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Original Article

A new exponential estimator for finite population mean under double sampling using one auxiliary variable

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Abstract

This paper proposes an exponential estimator of a population mean under double sampling using one auxiliary variable. The proposed estimator is modified from the estimator of Khatua and Misha (2013) and Ozgul and Cingi (2014). The bias and mean square error of the proposed estimator are derived under the second order of approximation. The efficiency of the proposed estimator is compared with the existing estimators based on mean square error. The results indicate that the proposed estimator is more efficient than others under empirical study with real datasets.

Keywords: auxiliary variable, exponential estimator, double sampling

1. Introduction

In the sampling technique, the researcher considers data about the target population. The population of the study may have only one or several groups depending on the objective of the research. Selecting a representative sample of the population, relies on a statistical method to draw samples from the population. Under simple random sampling without replacement of size *n* from population *N*, a sample mean $\bar{y}=(1/n)\sum_{i=1}^{n}y_i$ is an usual unbiased estimator of the population mean $\bar{Y}=(1/N)\sum_{i=1}^{N}y_i$ where y is a study variable. Mean squared error (MSE) of $\bar{y}$ is $\mathrm{MSE}(\bar{y})=\frac{1-f}{n}\bar{Y}^2C_y^2$,

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where $f = n/N$ and C_y is the coefficient of variation of study variable y .

In addition, there is another form of estimators that use the information of auxiliary variable x to increase efficiency in estimating the population mean. The ratio estimator is used to estimate the population mean when the correlation between the auxiliary and the study variable is positive. Murthy (1964) proposed the ratio estimator with known population mean of auxiliary variable as

$$\bar{y}_R = \bar{y}\left(\bar{X}/\bar{x}\right)$$

where $\bar{X}$ is population mean of the auxiliary variable x . When the correlation between the auxiliary and the study variable is negative, the product estimator with known population mean of auxiliary variable is presented as

$$\bar{y}_P = \bar{y}\left(\bar{x}/\bar{X}\right).$$

Moreover, many authors considered estimators under exponential estimators. Bahl and Tuteja (1991) proposed the ratio and product type exponential estimators, respectively as

$$\bar{y}_{\mathrm{Re}} = \bar{y}\exp\left(\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right) \text{ and } \bar{y}_{Pe} = \bar{y}\exp\left(\frac{\bar{x}-\bar{X}}{\bar{x}+\bar{X}}\right).$$

When the population mean of an auxiliary variable is unknown, then double sampling is utilized. There are two phases for double sampling. For a first phase, a sample of size n_1 of only auxiliary variable is drawn from the population of size N when $n_1 < N$. Then, a second phase sample of size n on auxiliary and study variables are drawn when $n < n_1$ where the second phase sample of size n is a sub-sample of the first phase sample of size n_1 . The ratio and product estimators with unknown population mean of auxiliary variable x in double sampling are respectively obtained by

$$\bar{y}_{Rd} = \bar{y}\left(\bar{x}'/\bar{x}\right) \text{ and } \bar{y}_{Pd} = \bar{y}\left(\bar{x}/\bar{x}'\right).$$

The MSE of the estimators $\bar{y}_{Rd}$ and $\bar{y}_{Pd}$ are given respectively as

$$\mathrm{MSE}\left(\bar{y}_{Rd}\right) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^*\left(C_x^2 - 2\rho C_x C_y\right)\right] \text{ and } \mathrm{MSE}\left(\bar{y}_{Pd}\right) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^*\left(C_x^2 + 2\rho C_x C_y\right)\right]$$

where $\lambda = \frac{1}{n} - \frac{1}{N}$, $\lambda' = \frac{1}{n_1} - \frac{1}{N}$, $\lambda^* = \lambda - \lambda'$, C_x is the coefficient of variation of the auxiliary variable x . ρ is the correlation coefficient between study variable y and auxiliary variable x .

For the recent study, exponential estimators have been widely studied by several authors. Singh and Vishwakarma (2007) proposed the exponential ratio and product estimators of population mean under double sampling,

$$\bar{y}_{\text{Re}Md} = \bar{y}\exp\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right) \text{ and } \bar{y}_{PeMd} = \bar{y}\exp\left(\frac{\bar{x} - \bar{x}'}{\bar{x} + \bar{x}'}\right),$$

respectively where $\bar{x}'$ is sample mean of the auxiliary variable x in the first phase sample of size n_1, $\bar{x}' = (1/n_1)\sum_{i=1}^{n_1} x_i$.

The MSE of the estimators $\bar{y}_{\text{Re}Md}$ and $\bar{y}_{PeMd}$ are given respectively as

$$\text{MSE}(\bar{y}_{\text{Re}Md}) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^*\left(\frac{1}{4}C_x^2 - \rho C_x C_y\right)\right] \text{ and}$$

$$\text{MSE}(\bar{y}_{PeMd}) = \bar{Y}^2\left[\lambda C_y^2 + \lambda^*\left(\frac{1}{4}C_x^2 + \rho C_x C_y\right)\right].$$

Khatua and Misha (2013) suggested exponential estimator under double sampling,

$$\bar{y}_{dge} = \bar{y}\left[d_1 + d_2(\bar{x}' - \bar{x})\right]\exp\left(\frac{\bar{x}' - \bar{x}}{\bar{x}' + \bar{x}}\right),$$

where the optimal value of d_1 and d_2 as $d_1 = \frac{1}{1 + C_y^2[\lambda - (\lambda - \lambda')\rho^2]}$ and $d_2 = \frac{(2\rho C_y - C_x)}{\{1 + C_y^2[\lambda - (\lambda - \lambda')\rho^2]\}2\bar{X}C_x}$,

respectively. The minimum MSE of the estimator $\bar{y}_{dge}$ is given as

$$\text{MSE}(\bar{y}_{dge}) = \frac{S_y^2\left[\lambda - (\lambda - \lambda')\rho^2\right]}{1 + C_y^2\left[\lambda - (\lambda - \lambda')\rho^2\right]}$$

where $S_y^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2$.

Ozgul and Cingi (2014) also proposed exponential regression cum ratio estimator under double sampling as

$$\bar{y}_{NH} = \left[k_1\bar{y} + k_2(\bar{x}' - \bar{x})\right]\exp\left(\frac{\bar{z}' - \bar{z}}{\bar{z} + \bar{z}'}\right)$$

where the optimal value of k_1 and k_2 as $k_1 = 1 - \frac{2 - \lambda^*\theta^2 C_x^2}{1 + (\lambda - \lambda^*\rho^2)}$, $k_2 = \frac{\bar{Y}}{\bar{X}}\left\{(\theta - 1) + \frac{2 - \lambda^*\theta^2 C_x^2}{1 + (\lambda - \lambda^*\rho^2)}(2\theta - K)\right\}$,

respectively, $K = \rho\frac{C_y}{C_x}$ and $\theta = \frac{a\bar{X}}{2(2\bar{X} + b)}$, and $\bar{z}' = a\bar{x}' + b$, $\bar{z} = a\bar{x} + b$ with $a \neq 0$ and b are any known constants.

The minimum MSE of the estimator $\bar{Y}_{NH}$ is given as

$$\text{MSE}(\bar{y}_{NH}) = \bar{Y}^2\frac{C_y^2(\lambda - \lambda^*\rho^2)(1 - \lambda^*\theta^2 C_x^2) - \left(\frac{\lambda^{*2}\theta^4 C_x^4}{4}\right)}{\left[1 + C_y^2(\lambda - \lambda^*\rho^2)\right]}.$$

In this paper, we propose the new estimator of the population mean under double sampling using the exponential method with minimum MSE. The paper is organized as follows. The proposed estimator and its bias and mean square error under the second order of approximation are presented in Section 2. Section 3 shows efficiency comparisons of the proposed estimator with respect to the existing estimators. Numerical study results are presented in Section 4. Finally, Section 5 contains our conclusions.

2. Proposed Exponential Estimator under Double sampling

The proposed estimator is motivated by exponential estimators under double sampling in Khatua and Misha (2013) and Ozgul and Cingi (2014),

$$\bar{y}_{WS} = \bar{y}\left[(1-\alpha)\left(\frac{\bar{x}'-\bar{x}}{\bar{x}}\right)+\alpha\exp\left(\frac{\bar{x}'}{\bar{x}}-1\right)\right] \tag{1}$$

where α is the optimal value and $0<\alpha<1$.

To obtain the bias and MSE of the proposed estimator, we suppose that $e_0=\frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $e_1=\frac{\bar{x}-\bar{X}}{\bar{X}}$, and $e'=\frac{\bar{x}'-\bar{X}}{\bar{X}}$ such that

$$E(e_0)=E(e_1)=E(e_1')=0,$$

$$E(e_0^2)=\lambda C_y^2,\ E(e_1^2)=\lambda C_x^2,\ E(e_1'^2)=\lambda' C_x^2,$$

$$E(e_0e_1)=\lambda\rho C_xC_y,\ E(e_1e_1')=\lambda' C_x^2 \text{ and } E(e_0e_1')=\lambda'\rho C_xC_y$$

where $\lambda=\frac{1}{n}-\frac{1}{N}$, $\lambda'=\frac{1}{n_1}-\frac{1}{N}$, $\lambda^*=\lambda-\lambda'$ (Singhand Vishwakarma, 2007).

We express the proposed estimator $\bar{y}_{WS}$ in terms of e's, and obtain

$$\bar{y}_{WS} = \bar{y}\left[(1-\alpha)\left(\frac{\bar{x}'-\bar{x}}{\bar{x}}\right)+\alpha\exp\left(\frac{\bar{x}'}{\bar{x}}-1\right)\right]$$

$$=\bar{Y}(1+e_0)\left\{(1-\alpha)\left[\frac{\bar{X}(1+e_1')-\bar{X}(1+e_1)}{\bar{X}(1+e_1)}\right]+\alpha\exp\left[\frac{\bar{X}(1+e_1')}{\bar{X}(1+e_1)}-1\right]\right\}$$

$$=\bar{Y}(1+e_0)\left\{(1-\alpha)\frac{(e_1'-e_1)}{(1+e_1)}+\alpha\exp\left[\frac{(1+e_1')}{(1+e_1)}-1\right]\right\}$$

$$=\bar{Y}(1+e_0)\left\{(1-\alpha)\frac{(e_1'-e_1)}{(1+e_1)}+\alpha\exp\left(\frac{e_1'-e_1}{1+e_1}\right)\right\}$$

$$=\bar{Y}(1+e_0)\left\{(1-\alpha)\frac{(e_1'-e_1)}{(1+e_1)}+\alpha f(e_1,e_1')\right\},$$

where $f\left(e_1,e_1'\right)=\exp\left(\dfrac{e_1'-e_1}{1+e_1}\right)$.

To solve the exponential term $f\left(e_1,e_1'\right)=\exp\left(\dfrac{e_1'-e_1}{1+e_1}\right)$, we use Taylor series method (Spiegel, 1968), as

$$f\left(e_1,e_1'\right)=1+\frac{\partial f\left(e_1,e_1'\right)}{\partial e_1}\bigg|_{e_1=e_1'=0}e_1+\frac{\partial f\left(e_1,e_1'\right)}{\partial e_1'}\bigg|_{e_1=e_1'=0}e_1'+\frac{1}{2!}\frac{\partial^2 f\left(e_1,e_1'\right)}{\partial e_1^2}\bigg|_{e_1=e_1'=0}e_1^2$$

$$+\frac{1}{2!}\frac{\partial^2 f\left(e_1,e_1'\right)}{\partial e_1'^2}\bigg|_{e_1=e_1'=0}e_1'^2+\frac{2}{2!}\frac{\partial^2 f\left(e_1,e_1'\right)}{\partial e_1\partial e_1'}\bigg|_{e_1=e_1'=0}e_1e_1'+\ldots.$$

We obtain the exponential term as $f\left(e_1,e_1'\right)=1-e_1+e_1'+\frac{3}{2}e_1^2+\frac{1}{2}e_1'^2-2e_1e_1'+\ldots$.

We have

$$\bar{y}_{WS}=\bar{Y}(1+e_0)\left\{(1-\alpha)\frac{(e_1'-e_1)}{(1+e_1)}+\alpha\left(1-e_1+e_1'+\frac{3}{2}e_1^2+\frac{1}{2}e_1'^2-2e_1e_1'+\ldots\right)\right\}$$

$$=\bar{Y}(1+e_0)\left\{(1-\alpha)\left(e_1'-e_1\right)\left(1-e_1+e_1^2+\ldots\right)+\alpha\left(1-e_1+e_1'+\frac{3}{2}e_1^2+\frac{1}{2}e_1'^2-2e_1e_1'+\ldots\right)\right\}. \tag{2}$$

Expanding the right-hand side of (2), obtaining up to the second order approximation of e's, we get

$$\bar{y}_{WS}=\bar{Y}\left[\alpha+\alpha e_0-e_1+e_1'-e_0e_1+e_0e_1'-\left(1+\alpha\right)e_1e_1'+\left(1+\frac{\alpha}{2}\right)e_1^2+\frac{\alpha}{2}e_1'^2\right]$$

$$\bar{y}_{WS}-\bar{Y}=\bar{Y}\left[\alpha-1+\alpha e_0-e_1+e_1'-e_0e_1+e_0e_1'-\left(1+\alpha\right)e_1e_1'+\left(1+\frac{\alpha}{2}\right)e_1^2+\frac{\alpha}{2}e_1'^2\right]. \tag{3}$$

From (3), the bias and MSE of the estimator are obtained by using $\text{Bias}\left(\bar{y}_{WS}\right)=E\left(\bar{y}_{WS}-\bar{Y}\right)$ and $\text{MSE}\left(\bar{y}_{WS}\right)=E\left(\bar{y}_{WS}-\bar{Y}\right)^2$, respectively.

Taking expectation on both sides of (3), the bias of estimator is given as

$$\text{Bias}\left(\bar{y}_{WS}\right)=\bar{Y}\left[\alpha-1+\alpha E(e_0)-E(e_1)+E(e_1')-E(e_0e_1)+E(e_0e_1')-\left(1+\alpha\right)E(e_1e_1')+\left(1+\frac{\alpha}{2}\right)E(e_1^2)+\frac{\alpha}{2}E(e_1'^2)\right]$$

$$=\bar{Y}\left[\alpha-1-\lambda\rho C_xC_y+\lambda'\rho C_xC_y-\left(1+\alpha\right)\lambda'C_x^2+\left(1+\frac{\alpha}{2}\right)\lambda C_x^2+\frac{\alpha}{2}\lambda'C_x^2\right]$$

$$=\bar{Y}\left[\alpha-1+\left(1+\frac{\alpha}{2}\right)\lambda^*C_x^2-\lambda^*\rho C_xC_y\right]. \tag{4}$$

It turns out that the proposed estimator is a biased estimator. The existing estimators are biased estimators as well.

The MSE of estimator is given as

$$\begin{aligned}\text{MSE}(\bar{y}_{WS}) &= \bar{Y}^2\Big[(\alpha-1)^2+2\alpha(\alpha-1)E(e_0)+2(\alpha-1)E(e_1)-2(\alpha-1)E(e_1')+\alpha^2E(e_0^2)\\ &\quad+(\alpha^2+\alpha-1)E(e_1^2)+(\alpha^2-\alpha+1)E(e_1'^2)-2(2\alpha-1)E(e_0e_1)\\ &\quad+2(2\alpha-1)E(e_0e_1')-2\alpha^2E(e_1e_1')\Big]\\ &= \bar{Y}^2\Big[(\alpha-1)^2+\alpha^2\lambda C_y^2+(\alpha^2+\alpha-1)\lambda^*C_x^2+2(1-2\alpha)\lambda^*\rho C_xC_y\Big]. \end{aligned} \quad (5)$$

The optimal value α can be obtained by minimizing $\text{MSE}(\bar{y}_{WS})$ in (5). Setting the first derivative of $\text{MSE}(\bar{y}_{WS})$ with respect to α equal to zero and solving yields the solution,

$$\alpha^* = \frac{2-\lambda^*C_x^2+4\lambda^*\rho C_xC_y}{2+2\lambda C_y^2+2\lambda C_x^2}. \quad (6)$$

The minimum MSE of proposed estimator is obtained by

$$\text{MSE}_{\min}(\bar{y}_{WS}) = \bar{Y}^2\Big[(\alpha^*-1)^2+\alpha^{*2}\lambda C_y^2+(\alpha^{*2}+\alpha^*-1)\lambda^*C_x^2+2(1-2\alpha^*)\lambda^*\rho C_xC_y\Big].$$

3. Efficiency Comparison

When conditions are satisfied, the proposed estimator ($\bar{y}_{WS}$) is more efficient than the estimators $\bar{y}$, $\bar{y}_{Rd}$, $\bar{y}_{\text{Re}Md}$, $\bar{y}_{dge}$ and $\bar{y}_{NH}$.

(i) $\text{MSE}_{\min}(\bar{y}_{WS}) < \text{MSE}(\bar{y})$ if

$$\rho < \frac{\frac{1-f}{n}C_y^2-(\alpha^*-1)^2-\alpha^{*2}\lambda C_y^2-(\alpha^{*2}+\alpha^*-1)\lambda^*C_x^2}{2(1-2\alpha^*)\lambda^*C_xC_y} \text{ when } 0<\alpha^*<1/2,$$

$$\frac{4(1-f)}{n}C_y^2-\lambda C_y^2+5\lambda^*C_x^2>1 \text{ when } \alpha^*=1/2,$$

$$\rho > \frac{\frac{1-f}{n}C_y^2-(\alpha^*-1)^2-\alpha^{*2}\lambda C_y^2-(\alpha^{*2}+\alpha^*-1)\lambda^*C_x^2}{2(1-2\alpha^*)\lambda^*C_xC_y} \text{ when } 1/2<\alpha^*<1.$$

(ii) $\text{MSE}_{\min}(\bar{y}_{WS}) < \text{MSE}(\bar{y}_{Rd})$ if

$$\rho < \frac{(1-\alpha^{*2})\lambda C_y^2-(\alpha^{*2}+\alpha^*-2)\lambda^*C_x^2-(\alpha^*-1)^2}{4(1-\alpha^*)\lambda^*C_xC_y}.$$

(iii) $\text{MSE}_{\min}(\bar{y}_{WS}) < \text{MSE}(\bar{y}_{\text{Re}Md})$ if

$$\rho < \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3-4\alpha^*\right)\lambda^* C_x C_y} \text{ when } 0 < \alpha^* < 3/4,$$

$7\lambda C_y^2 + 23\lambda^* C_x^2 > 1$ when $\alpha^* = 3/4$,

$$\rho > \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3-4\alpha^*\right)\lambda^* C_x C_y} \text{ when } 3/4 < \alpha^* < 1.$$

(iv) $\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{dge}\right)$ if

$$\rho < \frac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} \text{ when } 0 < \alpha^* < 1/2,$$

$[4\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \lambda C_y^2 + 5\lambda^* C_x^2 > 1$ when $\alpha^* = 1/2$,

$$\rho > \frac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} \text{ when } 1/2 < \alpha^* < 1.$$

(v) $\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{NH}\right)$ if

$$\rho < \frac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} \text{ when } 0 < \alpha^* < 1/2,$$

$[4\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \lambda C_y^2 + 5\lambda^* C_x^2 > 1$ when $\alpha^* = 1/2$,

$$\rho > \frac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} \text{ when } 1/2 < \alpha^* < 1.$$

4. Numerical Example

To consider the performance of the proposed estimator, we calculate the percent relative efficiency (PRE: %) of the estimators $\bar{y}_{\text{R}d}$, $\bar{y}_{\text{Re}Md}$, $\bar{y}_{dge}$, $\bar{y}_{NH}$ and $\bar{y}_{WS}$ with respect to $\bar{y}$. The percent relative efficiency of the proposed estimator $\bar{y}_{WS}$ with respect to $\bar{y}$ can be computed as

$$\text{PRE}\left(\bar{y}_{WS}, \bar{y}\right) = \frac{\text{MSE}\left(\bar{y}\right)}{\text{MSE}\left(\bar{y}_{WS}\right)} \times 100 \cdot$$

We use four datasets to exemplify the performance of the proposed estimator. The following data are considered with a positive correlation between x and y.

Population I: The data on annual wages and years of education for 100 workers (Lind *et al.*, 2010). The population data is summarized as follows.

y : The annual wage.

x : The years of education.

$N = 100$, $n_1 = 30$, $n = 10$, $\bar{Y} = 30{,}833.46$, $\bar{X} = 12.73$, $C_y = 0.5441$, $C_x = 0.2171$ and $\rho = 0.408$.

Population II: The data on the level of apple production and the number of apple trees in 104 villages in the East Anatolia region of Turkey in 1999 (Kadilar & Cingi, 2006). The population data is summarized as follows.

y : Level of apple production.

x : Number of apple trees.

$N = 104$, $n_1 = 40$, $n = 20$, $\bar{Y} = 625.37$, $\bar{X} = 13.93$, $C_y = 1.866$, $C_x = 1.653$ and $\rho = 0.865$.

Population III: The population data is summarized as follows (see Khan, 2015).

y : Estimated number of fish caught during 1995.

x : Estimated number of fish caught during 1994.

$N = 69$, $n_1 = 36$, $n = 20$, $\bar{Y} = 4514.899$, $\bar{X} = 4954.435$, $C_y = 1.3509$, $C_x = 1.4248$ and $\rho = 0.9601$.

Population IV: The population data is summarized as follows (see Khan, 2015).

y : Estimated number of fish caught during 1995.

x : Estimated number of fish caught during 1992.

$N = 69$, $n_1 = 35$, $n = 20$, $\bar{Y} = 4514.899$, $\bar{X} = 4230.174$, $C_y = 1.3509$, $C_x = 1.3164$ and $\rho = 0.9538$.

We have checked the conditions in section 3.

Population I:

Condition (i): We obtained $1/2 < \alpha^* = 0.9758 < 1$ and

$$\rho = 0.4080 > \frac{\frac{1-f}{n} C_y^2 - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1 - 2\alpha^*\right)\lambda^* C_x C_y} = 0.1486,$$ the condition (i) is satisfied, then

$\mathrm{MSE}_{\min}\left(\bar{y}_{WS}\right) < \mathrm{MSE}\left(\bar{y}\right)$.

Condition (ii): We obtained

$$\rho = 0.4080 < \frac{\left(1 - \alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 2\right)\lambda^* C_x^2 - \left(\alpha^* - 1\right)^2}{4\left(1 - \alpha^*\right)\lambda^* C_x C_y} = 1.1994,$$ the condition (ii) is satisfied, then

$\mathrm{MSE}_{\min}\left(\bar{y}_{WS}\right) < \mathrm{MSE}\left(\bar{y}_{Rd}\right)$.

Condition (iii): We obtained $3/4 < \alpha^* = 0.9758 < 1$ and

$$\rho = 0.4080 > \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3-4\alpha^*\right)\lambda^* C_x C_y} = -0.6595$$, the condition (iii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{\text{Re}Md}\right)$.

Condition (iv): We obtained $1/2 < \alpha^* = 0.9758 < 1$ and

$$\rho = 0.4080 > \frac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.4034$$, the condition (iv) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{dge}\right)$.

Condition (v): We obtained $1/2 < \alpha^* = 0.9758 < 1$ and

$$\rho = 0.4080 > \frac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.4045$$, the condition (v) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{NH}\right)$.

Population II:

Condition (i): We obtained $1/2 < \alpha^* = 0.9093 < 1$ and

$$\rho = 0.8650 > \frac{\frac{1-f}{n} C_y^2 - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.2705$$, the condition (i) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}\right)$.

Condition (ii): We obtained

$$\rho = 0.8650 < \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 2\right)\lambda^* C_x^2 - \left(\alpha^* - 1\right)^2}{4\left(1-\alpha^*\right)\lambda^* C_x C_y} = 1.2206$$, the condition (ii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{Rd}\right)$.

Condition (iii): We obtained $3/4 < \alpha^* = 0.9093 < 1$ and

$$\rho = 0.8650 > \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3-4\alpha^*\right)\lambda^* C_x C_y} = -2.1810$$, the condition (iii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{\text{Re}Md}\right)$.

Condition (iv): We obtained $1/2 < \alpha^* = 0.9093 < 1$ and

$$\rho = 0.8650 > \frac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.8284$$, the condition (iv) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{dge}\right)$.

Condition (v): We obtained $1/2 < \alpha^* = 0.9093 < 1$ and

$\rho = 0.8650 > \dfrac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1 - 2\alpha^*\right)\lambda^* C_x C_y} = 0.8377$, the condition (v) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{NH}\right)$.

Population III:

Condition (i): We obtained $1/2 < \alpha^* = 0.9547 < 1$ and

$\rho = 0.9601 > \dfrac{\frac{1-f}{n} C_y^2 - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1 - 2\alpha^*\right)\lambda^* C_x C_y} = 0.4548$, the condition (i) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}\right)$.

Condition (ii): We obtained

$\rho = 0.9601 < \dfrac{\left(1 - \alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 2\right)\lambda^* C_x^2 - \left(\alpha^* - 1\right)^2}{4\left(1 - \alpha^*\right)\lambda^* C_x C_y} = 1.2543$, the condition (ii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{Rd}\right)$.

Condition (iii): We obtained $3/4 < \alpha^* = 0.9547 < 1$ and

$\rho = 0.9601 > \dfrac{\left(1 - \alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3 - 4\alpha^*\right)\lambda^* C_x C_y} = -1.7716$, the condition (iii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{\text{Re}Md}\right)$.

Condition (iv): We obtained $1/2 < \alpha^* = 0.9547 < 1$ and

$\rho = 0.9601 > \dfrac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1 - 2\alpha^*\right)\lambda^* C_x C_y} = 0.9448$, the condition (iv) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{dge}\right)$.

Condition (v): We obtained $1/2 < \alpha^* = 0.9547 < 1$ and

$\rho = 0.9601 > \dfrac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1 - 2\alpha^*\right)\lambda^* C_x C_y} = 0.9491$, the condition (v) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{NH}\right)$.

Population IV:

Condition (i): We obtained $1/2 < \alpha^* = 0.9566 < 1$ and

$$\rho = 0.9538 > \frac{\frac{1-f}{n}C_y^2 - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.4131$$, the condition (i) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}\right)$.

Condition (ii): We obtained

$$\rho = 0.9538 < \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 2\right)\lambda^* C_x^2 - \left(\alpha^* - 1\right)^2}{4\left(1-\alpha^*\right)\lambda^* C_x C_y} = 1.2674$$, the condition (ii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{Rd}\right)$.

Condition (iii): We obtained $3/4 < \alpha^* = 0.9566 < 1$ and

$$\rho = 0.9538 > \frac{\left(1-\alpha^{*2}\right)\lambda C_y^2 - \left(\alpha^* - 1\right)^2 - \left(\alpha^{*2} - \alpha^* - \frac{5}{4}\right)\lambda^* C_x^2}{\left(3-4\alpha^*\right)\lambda^* C_x C_y} = -1.6377$$, the condition (iii) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{\text{Re}Md}\right)$.

Condition (iv): We obtained $1/2 < \alpha^* = 0.9566 < 1$ and

$$\rho = 0.9538 > \frac{[\text{MSE}\left(\bar{y}_{dge}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.9362$$, the condition (iv) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{dge}\right)$.

Condition (v): We obtained $1/2 < \alpha^* = 0.9566 < 1$ and

$$\rho = 0.9538 > \frac{[\text{MSE}\left(\bar{y}_{NH}\right)/\bar{Y}^2] - \left(\alpha^* - 1\right)^2 - \alpha^{*2}\lambda C_y^2 - \left(\alpha^{*2} + \alpha^* - 1\right)\lambda^* C_x^2}{2\left(1-2\alpha^*\right)\lambda^* C_x C_y} = 0.9403$$, the condition (v) is satisfied, then

$\text{MSE}_{\min}\left(\bar{y}_{WS}\right) < \text{MSE}\left(\bar{y}_{NH}\right)$.

The MSEs and percent relative efficiency of the estimators are shown in Table 1.

From Table 1, the proposed estimator $\bar{y}_{WS}$ is more efficient than the estimators $\bar{y}$, $\bar{y}_{\text{R}d}$, $\bar{y}_{\text{Re}Md}$, $\bar{y}_{dge}$ and $\bar{y}_{NH}$ by considering MSE and percent relative efficiency. Thus, the proposed estimator $\bar{y}_{WS}$ is recommended for using in practical applications under double sampling.

Table 1. MSE values of estimators and percent relative efficiency with respect to $\bar{y}$.

Estimators	Population I		Population II		Population III		Population IV	
	MSE	PRE	MSE	PRE	MSE	PRE	MSE	PRE
$\bar{y}$	25,333,986.975	100	54,933.748	100	1,320,845.873	100	1,320,845.873	100
$\bar{y}_{\mathrm{R}d}$	22,211,646.736	114.057	29,536.167	186.191	566,242.477	233.265	596,011.049	221.614
$\bar{y}_{\mathrm{Re}Md}$	23,025,808.879	110.024	35,586.136	154.537	713,654.423	185.082	769,192.876	171.718
$\bar{y}_{dge}$	21,703,113.206	116.730	27,449.342	200.346	543,932.662	242.833	578,758.096	228.221
$\bar{y}_{NH}$	21,687,201.832	116.815	26,989.473	203.760	537,168.937	245.890	572,960.011	230.530
$\bar{y}_{WS}$	21,637,648.258	117.083	25,644.686	214.445	519,712.787	254.149	553,734.702	238.534

5. Conclusions

We propose the exponential estimator by using one auxiliary variable under double sampling scheme. The bias and MSE of the proposed estimator are derived under the second order of approximation and compared with the existing estimators. In addition, the numerical examples show that the efficiency of the proposed exponential estimator is better than others. Moreover, it turns out that when the population mean of an auxiliary variable is unknown, and then double sampling is utilized. The proposed estimator is recommended for using in practical applications.

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