(i, j, r), D(i, j, r) = Q(i, j, r) = -\delta(i, j, r).$$

By applying DTM and Kronecker symbols to the given equation of motion for the vibration of a membrane, we have:

$$\begin{aligned}
 W(k, h, m + 2) = & [(k + 2)(k + 1)(h + 2)(h + 1)W(k + 2, h, m)W(k, h + 2, m) \\
 & - (h + 1)W(k, h + 1, m) \\
 & - \sum_{k_2=0}^k \sum_{k_1=0}^{k_2} \sum_{h_2=0}^h \sum_{h_1=0}^{h_2} \sum_{m_2=0}^m \sum_{m_1=0}^{m_2} W_1(k_1, h_1, m_1)W_2(k_2 - k_1, h_2 - h_1, m_2 - m_1) \\
 & W_3(k - k_2, h - h_2, m - m_2)] / (m + 2)(m + 1).
 \end{aligned}
 \tag{27}$$

For each  $k, h, m$  substituting (19)-(21) and by recursion in (27), we obtain the coefficients  $W(k, h, m)$  for the series solution.

That is:  $w(x, y, t) = -\frac{1}{2}\pi^2 t^2 x + 1/6\pi^4 t^4 x + \frac{1}{30}(-\pi^3) + \frac{1}{12}(\pi^2 - 2\pi^6)t^6 x + \frac{1}{12}\pi^4 t^2 x^3 - \frac{1}{36}\pi^6 t^4 x^3 + \dots$

**5. Results and Discussion**

Next, we will assess the results from examples 4.1 to– 4.4. In Figure 1, at the initial time, the approximate solution (dashed blue lines) obtained from DTM in Example 4.1 is close to the analytical solution (solid orange line) calculated from equation (13). As time progresses the results continue to stay close to each other.

The graphical results of the approximate solution from DTM of the vibration of a vibrating membrane with a damping term (Example 4.2) are shown in Figure 2. We found that the amplitude of vibration is less than that without a damping term, as shown in Table 2.

The exact solution, approximate solution with a damping term and error for the vibration of membrane are shown in Table 2.

The graphical results of the approximate solution from DTM of the vibration of a membrane with external force (Example 4.3) are shown in Figure 3. We found that the amplitude of vibration is less than that of the vibrating membrane without external force, as shown by the results in Table 3.

The graphical results of the approximate solution from DTM of the vibration a membrane with a damping term and external force (Example 4.4) are shown in Figure 4. We found that the amplitude of vibration is less than that of the vibrating membrane without a damping term and external force as shown by the results in Table 3.

The approximate solution for the vibration of a membrane with external force, with external force and a damping term, and error for the vibration of a membrane are shown in Table.

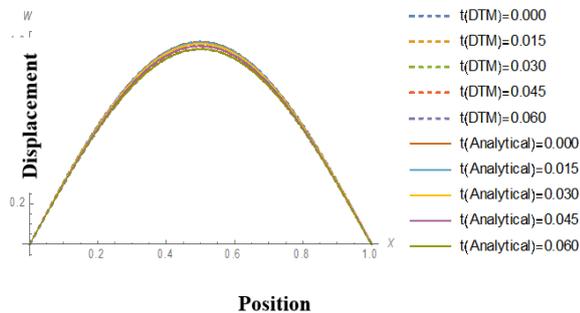


Figure 1. Graphical comparison between the amplitude of the analytical solution and the approximate solution from DTM of the vibration of a membrane.

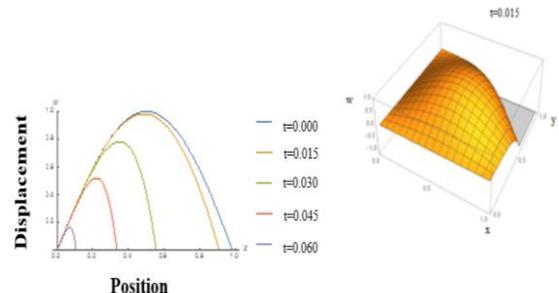


Figure 3. Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with an external force.

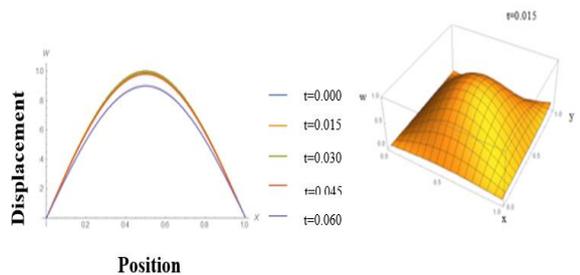


Figure 2. Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term.

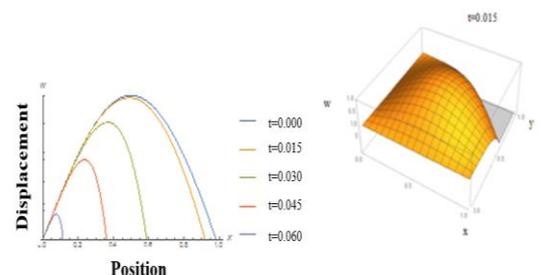


Figure 4. Graphical result of the amplitude of the approximate solution from DTM of the vibration of a membrane with a damping term and external force.

Table 2. Data Value at  $x=0.5$ ,  $y=0.5$ .

t(sec)	w(exact solution)	w(DTM)	w(damping)	$ w(\text{exact})-w(\text{DTM}) $	$ w(\text{DTM})-w(\text{damping}) $
$t=0.000$	1.00000	1.000101	1.0001078	0.000101	0.0007806
$t=0.015$	0.99113	0.997883	0.9970827	0.000103	0.0008003
$t=0.030$	0.980081	0.991239	0.9780586	0.000109	0.0131804
$t=0.045$	0.980081	0.980198	0.9133756	0.000117	0.0668224
$t=0.060$	0.964679	0.96481	0.7544184	0.000149	0.2103916

Table 3. Data Value at  $x=0.5$ ,  $y=0.5$ .

t(sec)	w(DTM)	w(external force)	w(damping+external force)	$ w(\text{DTM})-w(\text{external force}) $	$ w(\text{DTM})-w(\text{damping+external force}) $
$t=0.000$	1.00000	0.999787	0.999787	0.000313	0.000313
$t=0.015$	0.99113	0.9810904	0.9834975	0.0167926	0.0143855
$t=0.030$	0.980081	0.4118883	0.5655184	0.5793507	0.4257206
$t=0.045$	0.980081	-5.3055571	-3.556495	6.2857551	4.536693
$t=0.060$	0.902917	-34.028812	-24.203159	34.9936223	25.1679693

## 6. Conclusions

We found that the analytical solutions and approximate solutions are very similar in case of a problem with no damping or external force. In the case with damping and external force, analytical solutions are not available. Therefore, DTM was used to find approximate solutions. The obtained results show that the addition of a damping term, or addition of external force to the equation of motion of the membrane, can reduction the amplitude of membrane vibrations.

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