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THE METHOD OF BOUNDARY STATES IN PROBLEMS OF TORSION OF ANISOTROPIC CYLINDERS OF FINITE LENGTH

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ABSTRACT

The work is devoted to the development of the boundary state method for the class of problems of torsion of cylindrical bodies with a nontrivial cross-sectional shape made from anisotropic materials. At the ends of the final cylinder, the forces are specified, resulting in torsion moments. The concepts of the spaces of internal and boundary states for an anisotropic medium are formulated. The theory of constructing bases of these spaces was developed using the general solution of Lekhnitsky. The basis of internal states includes the components of the displacement vector, the strain tensor, and the stress tensor. The basis of the boundary states includes the forces at the boundary of the cylinder, and the displacement of the boundary points. Scalar products are introduced in each of the spaces. In the basis of internal states, the scalar product expresses the internal energy of elastic deformation. In the basis of boundary states, it expresses the work of external forces. An isomorphism of the state space is established, which establishes a one-to-one correspondence between their elements. Isomorphism allows the search for the internal state to be reduced to the study of the boundary state that is isomorphic to it. The state spaces are orthogonalized and the desired state is decomposed into a Fourier series in terms of the orthonormal basis elements, where the given surface forces act as coefficients. The problem is solved for a cylinder whose cross section is in the shape of an I-beam made of anisotropic material. Signs of convergence of the solution are given. The main features of the problem solution are formulated. The results are presented in graphical form.

1. INTRODUCTION

In solid mechanics, the determination of the characteristics of the stress-strain state of a rod during its torsion is a difficult task even for an isotropic body, since there is a non-symmetric distribution of stresses, failure of cross-sections, etc. To solve the problems of torsion of rods from a material having an anisotropy of a common type; these include polymers, reinforced fiberglass,

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glass reinforcement, etc., applied the method of boundary states.

The problems of torsion of rods were considered in works of various directions in mechanics. For example, in [1] the particle method was applied to the problems of torsion of a rod to the stage of destruction. An experimental diagram of the torsion of a fluoroplastic sample was constructed, and microscopy of the rupture section was performed. In [2], the torsion of an ideal-plastic rectangular prismatic rod with inclusion was considered. The stress state of the rod was determined, the lines of breaking of stresses were found, the field of characteristics was constructed. In [3], the problems of free and constrained torsion of an isotropic rod of a continuous circular cross section were solved numerically on the basis of the tensor-linear defining relation written through energetically consistent Cauchy stress tensors and Henki logarithmic deformations. In [4], the problems of torsion of a rod in an elastoplastic formulation by the method of boundary elements were solved. A study of the convergence of the solution depending on the parameters of the problem. In [5], a variant of flow theory was developed for the case of materials with large anisotropic elastoplastic deformations. The corresponding dynamic problem was formulated and a numerical method for solving two-dimensional axisymmetric problems was developed.

A number of papers are devoted to the torsion of bodies from anisotropic materials, for example, in [6], the stress-strain state of anisotropic cylindrical and prismatic rods was investigated under an arbitrary plasticity condition. In [7], a study was made of the peculiarities of the distribution of stresses and displacements in individual layers of a multilayer anisotropic rod. In [8], a method is proposed for solving the problem of torsion of layered anisotropic rods by the finite element method. The problem of torsion of rods of rhomboid section and section of compressor is considered. In [9], an analysis of solutions of problems of torsion and stretching of nanotubes with two types of cylindrical anisotropy is given, the theory of which was constructed by S.G. Lehnitsky in the framework of the classical theory of elasticity.

The boundary state method [10] is a new method of mechanics of a deformable solid. To date, its application in mechanics concerned a narrow range of tasks: torsion of prismatic rods, hydrodynamics of ideal liquids, static problems of the theory of elasticity of isotropic bodies, as in the absence of mass -output, and if there are any, problems of the linear theory of elasticity for inhomogeneous bodies, flat and spatial problems of the theory of elasticity for anisotropic bodies, dynamic problems: the study of forced oscillations of elastic bodies solution-set boundary value problems with singularities geometric and physical nature.

2. FORMULATION OF THE PROBLEM

We consider the equilibrium of an elastic homogeneous body (figure 1), bounded by a cylindrical surface, in the general case not circular, with general anisotropy. The domain of the cross section is finite and simply connected; body length is finite.

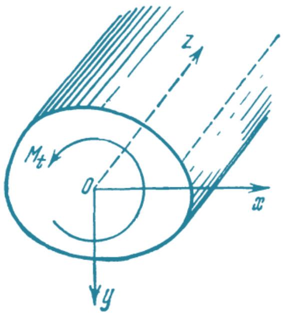

Figure1: Anisotropic cylinder

At the ends of the cylinder, there are forces $p_x(x,y)$ and $p_y(x,y)$, leading to torsion moments *M* relative to the *z*-axis. No initial stresses and bulk forces.

3. SOLUTION METHOD

To solve this problem, we use the boundary state method (MHS) [10]. MHS is a new energy method for solving problems of equations of mathematical physics. He showed his efficiency in solving boundary problems of the theory of elasticity, both for isotropic and anisotropic media, in solving problems of thermoelasticity, hydrodynamics of an ideal fluid, dynamics (oscillations) of isotropic bodies.

The foundation of the method is the space of internal Ξ and boundary $\tilde{A}$ states:

$$\Xi = \{\xi_1, \xi_2, \xi_3, \ldots, \xi_k, \ldots\}; \quad \tilde{A} = \{\gamma_1, \gamma_2, \gamma_3, \ldots, \gamma_k, \ldots\}.$$

The internal state is determined by the sets of components of the vector of displacements, tensors of deformations and stresses:

$$\xi_k = \{u_i^k, \varepsilon_{ij}^k, \sigma_{ij}^k\}. \qquad (1)$$

The main difficulty of forming a solution in the MHS is the design of the basis of internal states, which relies on a common or fundamental solution for the environment; It is also possible to use any private or special solutions. The method of constructing the basis of internal states will be described below.

Scalar product in the space of internal states Ξ is expressed through the internal energy of elastic deformation (hence the membership of the method in the energy class). For example, for the first and second internal state of the body occupying the *V* region:

$$(\xi_1, \xi_2) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 dV.$$

Moreover, due to the commutatively of the states of the medium:

$$(\xi_1, \xi_2) = (\xi_2, \xi_1) = \int_V \varepsilon_{ij}^1 \sigma_{ij}^2 dV = \int_V \varepsilon_{ij}^2 \sigma_{ij}^1 dV.$$

The boundary state is determined by the components of the vector of displacement of the points of the boundary and surface forces:

$$\gamma_k = \{u_i^k, p_i^k\}, \quad p_i^k = \sigma_{ij}^k n_j.$$

where n_j is a component of the normal to the boundary.

In the space of boundary states *Г*, the scalar product expresses the work of external forces on the surface of the body *S*, for example, for the first and second states:

$$(\gamma_1, \gamma_2) = \int_S p_i^1 u_i^2 dS .$$

Moreover, by virtue of the principle of possible movements:

$$(\gamma_1, \gamma_2) = (\gamma_2, \gamma_1) = \int_S p_i^1 u_i^2 dS = \int_S p_i^2 u_i^1 dS .$$

It is proved that in the case of a smooth boundary both state spaces are Hilbert and are conjugated by an isomorphism [10]. By definition, each element of $\xi_k \in \Xi$ corresponds to a single element of $\gamma_k \in \tilde{A}$, and this correspondence is one-to-one: $\xi_k \leftrightarrow \gamma_k$. This allows the search for the internal state to be reduced to the construction of a boundary state that is isomorphic to it. The latter essentially depends on the boundary conditions. In the case of the first and second main problems of mechanics, the problem reduces to a resolving system of equations for the Fourier coefficients, decomposition of the desired inner ξ and boundary γ states in a series in terms of the orthonormal basis elements:

$$\xi = \sum_{k=1}^{\infty} c_k \xi_k ; \quad \gamma = \sum_{k=1}^{\infty} c_k \gamma_k ,$$

or explicitly:

$$p_i = \sum_{k=1}^{\infty} c_k p_i^k ; \quad u_i = \sum_{k=1}^{\infty} c_k u_i^k ; \quad \sigma_{ij} = \sum_{k=1}^{\infty} c_k \sigma_{ij}^k ; \quad \varepsilon_{ij} = \sum_{k=1}^{\infty} c_k \varepsilon_{ij}^k .$$

The Fourier coefficients in the case of the first primary problem with the forces given by the ends of the cylinder $\mathbf{p} = \{p_{x0}, p_{y0}\}$ are:

$$c_k = (\mathbf{p}, \mathbf{u}^k) = \int_S (p_{x0} u^k + p_{y0} v^k) dS ,$$

where p_{x0} , and p_{y0} are set at the end of the effort, and $\mathbf{u}^k = \{u^k, v^k, w^k\}$ indicating displacement vector in the basic element $\gamma_k = \{u_i^k, p_i^k\}$.

4. CONSTRUCTION OF THE BASIS OF INTERNAL STATES

Lekhnitsky [11] received a general solution of the generalized Saint-Venant problem in the absence of mass forces:

$$\begin{aligned}
&\sigma_x = 2\,\mathrm{Re}[\mu_1^2 \Phi_1'(z_1) + \mu_2^2 \Phi_2'(z_2) + \mu_3^2 \lambda_3 \Phi_3'(z_3)];\\
&\sigma_y = 2\,\mathrm{Re}[\Phi_1'(z_1) + \Phi_2'(z_2) + \lambda_3 \Phi_3'(z_3)];\\
&\tau_{xy} = -2\,\mathrm{Re}[\mu_1 \Phi_1'(z_1) + \mu_2 \Phi_2'(z_2) + \mu_3 \lambda_3 \Phi_3'(z_3)];\\
&\tau_{yz} = -2\,\mathrm{Re}[\lambda_1 \Phi_1'(z_1) + \lambda_2 \Phi_2'(z_2) + \Phi_3'(z_3)] - \frac{\partial \psi_0}{\partial x};\\
&\tau_{xz} = 2\,\mathrm{Re}[\mu_1 \lambda_1 \Phi_1'(z_1) + \mu_2 \lambda_2 \Phi_2'(z_2) + \mu_3 \Phi_3'(z_3)] + \frac{\partial \psi_0}{\partial y};\\
&\sigma_z = \frac{1}{a_{33}}(Ax + By + C) - \frac{1}{a_{33}}\left(a_{13}\sigma_x + a_{23}\sigma_y + a_{34}\tau_{yz} + a_{35}\tau_{xz} + a_{36}\tau_{xy}\right);
\end{aligned} \qquad (2)$$

$$u = -\frac{A}{2}z^2 - \vartheta yz + U;$$
$$v = -\frac{B}{2}z^2 - \vartheta xz + V;$$
$$w = (Ax + By + C)z + W;$$

$$U = 2\operatorname{Re}\left[\sum_{k=1}^{3} p_k \Phi_k(z_k) + U_0\right]; \quad V = 2\operatorname{Re}\left[\sum_{k=1}^{3} q_k \Phi_k(z_k) + V_0\right]; \quad W = 2\operatorname{Re}\left[\sum_{k=1}^{3} r_k \Phi_k(z_k) + W_0\right],$$

here entered the notation:

$$p_k = \beta_{11}\mu_k^2 + \beta_{12} - \beta_{16}\mu_k + \lambda_k(\beta_{15}\mu_k - \beta_{14}); \quad p_3 = \lambda_3\left(\beta_{11}\mu_3^2 + \beta_{12} - \beta_{16}\mu_3\right) + \beta_{15}\mu_3 - \beta_{14};$$
$$q_k = \beta_{12}\mu_k - \beta_{26} + \frac{\beta_{22}}{\mu_k} + \lambda_k\left(\beta_{25}\mu_k - \frac{\beta_{24}}{\mu_k}\right); \quad q_3 = \lambda_3\left(\beta_{12}\mu_3 + \frac{\beta_{22}}{\mu_3} - \beta_{26}\right) + \beta_{25} - \frac{\beta_{24}}{\mu_3};$$
$$r_k = \beta_{14}\mu_k - \beta_{46} + \frac{\beta_{24}}{\mu_k} + \lambda_k\left(\beta_{45}\mu_k - \frac{\beta_{44}}{\mu_k}\right); \quad r_3 = \lambda_3\left(\beta_{14}\mu_3 + \frac{\beta_{24}}{\mu_3} - \beta_{46}\right) + \beta_{45} - \frac{\beta_{44}}{\mu_3};$$
$$(k = 1, 2);$$

$\beta_{ij} = a_{ij} - \frac{a_{i3}a_{j3}}{a_{33}}(i, j = 1, 2, 4, 5, 6)$ is reduced strain factors; a_{ij} is strain factors; ψ_0 is partial solution of an inhomogeneous system of differential equations [3]. The terms U_0, V_0, W_0 refer to particular solutions of differential equations [12], the corresponding ψ_0,

$$\psi_0 = \frac{-2\vartheta + (Aa_{34} - Ba_{35})}{4a_{33}} \frac{(\beta_{55}x^2 + 2\beta_{45}xy + \beta_{44}y^2)}{(\beta_{44}\beta_{55} - \beta_{45}^2)}.$$

Constants A, B, C, ϑ are determined from the equilibrium conditions at the ends:

$$CS = \iint\left(a_{13}\sigma_x + a_{23}\sigma_y + a_{36}\tau_{xy}\right)dxdy;$$
$$BI_1 = \iint\left(a_{13}\sigma_x + a_{23}\sigma_y + a_{36}\tau_{xy} + a_{35}\tau_{xz}\right)dxdy;$$
$$AI_2 = \iint\left(a_{13}\sigma_x + a_{23}\sigma_y + a_{36}\tau_{xy} + a_{34}\tau_{yz}\right)dxdy;$$
$$\iint\left(x\tau_{yz} - y\tau_{xz}\right)dxdy = M_t,$$

here I_1, I_2 are the main moments of inertia of the cross section (relative to the x and y-axes); M_t – torsion moment, to which the forces on the ends lead (Figure 1), where z_1, z_2, z_3 are the generalized complex variables with $z_1 = x + \mu_1 y$, $z_2 = x + \mu_2 y$, $z_3 = x + \mu_3 y$, μ_1, μ_2, μ_3 – various complex roots of the characteristic equation [2].

$$\Phi_1(z_1) = \frac{dF_1}{dz_1}; \quad \Phi_2(z_2) = \frac{dF_2}{dz_2}; \quad \Phi_3(z_3) = \frac{dF_3}{dz_3};$$
$$\Phi_1'(z_1) = \frac{d\Phi_1}{dz_1}; \quad \Phi_2'(z_2) = \frac{d\Phi_2}{dz_2}; \quad \Phi_3'(z_3) = \frac{d\Phi_3}{dz_3}; \qquad (3)$$
$$F = 2\operatorname{Re}[F_1(z_1) + F_2(z_2) + F_3(z_3)];$$

$$\psi = 2\,\mathrm{Re}\left[\lambda_1 F_1'(z_1) + \lambda_2 F_2'(z_2) + \frac{1}{\lambda_3} F_3'(z_3)\right];$$

where F and ψ are stress functions.

Base sets of internal states can be constructed by generating all sorts of options for the three analytical functions. For a simply connected domain, it has the form [13]:

$$\begin{pmatrix}\Phi_1(z_1)\\ \Phi_2(z_2)\\ \Phi_3(z_3)\end{pmatrix} \in \left\{ \begin{pmatrix} z_1^{\,k}\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ z_2^{\,k}\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ z_3^{\,k}\end{pmatrix}, \begin{pmatrix} iz_1^{\,k}\\ 0\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ iz_2^{\,k}\\ 0\end{pmatrix}, \begin{pmatrix} 0\\ 0\\ iz_3^{\,k}\end{pmatrix} : k = 1,2,\ldots \right\}. \qquad (4).$$

By giving the analytical functions (3) successively the values (4), all components of the elastic state (2) are determined, thereby determining the basis of the internal states (1).

Next comes the orthogonalization of the bases of the state spaces.

5. THE SOLUTION OF THE PROBLEM

As mentioned in section 3, MHS approach utilized for solving the problem. For instance, we considered the torsion problem of an anisotropic rod, the cross section of which is in the shape of an I-beam (Figure 2). The most general case of anisotropy is assumed, when the number of strain coefficients is 21. At the ends were set efforts [14]:

$$\begin{cases} p_x = -y;\, p_y = x;\, p_z = 0;\; z = -2; \\ p_x = y;\, p_y = -x;\, p_z = 0;\; z = 2. \end{cases}$$

There are no forces on the side surface.

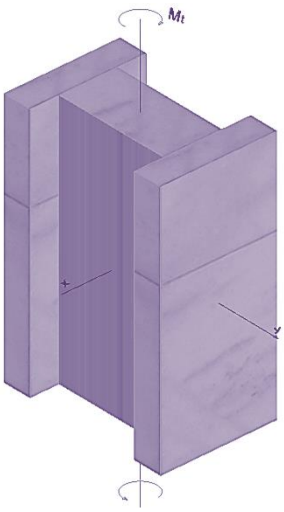

Figure 2: Boundary conditions

The problem is solved approximately; 240 elements of an orthonormal basis were used. The difficulty lies in the time spent on the process of orthonormalization of the basis of internal states and on the calculation of the Fourier coefficients, the headings of which are shown in Figure 3.

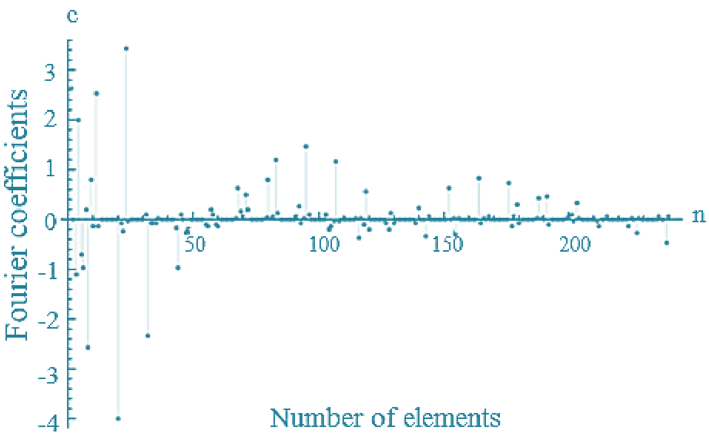

Figure 3: The values of the Fourier coefficients.

In Figure 4 shows the graph of saturation of the Bessel sum $b = \sum_{n=1}^{240} c_j^{\,2}$ (left side of the Parseval inequality).

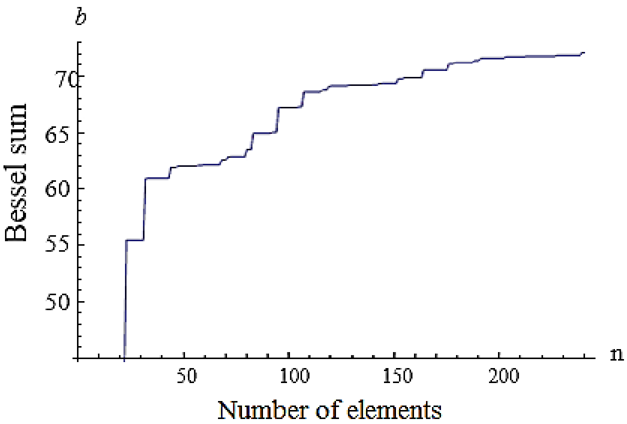

Figure 4: Bessel sum

In Figure 5 shows the contour of the deformed cylinder (due to small deformations, the contour is presented in a hypertrophied form).

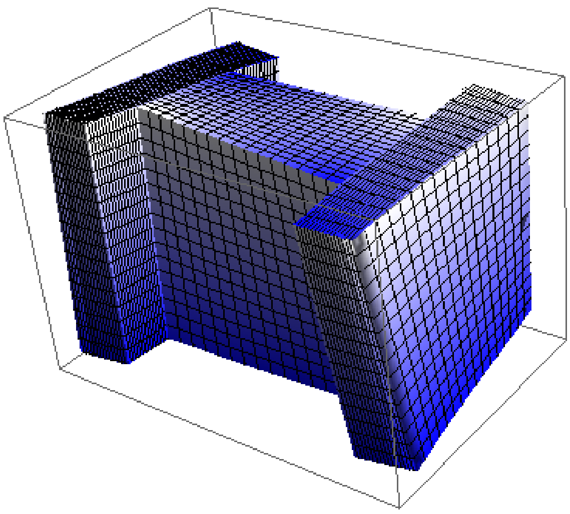

Figure 5: Contour of the deformed body

The integral value of the specified torque is 74.667, while the resulting solution is 74.693.

6. CONCLUSION

It should be noted that the singularity of the body geometry does not affect the convergence of the solution of the problems of torsion, bending and stretching of the rods. However, the picture changes dramatically for the worse if efforts are made on the side surface. To examine the boundary states, we consider torsion of circular bars with one end fixed and the other end free on which tractions that results in a pure torque are prescribed arbitrarily over the free end surface. Exact solutions that satisfy the prescribed boundary conditions point by point over the entire boundary surfaces are derived in a unified manner for problems of torsion of anisotropic cylinders with or without radial inhomogeneity. The following conclusions can be drawn from the analysis. (1) The classical solution based on the boundary states is useful for torsion of isotropic circular bars with or without radial inhomogeneity. The stress disturbance is confined to the local region near the end where the torsion load is applied. (2) The stresses at the fixed end of circular bars under torsion can be evaluated using the solution based on MHS except in the case of strong anisotropy

Results showed that by the MHS, the Fourier coefficients for calculating the boundary beam problem efficiently applied. The boundary state method was successfully implemented in terms of solving the torsion problem of anisotropic cylindrical bodies; the solution is reduced to the routine calculation of certain integrals. A specific solution for the problem of torsion for a complex contour body is constructed. When solving these problems, a rather “long” basis is required.

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