

## CHAPTER III

### MODEL PROBLEM

In this Chapter, we introduced the model problem, a semi-linear parabolic PDE with some assumptions used in this thesis. Thereafter, we formulated the variational problem and discretized the problem in order to use a finite element method.

#### 3.1 Model Problem

Let  $\Omega$  be a bounded polygonal domain in  $\mathbb{R}^2$  with boundary denoted by  $\Gamma = \partial\Omega$  and a final time  $T > 0$ . We consider a semi-linear parabolic PDE

$$\begin{aligned}\frac{\partial u}{\partial t} - \nabla \cdot (a(x)\nabla u) &= f(u(x, t)) && \text{in } \Omega \times (0, T) \\ u &= 0 && \text{on } \Gamma \times (0, T) \\ u &= u_0 && \text{on } \Omega \times \{t = 0\},\end{aligned}$$

where  $u_0 \in L^2(\Omega)$ ,  $a(x) \in L^\infty(\Omega)$  is a positive function ( $a(x) \geq \gamma$  for some  $\gamma > 0$ ) and the function  $f(u(\cdot, t)) \in L^2(\Omega)$  satisfying the Lipschitz condition, i.e., there exists a constant  $L > 0$  such that for each fixed  $t$ ,

$$\|f(u_1) - f(u_2)\|_0 \leq L\|u_1 - u_2\|_0 \quad \forall u_1(\cdot, t), u_2(\cdot, t) \in L^2(\Omega). \quad (3.1)$$

To obtain the weak form, we multiply the PDE by  $\varphi \in H_0^1(\Omega)$  and apply Green's theorem (see [9] page 459) to get

$$\left(\frac{\partial u}{\partial t}, \varphi\right)_0 + (a\nabla u, \nabla \varphi)_0 = (f, \varphi)_0, \quad (3.2)$$

where  $(v, w)_0 = \int_{\Omega} vw \, dx$ .

We define bilinear form  $b(\cdot, \cdot)$  by

$$b(v, w) = (a(x)\nabla v, \nabla w)_0 = \int_{\Omega} a(x)\nabla v \cdot \nabla w \, dx \quad \forall v, w \in H_0^1(\Omega).$$

**Lemma 3.1.** *The bilinear form  $b(\cdot, \cdot)$  is a continuous symmetric and coercive on  $H_0^1(\Omega)$ .*

*Proof.* First, we will show that a bilinear  $b(\cdot, \cdot)$  is a continuous. We need to show that there exists  $c > 0$  such that  $|b(u, v)| \leq c\|u\|_1 \|v\|_1$  for any  $u, v \in H_0^1(\Omega)$ .

Let  $u, v \in H_0^1(\Omega)$ . Since  $a(x) \in L^\infty(\Omega)$  in bounded domain  $\Omega$ , so

$$\begin{aligned} |b(u, v)| &= \left| \int_{\Omega} a(x)\nabla u \cdot \nabla v \, dx \right| \\ &\leq \int_{\Omega} \|a\|_{L^\infty(\Omega)} |\nabla u \cdot \nabla v| \, dx \\ &\leq \|a\|_{L^\infty(\Omega)} \int_{\Omega} |\nabla u| |\nabla v| \, dx \\ &\leq \|a\|_{L^\infty(\Omega)} \|u\|_1 \|v\|_1 \\ &\leq c\|u\|_1 \|v\|_1, \end{aligned}$$

where the last 2 steps follow from the Cauchy-Schwarz inequality and the norm equivalent of  $\|\cdot\|_1$  and  $|\cdot|_1$  on  $H_0^1(\Omega)$ , (see Theorem 2.6.) Note that the constant  $c := \|a\|_{L^\infty(\Omega)}$  depends only on function  $a(x)$ .

Next, we will show that a bilinear form  $b(\cdot, \cdot)$  is a symmetric and coercive in  $H_0^1(\Omega)$ . It easy to see that  $b(\cdot, \cdot)$  is a symmetric by the definition. To show that  $b(\cdot, \cdot)$  is coercive in  $H_0^1(\Omega)$ . Let  $v \in H_0^1(\Omega)$ . Since  $a(x)$  is a positive function in  $L^\infty(\Omega)$ , so

$a(x) \geq \gamma > 0$  a.e.  $x \in \Omega$  and

$$\begin{aligned}
 b(v, v) &= \int_{\Omega} a(x) \nabla v \cdot \nabla v \, dx \\
 &\geq \gamma \int_{\Omega} \nabla v \cdot \nabla v \, dx \\
 &= \gamma \int_{\Omega} |\nabla v|^2 \, dx \\
 &= \gamma \|\nabla v\|_0^2 \\
 &= \gamma |v|_1^2
 \end{aligned}$$

Since semi-norm  $|\cdot|_1$  and norm  $\|\cdot\|_1$  on  $H_0^1(\Omega)$  are equivalent, so by Theorem 2.6 with  $m = 1$ ,  $|v|_1 \geq \frac{1}{(1+s)} \|\nabla v\|_1$  and

$$b(v, v) \geq \frac{\gamma}{(1+s)^2} \|\nabla v\|_1^2.$$

Hence,  $b(\cdot, \cdot)$  is coercive in  $H_0^1(\Omega)$  with coercive constant  $\alpha = \frac{\gamma}{(1+s)^2}$ . □

Since  $b(\cdot, \cdot)$  is coercive and continuous in  $H_0^1(\Omega)$ , the energy norm

$$\|\varphi\| := \sqrt{b(\varphi, \varphi)} \quad \forall \varphi \in H_0^1(\Omega),$$

is equivalent to  $\|\cdot\|_1$ -norm by Remark 2.8.

**Lemma 3.2.** *For any  $\varphi \in H_0^1(\Omega)$ , there exists a constant  $C_p > 0$  such that*

$$\|\varphi\|_0 \leq C_p \|\varphi\|.$$

*Proof.* Let  $\varphi \in H_0^1(\Omega)$ . By Theorem 2.5 and Remark 2.8,

$$\|\varphi\|_0 \leq s|\varphi|_1 \leq s\|\varphi\|_1 \leq sC_e \|\varphi\|$$



where  $C_p := sC_e$ . □

In order to approximate weak solution, we assume the uniqueness and existence of weak solution in (3.2). To obtain the discrete problem, we divided this procedure into 2 steps.

1. Discretization on time  $(0, T)$ .

First, we partition  $(0, T)$  into  $N$  subintervals  $(t^{n-1}, t^n)$ ,  $n = 1, 2, \dots, N$  where  $t^0 = 0$  and  $t^N = T$ .

We define the  $n$ -th time-step size by

$$\tau_n = t^n - t^{n-1}, \quad \text{for } n = 1, 2, \dots, N.$$

It follows that

$$t^m = \sum_{n=1}^m \tau_n, \quad \text{for } m = 1, 2, \dots, N,$$

Consider at the time  $t = t^n$ , from the weak form in (3.2)

$$\left( \frac{\partial u}{\partial t}(t^n), \varphi \right)_0 + b(u^n, \varphi) = (f^n, \varphi)_0 \quad \forall \varphi \in H_0^1(\Omega),$$

where  $u^n := u(x, t^n)$  and  $f^n := f(u^n)$ .

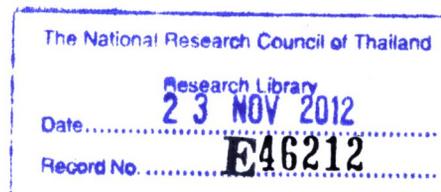
Next, we approximate  $\frac{\partial u}{\partial t}$  by the backward Euler, namely,  $\frac{\partial u}{\partial t} \Big|_{t=t^n} \approx \frac{u^n - u^{n-1}}{\tau_n}$ , so

$$\left( \frac{u^n - u^{n-1}}{\tau_n}, \varphi \right)_0 + b(u^n, \varphi) \approx (f^n, \varphi)_0 \quad \forall \varphi \in H_0^1(\Omega). \quad (3.3)$$

This approximation is used in the finite element scheme.

2. Discretization on space  $\Omega$ .

With a given initial triangulation  $\mathcal{M}^0$  of  $\Omega$ , for  $n \geq 1$ , let  $\{\mathcal{M}^n\}$  be a conforming



and shape-regular family of triangulations where  $\mathcal{M}^n$  is obtained from  $\mathcal{M}^{n-1}$ , does not need to be nested. Let  $V^n$  be a Lagrange finite element space of a continuous piecewise linear functions over the triangulation  $\mathcal{M}^n$  and  $V_0^n = V^n \cap H_0^1(\Omega)$ . Let  $\mathcal{P}_n : H_0^1(\Omega) \rightarrow V_0^n$  be a projection operator for mesh  $\mathcal{M}^n$  and define  $U_h^0 = \mathcal{P}_0 u_0$ . With initial information  $U_h^{n-1} \in V_0^{n-1}$ , we seek an approximation  $U_h^n \in V_0^n$  satisfying the discrete weak form

$$\left( \frac{U_h^n - U_h^{n-1}}{\tau_n}, v \right)_0 + b(U_h^n, v) = (f_h^n, v)_0 \quad \forall v \in V_0^n \quad (3.4)$$

where  $f_h^n := f(U_h^n)$ .

**Note 3.3.** We take  $U_h^{n-1} \in V_0^{n-1} \subset H_0^1(\Omega)$ , a solution from the  $(n-1)$ -th step to be an initial information for the  $n$ -th step in (3.4).

To approximate  $u(x, t)$  for  $t \in (t^{n-1}, t^n)$ , we interpolate linearly between  $U_h^{n-1}$  and  $U_h^n$ , namely, for each  $x \in \Omega$

$$U_h(x, t) := \frac{t^n - t}{\tau_n} U_h^{n-1}(x) + \frac{t - t^{n-1}}{\tau_n} U_h^n(x).$$

**Note 3.4.**  $U_h(x, t^n) = U_h^n(x)$ ,  $U_h(x, t^{n-1}) = U_h^{n-1}$  and  $\frac{\partial U_h}{\partial t} = \frac{U_h^n - U_h^{n-1}}{\tau_n}$  for  $t \in (t^{n-1}, t^n)$  which use in the next Chapter.