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Original Article

Periodic review inventory policy with variable ordering cost, lead time, and backorder rate

Nughthoh Arfawi Kurdhi[1*] and Rumi Iqbal Doewes[2]

[1] *Department of Mathematics, Faculty of Mathematics and Natural Science, Sebelas Maret University, Surakarta, 57126 Indonesia*

[2] *Faculty of Teacher Training and Education, Sebelas Maret University, Surakarta, 57126 Indonesia*

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Abstract

In this paper, a stochastic periodic review inventory model is developed. The backorder rate (backorder price discount), ordering cost (safety stock), lead time, and review period are treated as decision variables. The ordering cost and lead time can be controlled by using capital investment and crashing cost, respectively. It is assumed that shortages are allowed and partially backlogged. If an item is out of stock, the supplier may offer a negotiable price discount to the loyal, tolerant and obliged customers to pay off the inconvenience of backordering. Furthermore, it is assumed that the protection interval demand follows a normal distribution. Our objective is to develop an algorithm to determine the optimal decision variables, so that the total expected annual cost incurred has a minimum value. Finally, a numerical example is presented to illustrate the solution procedure, and sensitivity analysis is carried out to analyze the proposed model. The numerical results show that a significant amount of savings can be obtained by making decisions with capital investment in reducing ordering cost.

Keywords: periodic review, capital investment, price discount, ordering cost, stochastic demand, partial backlogging

1. Introduction

Optimal inventory policies have been subject to a lot of research in recent years. In traditional inventory systems, most of the literature treating inventory problems, either in the continuous review or periodic review models, the ordering (setup) cost, lead time, and backorder price discount, are regarded as prescribed constants and equal at the optimum. However, the experience of the Japanese indicates that this need not be the case. In practice, ordering cost may be controlled and reduced by virtue of various efforts, such as procedural changes, worker training, and specialized equipment acquisition. In the literature, Porteus (1985) first introduced the concept of investing in reducing the ordering cost in the classical economic order quantity (EOQ) model and determined an optimal ordering cost level. The framework has encouraged many researchers, such as Huang *et al.* (2011), Kurdhi *et al.* (2016), Lo (2013), Sarkar *et al.* (2015a), Vijayashree and Uthayakumar (2014, 2016), to examine ordering cost reduction. The papers have reported the relationship between the amount of capital investment and ordering cost level. If the ordering cost per order could be controlled and reduced effectively, the total relevant cost per unit time could be automatically improved. Therefore, this article deals with one important aspect of just-in-time (JIT) philosophy, i.e., reduction of ordering cost where the ordering cost varies as a function of capital expense.

Further, many companies recognize the significance of response time as a competitive weapon and have used time as a means of differentiating themselves in the marketplace (Pan & Hsiao, 2005). Lead time is the elapsed time between releasing an order and receiving it. Lead time usually consists of the following components: order preparation, order transit,

*Corresponding author
Email address: arfa@mipa.uns.ac.id

manufacture and assembly, transit, and uncrating, inspection and transport (Jaggi & Arneja, 2010; Joshi & Soni, 2011; Sana & Goyal, 2015; Tersine, 1982; Vijayashree & Uthayakumar, 2014; Yang *et al.*, 2016). In much of the literature, lead time is regarded as decision variable and can be decomposed into several components, each having a crashing cost function for the respective reduced lead time. According to Jaggi and Arneja (2010), the extra cost of reducing lead time involves transportation, administrative and supplier's speed-up costs. Hsu and Lee (2009) stated that crashing cost could be expenditures on information technology, equipment improvement, expedited order, or special shipping and handling. By shortening the lead time, the safety stock and stockout loss can be reduced, and the customer service level can be improved so as to gain competitive advantages in business. Chandra and Grabis (2008) indicated that short lead time could enhance the service level and lower inventory level effectively. As the Japanese example of just-in-time production has shown, consequently reducing lead times may increase productivity and improve the competitive position of the company (Tersine & Hummingbird, 1995; Vijayashree & Uthayakumar, 2015). Hence, lead time reduction has been one of the most offered themes for both practitioners and researchers. In the literature, lead time and ordering cost reductions in the continuous review inventory models have been continually modified (e.g., Gholami-Qadikolaei *et al.*, 2012; MA & QIU, 2012; Priyan & Uthayakumar, 2015) so as to accommodate more practical features of the real production or inventory systems. It is noted that the reduction of lead time and of ordering cost in a periodic review inventory model is quite sparse.

In classical inventory models dealing with the problem of shortages, it was often assumed that during the stockout period, shortages are either completely lost or completely backorderd. However, in many market situations, it can often be observed that some customers may refuse the backorder case, and some may prefer their demand to be backordered while shortages occur. In today's highly competitive market providing varieties of products to the customers due to globalization, partial backorder is a more realistic one (Bhowmick & Samanta, 2012). We can often observe that many fashionable products such as hi-fi equipment, certain brand gum shoes, and clothes may lead to a situation in which customers may prefer to wait for backorders when shortages occur. Moreover, the image of the selling shop is another one of the potential factors that can motivate customers intention to backorder. When a shortage occurs, there are some factors that motivate the customer to make backorders out of which price discount from the supplier is the major factor. By offering sufficient price discounts, the supplier can secure more backorders through negotiation. The higher the price discounts of a supplier, the higher the advantage of the customers, and hence, a higher backorder rate may result.

Jaggi and Arneja (2010) studied a periodic inventory model with backorder price discount, where shortages are partially backlogged. Lin (2015) explored the problem that the lead time and ordering cost reductions are inter-dependent in a periodic review inventory model with backorder price discount. Sarkar *et al.* (2015b) proposed a continuous review inventory model with order quantity, reorder point, backorder price discount, process quality, and lead time as decision variables. Kurdhi (2016) investigated an integrated inventory model with backorder price discount and variable lead time. Jindal and Solanki (2016) studied an integrated supply chain inventory model with quality improvement involving controllable lead time and backorder price discount. In this paper, the backorder price discount has been taken as one of the decision variables. The consideration is the unsatisfied demand during the shortages can lead to optimal backorder ratio by controlling the backorder price discount, and the supplier is to minimize the relevant total inventory cost (Table 1).

Table 1. Comparison between the contributions of different authors.

Author(s)	Periodic review	Price discount	Variable ordering cost	Variable lead time	Capital investment	Normal distribution
Chandra and Grabis (2009)				√		
Hsu and Lee (2009)	√			√		
Jaggi and Arneja (2010)	√	√		√		√
Joshi and Soni (2011)				√		
Huang *et al.* (2011)				√	√	√
MA and QIU (2012)			√	√	√	
Gholami-Qadikolaei *et al.* (2012)				√		√
Lin (2012)			√	√	√	√
Lo (2013)				√	√	√
Viyayashree and Uthayakumar (2014)				√	√	√
Lin (2015)	√	√		√	√	
Priyan and Uthayakumar (2015)		√	√	√	√	√
Sarkar (2015a)			√		√	
Sarkar (2015b)		√		√	√	√
Viyayashree and Uthayakumar (2015)				√	√	√
Sana and Goyal (2015)				√		√
Kurdhi (2016)	√	√		√		√
Kurdhi *et al.* (2016)			√	√	√	√
Jindal and Solanki (2016)		√		√		√
Vijayashree and Uthayakumar (2016)					√	
This model	√	√	√	√	√	√

The applications of the periodic inventory models can often be found in managing inventory cases such as smaller retail stores, grocery stores and drugstores. For this, in contrast to the continuous review inventory model, a periodic review model will be investigated by considering backorder price discount, controllable lead time, and capital investment to accommodate more practical feature of the real inventory system. In this paper, the logarithmic function of investment cost in need to analyze the effects of increasing investment to reduce the ordering cost. Besides, the lead time can be decomposed into several mutually independent components each having a different crashing cost for shortening lead time. Moreover, there is an option in which while a shortage occurs, a price discount can always be offered on the stockout to secure more backorders. The protection interval demand follows a normal distribution. Furthermore, a computational algorithm with the help of the software Mathematica 8 is furnished to find the optimal values of the decision-making variables. Finally, some numerical examples and sensitivity analysis are given to illustrate the solution procedure of the proposed model and the effects of the parameters.

2. Notations and Assumptions

The following notations are used throughout the paper in order to develop the mathematical model:

D : average demand (units per unit time)

A_0 : initial ordering cost per order

$I(A)$: capital investment required to achieve ordering cost A, $0<A\leq A_0$

h : inventory holding cost per unit per unit time

R : target stock level

β_0 : upper bound of the backorder rate

π_x : backorder price discount offered by the supplier per unit

π_0 : marginal profit (i.e. cost of lost demand) per unit

$C(L)$: total crashing cost of a cycle

X : demand during the protection interval, $T+L$, which has a probability density function (*p.d.f.*) f_X with finite mean $D(T+L)$ and standard deviation $\sigma\sqrt{T+L}$

η : fractional opportunity cost of capital per year

Z_α : safety factor

α : stock-out probability

ϕ : standard normal distribution

Φ : standard normal cumulative distribution function

$E(\cdot)$: mathematical expectation

x^+ : maximum value of x and 0, i.e., $x^+ = \max\{x, 0\}$.

Decision variables

A : ordering cost per order

β : backorder rate, $0 < \beta < 1$

T : length of a review period (unit time)

L : length of lead time (unit time)

In addition, the following assumptions are made.

a. The inventory level is reviewed every T units of time. A sufficient quantity is ordered up to the target stock level R, and the ordering quantity is received after L units of time. The length of the lead time L does not exceed an inventory cycle time T so that there is never more than a single order outstanding in any cycle.

b. The target stock level R = expected demand during the protection interval + safety stock (SS), and $SS = Z_\alpha \times$ (standard deviation of demand during protection interval $(T+L)$), i.e., $R = D(T+L) + Z_\alpha\sigma\sqrt{T+L}$, where $P(X > R) = \alpha$.

c. During the stock-out period, the backorder rate, β, is variable and is in proportion to the price discount offered by the supplier per unit π_x. The backorder rate is defined as $\beta = \beta_0\pi_x/\pi_0$, where $0 \leq \beta_0 < 1$ and $0 \leq \pi_x < \pi_0$, β_0 is the upper bound of the backorder rate, π_x is the backorder price discount offered by the supplier per unit, and π_0 is the marginal profit per unit.

d. The lead time L can be decomposed into n mutually independent components, each of which has a different crashing cost for reduced lead time. The i-th component has a normal duration b_i, minimum duration a_i, and crashing cost per unit time c_i. For convenience, c_i is assumed to be arranged such that $c_1 \leq c_2 \leq \ldots \leq c_n$. The components of lead time are crashed one at a time starting from the first component because it has the minimum unit crashing cost, and then the second component, and so on. Let G_i be the length of lead time with component i crashed to its minimum duration, $i = 1,2, \ldots, n$, then G_i can be expressed as:

$$G_i = \sum_{j=1}^{n} b_j - \sum_{j=1}^{i} (b_j - a_j).$$

The lead time crashing cost for a given $L \in [G_i, G_{i-1}]$ is given by:

$$C(L) = c_i(G_{i-1} - L) + \sum_{j=1}^{i-1} c_j(b_j - a_j).$$

e. The option of investing in reducing ordering cost is available. The investment required to reduce the ordering cost from initial ordering cost A_0 to a target level A is denoted by $I(A)$, where $I(A)$ is a convex and strictly decreasing function. It is assumed that the capital investment in reducing ordering cost is a logarithmic function of the ordering cost A. That is

$$I(A) = \frac{1}{\delta}\ln\left(\frac{A_0}{A}\right) \quad \text{for} \quad 0 < A \leq A_0,$$

where δ is a percentage decreased in ordering cost, A, per dollar and increased in investment $I(A)$. This function is consistent with the Japanese experience (Hall, 1983) and has been utilized by Lin (2012), Kurdhi *et al.* (2016), and others.

3. Model Development

In this section a quantitative model is provided for how managers should allocate investments in ordering cost reduction program. Using the same approach as in Montgomery *et al.* (1973) and Lin (2014) for the periodic review case, the expected net inventory level at the beginning of the period and at the end of the period are $R-DL+(1-\beta)E(X-R)^+$ and $R-DL-DT+(1-\beta)E(X-R)^+$, respectively, where $E(X-R)^+$ is the expected number of shortages at the end of the cycle. Thus, the expected annual holding cost is approximately $h\left[R-DL-\frac{DT}{2}+(1-\beta)E(X-R)^+\right]$, and the expected stock-out is $\frac{1}{T}[\beta\pi_x+(1-\beta)\pi_0]E(X-R)^+$. Hence, for the model without ordering cost reduction, the total expected annual cost, which is composed of ordering cost, holding cost, stock-out cost, and lead time crashing cost is expressed as

$$\begin{aligned} EAC(T,\beta,L) &= \frac{A}{T}+h\left[R-DL-\frac{DT}{2}+(1-\beta)E(X-R)^+\right] \\ &\quad +\frac{1}{T}(\beta\pi_x+(1-\beta)\pi_0)E(X-R)^+ +\frac{C(L)}{T}. \end{aligned} \tag{1}$$

Due to $\beta=\beta_0\pi_x/\pi_0$ (by assumption 3), the backorder price discount offered by a supplier, π_x, can be treated as a decision variable instead of the backorder ratio, β. Then, model (1) can be rewritten as

$$\begin{aligned} EAC(T,\pi_x,L) &= \frac{A}{T}+h[R-DL-\frac{DT}{2}+\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)E(X-R)^+] \\ &\quad +\frac{1}{T}\left(\frac{\beta_0\pi_x^2}{\pi_0}+\pi_0-\beta_0\pi_x\right)E(X-R)^+ +\frac{C(L)}{T}. \end{aligned} \tag{2}$$

The model proceed further by the assumption that the protection interval demand follows a normal distribution distribution with mean $D(T+L)$ and standard deviation $\sigma\sqrt{(T+L)}$. Since $R=D(T+L)+Z_\alpha\sigma\sqrt{(T+L)}$, the expected shortage quantity $E(X-R)^+$ at the end of the cycle can be expressed as

$$E(X-R)^+=\int_R^\infty (x-R)\frac{1}{\sigma\sqrt{2\pi(T+L)}}e^{-\frac{1}{2}\left(\frac{x-D(T+L)}{\sigma\sqrt{(T+L)}}\right)^2}dx=\sigma\sqrt{(T+L)}\psi(Z_\alpha), \tag{3}$$

where $\psi(Z_\alpha)=\phi(Z_\alpha)-k[1-\Phi(Z_\alpha)]$. Substituting (3) in (2), the total expected annual cost becomes

$$\begin{aligned} EAC(T,\pi_x,L) &= \frac{A+C(L)}{T}+h\left(\frac{DT}{2}+Z_\alpha\sigma\sqrt{(T+L)}\right) \\ &\quad +\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)+\frac{1}{T}\left(\frac{\beta_0\pi_x^2}{\pi_0}+\pi_0-\beta_0\pi_x\right)\right]\sigma\sqrt{(T+L)}\psi(Z_\alpha). \end{aligned} \tag{4}$$

In this section, the ordering cost, A, is considered as a decision variable and we seek to minimize the sum of the capital investment cost of reducing ordering cost and the inventory related costs (as expressed in (4)) by optimizing over T, π_x, L, and A, constrained on $0<A\le A_0$. Mathematically, the problem can be formulated as

$$\begin{aligned} \text{Min } EAC(T,\pi_x,A,L) &= \eta I(A)+\frac{A+C(L)}{T}+h\left(\frac{DT}{2}+Z_\alpha\sigma\sqrt{(T+L)}\right) \\ &\quad +[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right)+\frac{1}{T}\left(\frac{\beta_0\pi_x^2}{\pi_0}+\pi_0-\beta_0\pi_x\right)]\sigma\sqrt{(T+L)}\psi(Z_\alpha), \end{aligned} \tag{5}$$

subject to $0<A\le A_0$.

In this case, the ordering cost level is $A\in(0,A_0]$, which implies that if the optimal ordering cost obtained does not satisfy the restriction on A, then no ordering cost investment is made.

4. Solution Procedure

In order to determine the minimum cost for the nonlinear programming problem in (5), the restriction $0<A\le A_0$ can be ignored for the moment and the total relevant cost function over T, π_x, L, and A can be minimized by classical optimization techniques by taking the first partial derivatives of $EAC(T,\pi_x,A,L)$ with respect to T, π_x, A, and $L\in[G_i,G_{i-1}]$, respectively. It is obtained that

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial T} = -\frac{A}{T^2} + h\left(\frac{D}{2} + \frac{Z_\alpha\sigma}{2\sqrt{T+L}}\right) - \frac{C(L)}{T^2} - \frac{\sqrt{T+L}\sigma\left(\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}\right)\psi(Z_\alpha)}{T^2}$$
$$+ \frac{\sigma\left(h\left(1-\frac{\pi_x\beta_0}{\pi_0}\right) + \frac{\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}}{T}\right)\psi(Z_\alpha)}{2\sqrt{T+L}},$$
$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial \pi_x} = \sigma\beta_0\sqrt{T+L}\left(-\frac{h}{\pi_0} + \frac{1}{T}\left(\frac{2\pi_x}{\pi_0} - 1\right)\right)\psi(Z_\alpha), \tag{6}$$
$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial L} = -\frac{c_i}{T} + \frac{hZ_\alpha\sigma}{2\sqrt{T+L}} + \frac{\sigma\left(h\left(1-\frac{\pi_x\beta_0}{\pi_0}\right) + \frac{\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}}{T}\right)\psi(Z_\alpha)}{2\sqrt{T+L}}$$
$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial A} = \frac{1}{T} - \frac{\eta}{\delta A}.$$

By examining the second-order sufficient conditions, it can be shown that $EAC(T,\pi_x,A,L)$ is not a convex function of (T,π_x,A,L). However, for fixed (T,π_x,A), $EAC(T,\pi_x,A,L)$ is concave in $L \in [G_i,G_{i-1}]$, since

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial L^2} = -\frac{hZ_\alpha\sigma}{4(T+L)^{\frac{3}{2}}} - \frac{\sigma\left(h\left(1-\frac{\pi_x\beta_0}{\pi_0}\right) + \frac{\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}}{T}\right)\psi(Z_\alpha)}{4(T+L)^{\frac{3}{2}}} < 0.$$

Therefore, for fixed (T, π_x, A), the minimum total expected annual cost will occur at the end points of the interval $[G_i,G_{i-1}]$. Consequently, the problem is reduced to

$$\text{Min } EAC(T,\pi_x,A,G_i) = \eta I(A) + \frac{A+C(G_i)}{T} + h\left(\frac{DT}{2} + Z_\alpha\sigma\sqrt{(T+G_i)}\right)$$
$$+\left[h\left(1-\frac{\beta_0\pi_x}{\pi_0}\right) + \frac{1}{T}\left(\frac{\beta_0\pi_x^2}{\pi_0} + \pi_0 - \beta_0\pi_x\right)\right]\sigma\sqrt{(T+G_i)}\psi(Z_\alpha),$$

subject to $0 < A \le A_0$, $i = 0,1,2,\dots,n$.

On the other hand, for a given value of $L \in [G_i,G_{i-1}]$, by solving the equations $\frac{\partial EAC(T,\pi_x,A,L)}{\partial T} = 0$, $\frac{\partial EAC(T,\pi_x,A,L)}{\partial \pi_x} = 0$ and $\frac{\partial EAC(T,\pi_x,A,L)}{\partial A} = 0$, it is obtained that

$$\frac{A^*+C(L)}{T^{*2}} = h\frac{D}{2} + h\frac{\sigma}{2\sqrt{T^*+L}}\left(Z_\alpha + \left(1 - \frac{\beta_0}{\pi_0}\frac{hT^*+\pi_0}{2}\right)\psi(Z_\alpha)\right) + \sigma\psi(Z_\alpha)\frac{\left(\beta_0\frac{\left(\frac{hT^*+\pi_0}{2}\right)^2}{\pi_0} + \pi_0 - \beta_0\left(\frac{hT^*+\pi_0}{2}\right)\right)}{2T^*\sqrt{T^*+L}}$$
$$-\frac{\left(\beta_0\frac{\left(\frac{hT^*+\pi_0}{2}\right)^2}{\pi_0} + \pi_0 - \beta_0\left(\frac{hT^*+\pi_0}{2}\right)\right)}{T^{*2}}\sigma\sqrt{T^*+L}\psi(Z_\alpha), \tag{7}$$

$$\pi_x^* = \frac{T^*h+\pi_0}{2}, \tag{8}$$

and

$$A^* = \eta\alpha T^*. \tag{9}$$

Theoretically, for fixed $L \in [G_i, G_{i-1}]$, from (7), (8) and (9), the values of $T^*, \pi_x^*,$ and A^* can be obtained. Furthermore, the second-order sufficient conditions will be verified. For fixed $L \in [G_i, G_{i-1}]$, let us consider the Hessian matrix **H** as follows:

$$H = \begin{pmatrix} \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T^2} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T\,\partial \pi_x} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T\,\partial A} \\ \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial T} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x^2} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial A} \\ \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A\,\partial T} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A\,\partial \pi_x} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A^2} \end{pmatrix}. \tag{10}$$

Taking the second partial derivatives of $EAC(T, \pi_x, L, A)$ with respect to $T^*, \pi_x^*,$ and A, it is obtained that

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial T^2} = \frac{2(A+C(L))}{T^3} - \frac{\sigma\left(\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}\right)\psi(Z_\alpha)}{T^2\sqrt{L+T}} + \frac{2\sqrt{L+T}\sigma\left(\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}\right)\psi(Z_\alpha)}{T^3}$$

$$-\frac{hZ_\alpha\sigma}{4(L+T)^{\frac{3}{2}}} - \frac{\sigma\left(h\left(1-\frac{\pi_x\beta_0}{\pi_0}\right) + \frac{\pi_0 - \pi_x\beta_0 + \frac{\pi_x^2\beta_0}{\pi_0}}{T}\right)\psi(Z_\alpha)}{4(L+T)^{\frac{3}{2}}}$$

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial \pi_x^2} = \frac{2\sigma\beta_0\sqrt{T+L}\psi(Z_\alpha)}{T\pi_0},$$

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial A^2} = \frac{\eta}{\delta A^2},$$

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial T\,\partial \pi_x} = \frac{\partial EAC(T,\pi_x,L,A)}{\partial \pi_x\,\partial T}$$

$$= -\frac{h\sigma\beta_0\psi(Z_\alpha)}{2\pi_0\sqrt{T+L}} + \frac{\sigma\beta_0\pi_x\psi(Z_\alpha)}{T\sqrt{T+L}\pi_0} - \frac{\sigma\beta_0\psi(Z_\alpha)}{2T\sqrt{T+L}} - \frac{2\sigma\sqrt{T+L}\beta_0\pi_x\psi(Z_\alpha)}{T^2\pi_0} + \frac{\sigma\beta_0\sqrt{T+L}\psi(Z_\alpha)}{T^2},$$

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial T\,\partial A} = \frac{\partial EAC(T,\pi_x,L,A)}{\partial A\,\partial T} = -\frac{1}{T^2},$$

$$\frac{\partial EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial A} = \frac{\partial EAC(T,\pi_x,L,A)}{\partial A\,\partial \pi_x} = 0,$$

We proceed by evaluating the principal minor determinant of the Hessian matrix **H** at point (T^*, π_x^*, A^*). The first and second principal minor determinant of **H** then become

$$|H_{11}| = \frac{2(A+C(L))}{T^3} - \left[\frac{1}{4}h\sigma\left(T+L\right)^{-\frac{3}{2}}\left(Z_\alpha + \left(1-\frac{\beta_0\pi_x}{\pi_0}\right)\psi(Z_\alpha)\right)\right]$$

$$-\frac{1}{T^2}(T+L)^{-\frac{1}{2}}\left(\pi_0 - \beta_0\pi_x + \frac{\beta_0\pi_x^2}{\pi_0}\right)\sigma\psi(Z_\alpha) + \frac{2}{T^3}(T+L)^{-\frac{1}{2}}\left(\pi_0 - \beta_0\pi_x + \frac{\beta_0\pi_x^2}{\pi_0}\right)\sigma\psi(Z_\alpha)$$

$$> \frac{A(3T+4L)}{2T^3(T+L)} + \frac{(T+4L)\left[\pi_0 - \beta_0\pi_x + \frac{\beta_0\pi_x^2}{\pi_0}\right]\sigma\psi(Z_\alpha)}{2T^3\sqrt{T+L}} > 0$$

and

$$|H_{22}| = \begin{vmatrix} \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T^2} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T\,\partial \pi_x} \\ \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial T} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x^2} \end{vmatrix}$$

$$= \left[\frac{(T+L)\beta_0\sigma^2\psi^2(Z_\alpha)}{T^4\pi_0^2}\right]\left[\pi_0^2(1-\beta_0)+\left(\frac{3\beta_0 hT}{2}+\frac{3\beta_0\pi_0}{2}\right)\left(\pi_0-\frac{Th}{2}-\frac{\pi_0}{2}\right)\right]$$

$$= \left[\frac{(T+L)\beta_0\sigma^2\psi^2(Z_\alpha)}{T^4\pi_0^2}\right]\left[\pi_0^2(1-\beta_0)+3\beta_0\pi_x(\pi_0-\pi_x)\right] > 0.$$

Next, the third principal minor determinant of **H** is

$$|H_{33}| = \begin{vmatrix} \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T^2} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T\,\partial \pi_x} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial T\,\partial A} \\ \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial T} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x^2} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial \pi_x\,\partial A} \\ \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A\,\partial T} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A\,\partial \pi_x} & \frac{\partial^2 EAC(T,\pi_x,A,L)}{\partial A^2} \end{vmatrix}$$

$$= \frac{\eta\alpha}{A^2}\left(\frac{(T+L)\beta_0\sigma^2\psi^2(Z_\alpha)}{T^4\pi_0^2}\right)\left[\pi_0^2(4-\beta_0)-3\beta_0(hT)^2\right]-\left(\frac{2\beta_0\sigma\sqrt{T+L}\psi(Z_\alpha)}{T\pi_0}\right)\left(-\frac{1}{T^2}\right)^2$$
$$= \frac{\eta\alpha}{A^2}\left(\frac{(T+L)\beta_0\sigma^2\psi^2(Z_\alpha)}{T^4\pi_0^2}\right)\left[\pi_0^2(4-\beta_0)-3\beta_0(hT)^2\right]-\left(\frac{2\beta_0\sigma\sqrt{T+L}\psi(Z_\alpha)}{T^5\pi_0}\right).$$

It is difficult to determine mathematically the sign of $|H_{33}|$. However, since $|H_{11}|$ and $|H_{22}|$ are all positive, then (T^*, π_x^*, A^*) are the optimal solution if $|H_{33}| > 0$.

Furthermore, it is not possible to determine the closed-form solution for (T^*, π_x^*, A^*) from (7), (8) and (9). However, the optimal value of (T^*, π_x^*, A^*) can be obtained by adopting an iterative technique similar to that used in Kurdhi (2016), Lin (2015), and others. Thus, the following iterative algorithm is developed to find the optimal values for the length of a review period, backorder price discount, ordering cost, and lead time.

Algorithm 1

Step 1. For each $G_i, i = 0,1,2,\ldots,n$, and a given α (and hence, the value of safety factor Z_α can be found directly from the normal distribution table), perform (i) to (iv).

- (i) Start with $A_{i1} = A_0$.
- (ii) Substituting A_{i1} into (7) evaluates T_{i1}.
- (iii) Utilizing T_{i1} determines A_{i2} from (9).
- (iv) Repeat (ii) to (iii) until no change occurs in the values of T_i and A_i.

Step 2. Compare A_i and A_0.

- (i) If $A_i \le A_0$, then A_i is feasible and the solution found in Step 1 for given G_i is denoted by (T_{G_i}, A_{G_i}).
- (ii) If $A_i > A_0$, then A_i is not feasible and for given G_i, take $A_{G_i} = A_0$ and the corresponding value of T_{G_i} can be obtained by substituting A_{G_i} into (7).

Step 3. Compute π_{x_i} from (8) and then compare π_{x_i} and π_0.

- (i) If $\pi_{x_i} \le \pi_0$, then π_{x_i} is feasible and the solution found in Step 2 for given G_i is denoted by $(T_{G_i}, \pi_{x_{G_i}}, A_{G_i})$.
- (ii) If $\pi_{x_i} > \pi_0$, then π_{x_i} is not feasible and for given G_i, take $\pi_{x_i} = \pi_0$.

Step 4. For each $(T_{G_i}, \pi_{x_{G_i}}, A_{G_i}, G_i), i = 1,2,\ldots,n$, compute the corresponding total expected annual cost $EAC(T_{G_i}, \pi_{x_{G_i}}, A_{G_i}, G_i)$, utilizing (5).

Step 5. Find $\min_{i=1,2,\ldots,n} EAC(T_{G_i}, \pi_{xG_i}, A_{G_i}, G_i)$.

If $EAC(T_s, \pi_{x_S}, A_s, L_s) = \min_{i=1,2,\ldots,n} EAC(T_{G_i}, \pi_{xG_i}, A_{G_i}, G_i)$, then $(T_s, \pi_{x_S}, A_s, L_s)$ is the optimal solution.

Note that, once $(T_s, \pi_{x_S}, A_s, L_s)$ is obtained, the optimal target level $R_s = D(T_s + L_s) + k\sigma\sqrt{T_s + L_s}$ and the optimal backorder rate $\beta_s = \beta_0 \pi_{x_S}/\pi_0$ follow.

5. Numerical Example

The numerical examples given below illustrate the above solution procedure. Consider a periodic review inventory system with the following data: D = 600 units/year, A_0 = \$200/order, h = \$20/unit/year, π_0 = \$150/unit, σ = 7 units/week, and the lead time has three components with data shown in Table 2.

Table 2. Lead time data

Lead time component, i	Normal duration, b_i (days)	Minimum duration, a_i (days)	Unit crashing cost, c_i (\$/day)
1	20	6	0.4
2	20	6	1.2
3	16	9	5.0

Suppose that the protection interval demands follows a normal distribution. Consider the cases when the upper bounds of the backorder rate β_0 = 0.2, 0.35, 0.5, 0.65, 0.8 and 0.95, and α = 0.2 (in this situation, the value of safety factor, Z_α, can be found directly from the standard normal distribution table and is 0.845). Applying the proposed algorithm procedure yields the results shown in Table 3. From this table, the optimal inventory policy can easily be found by comparing $EAC(T_i, \pi_{x_i}, A_i, G_i)$, for $i = 0, 1, 2, 3$, and these are summarized in Table 4. Moreover, the optimal results of the fixed ordering cost model are listed in the same table to illustrate the effects of investing in ordering cost reduction. From Table 4, it can be observed that increasing

Table 3. Solution procedures (T_i, L_i in week)

β_0	i	L_i	T_i	A_i	π_{x_i}	R_i	$EAC(T_i, \pi_{x_i}, L_i, A_i)$	$\lvert H_{33}\rvert$
0.2	0	8	13.31	90.61	77.58	112.2	3512.12	30.79
	1	6	12.73	85.85	77.45	86.84	3302.80	31.71
	2	4	12.27	81.25	77.32	61.00	3092.35	29.08
	3	3	12.54	81.98	77.34	47.95	3088.15*	20.36
0.35	0	8	13.06	89.04	77.54	112.2	3453.67	57.24
	1	6	12.50	84.41	77.41	86.79	3249.01	58.72
	2	4	12.06	80.00	77.28	60.95	3045.45*	53.33
	3	3	12.34	80.91	77.31	47.91	3047.65	36.73
0.5	0	8	12.81	87.43	77.49	112.1	3394.11	54.10
	1	6	12.26	82.93	77.36	86.73	3194.22	79.92
	2	4	11.84	78.72	77.24	60.90	2997.76*	88.94
	3	3	12.15	79.82	77.28	47.87	3006.57	87.04
0.65	0	8	12.56	85.79	77.45	112.1	3333.37	120.7
	1	6	12.02	81.42	77.32	86.68	3138.38	122.8
	2	8	11.62	77.41	77.21	60.85	2949.22*	109.1
	3	3	11.95	78.71	77.24	47.83	2964.90	72.50
0.8	0	8	12.30	84.11	77.40	112.0	3271.38	158.8
	1	6	11.78	79.88	77.28	86.62	3081.42	160.9
	2	4	11.39	76.08	77.17	60.80	2899.80*	141.1
	3	3	11.75	77.58	77.21	47.78	2922.60	91.97
0.95	0	8	12.03	82.40	77.35	111.9	3208.05	112.5
	1	6	11.53	78.30	77.23	86.56	3023.25	176.4
	2	4	11.17	74.72	77.13	60.75	2849.43*	203.8
	3	3	11.54	76.43	77.18	47.74	2879.63	202.3

Table 4. Summary of the optimal solutions

Ordering cost reduction model							Fixed ordering cost model ($A = 200$)	
β_0	L_s	T_s	π_{x_s}	A_s	R_s	$EAC(\cdot)$	$EAC(\cdot)$	Savings (%)
0.20	3	12.54	77.34	81.89	47.95	3088.15	4746.27	34.93
0.35	4	12.06	77.28	80.00	60.95	3045.45	4672.85	34.82
0.50	4	11.84	77.24	78.72	60.90	2997.76	4598.94	34.81
0.65	4	11.62	77.21	77.41	60.85	2949.22	4524.55	34.81
0.80	4	11.39	77.17	76.08	60.80	2899.80	4449.66	34.83
0.95	4	11.17	77.13	74.72	60.75	2849.43	4374.24	34.85

the value of upper bound of the backorder rate will results in a decrease in the review period, the backorder price discount, ordering cost, and the total expected annual cost. Furthermore, comparing our model with that of the fixed ordering cost case, it can be observed that the savings range from 34.81% to 34.93%, which shows that significant savings can be achieved due to controlling the ordering cost.

In addition, the effects of changes in the system parameters h, D, and σ on the optimal review period, backorder price discount, lead time, ordering cost, and minimum total expected annual cost, will be examined. Using the same data and assumptions, β_0 is fixed at 0.95 and a sensitivity analysis is performed by changing each of the parameters by +50%, +25%, -25%, and -50%, taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Table 5. On the basis of the results of the Table 5, the following observations can be made.

(1) T_s and A_s decrease while π_{x_s} and $EAC(T_s, \pi_{x_s}, A_s, L_s)$ increase with an increase in the value of the holding cost parameter, h. Besides, it can be observed that as the value h changes, the value L_s is not influenced.

(2) T_s, π_{x_s} and A_s decrease, whereas $EAC(T_s, \pi_{x_s}, A_s, L_s)$ increases with an increase in the value of the demand parameter, D. Besides, the value L_s is not influenced by changes in the value of D.

(3) T_s, π_{x_s} A_s, and $EAC(T_s, \pi_{x_s}, A_s, L_s)$ increase while L_s decreases with an increase in the value of the model parameter σ.

Table 5. Effects of change in the parameters

Parameters' value	% of change	Optimum value $(T_s, \pi_{x_s}, L_s, A_s)$	$EAC(T_s, \pi_{x_s}, L_s, A_s)$	Percentage of influence
h=30	+50	(8.68, 77.54, 4, 59.27)	3464.53	+17.75%
25	+25	(9.71, 77.35, 4, 65.73)	3169.05	+10.09%
20	0	(11.17, 77.13, 4, 74.73)	2849.43	0%
15	-25	(13.41, 76.89, 4, 88.41)	2496.10	-14.15%
10	-50	(17.49, 76.61, 4, 112.77)	2090.73	-36.29%
D=900	+50	(8.90, 76.69, 4, 59.33)	3335.95	+14.58%
750	+25	(9.85, 76.88, 4, 65.77)	3104.65	+8.22%
600	0	(11.17, 77.14, 4, 74.73)	2849.43	0%
450	-25	(13.13, 77.52, 4, 88.33)	2560.85	-11.27%
300	-50	(16.50, 78.21, 4, 112.44)	2220.94	-28.30%
σ=10.5	+50	(13.15, 77.49, 3, 87.20)	3380.95	+15.72%
8.75	+25	(12.38, 77.34, 3, 82.03)	3137.28	+9.17%
7	0	(11.17, 77.14, 4, 74.73)	2849.43	0%
5.25	-25	(10.08, 76.92, 4, 67.36)	2524.05	-12.89%
3.5	-50	(8.81, 76.68, 4, 58.86)	2160.50	-31.89%

6. Conclusions

The purpose of this paper is to investigate a mixture inventory policy on a controlling ordering cost in the stochastic periodic review involving controllable backorder price discount and variable lead time for protection interval demand with normal distribution. By analyzing the total expected annual cost, an algorithm is developed to determine the optimal review period, backorder price discount, ordering cost, and lead time so that the total expected annual cost incurred has the minimum value. The results of the numerical examples indicate that by making decisions with capital investment in reducing ordering cost, it would help to lower the system cost, and a significant amount of savings can be obtained. To understand the effects of the optimal solution on changes in the value of the different parameters associated with the inventory system, sensitivity analysis is performed. Furthermore, it can be observed from the sensitivity analysis that the total expected annual cost is more highly sensitive to the changes in the value of holding cost parameter, h, than to the changes in D and σ. This paper is limited in the known demand distribution. In real situations, we often get difficulty in providing a precise estimation on the probability density function due to the insufficiency of historical data. Therefore, for further consideration on this problem, it would be interested to propose a distribution-free model according to the mean and the standard deviation of demand. Moreover, we can deal with a mixed stochastic inventory model in which the stock-out term in the objective function is replaced by a service level constraint. Another possible direction may be followed by considering stochastic periodic review inventory models with controllable safety factor or incorporating defective items and inspection errors in the future extension of the present article.

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