รายงานฉบับสมบูรณ์

โครงการวิจัย

การวางนัยทั่วไปของปัญหาอสมการแปรผันไม่คอนเวกซ์แยกและปัญหาจุดตรึง

Generalized split nonconvex variational inequalities problem and fixed point problem

โดย รองศาสตราจารย์ ดร.อิสระ อินจันทร์

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ABSTRACTS

The main objective of this paper is to introduced a general method for a split variationalinclusion and asymptotically nonexpansive semigroups in Hilbert space. We prove that the sequence generate by the iterative scheme converge strongly to a common solution of the set of solution of a split variational inclusion and the set of common fixed points of one-parameter asymptotically nonexpansive semigroups. The results presented in this paper extend and improvement of previously known results in this research area.

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CHAPTER I

Introduction

Let H be a real Hilbert Space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [1] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C|Tx = x\}$. Let $A : H_1 \to H_2$ be a mapping then $A^* : H_2 \to H_1$ is an adjoint operator of A if and only if $\langle A^*y, x\rangle = \langle y, Ax\rangle$ for $x \in H_1, y \in H_2$.

Recall also that a one-parameter family $\mathfrak{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [illegible]) if the following conditions are satisfied:

(i) $T(0)x = x, x \in C$

(ii) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(iii) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(iv) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\{x - y\|, x, y \in C.$$

A Lipschitzian semigroup $\mathcal{T}$ is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(\mathcal{T})$ to denote the common fixed point set of the semigroup; that is $Fix(\mathcal{T}) = \{x \in C|T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [illegible]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [illegible].

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [illegible] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [illegible], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In 2006, Marino and Xu [ ], introduced the following general iterative methods to approximate a fixed point of a nonexpansive mapping:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \tag{1.1}$$

where $\{\alpha_n\} \subseteq [0,1]$ satisfies certain conditions, f is a contraction of H into itself, and B is a strongly positive bounded linear operator on H. Moreover, they prove that $\{x_n\}$ converges strongly to $x^* \in Fix(T)$, the unique solution of the following variational inequality:

$$\langle (B - \gamma f)x^*, x^* - w\rangle \leq 0, \forall w \in Fix(T), \tag{1.2}$$

which is also the optimality condition of the minimization problem.

Recall also that a multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in H_1, u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0. \tag{1.3}$$

A monotone mapping M is maximal if the Graph(M) is not property contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1, \langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in Graph(M)$ implies that $u \in Mx$.

From a monotone mapping M the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in H_1, \tag{1.4}$$

for some $\lambda > 0$, where I is the identity mapping on H_1. Note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive.

In 2011, Moudafi [ ] introduce the split monotone variational inclusion problem: find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in H_2 : 0 \in f_2(y^*) + B_2(y^*). \end{cases}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

In 2015 Wen and Chen [ ] introduce a modified general iterative method for a split variational inclusion and nonexpansive semigroups, which is defined tn the following way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds. \tag{1.5}$$

where $\gamma \in [0,1]$ and $\{\alpha_n\} \subseteq [0,1]$, B is a strongly positive bounded linear operator on H_1.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.0.1. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathfrak{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x - y\|,$$

then we have $\mathfrak{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathfrak{T}$ is asymptotically nonexpansive semigroup.

Example 1.0.2. Let $H_1 = H_2 = \mathbb{R}$ and let $\mathfrak{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x - y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathfrak{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \nleq 1$, then $\mathfrak{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathfrak{T}$ is a nonexpansive semigroup then $\mathfrak{T}$ is asymptotically nonexpansive semigroup. But $\mathfrak{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

In this work we extend the results of Wen and Chen [ ] for $\mathfrak{T}$ is an asymptotically nonexpansive semigroup then we consider

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \tag{1.6}$$

where $\gamma \in [0, 1]$ and $\{\alpha_n\} \subseteq [0, 1]$, B is a strongly positive bounded linear operator on H_1.

CHAPTER II
Preliminaries

In this chapter, we give some defnitions, notations, and some useful results that will be used in the later chapters.

2.1 Useful lemmas.

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ respectively. Let C be a closed convex subset of H, let P_C be the matric projection of H onto C i.e. for $x \in H. P_C x$ satisfies the property

$$\|x - P_C x\| = \min_{y \in C} \|x - y\|.$$

It is know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0, \forall y \in C.$$

Lemma 2.1.1. Let H be a real Hilbert space, then the following hold:

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y) \rangle, \forall x, y \in H$:

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, t \in [0, 1]. \forall x, y \in H.$

Lemma 2.1.2. [ ] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, If $\{x_n\}$ is a sequence in C satisfying the properties:

(i) $x_n \rightharpoonup z$; and

(ii) $\limsup_{t \to \infty} \limsup_{n \to \infty} \|T(t)x_n - x_n\| = 0$,

then $z \in Fix(\mathcal{T})$.

Lemma 2.1.3. [ ] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,

$$\limsup_{u \to \infty} \limsup_{t \to \infty} \sup_{x \in C} \|\frac{1}{t}\int_0^t T(s)x ds - T(u)(\frac{1}{t}\int_0^t T(s)x ds)\| = 0.$$

Lemma 2.1.4. [ ] Let B be a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\overline{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \leq 1 - \varrho\overline{\gamma}$.

Lemma 2.1.5. [ ] Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0, 1)$ and B is a strongly positive bounded linear operator with a coefficient $\overline{\gamma} > 0$. Then for $0 < \gamma < \frac{\overline{\gamma}}{\rho}$,

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y \rangle \geq (\overline{\gamma} - \gamma\rho)\|x - y\|^2, \forall x, y \in H.$$

That is $B - \gamma f$ is strongly monotone with coefficient $\overline{\gamma} - \gamma\rho$.

Lemma 2.1.6. [[illegible]] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n b_n + \sigma_n,$$

where $\{\gamma_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $\{b_n\}_{n=1}^{\infty}, \{\sigma_n\}_{n=1}^{\infty}$ are sequence in $\mathbb{R}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$;

(iii) $\sigma_n \geq 0$ and $\Sigma_{n=1}^{\infty}\sigma_n < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.1.7. [[illegible], [illegible]] Let $S : H \to H$ be averaged and $T : H \to H$ be nonexpansive have:

(i) $W = (1-\alpha)S + \alpha T$ is averaged, where $\alpha \in (0,1)$.

(ii) The composite of finitely many averaged mapping is averaged.

Theorem 2.1.8. [[illegible]] Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0,1)$ and $T : H_1 \to H_1$ be a nonexpansive mapping such that $\Omega = Fix(T) \cap T \neq \emptyset$. For a given $x_0 \in H_1$ arbitrary, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n. \end{cases} \tag{2.7}$$

where $\lambda > 0$ and $\epsilon \in (0, 1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A; $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$. Then the sequens $\{u_n\}$ and $\{x_n\}$ both convergence strongly to $z \in \Omega$, where $z = P_\Omega(z)$.

Lemma 2.1.9. [[illegible]] The split variational inclusion problem (2.7) is equivalent to finding $x^* \in H_1$ such that $y^* = Ax^* \in H_2 : x^* = J_\lambda^{B_1}$ and $y^* = J_\lambda^{B_2}(y^*)$ for some $\lambda > 0$.

CHAPTER III

Main Results

3.1 SYSTEM OF NONCONVEX VARIATIONAL INEQUALITIES

In the first Theorem in this section we prove the unique fixed point by Banach contraction principle of Φ. The second Theorem we prove the strong convergence of modified general iterative method for a split variational inclusion and asymptotically nonexpansive semigroups to $q \in \Omega$ which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

Theorem 3.1.1. Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$. For any $\alpha \in (0, 1)$, define the mapping Φ on H_1 by

$$\Phi(x) = \alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds,$$

where $t > 0$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A and $1 < \frac{1}{t}\int_0^t L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$. Then the mapping Φ is a contraction and has a unique fixed point.

พิสูจน์. Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive, they are averaged. For $\epsilon \in (0, \frac{1}{L})$, the mapping $I + \epsilon A^*(J_\lambda^{B_2} - I)A$ is averaged; see e.g.[11]. It follows from Lemma2.1.3 (ii) that the mapping $J_\lambda^{B_1}(I + \epsilon A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive. By Lemma2.1.4, for any $x, y \in H_1$, we have

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \|\alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\alpha\gamma f(y) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\| \\
&\leq \alpha\gamma\|f(x) - f(y)\| + (1 - \alpha\overline{\gamma})\|\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\| \\
&\leq \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax] \\
&\quad -J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]\| \\
&\leq \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|x - y\| \\
&= \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})a\|x - y\| \\
&\leq [a - \alpha(\overline{\gamma}a - \gamma\rho)]\|x - y\|
\end{aligned}$$

From $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$, we have $[a - \alpha(\overline{\gamma}a - \gamma\rho)] < 1$. It follows that Φ is a contraction mapping. By the Banach contraction principle, $\Phi(x)$ has a unique fixed point x_α, that is

$$x_\alpha = \alpha\gamma f(x_\alpha) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x_\alpha + \epsilon A^*(J_\lambda^{B_2} - I)Ax_\alpha]ds.$$

□

Next Theorem we modified general iterative method for a split variational inclusion and asymptotically nonexpansive semigroups and prove the strong convergence of iterative to $q \in \Omega$ which is the unique solution of the following variational inequality: $\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega$.

Theorem 3.1.2. Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0, 1), \{t_n\} \subseteq (0, \infty)$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ satisfy:

(i) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty}\frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q \in \Omega$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

พิสูจน์. Let $p \in \Omega$, we have $p = J_\lambda^{B_1}p, J_\lambda^{B_2}(Ap) = Ap$ and $T(s)p = p$. From (1.6), let $u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]$, and Lemma 2.1.9, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n] - J_\lambda^{B_1}p\|^2 \\ &\leq \|x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\leq \|x_n - p\|^2 + 2\epsilon\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle + \epsilon^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2. \quad (3.8)\end{aligned}$$

By the definition of A and A^*, we obtain

$$\begin{aligned}\epsilon^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 &= \epsilon^2\langle A^*(J_\lambda^{B_2} - I)Ax_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &= \epsilon^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &\leq L\epsilon^2\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= L\epsilon^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \quad (3.9)\end{aligned}$$

And we have

$$\begin{aligned}
2\epsilon\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle &= 2\epsilon\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\
&= 2\epsilon\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\
&= 2\epsilon(\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\
&\quad -\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\
&= 2\epsilon[\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2} - I)Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2] \\
&\leq 2\epsilon[\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\| - \|(J_\lambda^{B_2} - I)Ax_n\|^2] \\
&= -\epsilon\|(J_\lambda^{B_2} - I)Ax_n\|^2. \qquad (3.10)
\end{aligned}$$

From (3.8), (3.9), (3.10) and $\epsilon \in (0, \frac{1}{L})$, it follows that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 + \epsilon(L\epsilon - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \leq \|x_n - p\|^2. \qquad (3.11)$$

Next, we set $w_n = \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds$ for $n \geq 0$, since $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ and from (3.11), we have

$$\begin{aligned}
\|w_n - p\| &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(s)p\| \\
&\leq \frac{1}{t_n}\int_0^{t_n} \|T(s)u_n - T(s)p\| ds \\
&\leq \frac{1}{t_n}\int_0^{t_n} L_s ds\|u_n - p\| \\
&\leq \frac{1}{t_n}\int_0^{t_n} L_s ds\|x_n - p\| \\
&\leq a\|x_n - p\|, \qquad (3.12)
\end{aligned}$$

where $a = \sup_{n\geq 1}\{a_n\}$. It follows from (1.6), (3.12) and Lemma 2.1.4, that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n\gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds - p\| \\
&= \|\alpha_n(\gamma f(x_n) - Bp) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n}(T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n] - T(s)p)ds\| \\
&\leq \alpha_n\|\gamma f(x_n) - Bp\| + (1 - \alpha_n\overline{\gamma})\frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)p\|ds \\
&\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + (1 - \alpha_n\overline{\gamma})a\|x_n - p\| \\
&\leq \alpha_n\gamma\rho\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1 - \alpha_n\overline{\gamma})a\|x_n - p\| \\
&= [1 - \alpha_n(\overline{\gamma}a - \gamma\rho)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|. \qquad (3.13)
\end{aligned}$$

Since $1 < a \leq \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ and $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, we have $\overline{\gamma}a - \gamma\rho > 0$. By a simple induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{\overline{\gamma}a - \gamma\rho}\|\gamma f(p) - Bp\|\}. \qquad (3.14)$$

Therefor, $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{w_n\}$. Next, we show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (1.6), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\alpha_n\gamma f(x_n)+(I-\alpha_n B)w_n-\alpha_n\gamma f(x_{n-1})+(I-\alpha_{n-1}B)w_{n-1}\| \\
&= \|\alpha_n\gamma[f(x_n)-f(x_{-1})]+(\alpha_n-\alpha_{n-1})\gamma f(x_{n-1})+(I-\alpha_n B)(w_n-w_{n-1}) \\
&\quad -(\alpha_n-\alpha_{n-1})Bw_{n-1}\| \\
&\leq \alpha_n\gamma\|f(x_n)-f(x_{-1})\|+|\alpha_n-\alpha_{n-1}|\|\gamma f(x_{n-1})\| \\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n-w_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Bw_{n-1}\| \\
&\leq \alpha_n\gamma\rho\|x_n-x_{-1}\|+(1-\alpha_n\overline{\gamma})\|w_n-w_{n-1}\| \\
&\quad +|\alpha_n-\alpha_{n-1}|[\gamma\|f(x_{n-1})\|+\|Bw_{n-1}\|]. \qquad (3.15)
\end{aligned}$$

Since $\frac{1}{t_n}\int_0^{t_n}T(s)pds=\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)pds$, we consider

$$\begin{aligned}
\|w_n-w_{n-1}\| &= \|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)u_{n-1}ds\| \\
&= \|\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-\frac{1}{t_n}\int_0^{t_n}T(s)u_{n-1}ds+\frac{1}{t_n}\int_0^{t_n}T(s)u_{n-1}ds+\frac{1}{t_n}\int_0^{t_{n-1}}T(s)u_{n-1}ds \\
&\quad -\frac{1}{t_n}\int_0^{t_{n-1}}T(s)u_{n-1}ds-\frac{1}{t_n}\int_0^{t_{n-1}}T(s)pds+\frac{1}{t_n}\int_0^{t_{n-1}}T(s)pds-\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)pds \\
&\quad +\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)pds-\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)u_{n-1}ds\| \\
&= \|(\frac{1}{t_n}\int_0^{t_n}T(s)u_nds-\frac{1}{t_n}\int_0^{t_n}T(s)u_{n-1}ds) \\
&\quad +(\frac{1}{t_n}\int_0^{t_{n-1}}T(s)u_{n-1}ds-\frac{1}{t_n}\int_0^{t_{n-1}}T(s)pds-\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)u_{n-1}ds \\
&\quad +\frac{1}{t_{n-1}}\int_0^{t_{n-1}}T(s)pds)+(\frac{1}{t_n}\int_0^{t_{n-1}}T(s)u_{n-1}ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^0 T(s)u_{n-1}ds)-(\frac{1}{t_n}\int_0^{t_{n-1}}T(s)pds+\frac{1}{t_n}\int_{t_{n-1}}^0 T(s)pds)\| \\
&= \|\frac{1}{t_n}\int_0^{t_n}[T(s)u_n-T(s)u_{n-1}]ds+(\frac{1}{t_n}-\frac{1}{t_{n-1}})\int_0^{t_{n-1}}[T(s)u_{n-1}-T(s)p]ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n}T(s)u_{n-1}ds-\frac{1}{t_n}\int_{t_{n-1}}^{t_n}T(s)pds\| \\
&= \|\frac{1}{t_n}\int_0^{t_n}[T(s)u_n-T(s)u_{n-1}]ds+(\frac{1}{t_n}-\frac{1}{t_{n-1}})\int_0^{t_{n-1}}[T(s)u_{n-1}-T(s)p]ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n}[T(s)u_{n-1}-T(s)p]ds\| \\
&\leq \frac{1}{t_n}\int_0^{t_n}\|T(s)u_n-T(s)u_{n-1}\|ds+|\frac{1}{t_n}-\frac{1}{t_{n-1}}|\int_0^{t_{n-1}}\|T(s)u_{n-1}-T(s)p\|ds \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n}\|T(s)u_{n-1}-T(s)p\|ds\| \\
&\leq \frac{1}{t_n}\int_0^{t_n}L_sds\|u_n-u_{n-1}\|+\frac{|t_{n-1}-t_n|}{t_n}\frac{1}{t_{n-1}}\int_0^{t_{n-1}}L_sds\|u_{n-1}-p\| \\
&\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n}L_sds\|u_{n-1}-p\|.
\end{aligned}$$

Now, we taking $\lim_{s\to\infty} L_s = 1$, it follows that $\frac{1}{t_n}\int_0^{t_n} L_s ds \to \frac{1}{t_n}\int_0^{t_n} ds$, and hence

$$\|w_n - w_{n-1}\| \leq \|u_n - u_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|. \tag{3.16}$$

From $\epsilon \in (0, \frac{1}{L})$ and mapping $J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]$ is averaged and hence nonexpansive, then we have

$$\|u_n - u_{n-1}\| = \|J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_n - J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{3.17}$$

From (3.15), (3.16) and (3.17), we have

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \alpha_n\gamma\rho\|x_n - x_{n-1}\| + (1 - \alpha_n\overline{\gamma})(\|x_n - x_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|) \\ &\quad + |\alpha_n - \alpha_{n-1}|[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|] \\ &\leq [1 - \alpha_n(\overline{\gamma} - \gamma\rho)]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \frac{2|t_{n-1} - t_n|}{t_n})M. \end{aligned} \tag{3.18}$$

where $m = \max\{sup_{n\in\mathbb{N}}[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|], sup_{n\in\mathbb{N}}\|u_{n-1} - p\|\}$. It follows from condition $(i) - (ii)$ and Lemma 2.1.6, hence

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.19}$$

Consider,

$$\begin{aligned}\|x_n - w_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n\gamma f(x_n) + (I - \alpha_n B)w_n - w_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) + Bw_n\|. \end{aligned} \tag{3.20}$$

From condition (i) and (3.19), we have

$$\lim_{n\to\infty} \|x_n - w_n\| = 0, \tag{3.21}$$

and that

$$\lim_{n\to\infty} \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| = 0. \tag{3.22}$$

For any $u \geq 0$, we have

$$\begin{aligned}\|x_n - T(u)x_n\| &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\ &\quad + \|T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)x_n\| \\ &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\ &\quad + L_u\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - x_n\|. \end{aligned}$$

From (3.22), Lemma 2.1.2 and $\limsup_{u\to\infty} L_u \leq 1$, we have

$$\lim_{n\to\infty} \|x_n - T(u)x_n\| = 0. \tag{3.23}$$

By the definition of x_n, (3.10), (3.11) and Lemma 2.1.1, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|\alpha_n\gamma f(x_n)+(1-\alpha_n B)w_n-p\|^2\\
&= \|(w_n-p)+\alpha_n(\gamma f(x_n)-Bw_n)\|^2\\
&\leq \|w_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Bw_n,x_{n+1}-p\rangle\\
&\leq \|u_n-p\|^2+2\alpha_n\langle\gamma f(x_n)-Bw_n,x_{n+1}-p\rangle\\
&\leq [\|x_n-p\|^2+\epsilon(L\epsilon-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2]\\
&\quad +2\alpha_n\langle\gamma f(x_n)-Bw_n,x_{n+1}-p\rangle\\
&\leq \|x_n-p\|^2-\epsilon(1-L\epsilon)\|(J_\lambda^{B_2}-I)Ax_n\|^2+2\alpha_n M_2^2, \qquad (3.24)
\end{aligned}$$

where $M_2=\max\{\sup_{n\in\mathbb{N}}\|\gamma f(x_n)-Bw_n\|,\sup_{n\in\mathbb{N}}\|x_{n+1}-p\|\}$ and $\epsilon\in(0,\frac{1}{L})$, it implies that

$$\begin{aligned}
\epsilon(1-L\epsilon)\|(J_\lambda^{B_2}-I)Ax_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n M_2^2\\
&\leq \|x_{n+1}-x_n\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\alpha_n M_2^2. \qquad (3.25)
\end{aligned}$$

From (3.19), we obtain

$$\lim_{n\to\infty}\|(J_\lambda^{B_2}-I)Ax_n\|=0. \qquad (3.26)$$

From (3.8), (3.10) and $\epsilon\in(0,\frac{1}{L})$, that

$$\begin{aligned}
\|u_n-p\|^2 &= \|J_\lambda^{B_1}[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n]-J_\lambda^{B_1}p\|^2\\
&\leq \langle u_n-p,x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p\rangle\\
&= \frac{1}{2}\{\|u_n-p\|^2+\|x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p\|^2\\
&\quad -\|u_n-p-[x_n+\epsilon A^*(J_\lambda^{B_2}-I)Ax_n-p]\|^2\}\\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2+\epsilon(L\epsilon-1)\|(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad -\|u_n-x_n-\epsilon A^*(J_\lambda^{B_2}-I)Ax_n\|^2\}\\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-[\|u_n-x_n\|^2+\epsilon^2\|A^*(J_\lambda^{B_2}-I)Ax_n\|^2\\
&\quad -2\epsilon\langle u_n-x_n,A^*(J_\lambda^{B_2}-I)Ax_n\rangle]\}\\
&\leq \frac{1}{2}\{\|u_n-p\|^2+\|x_n-p\|^2-\|u_n-x_n\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|\},
\end{aligned}$$

which implies that

$$\|u_n-p\|^2\leq\|x_n-p\|^2-\|u_n-x_n\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|. \qquad (3.27)$$

It follows from (3.24) and (3.27) that

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \|u_n-p\|^2+2\alpha_n M_2^2\\
&\leq \|x_n-p\|^2-\|u_n-x_n\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|+2\alpha_n M_2^2,
\end{aligned}$$

that is

$$\begin{aligned}
\|u_n-x_n\|^2 &\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|+2\alpha_n M_2^2\\
&\leq \|x_n-x_{n+1}\|(\|x_n-p\|+\|x_{n+1}-p\|)+2\epsilon\|A(u_n-x_n)\|\|(J_\lambda^{B_2}-I)Ax_n\|+2\alpha_n M_2^2.
\end{aligned}$$

From (3.19), (3.27) and condition (i), it follows that

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{3.28}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded, there exists weak limit w of $\{x_n\}$. Without loss of generality, we may assume that subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which is $x_{n_j} \rightharpoonup w$. From (3.28), we have subsequence $\{u_{n_j}\}$ of $\{u_n\}$, which is $u_{n_j} \rightharpoonup w$. Moreover, $u_{n_j} = J_\lambda^{B_1}[x_{n_j} + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}]$ with

$$\frac{(x_{n_j} - u_{n_j}) + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}}{\lambda} \in B_1 u_{n_j} \tag{3.29}$$

By taking limit $j \to \infty$, and (3.26), (3.28) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $Ax_{n_j} \rightharpoonup Aw$. From (3.26) and the the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain $Aw \in B_2(Aw)$. It follows from Lemma 2.1.9 that $w \in \mathcal{T}$.

Next, we show that $\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle \le 0$, where $q = P_\Omega(I - B + \gamma f)q$. From the sequence $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \lim_{j\to\infty}\langle \gamma f(q) - Bq, x_{n_j} - q\rangle. \tag{3.30}$$

Assume that $w \neq T(u)w$. From (3.22) and Opial's property, we have

$$\begin{aligned} \liminf_{j\to\infty}\|x_{n_j} - w\| &< \liminf_{j\to\infty}\|x_{n_j} - T(u)w\| \\ &\le \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + \|T(u)x_{n_j} - T(u)w\|) \\ &\le \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + L_u\|x_{n_j} - w\|) \\ &\le \liminf_{j\to\infty} L_u\|x_{n_j} - w\|. \end{aligned}$$

If we letting $u \to \infty$, we have $\limsup_{u\to\infty} L_u \le 1$, it follows that

$$\liminf_{j\to\infty}\|x_{n_j} - w\| < \liminf_{j\to\infty}\|x_{n_j} - w\|.$$

This is a contradiction. Then $w \in Fix(\mathcal{T})$. Consequently, $w \in \Omega = Fix(\mathcal{T}) \cap \mathfrak{F}$. It follows from (3.30) that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Bq, x_n - q\rangle = \langle \gamma f(q) - Bq, w - q\rangle \le 0. \tag{3.31}$$

On the other hand, we shall show that the uniqueness of a solution of the variational inequality

$$\langle (B - \gamma f)x, x - w\rangle \le 0, \forall w \in \Omega. \tag{3.32}$$

Suppose that $q, \widehat{q} \in \Omega$ are solution to (3.32), then

$$\langle (B - \gamma f)q, q - \widehat{q}\rangle \le 0, \tag{3.33}$$

and

$$\langle (B - \gamma f)\widehat{q}, \widehat{q} - q\rangle \le 0. \tag{3.34}$$

From (3.33) and (3.34), we have

$$\langle (B-\gamma f)q-(B-\gamma f)\widehat{q}, q-\widehat{q}\rangle \leq 0. \tag{3.35}$$

By Lemma 2.1.5, the strong monotone of $B-\gamma f$, we obtain $q=\widehat{q}$. Finally, we show that $\{x_n\}$ converges strongly to q as $n \to \infty$. From (1.6), (3.11) and Lemma 2.1.1, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \langle \alpha_n\gamma f(x_n)+(I-\alpha_n B)w_n - q, x_{n+1}-q\rangle \\
&= \alpha_n\langle \gamma f(x_n)-Bq, x_{n+1}-q\rangle + \langle (I-\alpha_n B)(w_n-q), x_{n+1}-q\rangle \\
&\leq \alpha_n\langle \gamma f(x_n-f(q), x_{n+1}-q\rangle + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n-q\|\|x_{n+1}-q\| \\
&\leq \alpha_n\gamma\rho\|x_n-q\|\|x_{n+1}-q\| + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\quad +(1-\alpha_n\overline{\gamma})\|w_n-q\|\|x_{n+1}-q\| \\
&= [1-\alpha_n(\overline{\gamma}-\gamma\rho)]\|x_n-q\|\|x_{n+1}-q\| + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}(\|x_n-q\|^2+\|x_{n+1}-q\|^2) + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle \\
&\leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}\|x_n-q\|^2 + \frac{1}{2}\|x_{n+1}-q\|^2 + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle.
\end{aligned}$$

it follows that

$$\|x_{n+1}-q\|^2 \leq \frac{1-\alpha_n(\overline{\gamma}-\gamma\rho)}{2}\|x_n-q\|^2 + \alpha_n\langle \gamma f(q)-Bq, x_{n+1}-q\rangle. \tag{3.36}$$

From $0<\gamma<\frac{\overline{\gamma}}{\rho}$, condition (i) and (3.31), from Lemma 2.1.6, we obtain that $\lim_{n\to\infty}\|x_n-q\|=0$ and then $\{x_n\}$ converges strongly to q, which is the unique solution of the following variational inequality:

$$\langle (B-\gamma f)q, q-w\rangle \leq 0, \forall w \in \Omega.$$

This completes the proof. □

Theorem 3.1.3. [ ] Let H_1 and H_2 be two Hilbert space, let $A: H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma}>0$. Let $B_1: H_1 \to 2^{H_1}$, $B_2: H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s): 0 \leq s < \infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f: H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0,1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0,1), \{t_n\} \subseteq (0,\infty)$ satisfy:

(i) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n-\alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty}t_n = \infty$ and $\lim_{n\to\infty}\frac{|t_n-t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q \in \Omega$, which is the unique solution of the following variational inequality:

$$\langle (B-\gamma f)q, q-w\rangle \leq 0, \forall w \in \Omega.$$

Proof. From examples 1.0.1 and 1.0.2, we see that a nonexpansive semigroups is an asymptotically nonexpansive semigroups then from Theorem 3.1.2 can be prove this theorem. □

Theorem 3.1.4. Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0,1)$, $\gamma \in (0, \frac{\bar{1}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0,1), \{t_n\} \subseteq (0,\infty)$, define $\{x_n\}$ in the following manner:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s) J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds. \tag{3.37}$$

and satisfies the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ generated by (3.37) converge strongly to $q = P_\Omega(q)$, which is the unique solution of the following variational inequality:

$$\langle (I-f)q, q-w\rangle \le 0, \forall w \in \Omega.$$

Proof. Putting $\gamma = 1$ and $B = I$, iterative scheme (1.6) reduces to (3.37). The desired conclusion follows immediately from Theorem 3.1.2. This complete the proof. □

Next, we give some examples and numerical results to illustrate our algorithm and main result of this paper.

CHAPTER IV

CONCLUSIONS

From chapter 3 we have 2 theorems for submitted to thai journals of mathematics.

4.1 Outputs Results.

Theorem 4.1.1. Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$. For any $\alpha \in (0, 1)$, define the mapping Φ on H_1 by

$$\Phi(x) = \alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds.$$

where $t > 0$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A and $1 < \frac{1}{t}\int_0^t L_s ds < \alpha < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$. Then the mapping Φ is a contraction and has a unique fixed point.

Theorem 4.1.2. Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0, 1), \{t_n\} \subseteq (0, \infty)$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ satisfy:

(i) $\lim_{n\to\infty} \alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q \in \Omega$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

4.2 Outputs 1 paper.

1. Split variational inclusion and fixed point problem for asymptotically nonexpansive semigroup in Hilbert spaces. Submitted to Thai Journal of Mathematics.

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APPENDIX

Split variational inclusion and fixed point problem for asymptotically nonexpansive semigroup in Hilbert spaces*

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Abstract

The main objective of this paper is to introduced a general method for a split variational inclusion and asymptotically nonexpansive semigroups in Hilbert space. We prove that the sequence generate by the iterative scheme converge strongly to a common solution of the set of solution of a split variational inclusion and the set of common fixed points of one-parameter asymptotically nonexpansive semigroups. The results presented in this paper extend and improvement of previously known results in this research area.

1 Introduction

Let H be a real Hilbert Space, C a nonempty closed convex subset of H and $T : C \to C$ a mapping. Recall that a self mapping f of C is a contraction if $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for some $\alpha \in (0, 1)$ and T is a nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$, and T is asymptotically nonexpansive [ ] if there exists a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x - T^n y\| \leq k_n\|x - y\|$ for all $n \geq 1$ and $x, y \in C$. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $Fix(T)$ the set of fixed points of T; that is, $Fix(T) = \{x \in C : Tx = x\}$. Let $A : H_1 \to H_2$ be a mapping then $A^* : H_2 \to H_1$ is an adjoint operator of A if and only if $\langle A^*y, x\rangle = \langle y, Ax\rangle$ for $x \in H_1, y \in H_2$.

Recall also that a one-parameter family $\mathcal{T} = \{T(t)|0 \leq t < \infty\}$ of self-mappings of a nonempty closed convex subset C of a Hilbert space H is said to be a (continuous) Lipschitian semigroup on C (see, e. g., [ ]) if the following conditions are satisfied:

(*i*) $T(0)x = x, x \in C$

(*ii*) $T(s + t)(x) = T(s)T(t), s, t \geq 0, x \in C$

(*iii*) for each $x \in C$, the maps $t \mapsto T(t)x$ is continuous on $[0, \infty)$

(*iv*) there exists a bounded measurable function $L : [0, \infty) \to [0, \infty)$ such that, for each $t > 0$

$$\|T(t)x - T(t)y\| \leq L_t\|x - y\|, x, y \in C.$$

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A Lipschitzian semigroup T is called nonexpansive (or a contraction semigroup) if $L_t = 1$ for all $t > 0$, and asymptotically nonexpansive semigroup if $\limsup_{t\to\infty} L_t \leq 1$, respectively. We use $Fix(T)$ to denote the common fixed point set of the semigroup: that is $Fix(T) = \{x \in C : T(t)x = x, t > 0\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [, , ,]. However. Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [,].

The theory of variational inequalities is a branch of the mathematical sciences dealing with general equilibrium problems. It has a wide range of applications in economics, operations research, industry, physical, and engineering sciences. Many research papers have been written lately, both on the theory and applications of this field. Important connection with main areas of pure and applied science have been made, see for example [, ,] and the references cited therein.

Variational inequalities theory, which was introduce by Stampacchia [ ], provides us with a simple, natural general and unified framework to study a wide class of problems arising in pure and applied science. The development of variational inequality theory can be viewed as the simultaneous pursuit of two different lines of research. On the one hand, it reveals the fundamental facts on the qualitative aspects of the solutions to important classes of problems. On the other hand, it also enables us to develop highly efficient and powerful new numerical methods for solving, for example, obstacle, unilateral, free, moving, and complex equilibrium problems.

In 2006, Marino and Xu [ ], introduced the following general iterative methods to approximate a fixed point of a nonexpansive mapping:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)Tx_n, \tag{1.1}$$

where $\{\alpha_n\} \subseteq [0, 1]$ satisfies certain conditions, f is a contraction of H into itself, and B is a strongly positive bounded linear operator on H. Moreover, they prove that $\{x_n\}$ converges strongly to $x^* \in Fix(T)$, the unique solution of the following variational inequality:

$$\langle (B - \gamma f)x^*, x^* - w\rangle \leq 0, \forall w \in Fix(T), \tag{1.2}$$

which is also the optimality condition of the minimization problem.

Recall also that a multi-valued mapping $M : H_1 \to 2^{H_1}$ is called monotone if, for all $x, y \in H_1, u \in Mx$ and $v \in My$ such that

$$\langle x - y, u - v\rangle \geq 0. \tag{1.3}$$

A monotone mapping M is maximal if the Graph(M) is not property contained in the graph of any other monotone mapping. It is well known that a monotone mapping M is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x - y, u - v\rangle \geq 0$ for every $(y, v) \in Graph(M)$ implies that $u \in Mx$.

From a monotone mapping M the resolvent mapping $J_\lambda^M : H_1 \to H_1$ associated with M is defined by

$$J_\lambda^M(x) := (I + \lambda M)^{-1}(x), \forall x \in H_1, \tag{1.4}$$

for some $\lambda > 0$, where I is the identity mapping on H_1. Note that for all $\lambda > 0$ the resolvent operator J_λ^M is single-valued, nonexpansive and firmly nonexpansive.

In 2011, Moudafi [ ] introduce the split monotone variational inclusion problem: find $x^* \in H_1$ such that

$$\begin{cases} 0 \in f_1(x^*) + B_1(x^*), \\ y^* = Ax^* \in H_2 : 0 \in f_2(y^*) + B_2(y^*), \end{cases}$$

where $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

In 2015 Wen and Chen [ ] introduce a modified general iterative method for a split variational inclusion and nonexpansive semigroups, which is defined tn the following way:

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \tag{1.5}$$

where $\gamma \in [0, 1]$ and $\{\alpha_n\} \subseteq [0, 1]$. B is a strongly positive bounded linear operator on H_1.

Next, we studies some examples for relationship between a nonexpansive semigroup and an asymptotically nonexpansive semigroup for motivation of this work.

Example 1.1. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{1}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{1}{1+2s})x - (\frac{1}{1+2s})y\| = (\frac{1}{1+2s})\|x - y\|,$$

then we have $\mathcal{T}$ is nonexpansive semigroup. If $L_s = 1$ we have $\limsup_{s\to\infty} L_s = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup.

Example 1.2. *Let $H_1 = H_2 = \mathbb{R}$ and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$, where $T(s)x = \frac{2+2s}{1+2s}x, \forall x \in \mathbb{R}$. We see that for any $x, y \in \mathbb{R}$*

$$\|T(s)x - T(s)y\| = \|(\frac{2+2s}{1+2s})x - (\frac{2+2s}{1+2s})y\| = (\frac{2+2s}{1+2s})\|x - y\|,$$

put $L_s = (\frac{2+2s}{1+2s})$ we have $\limsup_{s\to\infty} L_s = \limsup_{s\to\infty}(\frac{2+2s}{1+2s}) = 1$ then $\mathcal{T}$ is asymptotically nonexpansive semigroup. If we let $s = 1$ we have $\frac{2+2s}{1+2s} = \frac{4}{3} \nleq 1$, then $\mathcal{T}$ is not necessary nonexpansive semigroup.

From above example we see that a mapping $\mathcal{T}$ is a nonexpansive semigroup then $\mathcal{T}$ is asymptotically nonexpansive semigroup. But $\mathcal{T}$ is an asymptotically nonexpansive semigroup is not necessary nonexpansive semigroup.

In this work we extend the results of Wen and Chen [ ] for $\mathcal{T}$ is an asymptotically nonexpansive semigroup then we consider

$$x_{n+1} = \alpha_n \gamma f(x_n) + (I - \alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \tag{1.6}$$

where $\gamma \in [0, 1]$ and $\{\alpha_n\} \subseteq [0, 1]$. B is a strongly positive bounded linear operator on H_1.

2 Preliminaries

In this section, we collect and give some useful lemmas that will be used for our main result in the next section.

Lemma 2.1. *Let H be a real Hilbert space, then the following hold:*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, (x + y)\rangle, \forall x, y \in H$;

(ii) $\|tx + (1 - t)y\|^2 = t\|x\|^2 + (1 - t)\|y\|^2 - t(1 - t)\|x - y\|^2, t \in [0, 1], \forall x, y \in H$.

Lemma 2.2. *[ ] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C. If $\{x_n\}$ is a sequence in C satisfying the properties:*

(i) $x_n \rightharpoonup z$; *and*

(ii) $\limsup_{t\to\infty} \limsup_{n\to\infty} \|T(t)x_n - x_n\| = 0$.

then $z \in Fix(\mathcal{T})$.

Lemma 2.3. *[ ] Let C be a nonempty bounded closed convex subset of real Hilbert space H and let $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ an asymptotically nonexpansive semigroup on C, then for any $u \geq 0$,*

$$\limsup_{u\to\infty} \limsup_{t\to\infty} \sup_{x\in C} \|\frac{1}{t}\int_0^t T(s)xds - T(u)(\frac{1}{t}\int_0^t T(s)xds)\| = 0.$$

Lemma 2.4. *[ ] Let B be a strongly positive linear bounded operator on a Hilbert space H with a coefficient $\bar{\gamma} > 0$ and $0 < \varrho < \|B\|^{-1}$. Then $\|I - \varrho B\| \leq 1 - \varrho\bar{\gamma}$.*

Lemma 2.5. *[ ] Let C be a nonempty closed convex subset of a Hilbert space H. Assume that $f : C \to C$ is a contraction with a coefficient $\rho \in (0,1)$ and B is a strongly positive bounded linear operator with a coefficient $\bar{\gamma} > 0$. Then for $0 < \gamma < \frac{\bar{\gamma}}{\rho}$,*

$$\langle x - y, (B - \gamma f)x - (B - \gamma f)y\rangle \geq (\bar{\gamma} - \gamma\rho)\|x - y\|^2, \forall x, y \in H.$$

That is $B - \gamma f$ is strongly monotone with coefficient $\bar{\gamma} - \gamma\rho$.

Lemma 2.6. *[ ] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \gamma_n)a_n + \gamma_n b_n + \sigma_n,$$

where $\{\gamma_n\}_{n=1}^{\infty} \subseteq (0,1)$ and $\{b_n\}_{n=1}^{\infty}, \{\sigma_n\}_{n=1}^{\infty}$ are sequence in $\mathbb{R}$ such that

(i) $\lim_{n\to\infty} \gamma_n = 0$ and $\Sigma_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} b_n \leq 0$;

(iii) $\sigma_n \geq 0$ and $\Sigma_{n=1}^{\infty}\sigma_n < \infty$.
Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.7. *[,] Let $S : H \to H$ be averaged and $T : H \to H$ be nonexpansive have:*

(i) $W = (1 - \alpha)S + \alpha T$ is averaged, where $\alpha \in (0,1)$.

(ii) The composite of finitely many averaged mapping is averaged.

Theorem 2.8. *[ ] Let H_1 and H_2 be two real Hilbert spaces and $A : H_1 \to H_2$ be a bounded linear operator. Let $f : H_1 \to H_1$ be a contraction mapping with constant $\rho \in (0,1)$ and $T : H_1 \to H_1$ be a nonexpansive mapping such that $\Omega = Fix(T) \cap \mathcal{T} \neq \emptyset$. For a given $x_0 \in H_1$ arbitrary, let the iterative sequences $\{u_n\}$ and $\{x_n\}$ be generated by*

$$\begin{cases} u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n], \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \end{cases} \tag{2.1}$$

where $\lambda > 0$ and $\epsilon \in (0, 1/L)$, L is the spectral radius of the operator A^*A, and A^* is the adjoint of A; $\{\alpha_n\}$ is a sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$. Then the sequens $\{u_n\}$ and $\{x_n\}$ both convergence strongly to $z \in \Omega$, where $z = P_\Omega(z)$.

Lemma 2.9. *[ ] The split variational inclusion problem (2.1) is equivalent to finding $x^* \in H_1$ such that $y^* = Ax^* \in H_2 : x^* = J_\lambda^{B_1}$ and $y^* = J_\lambda^{B_2}(y^*)$ for some $\lambda > 0$.*

3 Main Result

In the first Theorem in this section we prove the unique fixed point by Banach contraction principle of Φ. The second Theorem we prove the strong convergence of modified general iterative method for a split variational inclusion and asymptotically nonexpansive semigroups to $q \in \Omega$ which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

Theorem 3.1. *Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$. For any $\alpha \in (0, 1)$, define the mapping Φ on H_1 by*

$$\Phi(x) = \alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds,$$

where $t > 0$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A. and A^* is the adjoint of A and $1 < \frac{1}{t}\int_0^t L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$. Then the mapping Φ is a contraction and has a unique fixed point.

Proof. Since $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are firmly nonexpansive, they are averaged. For $\epsilon \in (0, \frac{1}{L})$, the mapping $I + \epsilon A^*(J_\lambda^{B_2} - I)A$ is averaged; see e.g.[ ]. It follows from Lemma2.3 (ii) that the mapping $J_\lambda^{B_1}(I + \epsilon A^*(J_\lambda^{B_2} - I)A)$ is averaged and hence nonexpansive. By Lemma2.4, for any $x, y \in H_1$, we have

$$\begin{aligned}
\|\Phi(x) - \Phi(y)\| &= \|\alpha\gamma f(x) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\alpha\gamma f(y) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\| \\
&\leq \alpha\gamma\|f(x) - f(y)\| + (1 - \alpha\overline{\gamma})\|\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax]ds \\
&\quad -\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]ds\| \\
&\leq \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|J_\lambda^{B_1}[x + \epsilon A^*(J_\lambda^{B_2} - I)Ax] \\
&\quad -J_\lambda^{B_1}[y + \epsilon A^*(J_\lambda^{B_2} - I)Ay]\| \\
&\leq \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})(\frac{1}{t}\int_0^t L_s ds)\|x - y\| \\
&= \alpha\gamma\rho\|x - y\| + (1 - \alpha\overline{\gamma})a\|x - y\| \\
&\leq [a - \alpha(\overline{\gamma}a - \gamma\rho)]\|x - y\|
\end{aligned}$$

From $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$ and $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a < \frac{1-\alpha\gamma\rho}{1-\alpha\overline{\gamma}}$, we have $[a - \alpha(\overline{\gamma}a - \gamma\rho)] < 1$. It follows that Φ is a contraction mapping. By the Banach contraction principle, $\Phi(x)$ has a unique fixed point x_α, that is

$$x_\alpha = \alpha\gamma f(x_\alpha) + (I - \alpha B)\frac{1}{t}\int_0^t T(s)J_\lambda^{B_1}[x_\alpha + \epsilon A^*(J_\lambda^{B_2} - I)Ax_\alpha]ds.$$

□

Next Theorem we modified general iterative method for a split variational inclusion and asymptotically nonexpansive semigroups and prove the strong convergence of iterative to $q \in \Omega$ which is the unique solution of the following variational inequality: $\langle(B - \gamma f)q, q - w\rangle \leq 0, \forall w \in \Omega$.

Theorem 3.2. *Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \leq s < \infty\}$ be a one-operator asymptotically nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A. and A^* is the adjoint of A. For a* *given* $x_1 \in H_1$, *and suppose that the sequence* $\{\alpha_n\} \subseteq (0,1)$, $\{t_n\} \subseteq (0,\infty)$ *and* $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\bar{\gamma}}$ *satisfy:*

(i) $\lim_{n\to\infty}\alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$, *and* $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty}\frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.

Then the sequence $\{x_n\}$ *generated by (1.6) converge strongly to* $q \in \Omega$, *which is the unique solution of the following variational inequality:*

$$\langle (B-\gamma f)q, q-w\rangle \le 0, \forall w \in \Omega.$$

Proof. Let $p \in \Omega$, we have $p = J_\lambda^{B_1}p$, $J_\lambda^{B_2}(Ap) = Ap$ and $T(s)p = p$. From (1.6), let $u_n = J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]$, and Lemma 2.9, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n] - J_\lambda^{B_1}p\|^2 \\ &\le \|x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 \\ &\le \|x_n - p\|^2 + 2\epsilon\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle + \epsilon^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.1}$$

By the definition of A and A^*, we obtain

$$\begin{aligned} \epsilon^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 &= \epsilon^2\langle A^*(J_\lambda^{B_2} - I)Ax_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &= \epsilon^2\langle (J_\lambda^{B_2} - I)Ax_n, AA^*(J_\lambda^{B_2} - I)Ax_n\rangle \\ &\le L\epsilon^2\langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= L\epsilon^2\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.2}$$

And we have

$$\begin{aligned} 2\epsilon\langle x_n - p, A^*(J_\lambda^{B_2} - I)Ax_n\rangle &= 2\epsilon\langle A(x_n - p), (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\epsilon\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n - (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\epsilon\langle A(x_n - p) + (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle - \langle (J_\lambda^{B_2} - I)Ax_n, (J_\lambda^{B_2} - I)Ax_n\rangle \\ &= 2\epsilon[\langle J_\lambda^{B_2}Ax_n - Ap, (J_\lambda^{B_2} - I)Ax_n\rangle - \|(J_\lambda^{B_2} - I)Ax_n\|^2] \\ &\le 2\epsilon[\frac{1}{2}\|(J_\lambda^{B_2} - I)Ax_n\| - \|(J_\lambda^{B_2} - I)Ax_n\|^2] \\ &= -\epsilon\|(J_\lambda^{B_2} - I)Ax_n\|^2. \end{aligned} \tag{3.3}$$

From (3.1), (3.2), (3.3) and $\epsilon \in (0, \frac{1}{L})$, it follows that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \epsilon(L\epsilon - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 \le \|x_n - p\|^2. \tag{3.4}$$

Next, we set $w_n = \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds$ for $n \ge 0$. since $1 < \frac{1}{t_n}\int_0^{t_n} L_s ds < a_n < \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\bar{\gamma}}$ and from (3.4), we have

$$\begin{aligned} \|w_n - p\| &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(s)p\| \\ &\le \frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)p\|ds \\ &\le \frac{1}{t_n}\int_0^{t_n} L_s ds\|u_n - p\| \\ &\le \frac{1}{t_n}\int_0^{t_n} L_s ds\|x_n - p\| \\ &\le a\|x_n - p\|. \end{aligned} \tag{3.5}$$

where $a = \sup_{n\geq 1}\{a_n\}$. It follows from (1.6), (3.5) and Lemma 2.4. that

$$\begin{aligned} \|x_{n+1}-p\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n B)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds - p\| \\ &= \|\alpha_n(\gamma f(x_n) - Bp) + (I-\alpha_n B)\frac{1}{t_n}\int_0^{t_n}(T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n] - T(s)p)ds\| \\ &\leq \alpha_n\|\gamma f(x_n) - Bp\| + (1-\alpha_n\overline{\gamma})\frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)p\|ds \\ &\leq \alpha_n(\|\gamma f(x_n) - \gamma f(p)\| + \|\gamma f(p) - Bp\|) + (1-\alpha_n\overline{\gamma})a\|x_n - p\| \\ &\leq \alpha_n\gamma\rho\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\| + (1-\alpha_n\overline{\gamma})a\|x_n - p\| \\ &= [1-\alpha_n(\overline{\gamma}a - \gamma p)]\|x_n - p\| + \alpha_n\|\gamma f(p) - Bp\|. \end{aligned} \tag{3.6}$$

Since $1 < a \leq \frac{1-\alpha_n\gamma\rho}{1-\alpha_n\overline{\gamma}}$ and $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, we have $\overline{\gamma}a - \gamma\rho > 0$. By a simple induction, we have

$$\|x_n - p\| \leq \max\{\|x_0 - p\|, \frac{1}{\overline{\gamma}a - \gamma\rho}\|\gamma f(p) - Bp\|\}. \tag{3.7}$$

Therefor, $\{x_n\}$ is bounded, and so are $\{u_n\}$ and $\{w_n\}$. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. From (1.6), we have

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|\alpha_n\gamma f(x_n) + (I-\alpha_n B)w_n - \alpha_n\gamma f(x_{n-1}) + (I-\alpha_{n-1}B)w_{n-1}\| \\ &= \|\alpha_n\gamma[f(x_n) - f(x_{-1})] + (\alpha_n - \alpha_{n-1})\gamma f(x_{n-1}) + (I-\alpha_n B)(w_n - w_{n-1}) - (\alpha_n - \alpha_{n-1})Bw_{n-1}\| \\ &\leq \alpha_n\gamma\|f(x_n) - f(x_{-1})\| + |\alpha_n - \alpha_{n-1}|\|\gamma f(x_{n-1})\| \\ &\quad +(1-\alpha_n\overline{\gamma})\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Bw_{n-1}\| \\ &\leq \alpha_n\gamma\rho\|x_n - x_{-1}\| + (1-\alpha_n\overline{\gamma})\|w_n - w_{n-1}\| + |\alpha_n - \alpha_{n-1}|[\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|]. \end{aligned} \tag{3.8}$$

Since $\frac{1}{t_n}\int_0^{t_n} T(s)pds = \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds$, we consider

$$\begin{aligned} \|w_n - w_{n-1}\| &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds\| \\ &= \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds + \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds + \frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds \\ &\quad -\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds - \frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds + \frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds \\ &\quad +\frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds\| \\ &= \|(\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - \frac{1}{t_n}\int_0^{t_n} T(s)u_{n-1}ds) \\ &\quad +(\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds - \frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds - \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)u_{n-1}ds + \frac{1}{t_{n-1}}\int_0^{t_{n-1}} T(s)pds) \\ &\quad +(\frac{1}{t_n}\int_0^{t_{n-1}} T(s)u_{n-1}ds + \frac{1}{t_n}\int_{t_{n-1}}^{0} T(s)u_{n-1}ds) - (\frac{1}{t_n}\int_0^{t_{n-1}} T(s)pds + \frac{1}{t_n}\int_{t_{n-1}}^{0} T(s)pds)\| \\ &= \|\frac{1}{t_n}\int_0^{t_n}[T(s)u_n - T(s)u_{n-1}]ds + (\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}}[T(s)u_{n-1} - T(s)p]ds \\ &\quad +\frac{1}{t_n}\int_{t_{n-1}}^{t_n} T(s)u_{n-1}ds - \frac{1}{t_n}\int_{t_{n-1}}^{t_n} T(s)pds\| \end{aligned} \tag{3.9}$$

$$\begin{aligned} &= \|\frac{1}{t_n}\int_0^{t_n}[T(s)u_n - T(s)u_{n-1}]ds + (\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}}[T(s)u_{n-1} - T(s)p]ds \\ &\quad + \frac{1}{t_n}\int_{t_{n-1}}^{t_n}[T(s)u_{n-1} - T(s)p]ds\| \\ &\leq \frac{1}{t_n}\int_0^{t_n}\|T(s)u_n - T(s)u_{n-1}\|ds + |\frac{1}{t_n} - \frac{1}{t_{n-1}}|\int_0^{t_{n-1}}\|T(s)u_{n-1} - T(s)p\|ds \\ &\quad + \frac{1}{t_n}\int_{t_{n-1}}^{t_n}\|T(s)u_{n-1} - T(s)p\|ds\| \\ &\leq \frac{1}{t_n}\int_0^{t_n}L_s ds\|u_n - u_{n-1}\| + \frac{|t_{n-1} - t_n|}{t_n}\frac{1}{t_{n-1}}\int_0^{t_{n-1}}L_s ds\|u_{n-1} - p\| \\ &\quad + \frac{1}{t_n}\int_{t_{n-1}}^{t_n}L_s ds\|u_{n-1} - p\|. \end{aligned}$$

Now, we taking $\lim_{s\to\infty} L_s = 1$, it follows that $\frac{1}{t_n}\int_0^{t_n} L_s ds \to \frac{1}{t_n}\int_0^{t_n} ds$. and hence

$$\|w_n - w_{n-1}\| \leq \|u_n - u_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|. \tag{3.10}$$

From $\epsilon \in (0, \frac{1}{L})$ and mapping $J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]$ is averaged and hence nonexpansive. then we have

$$\|u_n - u_{n-1}\| = \|J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_n - J_\lambda^{B_1}[I + \epsilon A^*(J_\lambda^{B_2} - I)A]x_{n-1}\| \leq \|x_n - x_{n-1}\|. \tag{3.11}$$

From (3.8), (3.10) and (3.11), we have

$$\begin{aligned} \|x_{n+1} - x_n\| &\leq \alpha_n\gamma\rho\|x_n - x_{n-1}\| + (1 - \alpha_n\overline{\gamma})(\|x_n - x_{n-1}\| + \frac{2|t_{n-1} - t_n|}{t_n}\|u_{n-1} - p\|) \\ &\quad + |\alpha_n - \alpha_{n-1}|(\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|) \\ &\leq [1 - \alpha_n(\overline{\gamma} - \gamma\rho)]\|x_n - x_{n-1}\| + (|\alpha_n - \alpha_{n-1}| + \frac{2|t_{n-1} - t_n|}{t_n})M. \end{aligned} \tag{3.12}$$

where $m = \max\{sup_{n\in\mathbb{N}}(\gamma\|f(x_{n-1})\| + \|Bw_{n-1}\|), sup_{n\in\mathbb{N}}\|u_{n-1} - p\|\}$. It follows from condition $(i) - (ii)$ and Lemma 2.6, hence

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.13}$$

Consider.

$$\begin{aligned} \|x_n - w_n\| &\leq \|x_n - x_{n+1}\| + \|x_{n+1} - w_n\| \\ &= \|x_n - x_{n+1}\| + \|\alpha_n\gamma f(x_n) + (I - \alpha_n B)w_n - w_n\| \\ &\leq \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) + Bw_n\|. \end{aligned} \tag{3.14}$$

From condition (i) and (3.13), we have

$$\lim_{n\to\infty}\|x_n - w_n\| = 0. \tag{3.15}$$

and that

$$\lim_{n\to\infty}\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| = 0. \tag{3.16}$$

For any $u \geq 0$, we have

$$\begin{aligned}
\|x_n - T(u)x_n\| &\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\
&\quad + \|T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)x_n\| \\
&\leq \|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| + \|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - T(u)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds\| \\
&\quad + L_u\|\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds - x_n\|.
\end{aligned}$$

From (3.16), Lemma 2.2 and $\limsup_{u\to\infty} L_u \leq 1$, we have

$$\lim_{n\to\infty} \|x_n - T(u)x_n\| = 0. \tag{3.17}$$

By the definition of x_n; (3.3), (3.1) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|\alpha_n\gamma f(x_n) + (1-\alpha_n B)w_n - p\|^2 \\
&= \|(w_n - p) + \alpha_n(\gamma f(x_n) - Bw_n)\|^2 \\
&\leq \|w_n - p\|^2 + 2\alpha_n\langle\gamma f(x_n) - Bw_n, x_{n+1} - p\rangle \\
&\leq \|u_n - p\|^2 + 2\alpha_n\langle\gamma f(x_n) - Bw_n, x_{n+1} - p\rangle \\
&\leq [\|x_n - p\|^2 + \epsilon(L\epsilon - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2] \\
&\quad + 2\alpha_n\langle\gamma f(x_n) - Bw_n, x_{n+1} - p\rangle \\
&\leq \|x_n - p\|^2 - \epsilon(1 - L\epsilon)\|(J_\lambda^{B_2} - I)Ax_n\|^2 + 2\alpha_n M_2^2.
\end{aligned} \tag{3.18}$$

where $M_2 = \max\{\sup_{n\in\mathbb{N}}\|\gamma f(x_n) - Bw_n\|, \sup_{n\in\mathbb{N}}\|x_{n+1} - p\|\}$ and $\epsilon \in (0, \frac{1}{L})$, it implies that

$$\begin{aligned}
\epsilon(1 - L\epsilon)\|(J_\lambda^{B_2} - I)Ax_n\|^2 &\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\alpha_n M_2^2 \\
&\leq \|x_{n+1} - x_n\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\alpha_n M_2^2.
\end{aligned} \tag{3.19}$$

From (3.13), we obtain

$$\lim_{n\to\infty} \|(J_\lambda^{B_2} - I)Ax_n\| = 0. \tag{3.20}$$

From (3.1), (3.3) and $\epsilon \in (0, \frac{1}{L})$, that

$$\begin{aligned}
\|u_n - p\|^2 &= \|J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n] - J_\lambda^{B_1}p\|^2 \\
&\leq \langle u_n - p, x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n - p\rangle \\
&= \frac{1}{2}\{\|u_n - p\|^2 + \|x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n - p\|^2 - \|u_n - p - [x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n - p]\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 + \epsilon(L\epsilon - 1)\|(J_\lambda^{B_2} - I)Ax_n\|^2 - \|u_n - x_n - \epsilon A^*(J_\lambda^{B_2} - I)Ax_n\|^2\} \\
&\leq \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - [\|u_n - x_n\|^2 + \epsilon^2\|A^*(J_\lambda^{B_2} - I)Ax_n\|^2 - 2\epsilon\langle u_n - x_n, A^*(J_\lambda^{B_2} - I)Ax_n\rangle]\} \\
&\leq \frac{1}{2}\{\|u_n - p\|^2 + \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|\},
\end{aligned}$$

which implies that

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\|. \tag{3.21}$$

It follows from (3.18) and (3.21) that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\leq \|u_n - p\|^2 + 2\alpha_n M_2^2 \\
&\leq \|x_n - p\|^2 - \|u_n - x_n\|^2 + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M_2^2.
\end{aligned}$$

that is

$$\begin{aligned}\|u_n - x_n\|^2 &\le \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M_2^2 \\ &\le \|x_n - x_{n+1}\|(\|x_n - p\| + \|x_{n+1} - p\|) + 2\epsilon\|A(u_n - x_n)\|\|(J_\lambda^{B_2} - I)Ax_n\| + 2\alpha_n M_2^2.\end{aligned}$$

From (3.13), (3.21) and condition (i), it follows that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.22}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded, there exists weak limit w of $\{x_n\}$. Without loss of generality, we may assume that subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which is $x_{n_j} \rightharpoonup w$. From (3.22), we have subsequence $\{u_{n_j}\}$ of $\{u_n\}$, which is $u_{n_j} \rightharpoonup w$. Moreover, $u_{n_j} = J_\lambda^{B_1}[x_{n_j} + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}]$ with

$$\frac{(x_{n_j} - u_{n_j}) + \epsilon A^*(J_\lambda^{B_2} - I)Ax_{n_j}}{\lambda} \in B_1 u_{n_j}, \tag{3.23}$$

By taking limit $j \to \infty$, and (3.20), (3.22) and the fact that the graph of a maximal monotone operator is weakly-strongly closed, we obtain $0 \in B_1(w)$. Furthermore, since $\{x_n\}$ and $\{u_n\}$ have the same asymptotical behavior, $Ax_{n_j} \rightharpoonup Aw$. From (3.20) and the the fact that the resolvent $J_\lambda^{B_2}$ is nonexpansive, we obtain $Aw \in B_2(Aw)$. It follows from Lemma 2.9 that $w \in \mathcal{T}$.

Next, we show that $\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle \le 0$, where $q = P_\Omega(I - B + \gamma f)q$. From the sequence $x_{n_j} \rightharpoonup w$ and

$$\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle = \lim_{j\to\infty}\langle\gamma f(q) - Bq, x_{n_j} - q\rangle. \tag{3.24}$$

Assume that $w \ne T(u)w$. From (3.16) and Opial's property, we have

$$\begin{aligned}\liminf_{j\to\infty}\|x_{n_j} - w\| &< \liminf_{j\to\infty}\|x_{n_j} - T(u)w\| \\ &\le \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + \|T(u)x_{n_j} - T(u)w\|) \\ &\le \liminf_{j\to\infty}(\|x_{n_j} - T(u)x_{n_j}\| + L_u\|x_{n_j} - w\|) \\ &\le \liminf_{j\to\infty} L_u\|x_{n_j} - w\|.\end{aligned}$$

If we letting $u \to \infty$, we have $\limsup_{u\to\infty} L_u \le 1$, it follows that

$$\liminf_{j\to\infty}\|x_{n_j} - w\| < \liminf_{j\to\infty}\|x_{n_j} - w\|.$$

This is a contradiction. Then $w \in Fix(T)$. Consequently, $w \in \Omega = Fix(T) \cap \mathfrak{F}$. It follows from (3.24) that

$$\limsup_{n\to\infty}\langle\gamma f(q) - Bq, x_n - q\rangle = \langle\gamma f(q) - Bq, w - q\rangle \le 0. \tag{3.25}$$

On the other hand, we shall show that the uniqueness of a solution of the variational inequality

$$\langle(B - \gamma f)x, x - w\rangle \le 0, \forall w \in \Omega. \tag{3.26}$$

Suppose that $q, \widehat{q} \in \Omega$ are solution to (3.26), then

$$\langle(B - \gamma f)q, q - \widehat{q}\rangle \le 0, \tag{3.27}$$

and

$$\langle(B - \gamma f)\widehat{q}, \widehat{q} - q\rangle \le 0. \tag{3.28}$$

From (3.27) and (3.28), we have

$$\langle(B - \gamma f)q - (B - \gamma f)\widehat{q}, q - \widehat{q}\rangle \le 0. \tag{3.29}$$

By Lemma 2.5, the strong monotone of $B - \gamma f$. we obtain $q = \widehat{q}$. Finally, we show that $\{x_n\}$ converges strongly to q as $n \to \infty$. From (1.6), (3.4) and Lemma 2.1, we have

$$\begin{aligned} \|x_{n+1} - q\|^2 &= \langle \alpha_n \gamma f(x_n) + (I - \alpha_n B)w_n - q, x_{n+1} - q\rangle \\ &= \alpha_n\langle \gamma f(x_n) - Bq, x_{n+1} - q\rangle + \langle (I - \alpha_n B)(w_n - q), x_{n+1} - q\rangle \\ &\le \alpha_n\langle \gamma f(x_n - f(q), x_{n+1} - q\rangle + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle + (1 - \alpha_n\overline{\gamma})\|w_n - q\|\|x_{n+1} - q\| \\ &\le \alpha_n\gamma\rho\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle + (1 - \alpha_n\overline{\gamma})\|w_n - q\|\|x_{n+1} - q\| \\ &= [1 - \alpha_n(\overline{\gamma} - \gamma\rho)]\|x_n - q\|\|x_{n+1} - q\| + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\ &\le \frac{1 - \alpha_n(\overline{\gamma} - \gamma\rho)}{2}(\|x_n - q\|^2 + \|x_{n+1} - q\|^2) + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle \\ &\le \frac{1 - \alpha_n(\overline{\gamma} - \gamma\rho)}{2}\|x_n - q\|^2 + \frac{1}{2}\|x_{n+1} - q\|^2 + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle, \end{aligned}$$

it follows that

$$\|x_{n+1} - q\|^2 \le \frac{1 - \alpha_n(\overline{\gamma} - \gamma\rho)}{2}\|x_n - q\|^2 + \alpha_n\langle \gamma f(q) - Bq, x_{n+1} - q\rangle. \tag{3.30}$$

From $0 < \gamma < \frac{\overline{\gamma}}{\rho}$, condition (i) and (3.25), from Lemma 2.6, we obtain that $\lim_{n\to\infty}\|x_n - q\| = 0$ and then $\{x_n\}$ converges strongly to q, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \le 0, \forall w \in \Omega.$$

This completes the proof. □

Theorem 3.3. *[ ] Let H_1 and H_2 be two Hilbert space. let $A : H_1 \to H_2$ be a bounded linear operator and B be a strongly positive bounded linear operator on H_1 with constant $\overline{\gamma} > 0$. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$. L is spectral radius of the operator A^*A. and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0, 1)$, $\{t_n\} \subseteq (0, \infty)$ satisfy:*

(i) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ and $\lim_{n\to\infty}\frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.
Then the sequence $\{x_n\}$ generated by (1.6) converge strongly to $q \in \Omega$, which is the unique solution of the following variational inequality:

$$\langle (B - \gamma f)q, q - w\rangle \le 0, \forall w \in \Omega.$$

Proof. From examples 1.1 and 1.2, we see that a nonexpansive semigroups is an asymptotically nonexpansive semigroups then from Theorem 3.2 can be prove this theorem. □

Theorem 3.4. *Let H_1 and H_2 be two Hilbert space, let $A : H_1 \to H_2$ be a bounded linear operator. Let $B_1 : H_1 \to 2^{H_1}$, $B_2 : H_2 \to 2^{H_2}$ be maximal monotone mapping and $\mathcal{T} := \{T(s) : 0 \le s < \infty\}$ be a one-operator nonexpansive semigroup on H_1 such the $\Omega = Fix(\mathcal{T}) \cap \mathfrak{F} \neq \emptyset$. Assume that $f : H_1 \to H_1$ is a contraction mapping with constant $\rho \in (0, 1)$, $\gamma \in (0, \frac{\overline{\gamma}}{\rho})$, and $\epsilon \in (0, \frac{1}{L})$, L is spectral radius of the operator A^*A, and A^* is the adjoint of A. For a given $x_1 \in H_1$, and suppose that the sequence $\{\alpha_n\} \subseteq (0, 1)$, $\{t_n\} \subseteq (0, \infty)$, define $\{x_n\}$ in the following manner:*

$$x_{n+1} = \alpha_n\gamma f(x_n) + (1 - \alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)J_\lambda^{B_1}[x_n + \epsilon A^*(J_\lambda^{B_2} - I)Ax_n]ds, \tag{3.31}$$

and satisfies the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0, \Sigma_{n=1}^{\infty}\alpha_n = \infty$, and $\Sigma_{n=1}^{\infty}|\alpha_n - \alpha_{n-1}| < \infty$;

(ii) $\lim_{n\to\infty} t_n = \infty$ *and* $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{\alpha_n t_n} = 0$.
Then the sequence $\{x_n\}$ *generated by (3.31) converge strongly to* $q = P_\Omega(q)$. *which is the unique solution of the following variational inequality:*

$$\langle (I - f)q, q - w\rangle \leq 0, \forall w \in \Omega.$$

Proof. Putting $\gamma = 1$ and $B = I$, iterative scheme (1.6) reduces to (3.31). The desired conclusion follows immediately from Theorem 3.2. This complete the proof. □

Next, we give some examples and numerical results to illustrate our algorithm and main result of this paper.

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