CHAPTER IV

A POSTERIORI ERROR ESTIMATES

In this Chapter, we derived the upper and local lower bounds for the errors using the standard residual technique. The upper bound gives the bound of the global error in term of the estimator to ensure that the finite element solution is acceptable. The local lower bound gives the relation between the local errors and their estimators with some other quantities.

To obtain a posteriori error estimates, we employed the standard residual technique. We used area-based residual on element $K \in \mathcal{M}^n$ and edge-based residual on edge e on the element K to estimate the error on the element K.

We defined the area-based residual for element $K \in \mathcal{M}^n$ at fixed $t = t^n$ by

$$R^n := f_h^n - \frac{U_h^n - U_h^{n-1}}{\tau_n} + \nabla \cdot (a \nabla U_h^n)$$

and the edge-based residual for interior side $e \in \mathcal{B}^n$ by

$$J_e^n = (a \nabla U_h^n|_{K_1} - a \nabla U_h^n|_{K_2}) \cdot \vec{n}_e, \quad \text{where } e = \partial K_1 \cap \partial K_2.$$

Note 4.1. Since U_h^n is a piecewise linear function, so $\triangle U_h^n = 0$ and

$$\nabla \cdot (a \nabla U_h^n) = \nabla a \cdot \nabla U_h^n + a \Delta U_h^n = \nabla a \cdot \nabla U_h^n.$$

Note that, we need $\nabla a(x)$ to be well defined, i.e., $a(x)$ is differentiable in K, for each

$K \in \mathcal{M}_h$, thus we need to assume in additional that $a(x)$ is piecewise differentiable on Ω, i.e., $a|_K$ is differentiable for all $K \in \mathcal{M}_h$.

We define the *local error indicator* η_K^n for any $K \in \mathcal{M}^n$ by

$$\eta_K^n = \left(h_K^2 ||R^n||_{0,K}^2 + \sum_{e \subset \partial K} h_e ||J_e^n||_{0,e}^2 \right)^{\frac{1}{2}}. \tag{4.1}$$

For each element $K \in \mathcal{M}^n$, we use η_K^n as an indicator for refinement or coarsening. To check the error of the approximation on Ω to ensure that the finite element solution is acceptable, we defined the global error estimator on the space for fixed $t = t^n$ by

$$\eta_{space}^n = \sqrt{\sum_{K \in \mathcal{M}^n} (\eta_K^n)^2}.$$

We use η_{space}^n as a stopping criteria of the current loop of discrete system at time $t = t^n$. To start the next discrete system at time $t = t^{n+1}$, we need to find the suitable time step size that is not too large or too small. So we defined error estimators η_{time}^n to control time step size by

$$(\eta_{time}^n)^2 = \frac{1}{3} |||U_h^n - U_h^{n-1}|||^2.$$

This η_{time}^n is used for finding a suitable τ_n.

4.1 Upper Bound

To analyze the upper bound, we measured the error by the energy norm in space and L^2−norm in time. First, we estimated the error at a fixed time $t = t^n$, and then combined for all time in $(0, T)$.

Note 4.2. Since a constant C in each inequalities can change from line to line, we will use the same C to indicate a constant for convenience.

Lemma 4.3. *For any integer $n \geq 1$,*

$$(\frac{\partial(u-U_h)}{\partial t}, \varphi)_0 + b(u - U_h^n, \varphi)$$
$$= (f - f_h^n, \varphi)_0 + \sum_{K \in \mathcal{M}^n} \int_K R^n(\varphi - v)\, dx + \sum_{e \in \mathcal{B}^n} \int_e J_e^n(\varphi - v)\, ds$$

for all $\varphi \in H_0^1(\Omega)$, $v \in V_0^n$.

Proof. Let $\varphi \in H_0^1(\Omega)$ and $v \in V_0^n$. From the discrete weak form (3.4)

$$\begin{aligned}
(\tfrac{U^n - U^{n-1}}{\tau_n}, \varphi)_0 + b(U_h^n, \varphi) & \\
= \;& (\tfrac{U_h^n - U_h^{n-1}}{\tau_n}, \varphi)_0 + b(U_h^n, \varphi) + (f_h^n, v)_0 - (\tfrac{U_h^n - U_h^{n-1}}{\tau_n}, v)_0 - b(U_h^n, v) \\
= \;& (f_h^n, \varphi)_0 - (f_h^n, \varphi - v)_0 + (\tfrac{U_h^n - U_h^{n-1}}{\tau_n}, \varphi - v)_0 + b(U_h^n, \varphi - v)
\end{aligned}$$

We apply Green's theorem to term $b(U_h^n, \varphi - v)$, on each element $K \in \mathcal{M}^n$,

$$b(U_h^n, \varphi - v) = -\sum_{K \in \mathcal{M}^n} \int_K \nabla \cdot (a \nabla U_h^n)(\varphi - v) dx - \sum_{e \in \mathcal{B}^n} \int_e J_e^n(\varphi - v)\, ds.$$

Substituting the above equality to get

$$\begin{aligned}
(\tfrac{U^n - U^{n-1}}{\tau_n}, \varphi)_0 \; & + b(U_h^n, \varphi) \\
= \;& (f_h^n, \varphi)_0 - (f_h^n, \varphi - v)_0 + (\tfrac{U_h^n - U_h^{n-1}}{\tau_n}, \varphi - v)_0 \\
& - \sum_{K \in \mathcal{M}^n} \int_K \nabla \cdot (a \nabla U_h^n)(\varphi - v)\, dx - \sum_{e \in \mathcal{B}^n} \int_e J_e^n(\varphi - v)\, ds \\
= \;& (f_h^n, \varphi)_0 - \sum_{K \in \mathcal{M}^n} \int_K R^n(\varphi - v)\, dx - \sum_{e \in \mathcal{B}^n} \int_e J_e^n(\varphi - v)\, ds
\end{aligned}$$

We subtract the weak form (3.2) by the above equation to complete the proof. □

Lemma 4.4. *For any $n \geq 1$,*

$$b(u - U_h^n, u - U_h) = \tfrac{1}{2}\left(|||u - U_h^n|||^2 + |||u - U_h|||^2 - |||U_h - U_h^n|||^2\right).$$

Proof.

$$
\begin{aligned}
&b(u - U_h^n, u - U_h) \\
&\quad = b(u - U_h^n, u - U_h^n) - b(u - U_h^n, U_h - U_h^n) \\
&\quad = \|u - U_h^n\|^2 - b(U_h - U_h^n, U_h - U_h^n) + b(U_h - u, U_h - U_h^n) \\
&\quad = \|u - U_h^n\|^2 - \|U_h - U_h^n\|^2 + b(U_h - u, U_h - u) \\
&\qquad -b(U_h - u, U_h^n - u) \\
&\quad = \|u - U_h^n\|^2 + \|u - U_h\|^2 - \|U_h - U_h^n\|^2 - b(u - U_h^n, u - U_h)
\end{aligned}
$$

Thus, $b(u - U_h^n, u - U_h) = \frac{1}{2}\left(\|u - U_h^n\|^2 + \|u - U_h\|^2 - \|U_h - U_h^n\|^2\right).$ □

Now, we use 2 above Lemmas to bound the error at time $t = t^n$ in the following Lemma.

Lemma 4.5. *For fixed time $t = t^n$, if $e^{-2(C_pL)^2t}||u - U_h||_0^2$ is an increasing function of t then there exists a constant $C_1 > 0$ such that*

$$
\frac{d}{dt}(e^{-2(C_pL)^2t}||u - U_h||_0^2) + \frac{1}{2}\|u - U_h^n\|^2 \;\le\; C_1(\eta_{space}^n)^2 + \|U_h - U_h^n\|^2,
$$

where L is the Lipschitz constant of the function $f(u)$ in (3.1).

Proof. By Clement's approximations, there exists the interpolation function $\mathcal{I}^n$: $H_0^1(\Omega) \to V_0^n$ satisfying Clement's inequalities (2.1) and (2.2).

Applying the Cauchy-Schwarz inequality to Lemma 4.3 and set $v = \mathcal{I}^n\varphi$, we get $(\frac{\partial(u-U_h)}{\partial t}, \varphi)_0 + b(u - U_h^n, \varphi)$

$$
\begin{aligned}
&\le ||f - f_h^n||_0||\varphi||_0 + \sum_{K\in\mathcal{M}^n} ||R^n||_{0,K}||\varphi - \mathcal{I}^n\varphi||_{0,K} \\
&\quad + \sum_{e\in\mathcal{B}^n} ||J_e^n||_{0,e}||\varphi - \mathcal{I}^n\varphi||_{0,e}.
\end{aligned}
$$

By the Lipschitz continuity of f and Clement's approximations (2.1) and (2.2),

$$
\begin{aligned}
(\tfrac{\partial(u-U_h)}{\partial t},\varphi)_0 + b(u-U_h^n,\varphi) &\leq L||u-U_h^n||_0||\varphi||_0 \\
&\quad + \sum_{K\in\mathcal{M}^n} Ch_K||R^n||_{0,K}||\nabla\varphi||_{0,\tilde{\omega}_K} + \sum_{e\in\mathcal{B}^n} Ch_e^{\frac{1}{2}}||J_e^n||_{0,e}||\nabla\varphi||_{0,\tilde{\omega}_K} \\
&\leq L||u-U_h^n||_0||\varphi||_0 + C\left(\sum_{K\in\mathcal{M}^n}(\eta_K^n)^2\right)^{\frac{1}{2}}||\nabla\varphi||_0 \\
&\leq L||u-U_h^n||_0||\varphi||_0 + C\eta_{space}^n|||\varphi|||
\end{aligned}
$$

where the second inequality follows from Cauchy-Schwarz inequality.

Set $\varphi = u - U_h$, then use the Lemma 4.4, we get

$$
\begin{aligned}
&\tfrac{d}{dt}||u-U_h||_0^2 + (|||u-U_h^n|||^2 + |||u-U_h|||^2) \\
&\quad \leq 2L||u-U_h^n||_0\cdot||u-U_h||_0 + \ C\eta_{space}^n|||u-U_h||| + |||U_h-U_h^n|||^2
\end{aligned}
\tag{4.2}
$$

By the Young's inequality, namely, for any $a, b > 0$

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{\varepsilon b^2}{2} \qquad \forall\varepsilon > 0,$$

we separate terms $2L||u-U_h||_0$ from $||u-U_h^n||_0$ and the terms $|||u-U_h|||$ from η_{space}^n, by

$$C\eta_{space}^n|||u-U_h||| \leq \frac{C^2}{4}(\eta_{space}^n)^2 + |||u-U_h|||^2, \tag{4.3}$$

$$2L||u-U_h^n||_0\cdot||u-U_h||_0 \leq 2(C_pL)^2||u-U_h||_0^2 + \frac{||u-U_h^n||_0^2}{2C_p^2}. \tag{4.4}$$

Note, in (4.3), we used $\varepsilon = 2$ and in (4.4) we used $\varepsilon = \frac{1}{C_p^2}$.

By Lemma 3.2, so $\frac{||u-U_h^n||_0^2}{2C_p^2} \leq \frac{|||u-U_h^n|||^2}{2}$.

Substituting them in main inequality and cancelling the term $|||u-U_h|||^2$ in both sides, we get

$$\frac{d}{dt}||u-U_h||_0^2+\frac{1}{2}|||u-U_h^n|||^2 \;\leq\; 2(C_pL)^2||u-U_h||_0^2+C_1(\eta_{space}^n)^2+|||U_h-U_h^n|||^2.$$

Since $0\leq \frac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)$ and

$$\begin{aligned}\tfrac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2) &\leq e^{2(C_pL)^2t}\tfrac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)\\ &= \tfrac{d}{dt}||u-U_h||_0^2-2(C_pL)^2||u-U_h||_0^2,\end{aligned}$$

then we obtain the result

$$\frac{d}{dt}(e^{-2(C_pL)^2t}||u-U_h||_0^2)+\frac{1}{2}|||u-U_h^n|||^2 \;\leq\; C_1(\eta_{space}^n)^2+|||U_h-U_h^n|||^2.$$

□

Corollary 4.6. *If $L<\frac{1}{\sqrt{2C_p^2}}$, then*

$$\frac{d}{dt}||u-U_h||_0^2+\frac{1}{2}|||u-U_h^n|||^2\leq C_2(\eta_{space}^n)^2+|||U_h-U_h^n|||^2$$

Proof. From the inequality (4.2) in the proof of Lemma (4.5), we have

$$\begin{aligned}&\tfrac{d}{dt}||u-U_h||_0^2+(|||u-U_h^n|||^2+|||u-U_h|||^2)\\ &\quad\leq\; 2L||u-U_h^n||_0\cdot||u-U_h||_0+C\eta_{space}^n|||u-U_h|||\;+|||U_h-U_h^n|||^2.\end{aligned}$$

We apply Young's inequality to the first 2 terms on the right side by

$$2L||u-U_h^n||_0\cdot||u-U_h||_0\leq\frac{L^2}{2\varepsilon_1}||u-U_h||_0^2+2\varepsilon_1||u-U_h^n||_0^2, \tag{4.5}$$

$$C\eta_{space}^n|||u-U_h|||\leq\frac{C}{2\varepsilon_2}(\eta_{space}^n)^2+\frac{\varepsilon_2C}{2}|||u-U_h|||^2 \tag{4.6}$$

where we choose ε_1 and ε_2 such that $\frac{(C_pL)^2}{2\varepsilon_1}+\frac{C\varepsilon_2}{2}\leq 1$. This implies that we have to choose

$$\varepsilon_1 > \frac{(C_pL)^2}{2}.$$

By (4.5), (4.6) and Lemma 3.2 we get,

$$\frac{d}{dt}||u - U_h||_0^2 + |||u - U_h^n|||^2 \leq 2C_p^2\varepsilon_1|||u - U_h^n|||^2 + \frac{C}{2\varepsilon_2}(\eta_{space}^n)^2 + |||U_h - U_h^n|||^2$$

Since $L < \frac{1}{\sqrt{2C_p^2}}$, so $\frac{(C_pL)^2}{2} < \frac{1}{4C_p^2}$, and by choosing $\varepsilon_1 = \frac{1}{4C_p^2}$, then

$$\frac{d}{dt}||u - U_h||_0^2 + \frac{1}{2}|||u - U_h^n|||^2 \leq C_2(\eta_{space}^n)^2 + |||U_h - U_h^n|||^2$$

□

Theorem 4.7. *(Upper Bound) For any integer $1 \leq m \leq N$, under the assumption of Lemma 4.5, there exists a constant $C_1 > 0$ depending only on the shape constant κ of meshes $\mathcal{M}^n$, the coefficient $a(x)$, Lipschitz constant L and domain Ω such that the following error estimate holds*

$$\begin{aligned} e^{-2(C_pL)^2t^m}||u^m - U_h^m||_0^2 &+ \frac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}|||u - U_h^n|||^2dt \\ &\leq ||u_0 - U_h^0||_0^2 + \sum_{n=1}^{m}\tau_n(\eta_{time}^n)^2 + C_1\sum_{n=1}^{m}\tau_n(\eta_{space}^n)^2. \end{aligned}$$

Proof. From the Lemma 4.5, we combined the errors from time $t = 0$ to time $t = t^n$. Integrating to collect the error from $t = 0$ to $t = t^m$, we get

$$\begin{aligned} e^{-2(C_pL)^2t^m}||u^m - U_h^m||_0^2 &+ \frac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}|||u - U_h^n|||^2dt \\ &\leq ||u_0 - U_h^0||_0^2 + \sum_{n=1}^{m}\int_{t^{n-1}}^{t^n}|||U_h - U_h^n|||^2dt + C_1\sum_{n=1}^{m}\tau_n(\eta_{space}^n)^2. \end{aligned}$$

Note that $\int_{t^{n-1}}^{t^n} \|U_h - U_h^n\|^2 dt = \int_{t^{n-1}}^{t^n} (\frac{t-t^n}{\tau_n})^2 \|U_h^n - U_h^{n-1}\|^2 dt = \tau_n (\eta_{time}^n)^2.$ □

Corollary 4.8. *If we assume $L < \frac{1}{\sqrt{2}C_p^2}$, we obtain a sharper estimate,(without the assumption of Lemma 4.5)*

$$||u^m - U_h^m||_0^2 + \frac{1}{2}\sum_{n=1}^{m}\int_{t^{n-1}}^{t^n} \|u - U_h^n\|^2 dt \leq ||u_0 - U_h^0||_0^2 + \sum_{n=1}^{m} \tau_n (\eta_{time}^n)^2 + C_2 \sum_{n=1}^{m} \tau_n (\eta_{space}^n)^2.$$

Proof. We integrate Corollary 4.6 from $t = 0$ to $t = t^m$ to get the result. □

4.2 Local Lower Bound

The local lower bound is used for improving the finite element solutions at the fixed time $t = t^n$, with the given initial data as the solution from the previous time step $U_h^{n-1} \in V_0^{n-1}$. To compare the error, we consider $U_*^n \in H_0^1(\Omega)$, a solution of the auxiliary problem

$$(\frac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi)_0 + b(U_*^n, \varphi) = (f_*^n, \varphi)_0 \qquad \forall \varphi \in H_0^1(\Omega), \tag{4.7}$$

where $f_*^n := f(U_*^n)$.

Note 4.9. The equation in (4.7) is the corresponding weak form for the discrete problem (3.4) where $H_0^1(\Omega)$ is approximated by V_0^n.

Again, we measured the local error $U_*^n - U_h^n$ using the L^2-norm. Since error indicators η_K^n consist of 2 parts, the area-based and edge-based residuals, to bound the error indicators, we estimated the two residuals using the idea of element and edge bubble functions.

For convenience, we denote the square of error on element $K \in \mathcal{M}^n$ by

$$err_n^2(K) = \frac{h_K^2 ||U_*^n - U_h^n||_{0,K}^2}{\tau_n^2} + |U_*^n - U_h^n|_{1,K}^2.$$

Lemma 4.10. *(Error Representation) For any $\varphi \in H_0^1(\Omega)$,*

$$b(U_*^n - U_h^n, \varphi) = (f_*^n - f_h^n, \varphi)_0 - (\frac{U_*^n - U_h^n}{\tau_n}, \varphi)_0 + \sum_{K \in \mathcal{M}^n} \int_K R^n \varphi dx + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds$$

Proof. Let $\varphi \in H_0^1(\Omega)$.

$$\begin{aligned} &b(U_*^n - U_h^n, \varphi) \\ &= b(U_*^n, \varphi) - b(U_h^n, \varphi) \\ &= \left[(f_*^n, \varphi)_0 - (\tfrac{U_*^n - U_h^{n-1}}{\tau_n}, \varphi)_0\right] + \sum_{K \in \mathcal{M}^n} \int_K \nabla \cdot (a \nabla U_h^n) \varphi \, dx \\ &\quad + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds \\ &= (f_*^n - f_h^n, \varphi)_0 - (\tfrac{U_*^n - U_h^n}{\tau_n}, \varphi)_0 + \sum_{K \in \mathcal{M}^n} \int_K R^n \varphi dx + \sum_{e \in \mathcal{B}^n} \int_e J_e^n \varphi \, ds \end{aligned}$$

□

4.2.1 Estimate of R^n

First, let $\mathcal{P}_K : L^2(K) \to \mathbb{P}_l(K)$ be a L^2-projection to a space of polynomials of degree $\leq l$ on K.

Lemma 4.11. *For $n \geq 1$ and $K \in \mathcal{M}^n$, we have the estimate, there exist constants $c_1, c_2 > 0$ such that*

$$\begin{aligned} h_K^2 ||\mathcal{P}_K R^n||_{0,K}^2 &\leq c_1 (h_K^2 ||\mathcal{P}_K R^n - R^n||_{0,K}^2 + h_K^2 ||f_h^n - f_*^n||_{0,K}^2) \\ &\quad + c_2 err_n^2(K). \end{aligned}$$

Proof. Let $K \in \mathcal{M}^n$ and ψ_K be the element bubble function for the element K. Define $w = \psi_K \cdot \mathcal{P}_K R^n$. Note that $w \in \mathbb{P}_l(K)$ since ψ_K and $\mathcal{P}_K R^n$ are polynomials.

By proposition of bubble function, so

$$C^{-1}||\mathcal{P}_K R^n||_{0,K}^2 \le ||\psi_K^{\frac{1}{2}} \mathcal{P}_K R^n||_{0,K}^2 = (\mathcal{P}_K R^n, w)_{0,K}.$$

Since $w|_{\partial K} = 0$, we can extend w to the full domain Ω by letting $w = 0$ outside element K, so that $w \in H_0^1(\Omega)$.

Thus, $(\mathcal{P}_K R^n, w)_0 = (\mathcal{P}_K R^n, w)_{0,K}$.

By the Lemma 4.10, we set $\varphi = w$, so

$$(R^n, w)_{0,K} = (f_h^n - f_*^n, w)_{0,K} + (\frac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}$$

and

$$\begin{aligned}
(\mathcal{P}_K R^n, w)_{0,K} & \\
&= (\mathcal{P}_K R^n - R^n, w)_{0,K} + (R^n, w)_{0,K} \\
&= (\mathcal{P}_K R^n - R^n, w)_{0,K} + (f_h^n - f_*^n, w)_{0,K} \\
&\quad + (\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}
\end{aligned}$$

Thus, we get the inequality

$$\begin{aligned}
C^{-1}||\mathcal{P}_K R^n||_{0,K}^2 &\le (\mathcal{P}_K R^n - R^n, w)_{0,K} + (f_h^n - f_*^n, w)_{0,K} \\
&\quad + (\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,K} + (a\nabla(U_*^n - U_h^n), \nabla w)_{0,K}
\end{aligned}$$

Then we apply Cauchy-Schwarz inequality to the above inequality.

By proposition of the bubble function and $w \in \mathbb{P}_l$, so $||\nabla w||_{0,K} \le c h_K^{-1} ||w||_{0,K}$ and $||w||_{0,K} \le ||\mathcal{P}_K R^n||_{0,K}$.

Apply Cauchy-Schwarz and get

$$\begin{aligned}
||\mathcal{P}_K R^n||_{0,K} &\le C\left(||\mathcal{P}_K R^n - R^n||_{0,K} + ||f_h^n - f_*^n||_{0,K}\right) \\
&\quad + C\left(||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K} + h_K^{-1}|U_*^n - U_h^n|_{1,K}\right).
\end{aligned}$$

We multiply the inequality by h_K and get

$$\begin{aligned} h_K\|\mathcal{P}_K R^n\|_{0,K} &\leq C\left(h_K\|\mathcal{P}_K R^n - R^n\|_{0,K} + h_K\|f_h^n - f_*^n\|_{0,K}\right) \\ &+C\left(h_K\|\tfrac{U_*^n - U_h^n}{\tau_n}\|_{0,K} + |U_*^n - U_h^n|_{1,K}\right). \end{aligned}$$

From the fact, if $a, b, c \geq 0$ and $a \leq b + c$ then $a^2 \leq 2(b^2 + c^2)$. We square the both sides of the inequality to get

$$\begin{aligned} h_K^2\|\mathcal{P}_K R^n\|_{0,K}^2 &\leq C\left(h_K^2\|\mathcal{P}_K R^n - R^n\|_{0,K}^2 + h_K^2\|f_h^n - f_*^n\|_{0,K}^2\right) \\ &+C\left(\tfrac{h_K^2}{\tau_n^2}\|U_*^n - U_h^n\|_{0,K}^2 + |U_*^n - U_h^n|_{1,K}^2\right). \end{aligned}$$

Now, by definition of $err_n^2(K)$, we complete the proof. □

Lemma 4.12. *For $n \geq 1$ and $K \in \mathcal{M}^n$, we have the estimate, there exist constants $c_3, c_4 > 0$ such that*

$$\begin{aligned} h_K^2\|R^n\|_{0,K}^2 &\leq c_3(h_K^2\|\mathcal{P}_K R^n - R^n\|_{0,K}^2 + h_K^2\|f_h^n - f_*^n\|_{0,K}^2) \\ &+c_4 err_n^2(K). \end{aligned}$$

Proof. By triangle inequality,

$$\|R^n\|_{0,K} = \|\mathcal{P}_K R^n + (R^n - \mathcal{P}_K R^n)\|_{0,K} \leq \|\mathcal{P}_K R^n\|_{0,K} + \|R^n - \mathcal{P}_K R^n\|_{0,K}.$$

We multiply the inequality by h_K and square on both side to get

$$h_K^2\|R^n\|_{0,K}^2 \leq 2(h_K^2\|\mathcal{P}_K R^n\|_{0,K}^2 + h_K^2\|R^n - \mathcal{P}_K R^n\|_{0,K}^2).$$

Apply the Lemma 4.11 and complete the proof. □

4.2.2 Estimate of J_e^n

Let $\mathcal{P}_e : L^2(e) \to \mathbb{P}_l(e)$ be a L^2-projection onto the space of polynomials on e of degree $\leq l$.

Lemma 4.13. *For any $n \geq 1$ and $e \in \mathcal{B}^n$, the $h_e||\mathcal{P}_e J_e^n||_{0,e}^2$ can be bounded by*

$$\begin{aligned} h_e||\mathcal{P}_e J_e^n||_{0,e}^2 &\leq c_5 h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + c_6 \sum_{K' \subset \omega_e} err_n^2(K') \\ &\quad + c_7 \sum_{K' \subset \omega_e} \left(h_{K'}^2 ||R^n||_{0,K'}^2 + h_{K'}^2 ||f_h^n - f_*^n||_{0,K}^2 \right) . \end{aligned}$$

Proof. Let $e \in \mathcal{B}^n$ and ψ_e be the bubble function for the edge e.

Since J_e^n is a function define on the edge e, we can extend J_e^n constantly along the normal of e to ω_e.

Define $w = \psi_e \cdot \mathcal{P}_e J_e^n$. Since supp $w = \omega_e$, we can extend w by $w = 0$ outside ω_e, so that $w \in H_0^1(\Omega)$. Note that $w \in \mathbb{P}_l(\omega_e)$ since ψ_e and $\mathcal{P}_e J_e^n$ are both polynomials.

By proposition of bubble function for the edge e,

$$C^{-1}||\mathcal{P}_e J_e^n||_{0,e}^2 \leq ||\psi_e^{\frac{1}{2}} \mathcal{P}_e J_e^n||_{0,e}^2 = (\mathcal{P}_e J_e^n, w)_{0,e}.$$

By the Lemma 4.10, we set $\varphi = w$, so

$$\begin{aligned} (J_e^n, w)_{0,e} &= (f_*^n - f_h^n, w)_{0,\omega_e} + (R^n, w)_{0,\omega_e} \\ &\quad -(\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,\omega_e} - (a\nabla(U_*^n - U_h^n), \nabla w)_{0,\omega_e}, \end{aligned}$$

so

$$\begin{aligned} C^{-1}||\mathcal{P}_eJ_e^n||_{0,e}^2 &\leq (\mathcal{P}_eJ_e^n, w)_{0,e} \\ &= (\mathcal{P}_eJ_e^n - J_e^n, w)_{0,e} + (J_e^n, w)_{0,e} \\ &= (\mathcal{P}_eJ_e^n - J_e^n, w)_{0,e} + (f_*^n - f_h^n, w)_{0,\omega_e} + (R^n, w)_{0,\omega_e} \\ &\quad -(\tfrac{U_*^n - U_h^n}{\tau_n}, w)_{0,\omega_e} - (a\nabla(U_*^n - U_h^n), \nabla w)_{0,\omega_e} \end{aligned}$$

We apply Cauchy-Schwarz inequality and proposition of bubble function such that

$$\begin{aligned} ||\nabla w||_{0,K} &\leq ch_K^{-1}||w||_{0,K} \\ ||w||_{0,K} &\leq ch_e^{\frac{1}{2}}||\mathcal{P}_eJ_e^n||_{0,e} \\ ||w||_{0,e} &\leq ||\mathcal{P}_eJ_e^n||_{0,e}, \end{aligned}$$

then

$$\begin{aligned} ||\mathcal{P}_eJ_e^n||_{0,e} &\leq C||\mathcal{P}_eJ_e^n - J_e^n||_{0,e} + C\sum_{K'\subset\omega_e} h_{K'}^{\frac{1}{2}}(||f_*^n - f_h^n||_{0,K'} + ||R^n||_{0,K'}) \\ &\quad + C\sum_{K'\subset\omega_e}\left(h_{K'}^{\frac{1}{2}}||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K'} + h_e^{-\frac{1}{2}}|U_*^n - U_h^n|_{1,K'}\right). \end{aligned}$$

We multiply the above inequality by $h_e^{\frac{1}{2}}$ and get

$$\begin{aligned} h_e^{\frac{1}{2}}||\mathcal{P}_eJ_e^n||_{0,e} &\leq Ch_e^{\frac{1}{2}}||\mathcal{P}_eJ_e^n - J_e^n||_{0,e} + C\sum_{K'\subset\omega_e} h_{K'}(||f_*^n - f_h^n||_{0,K'} + ||R^n||_{0,K'}) \\ &\quad + C\sum_{K'\subset\omega_e}\left(h_{K'}||\tfrac{U_*^n - U_h^n}{\tau_n}||_{0,K'} + |U_*^n - U_h^n|_{1,K'}\right). \end{aligned}$$

Square the both sides of inequality and get

$$\begin{aligned} h_e||\mathcal{P}_eJ_e^n||_{0,e}^2 &\leq Ch_e||\mathcal{P}_eJ_e^n - J_e^n||_{0,e}^2 + C\sum_{K'\subset\omega_e} h_{K'}^2\left(||f_*^n - f_h^n||_{0,K'}^2 + ||R^n||_{0,K'}^2\right) \\ &\quad + C\sum_{K'\subset\omega_e}\left(\tfrac{h_e^2}{\tau_n^2}||U_*^n - U_h^n||_{0,K'}^2 + |U_*^n - U_h^n|_{1,K'}^2\right). \end{aligned}$$

By definition of $err_n^2(K')$, we complete the proof. □

Lemma 4.14. *For any $n \geq 1$ and $e \in \mathcal{B}^n$, the $h_e||J_e^n||_{0,e}^2$ can be bounded by*

$$h_e||J_e^n||_{0,e}^2 \leq c_8 h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e} + c_9 \sum_{K' \subset \omega_e} err_n^2(K')$$
$$+c_{10} \sum_{K' \subset \omega_e} \left(h_{K'}^2||R^n||_{0,K'}^2 + h_{K'}^2||f_h^n - f_*^n||_{0,K}^2\right).$$

Proof. By triangle inequality, so

$$||J_e^n||_{0,e} = ||\mathcal{P}_e J_e^n + (J_e^n - \mathcal{P}_e J_e^n)||_{0,e} \leq ||\mathcal{P}_e J_e^n||_{0,e} + ||J_e^n - \mathcal{P}_e J_e^n||_{0,e}.$$

We multiply the inequality by $h_e^{\frac{1}{2}}$ and square on both sides to get

$$h_e||J_e^n||_{0,e}^2 \leq 2(h_e||\mathcal{P}_e J_e^n||_{0,K}^2 + h_e||J_e^n - \mathcal{P}_e J_e^n||_{0,K}^2).$$

Apply the Lemma 4.13 and complete the proof. □

4.2.3 Estimate of the error indicator η_K^n

Define an oscillation on $K \in \mathcal{M}^n$ by

$$osc^2(K) = h_K^2||\mathcal{P}_K R^n - R^n||_{0,K}^2 + \sum_{e \subset \partial K} h_e||\mathcal{P}_e J_e^n - J_e^n||_{0,e}^2$$

and

$$osc^2(\omega_K) = \sum_{K \subset \omega_K} osc^2(K).$$

Theorem 4.15. *(Local Lower Bound) There exist constants $\hat{C}_1, \hat{C}_2 > 0$ depending on Lipschitz constant L, such that for any $K \in \mathcal{M}^n$, the following estimate holds*

$$(\eta_K)^2 \leq \hat{C}_1 osc^2(\omega_K) + \hat{C}_2 \sum_{K' \subset \omega_K} err_n^2(K')$$

Proof. By definition of η_K^n in (4.1) and Lemma 4.14, we get

$$
\begin{aligned}
(\eta_K^n)^2 \;\le\;& h_K^2 ||R^n||_{0,K}^2 + C \sum_{e\subset\partial K} \Bigg\{ h_e ||\mathcal{P}_e J_e^n - J_e^n||_{0,e}^2 \\
&+ \sum_{K'\subset\omega_e} \left[h_{K'}^2 \left(||f_*^n - f_h^n||_{0,K'}^2 + ||R^n||_{0,K'}^2 \right) + err_n^2(K') \right] \Bigg\} .
\end{aligned}
$$

Since $\omega_K = \bigcup_{e\subset\partial K} \omega_e$,

$$
\begin{aligned}
(\eta_K^n)^2 \;\le\;& C \sum_{K'\subset\omega_K} h_{K'}^2 ||R^n||_{0,K'}^2 + C \sum_{e\subset\partial K} h_e ||\mathcal{P}_e J_e^n - J_e^n||_{0,e}^2 \\
&+ C \sum_{K'\subset\omega_K} h_K^2 ||f_*^n - f_h^n||_{0,K'}^2 + C \sum_{K'\subset\omega_K} err_n^2(K') .
\end{aligned}
$$

By Lemma 4.12 and Lipschitz condition of function f, we get

$$
\begin{aligned}
(\eta_K^n)^2 \;\le\;& CL^2 \sum_{K'\subset\omega_K} h_{K'}^2 ||U_*^n - U_h^n||_{0,K'}^2 \\
&+ \hat{C}_1 osc^2(\omega_K) + \hat{C}_2 \sum_{K'\subset\omega_K} err_n^2(K') . \\
\le\;& \hat{C}_1 osc^2(\omega_K) + \hat{C}_2 \sum_{K'\subset\omega_K} err_n^2(K')
\end{aligned}
$$

□