CHAPTER II

PRELIMINARY

In this Chapter, we provided some basic knowledge of finite element analysis including definitions and theorems used in the proof of the main results. The proofs of theorems in this Chapter are omitted but can be found in the provided references. This Chapter consists of 3 parts: the Sobolev spaces, the construction of the finite element space, and some approximation results.

2.1 Sobolev Spaces

This section provides some basic knowledge about Sobolev spaces required later in this thesis. To obtain the variational problem from the given PDE problem one need to use functions in some Sobolev spaces. More details about Sobolev spaces can be found in Chapter 2 of [3].

Let Ω be an open subset of $\mathbb{R}^d$ with piecewise smooth boundary. $L^2(\Omega)$ is a set of function $u(x)$ which is square-integrable in the Lebesgue sense over Ω. It is known that $L^2(\Omega)$ is a Hilbert space with inner product [13]

$$(u, v)_0 = \int_\Omega uv\, dx \qquad \forall\, u, v \in L^2(\Omega),$$

with the norm defined by

$$||u||_0 = \sqrt{(u, u)_0}.$$

Definition 2.1. *Given an integer $m \geq 0$, let $H^m(\Omega)$ be the set of all functions u in $L^2(\Omega)$ which possess weak derivatives $\partial^\alpha u$ for all $|\alpha| \leq m$. We can define a scalar product on $H^m(\Omega)$ by*

$$(u, v)_m := \sum_{|\alpha| \leq m} (\partial^\alpha u, \partial^\alpha v)_0,$$

with the norm

$$||u||_m = \sqrt{(u, u)_m} = \sqrt{\sum_{|\alpha| \leq m} ||\partial^\alpha u||_0^2}.$$

And the semi-norm

$$|u|_m = \sqrt{\sum_{|\alpha| = m} ||\partial^\alpha u||_0^2}.$$

In this thesis, we are interested in functions in $H^1(\Omega)$.

Definition 2.2. *The completion of $C_0^\infty(\Omega)$ with respect to the Sobolev norm $||\cdot||_m$ is denoted by $H_0^m(\Omega)$.*

Note 2.3. $H^m(\Omega)$ and $H_0^m(\Omega)$ are Hilbert spaces.

Note 2.4. In this thesis, we only use the spaces $H^1(\Omega)$ and $H_0^1(\Omega)$.

Theorem 2.5. *Suppose Ω is bounded and contained in a d-dimensional cube with side length s. Then*

$$||v||_0 \leq s|v|_1 \qquad \forall\, v \in H_0^1(\Omega).$$

Proof. The proof can be found in the book by D. Braess [3]. □

Theorem 2.6. *If Ω is bounded, then $|\cdot|_m$ is a norm on $H_0^m(\Omega)$ which is equivalent to $||\cdot||_m$. In addition, if Ω is contained in a cube with side length s, then*

$$|v|_m \leq ||v||_m \leq (1+s)^m |v|_m \qquad \forall v \in H_0^1(\Omega).$$

Proof. The proof can be found in the book by D. Braess [3]. □

Definition 2.7. *Let H be a Hilbert space with norm $||\cdot||_H$.*

A bilinear form $b : H \times H \longrightarrow \mathbb{R}$ is called continuous provided there exists $c > 0$ such that

$$|b(u,v)| \leq c||u||_H \, ||v||_H \qquad \forall u, v \in H.$$

A bilinear form $b(\cdot,\cdot)$ is called coercive for a subspace V in H, provided for some $\alpha > 0$,

$$b(v,v) \geq \alpha ||v||_V^2 \qquad \forall v \in V$$

Remark 2.8. *We can define an energy norm on V with coercive bilinear form $b(\cdot,\cdot)$ by $|||v|||_b = \sqrt{b(v,v)}$. The norm $|||\cdot|||_b$ is equivalent to the norm of the Hilbert space $||\cdot||_H$, namely, there exist a constant $C_e > 0$ such that*

$$\frac{1}{C_e}||\cdot||_H \leq |||\cdot|||_b \leq C_e||\cdot||_H.$$

2.2 Standard Finite Element

The goal for this section is to build a finite element space V, a finite dimensional subspace of $H_0^1(\Omega)$, and to introduce some approximation results.

Let Ω be a bounded polygonal domain in $\mathbb{R}^2$.

Definition 2.9. *A partition $\mathcal{M} = \{K_1, K_2, \ldots, K_N\}$ of Ω into triangular subdomains K_i is called a* **triangulation** *of Ω if the following properties holds:*

1. $\bar{\Omega} = \bigcup_{i=1}^{N} K_i$.

2. If $K_i \cap K_j$ consists of exactly one point, then it is a common vertex of K_i and K_j.

3. If for $i \neq j$, $K_i \cap K_j$ consists of more than one point, then $K_i \cap K_j$ is a common edge of K_i and K_j.

Definition 2.10. *A family of triangulation $\{\mathcal{M}_k\}_{k \geq 0}$ is called* ***shape regular*** *provided that there exists a number $\kappa > 0$ such that every K in $\mathcal{M}_k$ and for every k contains a circle of radius ρ_K with*

$$\rho_K \geq \frac{h_K}{\kappa},$$

where h_K is the diameter of element K.

To define a finite element space V, for fixed a non-negative integer h, let $\mathcal{M}_h$ be a shape-regular triangulation of $\Omega \subset \mathbb{R}^2$ and $\mathbb{P}_l$ denote the set of polynomials of degree $\leq l$. Let V be a finite element spaces consisting of continuous piecewise linear functions, defined by

$$V = \{v \in H^1(\Omega) \mid v|_K \in \mathbb{P}_1, \ \forall K \in \mathcal{M}_h\}.$$

Here, we use linear Lagrange elements with nodal basis functions, i.e., for each node x_j of element K, the nodal basis for node x_j is $\phi_j(x_i) = \delta_{ij}$. For each $v \in V$, $v(x) = \sum_{i=1}^{N} v(x_i)\phi_i(x)$ where N is the total number of node.

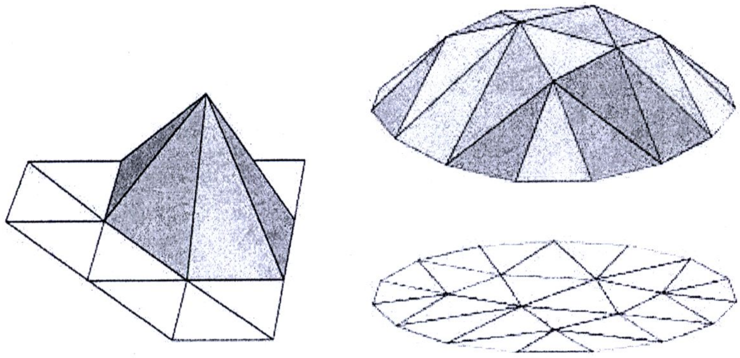

Figure 2.1: Example of nodal basis and a continuous piecewise linear function

2.3 Approximation Results

Let $\mathcal{B}$ be the set of all inter-element boundaries (interior sides) of all elements $K \in \mathcal{M}_h$. We denoted patches as follows:

$$\begin{aligned} \omega_e &= \bigcup\{K \in \mathcal{M}_h \,|\, e \subset \partial K\} && \forall\, e \in \mathcal{B}, \\ \omega_K &= \bigcup_{e \subset \partial K} \omega_e && \forall\, K \in \mathcal{M}_h \\ \tilde{\omega}_K &= \bigcup\{K' \in \mathcal{M}_h \,|\, K \cap K' \neq \varnothing\} && \forall\, K \in \mathcal{M}_h \end{aligned}$$

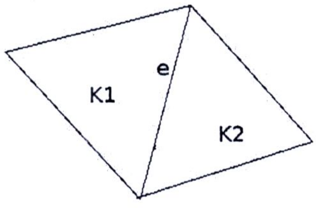

Figure 2.2: The example of the patch ω_e for the edge e

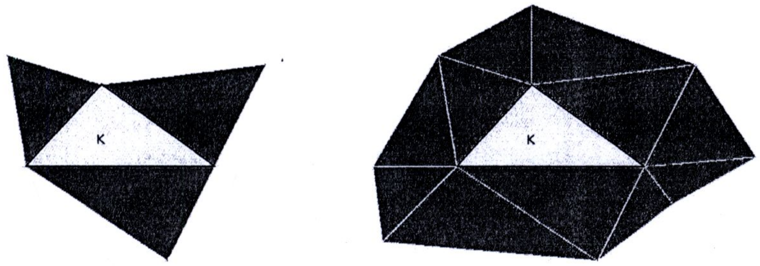

Figure 2.3: The left picture is the patch ω_K and the right picture is the patch $\widetilde{\omega}_K$

We state some important theorems and properties used in the proof of the main results as follows.

Theorem 2.11. *(Clement Interpolation Approximation) Let $\mathcal{M}_h$ be a shape-regular triangulation of Ω. Then there exists a linear mapping $\mathcal{I}_h : H^1(\Omega) \to V$ such that*

$$||v - \mathcal{I}_h v||_{0,K} \leq c h_K ||v||_{1,\tilde{\omega}_K} \qquad \forall v \in H^1(\Omega), K \in \mathcal{M}_h, \tag{2.1}$$

$$||v - \mathcal{I}_h v||_{0,e} \leq c h_K^{\frac{1}{2}} ||v||_{1,\tilde{\omega}_K} \qquad \forall v \in H^1(\Omega), e \subset \partial K, K \in \mathcal{M}_h. \tag{2.2}$$

Proof. The proof can be found in [6] by Ph. Clement . □

The Clement's interpolation approximations are the main ingredients for obtaining the upper bound in the error estimates. To obtain the local lower bound, we used the ideas of bubble functions. There are 2 types of bubble functions, element bubble functions and edge bubble functions. The definitions and properties are given below.

Definition 2.12. *Let $K \in \mathcal{M}_h$ and $e \in \mathcal{B}$. The functions ψ_K, ψ_e are the bubble functions corresponding to K and e, respectively, with properties:*

$$\psi_K \in \mathbb{P}_3,\ supp\ \psi_K = K,\ 0 \leq \psi_K \leq 1,\ \max \psi_K = 1,$$

and

$$\psi_e \in \mathbb{P}_2,\ supp\ \psi_e = \omega_e,\ 0 \le \psi_e \le 1,\ \max \psi_e = 1.$$

Proposition 2.13. *Let $\mathcal{M}_h$ be a shape-regular triangulation. Then there exists a constant c which depends only on the shape parameter κ such that*

$$\begin{aligned}
||\psi_K v||_{0,K} &\le ||v||_{0,K} && \forall v \in L^2(K),\\
||\psi_K^{\frac{1}{2}} p||_{0,K} &\ge c||p||_{0,K} && \forall p \in \mathbb{P}_l,\\
||\nabla(\psi_K p)||_{0,K} &\le ch_K^{-1}||\psi_K p||_{0,K} && \forall p \in \mathbb{P}_l,\\
||\psi_e^{\frac{1}{2}} \sigma||_{0,e} &\ge c||\sigma||_{0,e} && \forall \sigma \in \mathbb{P}_l,\\
ch_e^{\frac{1}{2}}||\sigma||_{0,e} \le ||\psi_e E\sigma||_{0,K} &\le ch_e^{\frac{1}{2}}||\sigma||_{0,K} && \forall \sigma \in \mathbb{P}_l,\\
||\nabla(\psi_e E\sigma)||_{0,K} &\le ch_K^{-1}||\psi_e E\sigma||_{0,K} && \forall \sigma \in \mathbb{P}_l,
\end{aligned}$$

where $E : L^2(e) \longrightarrow L^2(\omega_e)$ is an extension function on an edge e and h_e is the length of the edge e.

Proof. The proof can be found in [14] by R. Verfurth and [1] by M. Ainsworth and J.T. Oden. □