รายงานฉบับสมบูรณ์
โครงการวิจัย

ปัญหาอสมการแปรผันและจุดตรึงบนเซตไม่คอนเวกซ์
Nonconvex Variational Inequalities and Fixed Point Problems

โดย รองศาสตราจารย์ ดร.อิสระ อินจันทร์

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ABSTRACTS

Now we known that many authors studied and introduce the result of nonconvex variational inequalities. By relying on the prox-regularity notion, we introduce and establish the convergence of modified algorithm of system of strongly nonlinear nonconvex variational inequalities problems. Then we obtain as a particular case some known results in this field.

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CHAPTER I

INTRODUCTION

The theory of variational inequalities be first introduce by Stampacchia [2], provides simple and unified framwork to study a large number of problem arising in finance, economics, transportation, network and structural analysis, elasticity and optimization. Many research papers have been written lately, both on the theory and applications of this field, see for example [4, 7, 8] and the references cited therein.

The existence and iterative scheme of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator are convex sets. Recently, the concept of convex sets has been generalized in many differnt ways. It is known that the uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

In 2003, Bounkhel [3], 2004 Noor [10], Moudafi [9] and 2007 Pang et al. [12], considered the variational inequality probelm over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

Recently, in 2009, Noor [11] introduce and studied some new classes of variational and the Wiener-Hiof equations and established the equivalent between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equation, by using the projection technique. Noor also present some new projection iterative methods for solving the nonconvex variational inequalities and prove the convergence of iterative method under suitable conditions.

In the same year, Moudafi [9], introduce the convergence of two-step projection methods for a system of nonconvex variational inequalities problems for a mapping T is γ-strongly monotone and L-Lipschitz continuous.

Very recently, in 2013, Al-Shemas [1], introduce the strongly nonlinear general nonconvex variational inequalities which prove the convergence of the predictor-corrector method only requires pseudomonotonicity, which is weaker condition than monotonicity.

Motivated by [9] and [1], we introduced and studied the convergence of modified algorithm for the system of strongly nonlinear nonconvex variational inequalities problems for

CHAPTER II

PRELIMINARIES

In this chapter, we let C be a closed subset of a real Hilbert space H with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.0.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.0.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [5].

Lemma 2.0.3. Let C be a closed subset of a Hilbert space H. Then

$$z \in N_C^P(x) \text{ if and only if } \exists \sigma > 0, \langle z, y - x \rangle \leq \sigma \|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [6] and Poliquin et al. [11] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.0.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly r-prox-regular with respect to r if, for all $\overline{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \overline{x} \rangle \leq \frac{1}{2r} \|x - \overline{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.0.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [6, 14].

In this work let C be a closed subset of a real Hilbert space H with is uniformly r-prox-regular(nonconvex), set $C_r := \{x \in H : d(x, C) < r\}$. For given nonlinear mappings $T_1, T_2 : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ay^*, x - x^* \rangle, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 x^* + y^* - x^*, y - y^* \rangle + \lambda \|y - y^*\|^2 \geq \langle Ax^*, y - y^* \rangle, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the system of strongly nonlinear nonconvex variational inequalities(SSNNVI).

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0$ and $T_1 = T_2 = T$ then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle + \lambda \|y - y^*\|^2 \geq 0, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.6}$$

which is called the system of nonconvex variational inequalities(SNVI). We known that the inequalities (2.6) is equivalent as follows:

$$\begin{aligned} &y^* - x^* - \rho T y^* \in N^P_{C_r} x^*, \\ &x^* - y^* - \eta T x^* \in N^P_{C_r} y^*. \end{aligned} \tag{2.7}$$

Which is introduce by Moudafi [9].

If $A(x^*) \equiv 0, A(y^*) \equiv 0$, $T_1 = T_2 = T$ and $\lambda = 0$, then the problem (2.5) is equivalent to finding $x^*. y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle \geq 0, \forall y \in C_r, \eta > 0. \end{aligned} \tag{2.8}$$

Which is called system of variational inequalities(SVI), introduced by Verma [16].

If $T_1 = T_2 = T$, $x^* = y^*$ and $\rho = \eta = 1$, then the problem (2.5) is equivalent to finding $x^* \in C_r$ such that

$$\langle T x^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ax^*, x - x^* \rangle, \forall x \in C_r \tag{2.9}$$

which is known as the strongly nonlinear nonconvex variational inequality and studied by Noor [17].

In inequalities (2.9) if we let $A(x^*) \equiv 0$, we have to finding $x^* \in C_r$ such that

$$\langle Tx^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq 0, \forall x \in C_r, \tag{2.10}$$

which is called the nonconvex variational inequalities(NVI), introduced and studied by Bounkhel et. al.[3] and Noor [10, 11].

It is worth mentioning that if $C_r = C$ is convex set, then problem (2.10) is equivalent to finding $x^* \in C$ such that

$$\langle Tx^*, x - x^*\rangle \geq 0, \forall x \in C. \tag{2.11}$$

which is known as variational inequalities, introduced and studied by Stamphacia [2].

Now, if C_r is a nonconvex(uniform r-prox regular) set, then problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} 0 &\in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r}x^*, \\ 0 &\in \eta T_2 x^* + y^* - x^* - Ax^* + N^P_{C_r}y^*, \end{aligned} \tag{2.12}$$

which is $N^P_{C_r}u$ denote the normal cone of C_r at u. The problem (2.12) is called the the system of nonconvex variational inclusion problem associated with nonconvex variational inequalities.

Let H be a real Hilbert space. A mapping $T : H \to H$ is called $\gamma-$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq \gamma\|x - y\|^2. \tag{2.13}$$

for all $x, y \in H$. A mapping T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|Tx - Ty\| \leq \mu\|x - y\|, \tag{2.14}$$

for all $x, y \in H$.

Lemma 2.0.5. In a real Hilbert space H, there holds the inequality

1. $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \quad x, y \in H$ and $\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$,
2. $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, \forall t \in [0, 1]$.

Lemma 2.0.6. Let C be a nonempty closed subset of H, $r \in (0, +\infty]$ and set $C_r := \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:

(1) for all $x \in C_r, P_C(x) \neq \emptyset$,

(2) for all $s \in (0,r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

CHAPTER III

MAIN RESULTS

3.1 Convergence Theorems of system of strongly nonlinear Nonconvex variational Inequalities

In this section we introduce and establish the convergence of modified algorithm of system of strongly nonlinear nonconvex variational inequalities problems. Then we obtain as a particular case some known results in this field.

Lemma 3.1.1. For given $x^*, y^* \in C_r$ are solution of system of strongly nonlinear general nonconvex variational inequalities (2.5), if and only if

$$\begin{aligned} x^* &= P_{C_r}[y^* - \rho T_1 y^* + Ay^*], \\ y^* &= P_{C_r}[x^* - \eta T_2 x^* + Ax^*]. \end{aligned} \tag{3.15}$$

where $P_{C_r} = (I + N^P_{C_r})^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

พิสูจน์. Let $x^*, y^* \in C_r$ be a solution of (2.5), for a constant $\rho > 0$, we have

$$\langle \rho T_1 y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle Ay^*, x - x^* \rangle$$

if and only if

$$\langle Ay^* - \rho T_1 y^* - x^* + y^*, x - x^* \rangle \leq \lambda \|x - x^*\|^2.$$

Then,

$$Ay^* - \rho T_1 y^* - x^* + y^* \in N^P_{C_r} x^*$$

it implies that

$$\begin{aligned} 0 &\in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r} x^* = (I + N^P_{C_r})x^* - (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow (I + N^P_{C_r})x^* = (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow x^* = P_{C_r}[y^* - \rho T_1 y^* + Ay^*] \end{aligned}$$

where we have used the well-known fact that $P_{C_r} = (I + N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_{C_r}[x^* - \eta T_2 x^* + Ax^*].$$

This prove our assertions. □

Algorithm 3.1.2. For arbitrarily chosen initial points $x_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T_2 x_n + A x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_{C_r}[y_n - \rho T_1 y_n + A y_n], \rho > 0, \end{aligned} \tag{3.16}$$

where $\{\alpha_n\}$ is a sequence in $[0,1]$.

Now, we suggest and analyze the following explicit projection method (3.1.2) for solving the system of nonconvex variational inequalities (2.6).

Theorem 3.1.3. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous. If there exists constant $\rho, \eta > 0$ such that

$$|\rho - \frac{\gamma_1}{\mu_1^2}| < \frac{\sqrt{\gamma_1^2 t_s^2 - t_s\mu_1^2(t_s + t_s\beta - 1)}}{t_s\mu_1^2}, \tag{3.17}$$

$$|\eta - \frac{\gamma_2}{\mu_2^2}| < \frac{\sqrt{\gamma_2^2 t_s^2 - t_s\mu_2^2(t_s + t_s\beta - 1)}}{t_s\mu_2^2}, \tag{3.18}$$

which $\sqrt{1+\beta-\frac{1}{t_s}} < \frac{\gamma_1}{\mu_1}$ where $t_s = \frac{r}{r-s}$ for some $s \in (0,r)$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.1.2 converge to a solution of the system of nonconvex variational inequalities (2.6).

พิสูจน์. Let $x^*, y^* \in C_r$ be a solution of (2.6) and from Lemma 3.1.1, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T_1 y_n + A y_n] - x^*\| \\ &= \|(1-\alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T_1 y_n + A y_n] - P_C[y^* - \rho T_1 y^* + A y^*])\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - \rho T_1 y_n + A y_n) - (y^* - \rho T_1 y^* + A y^*)\| \\ &= (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*) + (A y_n - A y^*)\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \|A y_n - A y^*\|] \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \beta\|y_n - y^*\|] \end{aligned} \tag{3.19}$$

Since T_1 are both μ_1-Lipschitz continuous and γ_1-strongly monotone mapping and from Lemma 2.0.5, we consider

$$\begin{aligned} \|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\ &\le \|y_n - y^*\|^2 - 2\rho\gamma_1\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\ &= (1 - 2\rho\gamma_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2. \end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \le \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.20}$$

Replace (3.20) into (3.19), we have

$$\|x_{n+1} - x^*\| \le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2})\|y_n - y^*\|. \tag{3.21}$$

On the other hand, we can compute that

$$\begin{aligned} \|y_n - y^*\| &= \|P_C[x_n - \eta T_2 x_n + Ax_n] - y^*\| \\ &= \|P_C[x_n - \eta T_2 x_n + Ax_n] - P_C[x^* - \eta T_2 x^* + Ax^*]\| \\ &\le t_s\|(x_n - \eta T_2 x_n + Ax_n) - (x^* - \eta T_2 x^* + Ax^*)\| \\ &\le t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \|Ax_n - Ax^*\|] \\ &\le t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \beta\|x_n - x^*\|]. \end{aligned} \tag{3.22}$$

Similarly, from T_2 are both μ_2-Lipschitz continuous and γ_2-strongly monotone mapping, we have

$$\begin{aligned} \|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_2 x_n - T_2 x^*\rangle + \eta^2\|T_2 x_n - T_2 x^*\|^2 \\ &\le \|x_n - x^*\|^2 - 2\eta\gamma_2\|x_n - x^*\|^2 + \eta^2\mu_2^2\|x_n - x^*\|^2 \\ &= (1 - 2\eta\gamma_2 + \eta^2\mu_2^2)\|x_n - x^*\|^2. \end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| \le \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2}\|x_n - x^*\|. \tag{3.23}$$

Replace (3.23) into (3.22), we have

$$\|y_n - y^*\| \le t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})\|x_n - x^*\|. \tag{3.24}$$

Moreover, from (3.21) and (3.24) we put $\theta_1 = t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}), \theta_2 = t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\le (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\le \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|x_0 - x^*\|. \end{aligned} \tag{3.25}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and (3.29), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{3.26}$$

It follows from (3.26) and (3.25), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{3.27}$$

From (3.24) and (3.27), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \tag{3.28}$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.6). This completes the proof. □

Corollary 3.1.4. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \quad t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, \tag{3.29}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0,1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.30}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

พิสูจน์. From Theorem 3.1.3, if $T_1 = T_2 = T$ and $A(x) \equiv 0$, we have a result. □

Corollary 3.1.5. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \quad t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, \tag{3.31}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0, \end{aligned} \tag{3.32}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

พิสูจน์. From Theorem 3.1.3, if $T_1 = T_2 = T$, and $A(x) \equiv 0$ and $\alpha_n = 1$ for any $n \geq 1$, we have a result. □

3.2 Application of Convergence Theorems of system of strongly nonlinear Nonconvex variational Inequalities

In this section, we can applied Theorem 3.1.3 to the system of general nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*) \rangle + \lambda \|x - g(x^*)\|^2 \geq \langle A g(y^*), x - g(x^*) \rangle, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), y - g(y^*) \rangle + \lambda \|y - g(y^*)\|^2 \geq \langle A g(x^*), y - g(y^*) \rangle, \forall y \in C_r, \eta > 0 \end{aligned} \tag{3.33}$$

which is called the system of general strongly nonlinear nonconvex variational inequalities(SGSNNVI).

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0$ and $\lambda = 0$, then the problem (3.33) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*) \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), x - g(y^*) \rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{3.34}$$

which is called the system of general nonconvex variational inequalities.

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0, \lambda = 0$ and $T_1 = T_2 = T$, then the problem (3.33) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*) \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*) \rangle \geq 0, \forall x \in C_r, \eta > 0. \end{aligned} \tag{3.35}$$

Now, similarly of the proof of Lemma 3.1.1, we have the result.

Lemma 3.2.1. For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (3.33), if and only if

$$
\begin{aligned}
g(x^*) &= P_{C_r}[g(y^*) - \rho T_1 g(y^*) + Ag(y^*)],\\
g(y^*) &= P_{C_r}[g(x^*) - \eta T_2 g(x^*) + Ag(x^*)],
\end{aligned} \tag{3.36}
$$

where $P_{C_r} = (I + N_{C_r}^P)^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

Theorem 3.2.2. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous and let g be injective mapping. If there exists constant $\rho, \eta > 0$ such that

$$|\rho - \frac{\gamma_1}{\mu_1^2}| < \frac{\sqrt{\gamma_1^2 t_s^2 - t_s\mu_1^2(t_s + t_s\beta - 1)}}{t_s\mu_1^2}, \tag{3.37}$$

$$|\eta - \frac{\gamma_2}{\mu_2^2}| < \frac{\sqrt{\gamma_2^2 t_s^2 - t_s\mu_2^2(t_s + t_s\beta - 1)}}{t_s\mu_2^2}, \tag{3.38}$$

which $\sqrt{1 + \beta - \frac{1}{t_s}} < \frac{\gamma_1}{\mu_1}$ where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$
\begin{aligned}
g(y_n) &= P_{C_r}[g(x_n) - \eta T_2 g(x_n) + Ag(x_n)], \eta > 0\\
g(x_{n+1}) &= (1 - \alpha_n)g(x_n) + \alpha_n P_{C_r}[g(y_n) - \rho T_1 g(y_n) + Ag(y_n)], \rho > 0,
\end{aligned} \tag{3.39}
$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$ strongly converge to a solution of the system of nonconvex variational inequalities (3.33).

พิสูจน์. Similar the proof in Theorem 3.1.3, let $x^*, y^* \in C_r$ be a solution of (3.34) and from Lemma 3.2.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \leq \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{3.40}$$

where $\theta_1 = t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}), \theta_2 = t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})$. From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (3.37), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{3.41}$$

It follows from (3.40) and (3.41), we have

$$\lim_{n\to\infty} \|g(x_n) - g(x^*)\| = 0. \tag{3.42}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \le \theta_2\|g(x_n) - g(x^*)\|, \tag{3.43}$$

it follows that

$$\lim_{n\to\infty} \|g(y_n) - g(y^*)\| = 0. \tag{3.44}$$

From g is injective mapping, we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$ and $\lim_{n\to\infty} \|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (3.34). This complete the proof. □

Corollary 3.2.3. Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2}, \quad t_s\sqrt{t_s^2 - 1} < \frac{\gamma}{\mu}, \tag{3.45}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. Then the sequence $\{x_n\}$ and $\{y_n\}$ is generated by for $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0, \end{aligned} \tag{3.46}$$

strongly converge to a solution of the system of nonconvex variational inequalities (3.35).

พิสูจน์. From Theorem 3.2.2, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for any $n \ge 0$, we have a result. □

CHAPTER IV

CONCLUSIONS

4.1 Outputs 1 paper (Supported by Uttaradit Rajabhat University)

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APPENDIX

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Abstract: Now we known that many authors studied and introduce the result of nonconvex variational inequalities. By relying on the prox-regularity notion, we introduce and establish the convergence of modied algorithm of system of strongly nonlinear nonconvex variational inequalities problems. Then we obtain as a particular case some known results in this field.

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October 20, 2015

Dear Editor-in-Chief

Enclosed please find the pdf file of my paper entitled; "Convergence Theorems of Iterative Methods for System of Strongly Nonlinear Nonconvex Variational Inequalities" which I would like to submit in Nonlinear Analysis Hybrid Systems.

I would like to thank you in advance for your consideration.

Your sincerely,

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Convergence Theorems of Iterative Methods for System of Strongly Nonlinear Nonconvex Variational Inequalities

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Abstract

Now we known that many authors studied and introduce the result of nonconvex variational inequalities. By relying on the prox-regularity notion, we introduce and establish the convergence of modified algorithm of system of strongly nonlinear nonconvex variational inequalities problems. Then we obtain as a particular case some known results in this field.

---

1 Introduction

The theory of variational inequalities be first introduce by Stampacchia [ ], provides simple and unified framwork to study a large number of problem arising in finance, economics, transportation, network and structural analysis, elasticity and optimization. Many research papers have been written lately, both on the theory and applications of this field, see for example [[illegible], [illegible]] and the references cited therein.

The existence and iterative scheme of variational inequalities have been investigated over convex sets, and that is due to the fact that all techniques are mainly based on the properties of the projection operator are convex sets. Recently, the concept of convex sets has been generalized in many differnt ways. It is known that the uniformly prox-regular sets are an immediate consequence of the generalization of convex sets, these sets are nonconvex and include convex sets as a particular case.

In 2003, Bounkhel [[illegible]], 2004 Noor [[illegible]], Moudafi [[illegible]] and 2007 Pang et al. [[illegible]], considered the variational inequality probelm over these nonconvex sets. They suggested and analyzed some projection type iterative algorithms by using the prox-regular technique and auxiliary principle technique.

Recently, in 2009, Noor [[illegible]] introduce and studied some new classes of variational and the Wiener-Hiof equations and established the equivalent between the general nonconvex variational inequalities and the fixed point problems as well as the Wiener-Hopf equation, by using the projection technique. Noor also present some new projection iterative methods for solving the nonconvex variational inequalities and prove the convergence of iterative method under suitable conditions.

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In the same year, Moudafi [[illegible]], introduce the convergence of two-step projection methods for a system of nonconvex variational inequalities problems for a mapping T is γ-strongly monotone and L-Lipschitz continuous.

Very recently, in 2013, Al-Shemas [[illegible]], introduce the strongly nonlinear general nonconvex variational inequalities which prove the convergence of the predictor-corrector method only requires pseudomonotonicity, which is weaker condition than monotonicity.

Motivated by [[illegible]] and [[illegible]], we introduced and studied the convergence of modified algorithm for the system of strongly nonlinear nonconvex variational inequalities problems for two mappings satisfying strongly monotone and Lipschitz continuous. This work extended and improve some known results.

2 Preliminaries

Let C be a closed subset of a real Hilbert space H with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ respectively. Let us recall the following well-known definitions and some auxiliary results of nonlinear convex analysis and nonsmooth analysis.

Definition 2.1. Let $u \in H$ be a point not lying in C. A point $v \in C$ is called a closest point or a projection of u onto C if $d_C(u) = \|u - v\|$ when d_C is a usual distance. The set of all such closest points is denoted by $P_C(u)$; that is,

$$P_C(u) = \{v \in C : d_C(u) = \|u - v\|\}. \tag{2.1}$$

Definition 2.2. Let C be a subset of H. The proximal normal cone to C at x is given by

$$N_C^P(x) = \{z \in H : \exists \rho > 0; x \in P_C(x + \rho z)\}. \tag{2.2}$$

The following characterization of $N_C^P(x)$ can be found in [[illegible]].

Lemma 2.3. *Let C be a closed subset of a Hilbert space H. Then*

$$z \in N_C^P(x) \textit{ if and only if } \exists \sigma > 0, \langle z, y - x\rangle \leq \sigma\|y - x\|^2, \quad \forall y \in C. \tag{2.3}$$

Clark et al. [[illegible]] and Poliquin et al. [[illegible]] have introduced and atudied a new class of nonconvex sets, which are called uniformly prox-regular sets. This class or uniformly prox-regular sets has played an important part in many nonconvex applications such as optimization, dynamic systems, and differential inclusions.

Definition 2.4. For a given $r \in (0, +\infty]$, a subset C of H is said to be uniformly r-prox-regular with respect to r if, for all $\bar{x} \in C$ and for all $0 \neq z \in N_C^P(x)$, one has

$$\langle \frac{z}{\|z\|}, x - \bar{x}\rangle \leq \frac{1}{2r}\|x - \bar{x}\|^2, \quad \forall x \in C. \tag{2.4}$$

It is well known that a closed subset of a Hilbert space is convex if and only if it is proximally smooth of radius $r > 0$. Thus, in Definition 2.4, in the case of $r = \infty$, the uniform r-prox-regularity C is equivalent to convexity of C. Then, it is clear that the class of uniformly prox-regular sets is sufficiently large to include the class p-convex sets, $C^{1,1}$ submanifolds (possibly with boundary) of H, the images under a $C^{1,1}$ diffeomorphism of convex sets, and many other nonconvex sets; see [[illegible]].

In this work let C be a closed subset of a real Hilbert space H with is uniformly r-prox-regular(nonconvex), set $C_r := \{x \in H : d(x, C) < r\}$. For given nonlinear mappings $T_1, T_2 : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \geq \langle Ay^*, x - x^*\rangle, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 x^* + y^* - x^*, y - y^*\rangle + \lambda\|y - y^*\|^2 \geq \langle Ax^*, y - y^*\rangle, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.5}$$

which is called the *system of strongly nonlinear nonconvex variational inequalities*(SSNNVI).

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0$ and $T_1 = T_2 = T$ then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle + \lambda \|y - y^*\|^2 \geq 0, \forall y \in C_r, \eta > 0, \end{aligned} \tag{2.6}$$

which is called the *system of nonconvex variational inequalities*(SNVI). We known that the inequalities (2.6) is equivalent as follows:

$$\begin{aligned} &y^* - x^* - \rho T y^* \in N^P_{C_r} x^*, \\ &x^* - y^* - \eta T x^* \in N^P_{C_r} y^*. \end{aligned} \tag{2.7}$$

Which is introduce by Moudafi [illegible].

If $A(x^*) \equiv 0, A(y^*) \equiv 0$. $T_1 = T_2 = T$ and $\lambda = 0$, then the problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T y^* + x^* - y^*, x - x^* \rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T x^* + y^* - x^*, y - y^* \rangle \geq 0, \forall y \in C_r, \eta > 0. \end{aligned} \tag{2.8}$$

Which is called *system of variational inequalities*(SVI), introduced by Verma [14].

If $T_1 = T_2 = T$, $x^* = y^*$ and $\rho = \eta = 1$, then the problem (2.5) is equivalent to finding $x^* \in C_r$ such that

$$\langle T x^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq \langle A x^*, x - x^* \rangle, \forall x \in C_r \tag{2.9}$$

which is known as the strongly nonlinear nonconvex variational inequality and studied by Noor [15].

In inequalities (2.9) if we let $A(x^*) \equiv 0$, we have to finding $x^* \in C_r$ such that

$$\langle T x^*, x - x^* \rangle + \lambda \|x - x^*\|^2 \geq 0, \forall x \in C_r, \tag{2.10}$$

which is called the *nonconvex variational inequalities*(NVI), introduced and studied by Bounkhel et. al.[3] and Noor [10, 11].

It is worth mentioning that if $C_r = C$ is convex set, then problem (2.10) is equivalent to finding $x^* \in C$ such that

$$\langle T x^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{2.11}$$

which is known as *variational inequalities*, introduced and studied by Stamphacia [illegible].

Now, if C_r is a nonconvex(uniform r-prox regular) set, then problem (2.5) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &0 \in \rho T_1 y^* + x^* - y^* - A y^* + N^P_{C_r} x^*, \\ &0 \in \eta T_2 x^* + y^* - x^* - A x^* + N^P_{C_r} y^*, \end{aligned} \tag{2.12}$$

which is $N^P_{C_r} u$ denote the normal cone of C_r at u. The problem (2.12) is called the *the system of nonconvex variational inclusion problem associated with nonconvex variational inequalities*.

Let H be a real Hilbert space. A mapping $T : H \to H$ is called $\gamma -$ *strongly monotone* if there exists a constant $\gamma > 0$ such that

$$\langle T x - T y, x - y \rangle \geq \gamma \|x - y\|^2, \tag{2.13}$$

for all $x, y \in H$. A mapping T is called $\mu -$ *Lipschitz* if there exists a constant $\mu > 0$ such that

$$\|T x - T y\| \leq \mu \|x - y\|, \tag{2.14}$$

for all $x, y \in H$.

Lemma 2.5. *In a real Hilbert space H, there holds the inequality*

1. $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle \quad x, y \in H$ *and* $\|x-y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$.
2. $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2, \forall t \in [0,1]$.

Lemma 2.6. *Let C be a nonempty closed subset of H. $r \in (0, +\infty]$ and set $C_r := \{x \in H : d(x, C) < r\}$. If C is uniform r-uniformly prox-regular, then the following hold:*

(1) for all $x \in C_r, P_C(x) \neq \emptyset$,

(2) for all $s \in (0, r), P_C$ is Lipschitz continuous with constant $t_s = \frac{r}{r-s}$ on C_s,

(3) the proximal normal cone is closed as a set-valued mapping.

3 Main Results

In this section we first establish the equivalent between the system of nonconvex variational inequalities (2.6) with the projection technique.

Lemma 3.1. *For given $x^*, y^* \in C_r$ are solution of system of strongly nonlinear general nonconvex variational inequalities (2.5), if and only if*

$$\begin{aligned} x^* &= P_{C_r}[y^* - \rho T_1 y^* + Ay^*], \\ y^* &= P_{C_r}[x^* - \eta T_2 x^* + Ax^*], \end{aligned} \tag{3.1}$$

where $P_{C_r} = (I + N^P_{C_r})^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.5), for a constant $\rho > 0$, we have

$$\langle \rho T_1 y^* + x^* - y^*, x - x^*\rangle + \lambda\|x - x^*\|^2 \ge \langle Ay^*, x - x^*\rangle$$

if and only if

$$\langle Ay^* - \rho T_1 y^* - x^* + y^*, x - x^*\rangle \le \lambda\|x - x^*\|^2.$$

Then,

$$Ay^* - \rho T_1 y^* - x^* + y^* \in N^P_{C_r} x^*$$

it implies that

$$\begin{aligned} 0 &\in \rho T_1 y^* + x^* - y^* - Ay^* + N^P_{C_r} x^* = (I + N^P_{C_r})x^* - (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow (I + N^P_{C_r})x^* = (y^* - \rho T_1 y^* + Ay^*) \\ &\Leftrightarrow x^* = P_{C_r}[y^* - \rho T_1 y^* + Ay^*] \end{aligned}$$

where we have used the well-known fact that $P_{C_r} = (I + N^P_{C_r})^{-1}$.

Similarly, we obtain

$$y^* = P_{C_r}[x^* - \eta T_2 x^* + Ax^*].$$

□

This prove our assertions.

Algorithm 3.2. *For arbitrarily chosen initial points $x_0 \in C_r$, the sequence $\{x_n\}$ and $\{y_n\}$ in the following way:*

$$\begin{aligned} y_n &= P_{C_r}[x_n - \eta T_2 x_n + Ax_n], \eta > 0 \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P_{C_r}[y_n - \rho T_1 y_n + Ay_n], \rho > 0, \end{aligned} \tag{3.2}$$

where $\{\alpha_n\}$ is a sequence in $[0, 1]$.

Now, we suggest and analyze the following explicit projection method (3.2) for solving the system of nonconvex variational inequalities (2.6).

Theorem 3.3. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping, T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous. If there exists constant $\rho, \eta > 0$ such that*

$$|\rho - \frac{\gamma_1}{\mu_1^2}| < \frac{\sqrt{\gamma_1^2 t_s^2 - t_s\mu_1^2(t_s + t_s\beta - 1)}}{t_s\mu_1^2}, \tag{3.3}$$

$$|\eta - \frac{\gamma_2}{\mu_2^2}| < \frac{\sqrt{\gamma_2^2 t_s^2 - t_s\mu_2^2(t_s + t_s\beta - 1)}}{t_s\mu_2^2}, \tag{3.4}$$

which is $\sqrt{1 + \beta - \frac{1}{t_s}} < \frac{\gamma_1}{\mu_1}$ where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ obtained from Algorithm 3.2 converge to a solution of the system of nonconvex variational inequalities (2.6).

Proof. Let $x^*, y^* \in C_r$ be a solution of (2.6) and from Lemma 3.1, we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1 - \alpha_n)x_n + \alpha_n P_C[y_n - \rho T_1 y_n + Ay_n] - x^*\| \\
&= \|(1 - \alpha_n)(x_n - x^*) + \alpha_n(P_C[y_n - \rho T_1 y_n + Ay_n] - P_C[y^* - \rho T_1 y^* + Ay^*])\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - \rho T_1 y_n + Ay_n) - (y^* - \rho T_1 y^* + Ay^*)\| \\
&= (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*) + (Ay_n - Ay^*)\| \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \|Ay_n - Ay^*\|] \\
&\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s[\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| + \beta\|y_n - y^*\|].
\end{aligned} \tag{3.5}$$

Since T_1 are both μ_1-Lipschitz continuous and γ_1-strongly monotone mapping and from Lemma 2.5, we consider

$$\begin{aligned}
\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\|^2 &= \|y_n - y^*\|^2 - 2\rho\langle y_n - y^*, T_1 y_n - T_1 y^*\rangle + \rho^2\|T_1 y_n - T_1 y^*\|^2 \\
&\leq \|y_n - y^*\|^2 - 2\rho\gamma_1\|y_n - y^*\|^2 + \rho^2\mu_1^2\|y_n - y^*\|^2 \\
&= (1 - 2\rho\gamma_1 + \rho^2\mu_1^2)\|y_n - y^*\|^2.
\end{aligned}$$

It follows that

$$\|(y_n - y^*) - \rho(T_1 y_n - T_1 y^*)\| \leq \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2}\|y_n - y^*\|. \tag{3.6}$$

Replace (3.6) into (3.5), we have

$$\|x_{n+1} - x^*\| \leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2})\|y_n - y^*\|. \tag{3.7}$$

On the other hand, we can compute that

$$\begin{aligned}
\|y_n - y^*\| &= \|P_C[x_n - \eta T_2 x_n + Ax_n] - y^*\| \\
&= \|P_C[x_n - \eta T_2 x_n + Ax_n] - P_C[x^* - \eta T_2 x^* + Ax^*]\| \\
&\leq t_s\|(x_n - \eta T_2 x_n + Ax_n) - (x^* - \eta T_2 x^* + Ax^*)\| \\
&\leq t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \|Ax_n - Ax^*\|] \\
&\leq t_s[\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| + \beta\|x_n - x^*\|].
\end{aligned} \tag{3.8}$$

Similarly, from T_2 are both μ_2-Lipschitz continuous and γ_2-strongly monotone mapping, we have

$$\begin{aligned}
\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\|^2 &= \|x_n - x^*\|^2 - 2\eta\langle x_n - x^*, T_2 x_n - T_2 x^*\rangle + \eta^2\|T_2 x_n - T_2 x^*\|^2 \\
&\leq \|x_n - x^*\|^2 - 2\eta\gamma_2\|x_n - x^*\|^2 + \eta^2\mu_2^2\|x_n - x^*\|^2 \\
&= (1 - 2\eta\gamma_2 + \eta^2\mu_2^2)\|x_n - x^*\|^2.
\end{aligned}$$

It follows that

$$\|(x_n - x^*) - \eta(T_2 x_n - T_2 x^*)\| \leq \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2}\|x_n - x^*\|. \tag{3.9}$$

Replace (3.9) into (3.8), we have

$$\|y_n - y^*\| \leq t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})\|x_n - x^*\|. \tag{3.10}$$

Moreover, from (3.7) and (3.10) we put $\theta_1 = t_s(\beta + \sqrt{1 - 2\rho\gamma_1 + \rho^2\mu_1^2})$, $\theta_2 = t_s(\beta + \sqrt{1 - 2\eta\gamma_2 + \eta^2\mu_2^2})$, it follows that

$$\begin{aligned} \|x_{n+1} - x^*\| &\leq (1 - \alpha_n)\|x_n - x^*\| + \alpha_n\theta_1\theta_2\|x_n - x^*\| \\ &= (1 - (1 - \theta_1\theta_2)\alpha_n)\|x_n - x^*\| \\ &\leq \prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i)\|x_0 - x^*\|. \end{aligned} \tag{3.11}$$

Since $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and (3.15), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1 - (1 - \theta_1\theta_2)\alpha_i) = 0. \tag{3.12}$$

It follows from (3.12) and (3.11), we have

$$\lim_{n\to\infty}\|x_n - x^*\| = 0. \tag{3.13}$$

From (3.10) and (3.13), we have

$$\lim_{n\to\infty}\|y_n - y^*\| = 0. \tag{3.14}$$

Which is $x^*, y^* \in C_r$ satisfying the system of nonconvex variational inequalities (2.6). This completes the proof. □

Corollary 3.4. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2}, \ t_s\sqrt{t_s^2 - 1} < \frac{\gamma}{\mu}, \tag{3.15}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0, r)$. If the sequence of positive real number $\alpha_n \in [0, 1]$ with $\Sigma_{n=0}^{\infty}\alpha_n = 0$, then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n P_C[y_n - \rho T y_n], \rho > 0. \end{aligned} \tag{3.16}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. From Theorem 3.3, if $T_1 = T_2 = T$ and $A(x) \equiv 0$, we have a result. □

Corollary 3.5. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H, and let $T : C \to H$ be such that T are both μ-Lipschitz continuous and γ-strongly monotone mapping. If there exists constant $\rho, \eta > 0$ such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2 - 1)}}{t_s^2\mu^2}, \ t_s\sqrt{t_s^2 - 1} < \frac{\gamma}{\mu}, \tag{3.17}$$

where $t_s = \frac{r}{r-s}$ for some $s \in (0,r)$. Then the sequences $\{x_n\}$ and $\{y_n\}$ generated by for arbitrarily chosen initial points $x_0, y_0 \in C_r$.

$$\begin{aligned} y_n &= P_C[x_n - \eta T x_n], \eta > 0 \\ x_{n+1} &= P_C[y_n - \rho T y_n], \rho > 0. \end{aligned} \tag{3.18}$$

converge to a solution of the system of nonconvex variational inequalities (2.8).

Proof. From Theorem 3.3, if $T_1 = T_2 = T$. and $A(x) \equiv 0$ and $\alpha_n = 1$ for any $n \geq 1$. we have a result. □

4 Applications

In this section, we can applied Theorem 3.3 to the system of general nonconvex variational inequalities, for given nonlinear mappings $T, g : C_r \to H$, we consider the problem of finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle + \lambda \|x - g(x^*)\|^2 \geq \langle Ag(y^*), x - g(x^*)\rangle, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), y - g(y^*)\rangle + \lambda \|y - g(y^*)\|^2 \geq \langle Ag(x^*), y - g(y^*)\rangle, \forall y \in C_r, \eta > 0. \end{aligned} \tag{4.1}$$

which is called the *system of general strongly nonlinear nonconvex variational inequalities*(SGSNNVI).

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0$ and $\lambda = 0$, then the problem (1.1) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T_1 g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T_2 g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0, \end{aligned} \tag{4.2}$$

which is called the *system of general nonconvex variational inequalities.*

If $A(x^*) \equiv 0$, $A(y^*) \equiv 0, \lambda = 0$ and $T_1 = T_2 = T$. then the problem (1.1) is equivalent to finding $x^*, y^* \in C_r$ such that

$$\begin{aligned} &\langle \rho T g(y^*) + g(x^*) - g(y^*), x - g(x^*)\rangle \geq 0, \forall x \in C_r, \rho > 0 \\ &\langle \eta T g(x^*) + g(y^*) - g(x^*), x - g(y^*)\rangle \geq 0, \forall x \in C_r, \eta > 0. \end{aligned} \tag{4.3}$$

Now, similarly of the proof of Lemma 3.1, we have the result.

Lemma 4.1. *For given $x^*, y^* \in C_r$ is a solution of system of nonconvex variational inequalities (4.1), if and only if*

$$\begin{aligned} g(x^*) &= P_{C_r}[g(y^*) - \rho T_1 g(y^*) + Ag(y^*)], \\ g(y^*) &= P_{C_r}[g(x^*) - \eta T_2 g(x^*) + Ag(x^*)]. \end{aligned} \tag{4.4}$$

where $P_{C_r} = (I + N^P_{C_r})^{-1}$ is the projection of H onto the uniformly prox-regular set C_r.

Theorem 4.2. *Let C be a uniformly r-prox-regular closed subset of a Hilbert space H. and let $T_1, T_2, A : C \to H$ be such that T_1 is a μ_1-Lipschitz continuous and γ_1-strongly monotone mapping. T_2 is a μ_2-Lipschitz continuous and γ_2-strongly monotone mapping and A is a β-Lipschitz continuous and let g be injective mapping. If there exists constant $\rho, \eta > 0$ such that*

$$|\rho - \frac{\gamma_1}{\mu_1^2}| < \frac{\sqrt{\gamma_1^2 t_s^2 - t_s \mu_1^2 (t_s + t_s \beta - 1)}}{t_s \mu_1^2}, \tag{4.5}$$

$$|\eta - \frac{\gamma_2}{\mu_2^2}| < \frac{\sqrt{\gamma_2^2 t_s^2 - t_s \mu_2^2 (t_s + t_s \beta - 1)}}{t_s \mu_2^2}, \tag{4.6}$$

which is $\sqrt{1+\beta-\frac{1}{t_s}} < \frac{\gamma_1}{\mu_1}$ *where* $t_s = \frac{r}{r-s}$ *for some* $s \in (0,r)$. *If the sequence of positive real number* $\alpha_n \in [0,1]$ *with* $\Sigma_{n=0}^{\infty}\alpha_n = 0$, *then the sequences* $\{x_n\}$ *and* $\{y_n\}$ *is generated by for* $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_{C_r}[g(x_n) - \eta T_2 g(x_n) + Ag(x_n)], \eta > 0 \\ g(x_{n+1}) &= (1-\alpha_n)g(x_n) + \alpha_n P_{C_r}[g(y_n) - \rho T_1 g(y_n) + Ag(y_n)], \rho > 0, \end{aligned} \tag{4.7}$$

where $\{\alpha_n\}$ *is a sequence in* $[0,1]$ *strongly converge to a solution of the system of nonconvex variational inequalities (4.1).*

Proof. Similar the proof in Theorem 3.3, let $x^*, y^* \in C_r$ be a solution of (4.2) and from Lemma 1.1, we can compute that

$$\|g(x_{n+1}) - g(x^*)\| \leq \prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i)\|g(x_0) - g(x^*)\|. \tag{4.8}$$

where $\theta_1 = t_s(\beta + \sqrt{1-2\rho\gamma_1+\rho^2\mu_1^2}), \theta_2 = t_s(\beta + \sqrt{1-2\eta\gamma_2+\eta^2\mu_2^2})$. From $\Sigma_{n=0}^{\infty}\alpha_n = \infty$ and conditions (1.5), we obtain

$$\lim_{n\to\infty}\prod_{i=0}^{n}(1-(1-\theta_1\theta_2)\alpha_i) = 0. \tag{4.9}$$

It follows from (4.8) and (4.9), we have

$$\lim_{n\to\infty}\|g(x_n) - g(x^*)\| = 0. \tag{4.10}$$

And we can compute that

$$\|g(y_n) - g(y^*)\| \leq \theta_2\|g(x_n) - g(x^*)\|. \tag{4.11}$$

it follows that

$$\lim_{n\to\infty}\|g(y_n) - g(y^*)\| = 0. \tag{4.12}$$

From g is injective mapping. we have $\lim_{n\to\infty}\|x_n - x^*\| = 0$ and $\lim_{n\to\infty}\|y_n - y^*\| = 0$ satisfying the system of general nonconvex variational inequalities (1.2). This complete the proof. □

Corollary 4.3. *Let* C *be a uniformly* r*-prox-regular closed subset of a Hilbert space* H, *and let* $T : C \to H$ *be such that* T *are both* μ*-Lipschitz continuous and* γ*-strongly monotone mapping. If there exists constant* $\rho, \eta > 0$ *such that*

$$\frac{\gamma}{t_s^2\mu^2} - \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2} < \rho, \eta < \frac{\gamma}{t_s^2\mu^2} + \frac{\sqrt{\gamma^2 - t_s^2\mu^2(t_s^2-1)}}{t_s^2\mu^2}, \ t_s\sqrt{t_s^2-1} < \frac{\gamma}{\mu}, \tag{4.13}$$

where $t_s = \frac{r}{r-s}$ *for some* $s \in (0,r)$. *Then the sequence* $\{x_n\}$ *and* $\{y_n\}$ *is generated by for* $x_0, y_0 \in C_r$,

$$\begin{aligned} g(y_n) &= P_C[g(x_n) - \eta T g(x_n)], \eta > 0 \\ g(x_{n+1}) &= P_C[g(y_n) - \rho T g(y_n)], \rho > 0. \end{aligned} \tag{4.14}$$

strongly converge to a solution of the system of nonconvex variational inequalities (1.3).

Proof. From Theorem 4.2, if $T_1 = T_2 = T$ and $\alpha_n = 1$ for any $n \geq 0$, we have a result. □

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