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Algebraic, order, and analytic properties of Tracy-Singh sums for Hilbert space operators

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Abstract

We introduce the Tracy-Singh sum for operators on a Hilbert space, generalizing both the Tracy-Singh sum for matrices and the tensor sum for operators. The Tracy-Singh sum is shown to be compatible with algebraic operations and order relations. Then we establish a binomial theorem involving Tracy-Singh sums, and its consequences. We also investigate continuity, convergence, and norm bounds for Tracy-Singh sums. Moreover, we derive operator identities involving Tracy-Singh sums and certain operator functions defined by power series.

Keywords: operator matrix, tensor product, Tracy-Singh product, Tracy-Singh sum, analytic functions of operators

1. Introduction

It is well known that the Kronecker product and the Tracy-Singh product for matrices possess pleasing properties in algebraic, order, and analytic viewpoints. Such matrix products have applications in various fields, including quantum physics, thermodynamics, control and system theory, signal processing, image compression, computer science, statistics. See, e.g., Steeb and Hardy (2011) for an excellent source on this subject.

Let M_n denote the set of n-by-n complex matrices for each natural number n. For any $A \in M_m$ and $B \in M_n$, the Kronecker product of A and B is given by

$$A \hat{\otimes} B = \left[a_{ij} B \right]_{ij} . \tag{1}$$

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This matrix product was generalized to the Tracy-Singh product of partitioned matrices in Tracy and Singh (1972). Let $A = \left[A_{ij} \right]$ be partitioned with A_{ij} of order $m_i \times m_j$ as the (i, j) th submatrix and let $B = \left[B_{kl} \right]$ be a partitioned matrix with B_{kl} of order $n_k \times n_l$ as the (k, l) th submatrix, where $\sum_{i=1}^{r} m_i = m$ and $\sum_{k=1}^{s} n_k = n$. The Tracy-Singh product of A and B is defined by

$$A \hat{\boxtimes} B = \left[\left[A_{ij} \hat{\otimes} B_{kl} \right]_{kl} \right]_{ij} . \tag{2}$$

Tracy-Singh products for matrices have been developed by many authors in both theory and applications, see e.g., Tracy and Singh (1972), Tracy and Jinadasa (1989), Shuangzhe (1999). The Tracy-Singh sum, introduced by Al Zhour and Kilicman (2006), is defined for each $A \in M_n$ and $B \in M_m$ by

$$A \hat{\boxplus} B = A \hat{\boxtimes} I_n + I_m \hat{\boxtimes} B , \tag{3}$$

where I_m and I_n are block identity matrices of order $m \times m$ and $n \times n$, respectively. If both A and B consist of only one block, then (3) reduces to the Kronecker sum.

The notion of Kronecker product (sum) for matrices was then generalized to tensor product (sum) of bounded linear operators on a Hilbert space. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. The tensor product of two bounded linear operators $A : \mathcal{H} \to \mathcal{H}$ and $B : \mathcal{K} \to \mathcal{K}$ is the unique bounded linear operator from $\mathcal{H} \otimes \mathcal{K}$ into itself such that

$$(A \otimes B)(x \otimes y) = Ax \otimes By \tag{4}$$

for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. The theory of tensor product for operators was focused by many authors from the past until nowadays, see e.g. Zanni and Kubrusly (2015), Kubrusly and Levan (2011). The tensor sum for operators was investigated in Kubrusly and Levan (2011).

Recently, the notion of Tracy-Singh product for matrices was generalized to Tracy-Singh product for Hilbert space operators; see Ploymukda, Chansangiam, and Lewkee ratiyutkul (2018a). To define the Tracy-Singh product, we apply the projection theorem for Hilbert spaces to decompose

$$\mathcal{H} = \bigoplus_{i=1}^{m} \mathcal{H}_i \quad \text{and} \quad \mathcal{K} = \bigoplus_{k=1}^{n} \mathcal{K}_l$$

where all $\mathcal{H}_i$'s and $\mathcal{K}_k$'s are Hilbert spaces. Thus every bounded linear operator $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ can be expressed uniquely as operator matrices

$$A = \left[A_{ij} \right]_{i,j=1}^{m,m} \quad \text{and} \quad B = \left[B_{kl} \right]_{k,l=1}^{n,n}$$

where $A_{ij} \in \mathbb{B}(\mathcal{H}_j, \mathcal{H}_i)$ and $B_{kl} \in \mathbb{B}(\mathcal{K}_l, \mathcal{K}_k)$ for each i, j, k, l.

Definition 1. According to the above notations, the Tracy-Singh product of A and B is defined to be the operator matrix

$$A \boxtimes B = \left[\left[A_{ij} \otimes B_{kl} \right]_{kl} \right]_{ij}, \tag{5}$$

which is a bounded linear operator from $\bigoplus_{i,k=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_k$ into itself. Note that when $m = n = 1$, the Tracy-Singh product $A \boxtimes B$ reduces to the tensor product $A \otimes B$. Algebraic, order, and analytic properties of the Tracy-Singh product of operators were investigated in Ploymukda, Chansangiam, and Lewkeeratiyutkul (2018a) and (2018b).

In this paper, we generalize the notions of Tracy-Singh sum for matrices and the tensor sum of operators to the Tracy-Singh sum for operators. This kind of operator sum turns out to be compatible with certain algebraic operations and operator orderings. Binomial theorem involving Tracy-Singh sums and its consequences are then established. Moreover, we investigate continuity, convergence, and norm bounds for Tracy-Singh sums of operators. Finally, we derive

operator identities concerning Tracy-Singh sums and certain operator functions defined by power series.

This paper is organized as follows. In Section 2, we provide preliminaries on the Tracy-Singh product for operators. We introduce the Tracy-Singh sum for operators in Section 3 and then investigate its algebraic and order properties. In Section 4, we prove the binomial theorem involving Tracy-Singh sums and deduce some consequences. Analytic properties of Tracy-Singh sums are presented in Section 5.

2. Preliminaries on Tracy-Singh Products for Operators

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be complex Hilbert spaces. When $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces, denote by $\mathbb{B}(\mathcal{X}, \mathcal{Y})$ the Banach space of bounded linear operators from $\mathcal{X}$ into $\mathcal{Y}$, and abbreviate $\mathbb{B}(\mathcal{X}, \mathcal{X})$ to $\mathbb{B}(\mathcal{X})$. Unless otherwise stated, capital letters mean operators on a Hilbert space. In particular, I stands for the identity operator. The positive-semidefinite ordering between two Hermitian operators A and B in $\mathbb{B}(\mathcal{H})$ is defined as follows: $A \leq B$ means that $B - A$ is a positive operator.

Fundamental properties of the Tracy-Singh product for operators are summarized here; see Ploymukda, Chansangiam, and Lewkeeratiyutkul (2018a, 2018b). Recall that each $A \in M_m$ corresponds to a bounded linear operator $L_A : \mathbb{C}^m \to \mathbb{C}^m$, $x \mapsto Ax$.

Lemma 2. For complex matrices $A \in M_m$ and $B \in M_n$, we have

$$L_A \boxtimes L_B = L_{A \boxtimes B}. \tag{6}$$

Proposition 3. Let $A = \left[A_{ij} \right] \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices. Then

$$A \boxtimes B = \left[A_{ij} \boxtimes B \right]_{ij} = \begin{bmatrix} A_{11} \boxtimes B & \cdots & A_{1n} \boxtimes B \\ \vdots & \ddots & \vdots \\ A_{n1} \boxtimes B & \cdots & A_{nn} \boxtimes B \end{bmatrix}.$$

Lemma 4. The following properties hold, provided that all operators are compatible.

$$\left(A \boxtimes B \right)^* = A^* \boxtimes B^*, \tag{7}$$

$$\left(\alpha A \right) \boxtimes B = \alpha \left(A \boxtimes B \right) = A \boxtimes \left(\alpha B \right), \tag{8}$$

$$A \boxtimes \left(B + C \right) = A \boxtimes B + A \boxtimes C, \tag{9}$$

$$\left(B + C \right) \boxtimes A = B \boxtimes A + C \boxtimes A, \tag{10}$$

$$\left(A \boxtimes B \right)\left(C \boxtimes D \right) = AC \boxtimes BD. \tag{11}$$

Lemma 5. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If $A, B \geq 0$, then $A \boxtimes B \geq 0$.

The next lemma asserts the continuity of the map $(A, B) \mapsto A \boxtimes B$.

Lemma 6. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxtimes B_r \to A \boxtimes B$.

Lemma 7. For any operator $A = \left[A_{ij} \right]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{kl} \right]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$, we have

$$\frac{1}{mn} \|A\| \|B\| \ \leq \ \|A \boxtimes B\| \ \leq \ mn \|A\| \|B\|. \tag{12}$$

The next lemma is a direct consequence of Lemma 7.

Lemma 8. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then $A \boxtimes B = 0$ if and only if $A = 0$ or $B = 0$.

Lemma 9. Let $A \in \mathbb{B}(\mathcal{H})$.

(i) If f is an analytic function on a region containing the spectra of A and $I \boxtimes A$, then $f(I \boxtimes A) = I \boxtimes f(A)$.

(ii) If f is an analytic function on a region containing the spectra of A and $A \boxtimes I$, then $f(A \boxtimes I) = f(A) \boxtimes I$.

3. Algebraic and Order Properties of Tracy-Singh Sums for Operators

In this section, we define the Tracy-Singh sum for Hilbert space operators and investigate its algebraic and order properties. It turns out that the Tracy-Singh sum is compatible with the adjoint operation, the scalar multiplication, the ordinary sum and commutators. Moreover, positivity and positive-semidefinite ordering of operators are preserved by Tracy-Singh sums.

Definition 10. Let $A = \left[A_{ij} \right]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{kl} \right]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$. We define the **Tracy-Singh sum** of A and B as follows:

$$A \boxplus B = A \boxtimes I_{\mathcal{K}} + I_{\mathcal{H}} \boxtimes B, \tag{13}$$

which belongs to $\mathbb{B}\left(\bigoplus_{i,j=1}^{m,n} \mathcal{H}_i \otimes \mathcal{K}_j \right)$.

Note that if both A and B are 1 X 1 block operator matrices i.e. $m = n = 1$, their Tracy-Singh sum $A \boxplus B$ is known as the tensor sum (Kubrusly & Levan, 2011)

$$A \oplus B = A \otimes I_{\mathcal{K}} + I_{\mathcal{H}} \otimes B.$$

The next result asserts that the Tracy-Singh sum of two linear maps induced by matrices is just the linear map induced by the Tracy-Singh sum of those matrices.

Proposition 11. Let $A = \left[A_{ij} \right] \in M_n(\mathbb{C})$ and let $B = \left[B_{kl} \right] \in M_m(\mathbb{C})$ be complex partitioned matrices. Then

$$L_A \boxplus L_B \ = \ L_{A \hat{\boxplus} B}. \tag{14}$$

It follows that if $\mathcal{E}$ is a standard ordered basis of M_{mn}, then

$$[L_A \boxplus L_B]_{\mathcal{E}} = A \hat{\boxplus} B.$$

Proof. We know that the linear map induced by the identity matrix is the identity operator. By applying Lemma 2, we get

$$\begin{aligned} L_A \boxplus L_B \ &= \ L_A \boxtimes I + I \boxtimes L_B \ = \ L_A \boxtimes L_I + L_I \boxtimes L_B \\ &= \ L_{A \hat{\boxplus} I} + L_{I \hat{\boxplus} B} \ = \ L_{A \hat{\boxplus} I + I \hat{\boxplus} B} \\ &= \ L_{A \hat{\boxplus} B}. \end{aligned}$$

Proposition 11 says that the matrix representation of the Tracy-Singh sum of linear maps induced by two matrices with respect to the standard ordered basis is just the Tracy-Singh sum of these matrices.

Proposition 12. Let α and β be complex scalars. Then for any $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$, we have

$$(A \boxplus B)^* \ = \ A^* \boxplus B^*, \tag{15}$$

$$\alpha(A \boxplus B) \ = \ (\alpha A) \boxplus (\alpha B), \tag{16}$$

$$(\alpha + \beta)(A \boxplus B) \ = \ \alpha A \boxplus \beta B + \beta A \boxplus \alpha B, \tag{17}$$

$$(A + C) \boxplus (B + D) \ = \ A \boxplus B + C \boxplus D. \tag{18}$$

We call (18) the mixed sum property.

Proof. Using Lemma 4, we obtain (15)-(17). By applying properties (9) and (10) of Lemma 4, we get

$$\begin{aligned} (A + C) \boxplus (B + D) \ &= \ (A + C) \boxtimes I + I \boxtimes (B + D) \\ &= \ A \boxtimes I + C \boxtimes I + I \boxtimes B + I \boxtimes D \\ &= \ (A \boxtimes I + I \boxtimes B) + (C \boxtimes I + I \boxtimes D) \\ &= \ A \boxplus B + C \boxplus D. \end{aligned}$$

It follows from (15) that if A and B are Hermitian (skew-Hermitian), then so is $A \boxplus B$. The mixed sum property (18) implies that if $A_1 = X_1 + iY_1$ and $A_2 = X_2 + iY_2$ are the Cartesian decompositions of A_1 and A_2, respectively, then $A_1 \boxplus A_2 = X_1 \boxplus X_2 + i(Y_1 \boxplus Y_2)$ is the Cartesian decomposition of $A_1 \boxplus A_2$.

Lemma 13. Let $A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B_1, B_2 \in \mathbb{B}(\mathcal{K})$ be nonzero operators. Then $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $A_1 = \alpha A_2$ and $B_1 = \alpha^{-1} B_2$.

Proof. See Proposition 2.1 of Stochel (1996).

Theorem 14. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then $A \boxplus B = 0$ if and only if there exists $\alpha \in \mathbb{C} \setminus \{0\}$ such that $A = \alpha I$ and $B = -\alpha I$.

Proof. If $A = \alpha I$ and $B = -\alpha I$, then $A \boxplus B = 0$ by Lemma 4. Now, suppose that $A \boxplus B = 0$. Then by Lemma 3,

$$
0 = A \boxplus B = \begin{bmatrix} A_{11} \boxplus B & A_{12} \boxtimes I & \cdots & A_{1m} \boxtimes I \\ A_{21} \boxtimes I & A_{22} \boxplus B & \cdots & A_{2m} \boxtimes I \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} \boxtimes I & A_{m2} \boxtimes I & \cdots & A_{mm} \boxplus B \end{bmatrix}.
$$

For any $i, j \in \{1,\ldots,m\}$ such that $i \neq j$, since $A_{ij} \boxtimes I = 0$, by Lemma 8 we can deduce $A_{ij} = 0$. Let $i \in \{1,\ldots,m\}$ and consider

$$
\begin{bmatrix} A_{ii} \oplus B_{11} & I \otimes B_{12} & \cdots & I \otimes B_{1n} \\ I \otimes B_{21} & A_{ii} \oplus B_{22} & \cdots & I \otimes B_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ I \otimes B_{n1} & I \otimes B_{n2} & \cdots & A_{ii} \oplus B_{nn} \end{bmatrix} = 0.
$$

Lemma 8 implies that $B_{kl} = 0$ for any $k \neq l$. For $k \in \{1,\ldots,n\}$, we have $A_{ii} \oplus B_{kk} = 0$ if and only if $A_{ii} \otimes (-I) = I \otimes B_{kk}$. If $A_{ii} \otimes (-I) = I \otimes B_{kk}$, then $I \otimes B_{kk} = 0$ and hence $B_{kk} = 0$ for all $k = 1,\ldots,n$ by Lemma 8. It follows that $B = 0$ and $A \boxtimes I = A \boxplus B = 0$. Thus $A = 0$ by Lemma 8. Assume that $A_{ii} \neq 0$ for all $i = 1,\ldots,m$. By Lemma 13, there exists $\alpha_{ik} \in \mathbb{C} \setminus \{0\}$ such that $A_{ii} = \alpha_{ik} I$ and $-I = \alpha_{ik}^{-1} B_{kk}$. For any fixed i, we have $A_{ii} = \alpha_{ik} I$ for all $k = 1,\ldots,n$. Hence, $\alpha_{i1} = \alpha_{i2} = \ldots = \alpha_{in} = \alpha_i$. For any fixed $k \in \{1,\ldots,n\}$, we have $B_{kk} = -\alpha_i I$ for all $i = 1,\ldots,m$. This implies that $\alpha_1 = \alpha_2 = \ldots = \alpha_m = \alpha$. Thus $A_{ii} = \alpha I$ for all $i = 1,\ldots,m$ and $B_{kk} = -\alpha I$ for all $k = 1,\ldots,n$. Therefore $A = \alpha I$ and $B = -\alpha I$.

The next result provides formulas for the inverses of $A \boxplus B$ in terms of Tracy-Singh products.

Proposition 15. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be invertible operators. If $A \boxplus B$ and $A^{-1} \boxplus B^{-1}$ are invertible, then

$$
(A \boxplus B)^{-1} = \left(A^{-1} \boxtimes I\right)\left(A^{-1} \boxplus B^{-1}\right)^{-1}\left(I \boxtimes B^{-1}\right), \quad (19)
$$

$$
(A \boxplus B)^{-1} = \left(I \boxtimes B^{-1}\right)\left(A^{-1} \boxplus B^{-1}\right)^{-1}\left(A^{-1} \boxtimes I\right), \quad (20)
$$

$$
(A \boxplus B)^{-1} = \left(A^{-1} \boxplus B^{-1}\right)^{-1}\left(A^{-1} \boxtimes B^{-1}\right). \quad (21)
$$

Proof. Applying Lemma 4, we have

$$
\begin{aligned}
(A \boxplus B)^{-1} &= (A \boxtimes I + I \boxtimes B)^{-1} \\
&= \left(A^{-1} \boxtimes I\right)\left[\left(I \boxtimes B + A \boxtimes I\right)\left(A^{-1} \boxtimes I\right)\right]^{-1} \\
&= \left(A^{-1} \boxtimes I\right)\left[A^{-1} \boxtimes B + I \boxtimes I\right]^{-1} \\
&= \left(A^{-1} \boxtimes I\right)\left[\left(I \boxtimes B^{-1}\right)\left(A^{-1} \boxtimes B + I \boxtimes I\right)\right]^{-1}\left(I \boxtimes B^{-1}\right) \\
&= \left(A^{-1} \boxtimes I\right)\left[A^{-1} \boxtimes I + I \boxtimes B^{-1}\right]^{-1}\left(I \boxtimes B^{-1}\right) \\
&= \left(A^{-1} \boxtimes I\right)\left(A^{-1} \boxplus B^{-1}\right)^{-1}\left(I \boxtimes B^{-1}\right).
\end{aligned}
$$

Similarly, we obtain the properties (20) and (21).

Recall that the **commutator** and the **anticommutator** of $A, B \in \mathbb{B}(\mathcal{H})$ are defined respectively by

$$
[A, B] = AB - BA,
$$

$$
[A, B]_+ = AB + BA.
$$

Proposition 16. Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$. Then

$$
[A \boxplus B, C \boxplus D] = [A, C] \boxplus [B, D]. \quad (22)
$$

Proof. This follows directly from Lemma 4.

Recall that an operator $A \in \mathbb{B}(\mathcal{H})$ is said to be normal if $[A^*, A] = 0$. It follows from Propositions 12 and 16 that if $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ are normal, then so is $A \boxplus B$. The next result shows that the Tracy-Singh sum preserves positivity and order.

Proposition 17. Let $A, A_1, A_2 \in \mathbb{B}(\mathcal{H})$ and $B, B_1, B_2 \in \mathbb{B}(\mathcal{K})$.
(i)   If $A \geq 0$ and $B \geq 0$, then $A \boxplus B \geq 0$.
(ii) If $A_1 \geq A_2$ and $B_1 \geq B_2$, then $A_1 \boxplus B_1 \geq A_2 \boxplus B_2$.

Proof. This follows directly from Lemma 5 and Proposition 12.

Proposition 18. Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$ be positive operators. Then

$$
(A \boxplus B)(C \boxplus D) \geq AC \boxplus BD. \quad (23)
$$

Proof. Applying Lemma 4, we have

$$(A \boxplus B)(C \boxplus D) = AC \boxplus BD + A \boxtimes D + C \boxtimes B.$$

Since $A \boxtimes D + C \boxtimes B$ is positive, we get the result.

Corollary 19. Let $A, C \in \mathbb{B}(\mathcal{H})$ and $B, D \in \mathbb{B}(\mathcal{K})$ be positive operators. Then

$$\left[A \boxplus B, C \boxplus D \right]_+ \geq \left[A, C \right]_+ \boxplus \left[B, D \right]_+. \tag{24}$$

Proof. This follows directly from Propositions 12 and 18.

4. Binomial Theorem Involving Tracy-Singh Sums and its Consequences

In this section, we prove an operator version of the binomial theorem in which the sum and the product are replaced by the Tracy-Singh sum and the Tracy-Singh product, respectively. Consequently, we obtain certain operator inequalities, including a Bernoulli-type inequality. Binomial theorem is also used to treat the nilpotency of the Tracy-Singh sum of two operators.

Theorem 20. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be compatible operator matrices. Then for any integer $r \geq 2$,

$$(A \boxplus B)^r = A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k} \left(A^{r-k} \boxtimes B^k \right). \tag{25}$$

Proof. By using Lemma 4, we have

$$\begin{aligned}
(A \boxplus B)^2 &= \left(A \boxtimes I + I \boxtimes B \right)\left(A \boxtimes I + I \boxtimes B \right) \\
&= A^2 \boxtimes I + 2A \boxtimes B + I \boxtimes B^2 \\
&= A^2 \boxplus B^2 + 2A \boxtimes B.
\end{aligned}$$

This show that (25) is true for $r = 2$. Suppose that (25) holds for an integer $r \geq 2$. Then, by Lemma 4,

$$\begin{aligned}
(A \boxplus B)^{r+1} &= (A \boxplus B)^r (A \boxplus B) \\
&= \left(A^r \boxplus B^r + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \boxtimes B^k \right) \right)(A \boxplus B) \\
&= \left(A^r \boxplus B^r \right)(A \boxplus B) + \left(\sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \boxtimes B^k \right) \right)(A \boxplus B) \\
&= \left(A^r \boxtimes I \right)\left(A \boxtimes I \right) + \left(A^r \boxtimes I \right)\left(I \boxtimes B \right) + \left(I \boxtimes B^r \right)\left(A \boxtimes I \right) \\
&\quad + \left(I \boxtimes B^r \right)\left(I \boxtimes B \right) + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \boxtimes B^k \right)\left(A \boxtimes I \right) \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \boxtimes B^k \right)\left(I \boxtimes B \right) \\
&= A^{r+1} \boxtimes I + A^r \boxtimes B + A \boxtimes B^r + I \boxtimes B^{r+1} \\
&\quad + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k+1} \boxtimes B^k \right) + \sum_{k=1}^{r-1} \binom{r}{k}\left(A^{r-k} \boxtimes B^{k+1} \right)
\end{aligned}$$

$$\begin{aligned}
&= A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^{r} \binom{r}{k}\left(A^{(r+1)-k} \boxtimes B^k \right) \\
&\quad + \sum_{k=1}^{r} \binom{r}{k-1}\left(A^{(r+1)-k} \boxtimes B^k \right) \\
&= A^{r+1} \boxplus B^{r+1} + \sum_{k=1}^{r} \binom{r+1}{k}\left(A^{(r+1)-k} \boxtimes B^k \right).
\end{aligned}$$

Corollary 21. Let $A \geq 0$ and $B \geq 0$ be compatibly operator matrices. Then

$$(A \boxplus B)^r \geq A^r \boxplus B^r \tag{26}$$

for any $r \in \mathbb{N}$.

Proof. This follows immediately from Theorem 20 and Lemma 5.

The next result is a Bernoulli type inequality concerning Tracy-Singh sums.

Corollary 22. Let A be a positive operator. Then for any $r \in \mathbb{N}$,

$$(I \boxplus A)^r \geq I \boxplus (rA). \tag{27}$$

Proof. Since $A \geq 0$, we have $I \boxplus A^r \geq I \boxtimes I$. By Theorem 21, we obtain

$$\begin{aligned}
(I \boxplus A)^r &= I \boxplus A^r + \sum_{k=1}^{r-1} \binom{r}{k}\left(I \boxtimes A^k \right) \\
&= I \boxplus A^r + r\left(I \boxtimes A \right) + \sum_{k=2}^{r-1} \binom{r}{k}\left(I \boxtimes A^k \right) \\
&\geq I \boxplus A^r + r\left(I \boxtimes A \right) \\
&\geq I \boxtimes I + I \boxtimes (rA) \\
&= I \boxplus (rA).
\end{aligned}$$

Corollary 23. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. If A and B are nilpotent, then $A \boxplus B$ is also nilpotent.

Proof. Suppose $A^r = 0$ and $B^s = 0$ for some $r, s \in \mathbb{N}$. From Theorem 20, we get

$$\begin{aligned}
(A \boxplus B)^{r+s} &= \sum_{k=0}^{r+s} \binom{r+s}{k}\left(A^{r+s-k} \boxtimes B^k \right) \\
&= \sum_{k=0}^{s-1} \binom{r+s}{k}\left(A^{r+s-k} \boxtimes B^k \right) + \sum_{k=s}^{r+s} \binom{r+s}{k}\left(A^{r+s-k} \boxtimes B^k \right).
\end{aligned}$$

If $k \geq s$, then $B^k = 0$. If $k < s$, then $r + s - k > r$ and hence $A^k = 0$. Now,

$$(A \boxplus B)^{r+s} = \sum_{k=0}^{s-1} \binom{r+s}{k}(0 \boxtimes B^k) + \sum_{k=s}^{r+s} \binom{r+s}{k}\left(A^{r+s-k} \boxtimes 0\right)$$

$$= \sum_{k=0}^{s-1} \binom{r+s}{k}(0 \boxtimes B)^k + \sum_{k=s}^{r+s} \binom{r+s}{k}(A \boxtimes 0)^{r+s-k}$$

$$= 0.$$

This means that $A \boxplus B$ is nilpotent.

5. Analytic Properties of Tracy-Singh Sums

In this section, we investigate continuity, convergence, and norm bounds for Tracy-Singh sums of operators. We also discuss the relationship between Tracy-Singh sums and certain functions of operators defined by power series.

Proposition 24. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$ be operator matrices, and let $(A_r)_{r=1}^{\infty}$ and $(B_r)_{r=1}^{\infty}$ be sequences in $\mathbb{B}(\mathcal{H})$ and $\mathbb{B}(\mathcal{K})$, respectively. If $A_r \to A$ and $B_r \to B$, then $A_r \boxplus B_r \to A \boxplus B$.

Proof. By Lemma 6, we have $A_r \boxtimes I \to A \boxtimes I$ and $I \boxtimes B_r \to I \boxtimes B$. Thus $A_r \boxtimes I + I \boxtimes B_r \to A \boxtimes I + I \boxtimes B$, i.e. $A_r \boxplus B_r \to A \boxplus B$.

Proposition 24 asserts that the Tracy-Singh sum is (jointly) continuous with respect to the operator-norm topology. The next result is a triangle-like inequality for Tracy-Singh sums.

Proposition 25. For any $A = \left[A_{ij}\right]_{i,j=1}^{m,m} \in \mathbb{B}(\mathcal{H})$ and $B = \left[B_{kl}\right]_{k,l=1}^{n,n} \in \mathbb{B}(\mathcal{K})$, we have

$$\frac{1}{mn}\|A \boxplus B\| \leq \|A\| + \|B\|. \tag{28}$$

Proof. This follows from the norm estimate in Lemma 7.

In the rest of paper, we shall establish operator identities involving Tracy-Singh sums and functions of operators defined by power series. Recall that for any $T \in \mathbb{B}(\mathcal{H})$, we define

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k, \tag{29}$$

$$\sin(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} T^{2k+1}, \tag{30}$$

$$\cos(T) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} T^{2k}, \tag{31}$$

$$\sinh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} T^{2k+1}, \tag{32}$$

$$\cosh(T) = \sum_{k=0}^{\infty} \frac{1}{(2k)!} T^{2k}. \tag{33}$$

The series on the right hand sides of (29)-(33) converge in the norm topology of $\mathbb{B}(\mathcal{H})$. If T is positive and invertible, we define $\log T$ to be the operator X such that $e^X = T$. The following facts are well-known.

Lemma 26. For any $S, T \in \mathbb{B}(\mathcal{H})$ satisfying $ST = TS$, we have

$$e^{S+T} = e^S e^T,$$

$$\sin(S+T) = \sin S \cos T + \cos S \sin T,$$

$$\cos(S+T) = \cos S \cos T - \sin S \sin T,$$

$$\sinh(S+T) = \sinh S \cosh T + \cosh S \sinh T,$$

$$\cosh(S+T) = \cosh S \cosh T - \sinh S \sinh T.$$

If S and T are invertible positive operators such that $ST = TS$, then

$$\log ST = \log S + \log T.$$

Theorem 27. Let $A \in \mathbb{B}(\mathcal{H})$ and $B \in \mathbb{B}(\mathcal{K})$. Then

$$e^{A \boxplus B} = e^A \boxtimes e^B, \tag{34}$$

$$\sin(A \boxplus B) = \sin A \boxtimes \cos B + \cos A \boxtimes \sin B, \tag{35}$$

$$\cos(A \boxplus B) = \cos A \boxtimes \cos B - \sin A \boxtimes \sin B, \tag{36}$$

$$\sinh(A \boxplus B) = \sinh A \boxtimes \cosh B + \cosh A \boxtimes \sinh B, \tag{37}$$

$$\cosh(A \boxplus B) = \cosh A \boxtimes \cosh B - \sinh A \boxtimes \sinh B. \tag{38}$$

If A and B are invertible positive operators, then

$$\log(A \boxtimes B) = \log A \boxplus \log B. \tag{39}$$

Proof. By property (11) in Lemma 4, we have $(A \boxtimes I)(I \boxtimes B) = (I \boxtimes B)(A \boxtimes I)$. It follows that

$$e^{A \boxplus B} = e^{A \boxtimes I + I \boxtimes B} = e^{A \boxtimes I} e^{I \boxtimes B} \quad \text{(by Lemma 26)}$$

$$= \left(e^A \boxtimes I\right)\left(I \boxtimes e^B\right) \quad \text{(by Lemma 9)}$$

$$= e^A \boxtimes e^B \quad \text{(by Lemma 4, property (11)).}$$

Similarly, we get the identities (35)-(39).

Our results in this paper suggest that the Tracy-Singh sum is a "sum", while the Tracy-Singh product is a "product" for operators.

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