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Original Article

Rough statistical convergence on triple sequence of the Bernstein operator of random variables in probability

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Abstract

This paper aims to improve further on the work of Phu (2001), Aytar (2008), and Ghosal (2013). We propose a new apporach to extend the application area of rough statistical convergence usually used in triple sequence of the Bernstein operator of real numbers to the theory of probability distributions. The introduction of this concept in the probability of Bernstein polynomials of rough statistical convergence, Bernstein polynomials of rough strong Cesàro summable, Bernstein polynomials of rough lacunary statistical convergence, Bernstein polynomials of rough $N_\theta -$ convergence, Bernstein polynomials of rough $\lambda -$ statistical convergence, and Bernstein polynomials of rough strong $(V, \lambda) -$ summable to generalize the convergence analysis to accommodate any form of distribution of random variables. Among these six concepts in probability only three convergences are distinct Bernstein polynomials of rough statistical convergence: (1) Bernstein polynomials of rough lacunary statistical convergence, (2) Bernstein polynomials of rough $\lambda -$ statistical convergence where Bernstein polynomials of rough strong Cesàro summable is equivalent to Bernstein polynomials of rough statistical convergence, and (3) Bernstein polynomials of rough $N_\theta -$ convergence which is equivalent to Bernstein polynomials of rough lacunary statistical convergence. Bernstein polynomials of rough strong $(V, \lambda) -$ summable is equivalent to Bernstein polynomials of rough $\lambda -$ statistical convergence. Basic properties and interrelations of these three distinct convergences are investigated and some observations were made in these classes and in this way we demonstrated that rough statistical convergence in probability is the more generalized concept than the usual Bernstein polynomials of rough statistical convergence.

Keywords: rough statistical convergence, rough strong Cesàro summable, rough lacunary statistical convergence, rough $N_\theta -$ convergence, rough λ- statistical convergence, rough strong (V,λ)- summable, Bernstein polynomials

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1. Introduction

In probability theory, a new type of convergence called statistical convergence of random variables in pro-bability was introduced in Ghosal (2013). Let $(B_{mnk}(f,x))$ be a triple sequence of Bernstein operator of random variables where each $(B_{mnk}(f,x))$ is defined on the same sample spaces W (for each (m,n,k)) with respect to a given class of events Δ and a given probability function $P:\Delta \to \mathbb{R}^3$. Then the triple sequence of Bernstein operator of $(B_{mnk}(f,x))$ is said to be statistical convergence in probability to a random variable $(B_{mnk}(f,x)): W \to \mathbb{R}^3$ if for any $\varepsilon, \delta > 0$

$$lim_{uvw\to\infty}\frac{1}{uvw}|\{m \le u, n \le v, k \le w: P(|B_{mnk}(f,x) - f(x)| \ge \delta)\}| = 0.$$

In this case we write $B_{mnk}(f,x) \to^{S^P} f(x)$. The class of all triple sequences of the Bernstein opeator of random variables which are statistical convergence in probability is denoted by S^P.

In this paper we introduce new notions namely Bernstein polynomials of rough statistical convergence in probability, Bernstein polynomials of rough strong Cesàro summable in probability, Bernstein polynomials of rough lacunary statistical convergence in probability, Bernstein polynomials of rough $N_\theta -$ convergence in probability, Bernstein polynomials of rough strong $(V,\lambda) -$ summable in probability, and Bernstein polynomials of rough $\lambda -$ statistical convergence in probability. Among these six concepts in probability only three convergences are distinct-Bernstein polynomials of rough statistical convergence in probability: (1) Bernstein polynomials of rough lacunary statistical convergence in probability, (2) Bernstein polynomials of rough $\lambda -$ statistical convergence in probability, and (3) Bernstein polynomials of rough $N_\theta -$ convergence in probability which is equivalent to Bernstein polynomials of rough lacunary statistical convergence in probability. Bernstein polynomials of rough strong $(V,\lambda) -$ summable in probability is equivalent to Bernstein polynomials of rough $\lambda -$ statistical convergence in probability. Basic properties and interrelations between these three distinct convergences were investigated and some observations about these classes were made.

The idea of statistical convergence was introduced by Steinhaus and also independently by Fast for real or complex sequences. Statistical convergence is a generalization of the usual notion of convergence, which parallels the theory of ordinary convergence.

Let K be a subset of the set of positive integers $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, and let us denote the set $\{(m,n,k) \in K: m \le u, n \le v, k \le w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{uvw\to\infty}\frac{|K_{uvw}|}{uvw}$, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N}\backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \le \delta(K_2)$ (Tripathy & Goswami, 2016).

The Bernstein operator of order (r,s,t) is given by

$$B_{rst}(f,x) = \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} f\left(\frac{mnk}{rst}\right)\binom{r}{m}\binom{s}{n}\binom{t}{k} x^{m+n+k}(1-x)^{(m-r)+(n-s)+(k-t)}$$

where f is a continuous (real or complex valued) function defined on $[0,1]$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x)) \in \mathbb{R}, m,n,k \in \mathbb{N}$.

Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as $st - lim \quad x = 0$, provided that the set

$$K_\varepsilon := \{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \ge \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials. i.e., $\delta(K_\varepsilon) = 0$. That is,

$$lim_{rst\to\infty}\frac{1}{rst}|\{(m,n,k) \le (r,s,t): |B_{mnk}(f,x) - (f,x)| \ge \varepsilon\}| = 0.$$

In this case, we write $\delta - limB_{mnk}(f,x) = f(x)$ or $B_{mnk}(f,x) \to^{S_B} f(x)$.

Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric $(B_{mnk}(f,x), d)$. Consider a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ such that $(B_{mnk}(f,x)) \in \mathbb{R}, m,n,k \in \mathbb{N}$.

A triple sequence of Bernstein operator of $(B_{mnk}(f,x))$ is said to be statistically convergent to $f(x) \in \mathbb{R}$, written as $st - lim \quad B_{mnk}(f,x) = f(x)$, provided that the set

$$\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x), f(x)| \geq \varepsilon\}$$

has natural density zero for any $\varepsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$.

If a triple sequence of Bernstein polynomial is statistically convergent, then for every $\varepsilon > 0$, infinitely many terms of the sequence may remain outside the $\varepsilon -$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ satisfies some property P for all m,n,k except a set of natural density zero, then we say that the triple sequence of Bernstein polynomials $(B_{mnk}(f,x))$ satisfies P for almost all (m,n,k) and we abbreviate this by a.a. (m,n,k).

Let $\left(B_{m_i n_j k_\ell}(f,x)\right)$ be a sub sequence of (B_{mnk}) (f,x). If the natural density of the set $K = \{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i,j,\ell) \in \mathbb{N}^3\}$ is different from zero, then $\left(B_{m_i n_j k_\ell}(f,x)\right)$ is called a non thin sub sequence of a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$.

$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ provided that the natural density of the set

$$\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - c| < \varepsilon\}$$

is different from zero for every $\varepsilon > 0$. We denote the set of all statistical cluster points of the sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ by Γ_x.

A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x)|^{1/m+n+k} \geq M\}\right) = 0$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu (2001), who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar (2008) extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal, Chandra, and Dutta (2013) extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence.

We show that this set of statistical cluster points and the set of rough statistical limit points of a triple sequence of Bernstein polynomials.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner, Gurdal, and Duden (2007); Sahiner and Tripathy (2008); Esi (2014); Esi and Catalbas (2014); Esi and Savas (2015); Esi, Araci, and Acikgoz (2016); Dutta, Esi, and Tripathy (2013); Subramanian and Esi (2015); Debnath, Sarma, and Das (2015); Esi, Araci, and Esi (2017); Esi, Subramanian, and Esi (2017); Tripathy and Goswami (2014, 2015a, 2015b, 2016) and many others.

Throughout the paper let r be a nonnegative real number.

2. Triple Bernstein polynomials of rough statistical convergence in probability

2.1 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0}, \mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers and $\boldsymbol{r}$ be a non-negative real number is said to be rough convergent to $\boldsymbol{f(x)}$ with respect to the roughness of degree $\boldsymbol{r}$ (or shortly: $\boldsymbol{r} -$ convergent to $\boldsymbol{B_{mnk}(f,x)}$) if for every $\boldsymbol{\varepsilon > 0}$, there exists a natural number $(\boldsymbol{uvw})$ such that

$$|B_{mnk}(f,x) - f(x)| < r + \varepsilon \text{ for all } m \geq u, n \geq v, k \geq w$$

and we denote by $B_{mnk}(f,x) \to^r f(x)$. if we take $r = 0$, then we obtain the ordinary convergence.

2.2 Definition. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers and r be a non-negative real number is said to be rough statistically convergent to $f(x)$ with respect to the roughness of degree r (or shortly: $r-$ statistically convergent to $f(x)$) if for every $\varepsilon > 0$, the set

$$\{(m,n,k) \in \mathbb{N}^3 : |B_{mnk}(f,x) - f(x)| \geq r + \varepsilon\}$$

has asymptotic density zero or equivalently, if the condition

$S - lim_{mnk\to\infty} sup|B_{mnk}(f,x) - f(x)| \leq r$ is satis-fied and we denote by $B_{mnk}(f,x) \to_r^{S^P} f(x)$. If we take $r = 0$, then we obtain the ordinary statistical convergence.

2.3 Definition. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers and r be a non-negative real number is said to be rough statistically convergent in probability to a random variable $(B_{mnk}(f,x)) : W \to \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ statistically convergent in probability to $f(x)$) if for each $\varepsilon, \delta > 0$,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} |\{m \leq u, n \leq v, k \leq w : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| = 0,$$

or, equivalently,

$$\lim_{uvw\to\infty} \frac{1}{(uvw)} |\{m \leq u, n \leq v, k \leq w : 1 - P(|B_{mnk}(f,x) - f(x)| < r + \varepsilon) \geq \delta\}| = 0,$$

and we write $B_{mnk}(f,x) \to_r^{S^P} f(x)$. The class of all $r-$ statistically convergent triple squences of Bernstein polynomials of random variables in probability will be simply denoted by rS^P.

2.4 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers and if $B_{mnk}(f,x) \to_r^{S^P} f(x)$ and $B_{mnk}(f,y) \to_r^{S^P} f(y)$ then $P\{|f(x) - f(y)| \geq r\} = 0$.

Proof: Let ε, δ be any two positive real numbers and let

$$(uvw) \in \left\{(mnk) \in \mathbb{N}^3 : P\left(|B_{mnk}(f,x) - f(x)| \geq r + \frac{\varepsilon}{2}\right) < \frac{\delta}{2}\right\} \cap \left\{(mnk) \in \mathbb{N}^3 : P\left(|B_{mnk}(f,y) - f(y)| \geq r + \frac{\varepsilon}{2}\right) < \frac{\delta}{2}\right\}$$

(existence of (uvw) is guaranteed since asymptotic density of both the sets is equal to 1). Then

$$P(|f(x) - f(y)| \geq r + \varepsilon) \leq P\left(|B_{mnk}(f,x) - f(x)| \geq r + \frac{\varepsilon}{2}\right) + P\left(|B_{mnk}(f,y) - f(y)| \geq r + \frac{\varepsilon}{2}\right) < \delta.$$

This implies $P(|f(x) - f(y)| \geq r) = 0$.

So, we have

2.5 Remark. (i) $B_{mnk}(f,x) \to_r^{S^P} f(x)$ and $B_{mnk}(f,y) \to_r^{S^P} f(y)$then $P\{f(x) = f(y)\} = 1$ (here $r = 0$).
(ii) If $B_{mnk}(f,x) \to_r^{S^P} f(x)$ and $B_{mnk}(f,x) \to_r^{S^P} f(y)$ then $\{P\{f(x) - f(y)\} < r\} = 1$

2.6 Definition. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers and a discrete random variable $(B_{mnk}(f,x))$ is said to be one-point distribution at the point c if the spectrum consists of a single point c and $P(X = c) = 1$. Here c is a parameter of the one point distribution.

2.7 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x)) \to_r^S f(x)$ then $B_{mnk}(f,x) \to_r^{S^P} f(x)$.

Proof: Here for every $(uvw), (B_{mnk}(f,x))$ can be regarded as a random variable with one element $(B_{mnk}(f,x))$ in the corresponding spectrum. Let ε be a positive real number. Since $B_{mnk}(f,x) \to_r^S f(x)$ then

$$lim_{uvw\to\infty} \frac{1}{(uvw)} |\{m \leq u, n \leq v, k \leq w: |B_{mnk}(f,x) - f(x)| \geq r + \varepsilon\}| = 0,$$

$$\Rightarrow lim_{uvw\to\infty} \frac{1}{(uvw)} |\{m \leq u, n \leq v, k \leq w: |B_{mnk}(f,x) - f(x)| < r + \varepsilon\}| = 1.$$

Now the event $\{w: w \in W \ \ and \ \ |B_{mnk}(f,x,w) - f(x,w)| < r + \varepsilon\}$ is the same as the event $|B_{mnk}(f,x) - f(x)| < r + \varepsilon$ which is here the certain event W for all $(uvw) \in \{(mnk) \in \mathbb{N}^3: |B_{mnk}(f,x) - f(x)| < r + \varepsilon\}$. So,

$P(\{w: w \in W \ \ and \ \ |B_{mnk}(f,x,w) - f(x,w)| < r + \varepsilon\}) = P(|B_{mnk}(f,x) - f(x)| < r + \varepsilon) = P(W) = 1$ for all $(uvw) \in \{(mnk) \in \mathbb{N}^3: |B_{mnk}(f,x) - f(x)| < r + \varepsilon\}$. Thus for any $\delta > 0$,

$$\{(uvw) \in \mathbb{N}^3: 1 - P(|B_{uvw}(f,x) - f(x)| < r + \varepsilon)\} \subset \mathbb{N}^3 \backslash \{m \leq u, n \leq v, k \leq w: |B_{mnk}(f,x) - f(x)| < r + \varepsilon\} = \{m \leq u, n \leq v, k \leq w: |B_{mnk}(f,x) - f(x)| \geq r + \varepsilon\}.$$

In general, the converse is not true, i.e., if a triple sequence of Bernstein polynomials of random variables $(B_{mnk}(f,x))$ is a rough statistical convergence in probability to a real number $f(x)$ then each of $(B_{mnk}(f,x))$ may not have one point distribution so each $(B_{mnk}(f,x))$ cannot be treated as a constant which is rough statistical convergence to $f(x)$, i.e., rough statistical convergence in probability is the more generalized concept than usual rough statistical convergence.

Example: Let a triple sequence of Bernstein polynomials of random variables $(B_{mnk}(f,x))$ be defined by,

$$B_{mnk}(f,x) \in \begin{cases} \{-10,10\} \ \ with \ \ probability \ \ P(B_{mnk}(f,x) = -10) = P(B_{mnk}(f,x) = 10), \\ \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ \{0,1\} \ \ with \ \ probability \ \ P(B_{mnk}(f,x) = 0) = P(B_{mnk}(f,x) = 1), \\ \text{if } (m,n,k) \neq (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

Let $0 < \varepsilon < 1$ be given. Then

$$P(|B_{mnk}(f,x) - 1| \geq 2 + \varepsilon) = \begin{cases} 1 & \text{if } (m,n,k) = (u,v,w)^2 \text{ for some } (u,v,w) \in \mathbb{N}^3 \\ 0 & \text{if } (m,n,k) \neq (u,v,w)^2 \text{ for any } (u,v,w) \in \mathbb{N}^3. \end{cases}$$

This implies $B_{mnk}(f,x) \to_2^{S^P} f(x)$. But it is not an ordinary rough statistical convergence of a triple sequence of Bernstein polynomials of numbers to 1.

2.8 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers.

(i) $B_{mnk}(f,x) \to_r^{S^P} f(x) \Leftrightarrow B_{mnk}(f,x) - f(x) \to_r^{S^P} 0$,

(ii) $B_{mnk}(f,x) \to_r^{S^P} f(x) \Rightarrow cB_{mnk}(f,x) \to_{|c|r}^{S^P} c \quad f(x)$, where $c \in \mathbb{R}$,

(iii) $B_{mnk}(f,x) \to_r^{S^P} f(x)$ and $B_{mnk}(f,y) \to_r^{S^P} f(y) \Rightarrow B_{mnk}(f,x) + B_{mnk}(f,y) \to_r^{S^P} f(x) + f(y)$,

(iv) $B_{mnk}(f,x) \to_r^{S^P} f(x)$ and $B_{mnk}(f,y) \to_r^{S^P} f(y) \Rightarrow B_{mnk}(f,x) - B_{mnk}(f,y) \to_r^{S^P} f(x) - f(y)$,

(v) $B_{mnk}(f,x) \to_{r}^{S^P} 0 \Rightarrow B_{mnk}^2(f,x) \to_{r^2}^{S^P} 0,$

(vi) $B_{mnk}(f,x) \to_{r}^{S^P} f(x) \Rightarrow B_{mnk}^2(f,x) \to_{r^2+2|B_{mnk}(f,x)|r}^{S^P} f^2(x),$

(vii) $B_{mnk}(f,x) \to_{r}^{S^P} f(x)$ and $B_{mnk}(f,y) \to_{r}^{S^P} f(y) \Rightarrow B_{mnk}(f,x) \cdot B_{mnk}(f,y) \to_{\frac{r}{2}+\frac{r(|f(x)+f(y)|+|f(x)-f(y)|)}{2}}^{S^P} f(x) \cdot f(y),$

(viii) If $0 \le B_{mnk}(f,x) \le B_{mnk}(f,y)$ and $B_{mnk}(f,y) \to_{r}^{S^P} 0 \Rightarrow B_{mnk}(f,x) \to_{r}^{S^P} 0,$

(ix) $B_{mnk}(f,x) \to_{r}^{S^P} f(x)$, then for each $\varepsilon > 0$ there exists $(uvw) \in \mathbb{N}^3$ such that any $\delta > 0$

$$lim_{uvw\to\infty} \frac{1}{(uvw)} |\{m \le u, n \le v, k \le w: P(|B_{mnk}(f,x) - f_{uvw}(x)| \ge 2r + \varepsilon) \ge \delta\}| = 0.$$

which is called rough statistical Cauchy condition in probability.

Proof: Let ε, δ be any positive real numbers. Then for **(i)**, the proof is straightforward, hence omitted.

(ii) If $c = 0$ then the claim is obvious. So assuming $c \ne 0$, then

$$\{(m,n,k) \in \mathbb{N}^3: P(|c \quad B_{mnk}(f,x) - c \quad f(x)| \ge |c|r + \varepsilon) \ge \delta\} = \left\{(m,n,k) \in \mathbb{N}^3: P\left(|B_{mnk}(f,x) - f(x)| \ge r + \frac{\varepsilon}{|c|}\right) \ge \delta\right\}.$$

Hence, $cB_{mnk}(f,x) \to_{|c|r}^{S^P} cB_{mnk}(f,x)$.

(iii)

$$P\left(\left|\left(B_{mnk}(f,x) + B_{mnk}(f,y)\right) - \left(f(x) + f(y)\right)\right| \ge r + \varepsilon\right) = P\left(\left|\left(B_{mnk}(f,x) - f(x)\right) + \left(B_{mnk}(f,y) - f(y)\right)\right| \ge r + \varepsilon\right) \le P\left(|B_{mnk}(f,x) - f(x)| \ge r + \frac{\varepsilon}{2}\right) + P\left(|B_{mnk}(f,y) - f(y)| \ge r + \frac{\varepsilon}{2}\right).$$

This implies

$$\left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|\left(B_{mnk}(f,x) + B_{mnk}(f,y)\right) - \left(f(x) + f(y)\right)\right| \ge r + \varepsilon\right) \ge \delta\right\} \subseteq \left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|\left(B_{mnk}(f,x) - f(x)\right)\right| \ge r + \frac{\varepsilon}{2}\right) \ge \frac{\delta}{2}\right\} \cup \left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|\left(B_{mnk}(f,y) - f(y)\right)\right| \ge r + \frac{\varepsilon}{2}\right) \ge \frac{\delta}{2}\right\}.$$

Hence $B_{mnk}(f,x) + B_{mnk}(f,y) \to_{r}^{S^P} f(x) + f(y)$.

(iv) Similar to the proof of (iii) and therefore omitted.

(v)

$$\left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|B_{mnk}^2(f,x)\right| \ge r^2 + \delta\right)\right\} = \left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|B_{mnk}^2(f,x)\right| \ge r^2 + 2r\eta + \eta^2\right)\right\}$$

(where $\eta = -r + \sqrt{r^2 + \delta} > 0$)$= \left\{(m,n,k) \in \mathbb{N}^3: P\left(\left|B_{mnk}^2(f,x)\right| \ge r + \eta\right)\right\}$. Hence, $B_{mnk}^2(f,x) \to_{r^2}^{S^P} 0$.

(vi) $B_{mnk}^2(f,x) = \left(B_{mnk}(f,x) - f(x)\right)^2 + 2f(x)\left(B_{mnk}(f,x) - f(x)\right) + f(y)$, so $B_{mnk}^2(f,x) \to_{2r^2+2|l|r}^{S^P} f(y)$.

(vii)

$$\left(B_{mnk}(f,x) + B_{mnk}(f,y)\right)^2 \to_{r^2+2r|f(x)+f(y)|}^{S^P} \left(f(x) + f(y)\right)^2 \text{ and}$$

$$\left(B_{mnk}(f,x) - B_{mnk}(f,y)\right)^2 \to_{r^2+2r|f(x)-f(y)|}^{S^P} \left(f(x) - f(y)\right)^2$$

$$\Rightarrow B_{mnk}(f,x) \cdot B_{mnk}(f,y) = \frac{1}{4}\left\{\left\{\left(B_{mnk}(f,x) + B_{mnk}(f,y)\right)^2 - \left(B_{mnk}(f,x) - B_{mnk}(f,y)\right)^2\right\}\right\} \to_{\frac{r^2}{2}+\frac{r(|f(x)+f(y)|+|f(x)-f(y)|)}{2}}^{S^P} \frac{1}{4}\left\{\left(f(x) + f(y)\right)^2 - \left(f(x) - f(y)\right)^2\right\} = f(x) \cdot f(y).$$

(viii) The proof is straightforward, hence omitted.

(ix) Now choose $(u, v, w) \in \mathbb{N}^3$ such that $P\left(|B_{uvw}(f,x) - B_{mnk}(f,x)| \geq r + \frac{\varepsilon}{2}\right) < \frac{\delta}{2}$.

Then the claim is obvious from the inequality

$$P(|B_{mnk}(f,x) - B_{uvw}(f,x)| \geq 2r + \varepsilon)$$
$$\leq P\left(|B_{mnk}(f,x) - B_{mnk}(f,x)| \geq r + \frac{\varepsilon}{2}\right)$$
$$+ P\left(|B_{uvw}(f,x) - B_{mnk}(f,x)| \geq r + \frac{\varepsilon}{2}\right)$$
$$\leq \frac{\delta}{2} + P\left(|B_{mnk}(f,x) - B_{mnk}(f,x)| \geq r + \frac{\varepsilon}{2}\right).$$

2.9 Theorem. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of random variables of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers with the property that $\boldsymbol{B_{mnk}(f,x) - f(x) \rightarrow_r^{S^P} 0}$. If $\boldsymbol{m(B_{mnk}(f,x))}$ is a median of $\boldsymbol{B_{mnk}(f,x)}$ then $\boldsymbol{B_{mnk}(f,x) - m(B_{mnk}(f,x)) \rightarrow_r^{S^P} 0}$ and $\boldsymbol{f(x) - m(B_{mnk}(f,x)) \rightarrow_r^{S^P} 0}$.

Proof: The proof is straightforward, hence omitted.

2.10 Theorem. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers and $\boldsymbol{r > 0}$. Then $\boldsymbol{B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)} \Leftrightarrow$ there exists a triple sequence of random variables of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ such that $\boldsymbol{B_{mnk}(f,y) \rightarrow_r^{S^P} f(y)}$ and $\boldsymbol{S - lim_{mnk \to \infty} P(|B_{mnk}(f,x) - B_{mnk}(f,y)| > r) = 0}$.

Proof: Let $B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)$ and

$$A = \{(m,n,k) \in \mathbb{N}^3 : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}. \text{ Then } d(A) = 0.$$

Now we define

$$y_{mnk} = \begin{cases} f(x) & \text{if } (m,n,k) \in \mathbb{N}^3 \backslash A \\ B_{mnk}(f,x) + Z & \text{otherwise.} \end{cases}$$

(Where $B_{mnk}(f,x)$ is a random variable of Bernstein polynomials and $Z \in (-r, r)$ with probability $P(B_{mnk}(f,x) = -r) = P(B_{mnk}(f,x) = r)$.) Then it is very obvious that

$$d(\{(m,n,k) \in \mathbb{N}^3 : P(|B_{mnk}(f,y) - f(y)| \geq \varepsilon) \geq \delta\}) = 0 \text{ and}$$
$$d(\{(m,n,k) \in \mathbb{N}^3 : P(|B_{mnk}(f,x) - B_{mnk}(f,y)| \geq r + \varepsilon) \geq \delta\}) \leq$$
$$d\left(\left\{(m,n,k) \in \mathbb{N}^3 : P\left(|B_{mnk}(f,x) - f(x)| \geq r + \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right) +$$
$$d\left(\left\{(m,n,k) \in \mathbb{N}^3 : P\left(|B_{mnk}(f,y) - f(y)| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right) = 0.$$

Conversely, let $B_{mnk}(f,y) \rightarrow_r^{S^P} f(y)$ and $S - lim_{mnk \to \infty} P(|B_{mnk}(f,x) - B_{mnk}(f,y)| > r) = 0$. Then for each $\varepsilon, \delta > 0$,

$$\lim_{uvw \to \infty} \frac{1}{(uvw)} \left|\left\{m \leq u, n \leq v, k \leq w : P\left(|B_{mnk}(f,y) - f(y)| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| = 0 \text{ and}$$

$$\lim_{uvw \to \infty} \frac{1}{(uvw)} \left|\left\{m \leq u, n \leq v, k \leq w : P\left(|B_{mnk}(f,x) - B_{mnk}(f,y)| \geq r + \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\right\}\right| = 0.$$

We have the following inequality

$$P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \leq P\left(|B_{mnk}(f,y) - f(y)| > \frac{\varepsilon}{2}\right) + P\left(|B_{mnk}(f,x) - B_{mnk}(f,y)| \geq r + \frac{\varepsilon}{2}\right).$$

$\Rightarrow \{(m,n,k) \in \mathbb{N}^3 : P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon) \geq \delta\} \subseteq \Big\{(m,n,k) \in \mathbb{N}^3 : P\left(|B_{mnk}(f,y) - f(y)| \geq \frac{\varepsilon}{2}\right) \geq \frac{\delta}{2}\Big\} \cup \ \Big\{(m,n,k) \in \mathbb{N}^3 : P\Big(|B_{mnk}(f,x) - B_{mnk}(f,y)| \geq r + \frac{\varepsilon}{2}\Big) \geq \frac{\delta}{2}\Big\}.$

Hence $B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)$. In view of Theorem 2.4 of Debnath, Sarma, and Das (2015), we formulate the following result without proof.

2.11 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers**.** If $B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)$ and $g: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a continuous function on $\mathbb{R}^3$, then there exists a triple sequence of Bernstein polynomials of random variables $(B_{mnk}(f,y))$ such that $g(B_{mnk}(f,y)) \rightarrow_r^{S^P} f(y)$ and $g(P(|B_{mnk}(f,x) - B_{mnk}(f,y)| > r)) \rightarrow_r^{S^P} 0$.

3. Strong Cesàro Summable of a Triple Sequence of Bernstein Polynomials of Real Numbers

3.1 Definition. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is said to be strong Cesàro summable to $f(x)$ if

$$lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} |B_{mnk}(f,x) - f(x)| = 0.$$

In this case we write $B_{mnk}(f,x) \rightarrow^{[C,1,1]} f(x)$. The set of all strong Cesàro summable triple squences of Bernstein polynomials is denoted by $[C,1,1]$.

3.2 Definition. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of random variables of $(B_{mnk}(f,x))$ of real numbers and r be a non-negative real number is said to be rough strong Cesàro summable in probability to a random variable $B_{mnk}(f,x): W \rightarrow \mathbb{R}^3$ with respect to the roughness of degree r (or shortly: $r-$ strong Cesàro summable in probability to $f(x)$) of for each $\varepsilon > 0$,

$$lim_{uvw\rightarrow\infty} \frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon) = 0.$$

In this case we write $B_{mnk}(f,x) \rightarrow_r^{[C,1,1]^P} f(x)$.

The class of all $r-$ strong Cesàro summable triple sequences of Bernstein polynomials of random variables in probability will be simply denoted by $r[C,1,1]^P$.

3.3 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers. The followings are equivalent: (i) $B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)$ (ii) $B_{mnk}(f,x) \rightarrow_r^{[C,1,1]^P} f(x)$.

Proof: (i)⇒ (ii): First suppose that $B_{mnk}(f,x) \rightarrow_r^{S^P} f(x)$. Then we can write

$$\frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1}^{w} P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon) =$$
$$\frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1, P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\frac{\delta}{2}}^{w} P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon) +$$
$$\frac{1}{(uvw)} \sum_{m=1}^{u} \sum_{n=1}^{v} \sum_{k=1, P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)<\frac{\delta}{2}}^{w} P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon)$$

$$\leq \frac{1}{(uvw)} \left|\left\{m \leq u, n \leq v, k \leq w : P(|B_{mnk}(f,x) - f(x)| \geq r+\varepsilon) > \frac{\delta}{2}\right\}\right| + \frac{\delta}{2}.$$

(ii)⇒ (i) Next suppose that condition (ii) holds. then

$\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P(|B_{mnk}(f,x)-f(x)| \geq r+\varepsilon) \geq$
$\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1, P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\delta}^{w} P(|B_{mnk}-f(x)| \geq r+\varepsilon) \geq \delta|\{m \leq$
$u, n \leq v, k \leq w: P(|B_{mnk}(f,x)-f(x)| \geq r+\varepsilon) > \delta\}|.$

Therefore

$$\frac{1}{(uvw)}\sum_{m=1}^{u}\sum_{n=1}^{v}\sum_{k=1}^{w} P(|B_{mnk}(f,x)-f(x)| \geq r+\varepsilon) \geq \frac{1}{(uvw)}|\{m \leq u, n \leq v, k \leq w: P(|B_{mnk}(f,x)-f(x)| \geq r+\varepsilon) > \delta\}|.$$

Hence $B_{mnk}(f,x) \rightarrow_{r}^{S^{P}} f(x)$.

4. Triple Bernstein Polynomials of Rough Lacunary Statistical Convergence in Probability

4.1 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers, $\boldsymbol{\theta_{i,\ell,j}} = \{(\boldsymbol{m_i},\boldsymbol{n_\ell},\boldsymbol{k_j})\}$ is called triple Bernstein polynomials of lacunary if there exist three increasing sequences of integers such that

$$m_0 = 0, h_i = m_i - m_{r-1} \rightarrow \infty \text{ as } i \rightarrow \infty \text{ and}$$
$$n_0 = 0, \overline{h_\ell} = n_\ell - n_{\ell-1} \rightarrow \infty \text{ as } \ell \rightarrow \infty.$$
$$k_0 = 0, \overline{h_j} = k_j - k_{j-1} \rightarrow \infty \text{ as } j \rightarrow \infty.$$

Let $m_{i,\ell,j} = m_i n_\ell k_j, h_{i,\ell,j} = h_i\overline{h_\ell h_j}$, and $\theta_{i,\ell,j}$ is determine by

$$I_{i,\ell,j} = \{(m,n,k): m_{i-1} < m < m_i \quad and \quad n_{\ell-1} < n \leq n_\ell \quad and \quad k_{j-1} < k \leq k_j\}, q_i = \frac{m_i}{m_{i-1}}, \overline{q_\ell} = \frac{n_\ell}{n_{\ell-1}}, \overline{q_j} = \frac{k_j}{k_{j-1}}.$$

4.2 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers and $\boldsymbol{\theta} = \{\boldsymbol{m_r n_s k_t}\}_{(rst)\in\mathbb{N}\cup 0}$ is said to be triple Bernstein polynomials of lacunary statistically convergent to a real number $\boldsymbol{f}(\boldsymbol{x})$ (or shortly: $\boldsymbol{S_\theta}$- convergent to $\boldsymbol{f}(\boldsymbol{x})$) if for any $\boldsymbol{\varepsilon} > \mathbf{0}$,

$$lim_{rst\rightarrow\infty}\frac{1}{h_{rst}}|\{(m,n,k) \in I_{rst}: |B_{mnk}(f,x)-f(x)| \geq \varepsilon\}| = 0,$$

and it is denoted by $B_{mnk}(f,x) \rightarrow^{S_\theta} f(x)$, where

$$I_{r,s,t} = \{(m,n,k): m_{r-1} < m < m_r \quad and \quad n_{s-1} < n \leq n_s \quad and \quad k_{t-1} < k \leq k_t\}, q_r = \frac{m_r}{m_{r-1}}, \overline{q_s} = \frac{n_s}{n_{s-1}}, \overline{q_t} = \frac{k_t}{k_{t-1}}.$$

4.3 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers and $\boldsymbol{\theta} = \{\boldsymbol{m_r n_s k_t}\}$ be the triple Bernstein polynomials of lacunary is said to be $\boldsymbol{N_\theta} -$ convergent to a real number $\boldsymbol{f}(\boldsymbol{x})$ if for any $\boldsymbol{\varepsilon} > \mathbf{0}$,

$$lim_{rst\rightarrow\infty}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}|B_{mnk}(f,x)-f(x)| = 0.$$

In this case we write $B_{mnk}(f,x) \rightarrow^{N_\theta} f(x)$.

4.4 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of random variables $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers and $\boldsymbol{r}$ be a non-negative real number is said to be rough lacunary statistically convergent in probability to $\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}): \boldsymbol{W} \rightarrow \mathbb{R}^3$ with respect to the roughness of degree $\boldsymbol{r}$ (or shortly: $\boldsymbol{r} -$ lacunary statistically convergent in probability to $\boldsymbol{f}(\boldsymbol{x})$) if for any $\boldsymbol{\varepsilon},\boldsymbol{\delta} > \mathbf{0}$

$$lim_{rst\to\infty}\frac{1}{h_{rst}}|\{(m,n,k)\in I_{rst}: P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\delta\}|=0,$$

and we write $B_{mnk}(f,x)\to_r^{S_\theta^P} f(x)$. The class of all $r-$ triple Bernstein polynomials of lacunary statistically convergent sequences of random variables in probability will be denoted simply by rS_θ^P.

4.5 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of random variables of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers and $\boldsymbol{r}$ be a non-negative real number is said to be rough $\boldsymbol{N_\theta}-$ convergent in probability to $\boldsymbol{B_{mnk}(f,x)}: \boldsymbol{W}\to\mathbb{R}^3$ with respect to the roughness of degree $\boldsymbol{r}$ (or shortly: $\boldsymbol{r}-\boldsymbol{N_\theta}-$ convergent in probability to $\boldsymbol{f(x)}$) if for any $\boldsymbol{\varepsilon}>\mathbf{0}$,

$$lim_{rst\to\infty}\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)=0,$$

and we write $B_{mnk}(f,x)\to_r^{N_\theta^P} f(x)$. The class of all $r-N_\theta-$ convergent triple sequence of Bernstein polynomials of random variables in probability will be denoted simply by rN_θ^P.

4.6 Theorem. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers and $\boldsymbol{\theta}=\{\boldsymbol{m_r},\boldsymbol{n_s},\boldsymbol{k_t}\}$ be a triple Bernstein polynomials of lacunary sequence. Then the followings are equivalent:

(i) $(B_{mnk}(f,x))$ is a $r-$ triple Bernstein polynomials of lacunary statistically convergent in probability to $f(x)$.
(ii) $(B_{mnk}(f,x))$ is $r-N_\theta-$ convergent in probability to $f(x)$.

Proof: (i)⇒ (ii) : First suppose that $B_{mnk}(f,x)\to_r^{S_\theta^P} f(x)$. Then we can write

$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)=$$
$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t,P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\frac{\delta}{2}}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)+$$
$$\frac{1}{h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t,P(|B_{mnk}(f,x)-f(\ )|\geq r+\varepsilon)<\frac{\delta}{2}}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\leq$$
$$\frac{1}{h_{rst}}\left|\left\{(m,n,k)\in I_{rst}: P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\frac{\delta}{2}\right\}\right|.$$

(ii)⇒ (i): Next suppose that condition (ii) holds. Then

$$\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq$$
$$\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t,P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\delta}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq$$
$$\delta\quad|\{(m,n,k)\in I_{rst}: P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\delta\}|.$$

Therefore

$$\frac{1}{\delta\ \ h_{rst}}\sum_{m\in I_r}\sum_{n\in I_s}\sum_{k\in I_t}P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\frac{1}{h_{rst}}|\{(m,n,k)\in I_{rst}: P(|B_{mnk}(f,x)-f(x)|\geq r+\varepsilon)\geq\delta\}|.$$

Hence $B_{mnk}(f,x)\to_r^{S_\theta^P} f(x)$.

We formulate the following result in the light of Theorem 2.4.

4.7 Theorem. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}(f,x)})$ of real numbers. If $\boldsymbol{B_{mnk}(f,x)}\to_r^{S_\theta^P}\boldsymbol{f(x)}$ and $\boldsymbol{B_{mnk}(f,y)}\to_r^{S_\theta^P}\boldsymbol{f(y)}$ then $\boldsymbol{P}(|\boldsymbol{f(x)}-\boldsymbol{f(y)}|\geq\boldsymbol{r})=\mathbf{0}$.

5. Triple Bernstein Poloynomials of Rough-Λ-Statistical Convergence in Probability

Let $\lambda = (\lambda_{uvw})$ be a non-decreasing triple sequence of Bernstein polynomials of positive numbers tending to ∞ such that $\lambda_{uvw+1} \leq \lambda_{uvw} + 1, \lambda_{111} = 1$. The collection of all such triple sequence of Bernstein polynomials of λ is denoted by $\mathbb{D}$.

The generalized Delavale$e' -$ Pousin mean, f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers is defined by $t_{uvw}(x) = \frac{1}{\lambda_{uvw}} \sum_{(m,n,k) \in Q_{uvw}} B_{mnk}(f,x)$, where $Q_{uvw} = [(uvw) - \lambda_{uvw} + 1, (uvw)]$. A triple sequence of Bernstein poloynomials of $(B_{mnk}(f,x))$ of real numbers is said to be $[V,\lambda] -$ summable to $f(x)$, if $lim t_{uvw}(f,x) = f(x)$.

5.1 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers is said to be strong $[\boldsymbol{V},\boldsymbol{\lambda}] -$ summable (or shortly:$[\boldsymbol{V},\boldsymbol{\lambda}] -$ convergent) to $\boldsymbol{f}(\boldsymbol{x})$, if

$$lim_{uvw \to \infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k) \in Q_{uvw}} |B_{mnk}(f,x) - f(x)| = 0.$$

In this case we write $B_{mnk}(f,x) \to^{[V,\lambda]} f(x)$.

5.2 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers is said to be $\boldsymbol{\lambda} -$ statistically convergent (or shortly: $\boldsymbol{S_\lambda} -$ convergent) to $\boldsymbol{f}(\boldsymbol{x})$ if for any $\boldsymbol{\varepsilon} > \mathbf{0}$,

$$lim_{uvw \to \infty} \frac{1}{\lambda_{uvw}} |\{(m,n,k) \in Q_{uvw} : |B_{mnk}(f,x) - f(x)| \geq \varepsilon\}| = 0.$$

In this case we write $S_\lambda - lim B_{mnk}(f,x) = f(x)$ or by $B_{mnk}(f,x) \to^{S_\lambda} f(x)$.

5.3 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of random variables of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers and $\boldsymbol{r}$ be a non-negative real number is said to be rough $[\boldsymbol{V},\boldsymbol{\lambda}] -$ summable in probability to $\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}) : \boldsymbol{W} \to \mathbb{R}^3$ with respect to the roughness of degree $\boldsymbol{r}$ (or shortly: $\boldsymbol{r} - [\boldsymbol{V},\boldsymbol{\lambda}] -$ summable in probability to $\boldsymbol{f}(\boldsymbol{x})$) if for any $\boldsymbol{\varepsilon} > \mathbf{0}$,

$$lim_{uvw \to \infty} \frac{1}{\lambda_{uvw}} \sum_{(m,n,k) \in Q_{uvw}} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) = 0.$$

In this case we write $B_{mnk}(f,x) \to_r^{[V,\lambda]^P} f(x)$. The class all rough $[V,\lambda] -$ summable sequences of random variables in probability will be denoted simply by $r[V,\lambda]^P$.

5.4 Definition. Let $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers and $\boldsymbol{r}$ be a non-negative real number is said to be rough $\boldsymbol{\lambda} -$ statistically convergent in probability to $\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}) : \boldsymbol{W} \to \mathbb{R}^3$ with respect to the roughness of degree $\boldsymbol{r}$ (or shortly: $\boldsymbol{r} - \boldsymbol{\lambda} -$ statistically convergent in probability to $\boldsymbol{f}(\boldsymbol{x})$) if for any $\boldsymbol{\varepsilon}, \boldsymbol{\delta} > \mathbf{0}$,

$$lim_{uvw \to \infty} \frac{1}{\lambda_{uvw}} |\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| = 0.$$

In this case we write $B_{mnk}(f,x) \to_r^{S_\lambda^P} f(x)$. The class of all $r - \lambda -$ statistically convergent triple sequence of Bernstein polynomials of random variables in probability will be denoted simply by rS_λ^P.

5.5 Theorem. If $\boldsymbol{f}$ be a continuous function defined on the closed interval $[\mathbf{0},\mathbf{1}]$. A triple sequence of Bernstein polynomials of $(\boldsymbol{B_{mnk}}(\boldsymbol{f},\boldsymbol{x}))$ of real numbers then the following are equivalent:

(i) $(B_{mnk}(f,x))$ is $r - [V,\lambda] -$ summable in probability to $f(x)$.
(ii) $(B_{mnk}(f,x))$ is $r - \lambda -$ statistically convergent in probability to $f(x)$.

Proof: Similar to the proof of the Theorem 4.6, so omitted.

5.6 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers. If $B_{mnk}(f,x) \rightarrow_{r}^{S_{\lambda}^{P}} f(x)$ and $B_{mnk}(f,y) \rightarrow_{r}^{S_{\lambda}^{P}} f(y)$ then $P(|f(x)-f(y)| \geq r) = 0$.

Proof: Similar to the proof of the Theorem 2.4 and therefore omitted.

5.7 Theorem. Let f be a continuous function defined on the closed interval $[0,1]$. A triple sequence of Bernstein polynomials of $(B_{mnk}(f,x))$ of real numbers. If $\lambda \in \mathbb{D}$ is such that $lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, then $rS_{\lambda}^{P} \subset rS^{P}$.

Proof: Let $0 < \eta < 1$ be given. Since $lim\left(\frac{\lambda_{uvw}}{uvw}\right) = 1$, we can choose $(r,s,t) \in \mathbb{N}^3$ such that $\left|\frac{\lambda_{uvw}}{uvw} - 1\right| < \frac{\eta}{2}$ for all $u \geq r, v \geq s, w \geq t$. Now observe that for $\varepsilon, \delta > 0$

$$\frac{1}{(uvw)}|\{m \leq u, n \leq v, k \leq w : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| = \frac{1}{(uvw)}|\{m \leq u, n \leq v, k \leq w - \lambda_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| + \frac{1}{(uvw)}|\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}|$$

$$\leq \frac{(uvw) - \lambda_{uvw}}{(uvw)} + \frac{1}{(uvw)}|\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| \leq 1 - \left(1 - \frac{\eta}{2}\right) + \frac{1}{(uvw)}|\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| = \frac{\eta}{2} + \frac{\lambda_{uvw}}{(uvw)} \cdot \frac{1}{\lambda_{uvw}}|\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}| < \frac{\eta}{2} + \frac{2}{\lambda_{uvw}}|\{(m,n,k) \in Q_{uvw} : P(|B_{mnk}(f,x) - f(x)| \geq r + \varepsilon) \geq \delta\}|$$

hold for all $u \geq r, v \geq s, w \geq t$.

References

Aytar, S. (2008). Rough statistical Convergence. *Numerical Functional Analysis Optimization*, 29(3), 291-303.

Esi, A. (2014). On some triple almost lacunary sequence spaces defined by Orlicz functions. *Research and Reviews: Discrete Mathematical Structures*, 1(2), 16-25.

Esi, A., Araci, S., & Acikgoz, M. (2016). Statistical Convergence of Bernstein Operators. *Applied Mathematics & Information Sciences*, 10(6), 2083-2086.

Esi, A., Araci, S., & Esi, A. (2017). $\lambda-$Statistical convergence of Bernstein polynomial sequences. *Advances and Applications in Mathematical Sciences*, 16(1), 113-119.

Esi, A., & Catalbas, M. N. (2014). Almost convergence of triple sequences. *Global Journal of Mathematical Analysis*, 2(1), 6-10.

Esi, A., & Savas, E. (2015). On lacunary statistically convergent triple sequences in probabilistic normed space. *Applied Mathematics & Information Sciences*, 9(5), 2529-2534.

Esi, A., Subramanian, N. & Esi, A. (2017). Triple rough statistical convergence of sequence of Bernstein operators. *International Journal of Advanced and Applied Sciences*, 4(2), 28-34.

Dutta, A. J., Esi, A., & Tripathy, B. C. (2013). Statistically convergent triple sequence spaces defined by Orlicz function. *Journal of Mathematical Analysis*, 4(2), 16-22.

Debnath, S., Sarma, B., & Das, B. C. (2015). Some generalized triple sequence spaces of real numbers. *Journal of Nonlinear Analysis and Optimization*, 6(1), 71-79.

Ghosal, S. (2013). Statistical convergence of a sequence of random variables and limit theorems. *Applications of Mathematics*, 58(4), 423-437.

Pal, S. K., Chandra, D., & Dutta, S. (2013). Rough ideal Convergence. *Hacettepe Journal Mathematics and Statistics*, 42(6), 633-640.

Phu, H. X. (2001). Rough convergence in normed linear spaces. *Numerical Functional Analysis Optimization*, 22, 201-224.

Sahiner, A., Gurdal, M., & Duden, F. K. (2007). Triple sequences and their statistical convergence, *Selcuk Journal of Applied Mathematics*, 8(2), 49-55.

Sahiner, A., & Tripathy, B. C. (2008). Some I related properties of triple sequences. *Selcuk Journal of Applied Mathematics*, 9(2), 9-18.

Subramanian, N., & Esi, A. (2015). The generalized tripled difference of χ^3 sequence spaces. *Global Journal of Mathematical Analysis*, 3(2), 54-60.

Tripathy, B. C., & Goswami, R. (2014). On triple difference sequences of real numbers in probabilistic normed spaces. *Proyecciones Journal of Mathematics*, 33(2), 157-174.

Tripathy, B. C., & Goswami, R. (2015). Vector valued multiple sequences defined by Orlicz functions. *Boletim da Sociedade Paranaense de Matemática*, 33(1), 67-79.

Tripathy, B. C., & Goswami, R. (2015). Multiple sequences in probabilistic normed spaces. *Afrika Matematika*, 26(5-6), 753-760.

Tripathy, B. C., & Goswami, R. (2016). Statistically convergent multiple sequences in probabilistic normed spaces. *Politehnica University of Bucharest. Scientific Bulletin. Series A. Applied Mathematics and Physics, 78*(4), 83-94.