Songklanakarin J. Sci. Technol.
40 (6), 1368-1377, Nov. – Dec. 2018

http://www.sjst.psu.ac.th

Original Article

An approximation of average run length using the Markov chain approach of a generally weighted moving average chart to monitor the number of defects

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Received: 12 April 2017; Revised: 8 August 2017; Accepted: 23 August 2017

Abstract

The objective of this research is to propose an approximation average run length (ARL) using the Markov chain approach (MCA) of a generally weighted moving average chart (GWMA) when observations are based on an underlying binomial distribution. The numerical results obtained from the MCA were compared with the results obtained from a Monte Carlo (MC) simulation method and the efficiency of the ARL was measured by CPU time. The performances of the GWMA and exponentially weighted moving average (EWMA) charts were compared in terms of monitoring the change in the process mean as defined by an out-of-control average run length (ARL_1). The numerical results showed that the results of the ARL obtained from MCA were in good agreement with the results obtained from MC; however, the MCA took less CPU time than the MC simulation method. Furthermore, the performance of the GWMA chart was superior to the EWMA chart when the magnitudes of change were small ($\delta \leq 0.05$), otherwise the EWMA performed better than the GWMA chart.

Keywords: closed-form formulae, time-varying chart, time consuming, binomial distribution and stopping times

1. Introduction

The role of a control chart is to monitor a process, to identify any unusual causes and to make improvements for a change in a process. A variety of statistical methods have been developed in many processes that include health care, industry, business, engineering, and other applications. An important assumption in the design of control charts is that the measurable quality characteristic is normally distributed. However, in many situations we may have reasons to believe that an underlying distribution always deviates from a normal distribution. In practical applications, there are many situations in which the observations come from non-normal distributions such as exponential, Student-t or gamma distributions (Amhemed, 2010; Borror, 1999; Stoumbos & Reysnolds, 2000). The observations from these non-normal distributions need to be monitored by an appropriate control chart. The underlying statistical principles for a control chart of nonconforming proportions are based on a binomial distribution. Attribute control charts are an important tool in statistical process control to monitor count data or attribute data. When the quality characteristic cannot be measured on a continuous scale, an attribute control chart must be used; for example, p, np, c, and u charts. Examples of quality characteristics as attributes are the number of failures in a production run, the proportion of defective items, and the number of nonconformities in a production process. The p and np charts are widely used, primarily to monitor the fraction of non-conforming products.

Since the introduction of the cumulative sum (CUSUM) chart by Page (1954) and the exponentially weighted moving average (EWMA) chart by Roberts (1959), both charts have been widely employed to monitor a process mean due to their excellent performance in detecting small to moderate mean shifts and they have also been applied to discrete processes for attribute data (Montgomery, 2012). The EWMA is also relatively robust in the face of non-normal

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distributed quality characteristics. Borror, Montgomery, and Runger (1999) presented the EWMA chart to monitor Poisson observations and evaluate an average run length (ARL) using the Markov chain approach (MCA). The results showed that the performance of the EWMA chart was superior to the Shewhart *c* chart.

A double EWMA (DEWMA) chart was developed by Shamma and Shamma (1992). Later, Zhang, Govindaraju, Lai, and Bebbington (2003) studied the DEWMA chart when observations were from a Poisson distribution and found that this chart was more sensitive to small changes in the process mean than the EWMA chart. Recently, Sheu and Lin (2003) proposed and analyzed the generally weighted moving average control chart (GWMA), which was a generalization of the EWMA-$\bar{X}$ control chart for monitoring process changes. The authors evaluated the ARL of the EWMA and GWMA control charts using simulation. The GWMA control chart is more sensitive than the EWMA control chart especially for detecting a small shift. Sheu and Yang (2006) studied a GWMA chart when observations were Poisson distributions and found that the GWMA chart performed better than *c* and EWMA charts for large process changes.

One of the common characteristics of control charts for measurement and comparison of the performance is ARL. The expected numbers of observations are taken from a process until the control chart signals. Ideally, an acceptable in-control ARL (ARL_0) should be large enough. Otherwise, it should be small when the process is out-of-control, which is traditionally called an ARL_1.

Many methods for evaluating the ARL for control charts have been studied. Monte Carlo (MC) simulation is a simple approach that is often used to test accuracy with other methods. Roberts (1959) presented the ARL for EWMA charts using simulations for processes following a normal distribution and derived nomograms that could be used to find the ARL for a variety of parameter values. The MCA was first proposed by Brook and Evans (1972) to study the run length properties under the assumption of independent and identically distributed (i.i.d.) observations for a CUSUM chart. More references on the evaluation of characteristics by MCA are Champ and Woodall (1987), and Champ and Rigdon (1991). Crowder (1987) used numerical quadrature methods to solve exact integral equations for the ARL for the normal distribution. ARL for an EWMA control chart for an exponential distribution using differential equations was studied by Gan (1998). Recently, Phengsalae, Areepong, and Sukparungsee (2015) and Areepong and Sukparungsee (2016) derived the closed-form expression of ARL when observations were underlying Poisson and ZIB processes, respectively.

The aim of this paper is to propose the MCA to evaluate the ARL of a GWMA for a binomial distribution. Moreover, the performances of GWMA and EWMA charts were compared.

2. Control Charts and their Properties

Assume that the process observations $X_1, X_2, ..., X_n$ are identical and independently distributed random variables with a binomial distribution, where X_i is the number of nonconforming items in sample *i* of *n* samples of size *m*. The fraction of nonconforming items is defined as the ratio of the number of nonconforming items in the population to the total number of items in that population. The binomial mass function can be expressed as

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} \;;\; x = 0, 1, 2, ..., n,$$

where $f(x)$ is the probability of *x* successes out of *n* trials, *n* is the number of trials, and *p* is the probability of success in a given trial. In general, the mean of a binomial distribution with parameters *n* and *p* is

$$E(X) = np.$$

The variance of the binomial distribution can be calculated by

$$Var(X) = np(1-p).$$

It is assumed that $p = p_0$ while the process is in-control and $p = p_1 > p_0$ when the process goes out-of-control. It is assumed that there is a change-point time $\theta \le \infty$ at which the parameter changes from $p = p_0$ to $p = p_1$. Note, $\theta = \infty$ means that a process always remains in an in-control state.

Let $E_\theta(.)$ be the expectation that an in-control parameter $(p = p_0)$ has been changed to an out-of-control parameter $(p = p_1)$ for a distribution function $F(x; n, p)$ at the change-time point (θ), where $\theta \le \infty$. In the literature on quality controls, the quantity $E_\infty(\tau)$ is called the ARL of the control chart for a given process where τ is the stopping time.

A typical condition imposed on an ARL_0 is that:

$$ARL_0 = E_\infty(\tau) = T, \tag{1}$$

where *T* is given (usually large). For a given distribution function and chart, this condition then determines the choices for the control limits.

A typical definition of the ARL_1 is that

$$ARL_1 = E_1(\tau \mid \tau \ge 1), \tag{2}$$

for the change point occurs at the beginning $\theta = 1$.

2.1 Exponentially weighted moving average control chart: EWMA

The EWMA control chart was first introduced by Roberts (1959) to detect small shifts in the mean of a process. It is now widely implemented in process control. The EWMA statistics are as follows:

$$Z_t = \lambda X_t + (1-\lambda) Z_{t-1},\; t = 1,\; 2,\; \ldots, \tag{3}$$

where Z_t is the EWMA statistic at time t^{th} and $Z_0 = E(X_t)$ is the initial statistic value.

X_t is the binomial observations at the t^{th} time

λ is a smoothing parameter $(0 \le \lambda \le 1)$.

The mean and variance of EWMA statistics are $E(Z_t) = p_0$ and $Var(Z_t) = \sigma^2_{Z_t} = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)\left[1-(1-\lambda)^{2t}\right]$, respectively. Since $0 < (1-\lambda) < 1$, we have that $(1-\lambda)^{2t} \to 0$ as $t \to \infty$. Therefore, asymptotic variance is $Var(Z_t) = \sigma^2_{Z_t} = \sigma^2 \left(\frac{\lambda}{2-\lambda}\right)$. When constant upper and lower limits are preferred for detecting change-points, the standard deviation used in the limits is usually the asymptotic value. The upper and lower control limits of EWMA chart are

$$UCL = p_0 + H\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} = h_U \tag{4}$$

and

$$LCL = p_0 - H\sigma\sqrt{\frac{\lambda}{(2-\lambda)}} = h_L. \tag{5}$$

The corresponding stopping time for the EWMA procedure is given by

$$\tau = \inf\{\mathrm{t} > 0 : Z_t > UCL \text{ or } Z_t < LCL\}.$$

where H is the width of a control limit. Let $LCL = h_L = 0$ as we consider the EWMA chart for monitoring the case of an increasing mean while the fraction of non-conforming products cannot be less than 0.

2.2 Generally weighted moving average control chart: GWMA

The GWMA chart was first presented by Sheu and Lin (2003). This chart is a generalized extension model of the EWMA chart by adding an adjustment smoothing constant (*w*). If the weighted historical observation constant equals $q = 1-\lambda$ and $w = 1,$ then the GWMA chart coincides with the EWMA chart.

The GWMA statistic is expressed as follows

$$Y_t = \sum_{i=1}^{t} (q^{(i-1)^w} - q^{i^w}) X_{t-i+1} + q^{t^w} Y_0. \tag{6}$$

Using a geometric series, equation (6) can be rewritten as

$$Y_t = \frac{(1-q)(q-1)-(q-1)q(q-1)}{(q-1)(1-q)} X_{t-i+1} + q^w Y_0 \tag{7}$$

where Y_t is GWMA statistic at time t^{th}, where the initial statistic value $Y_0 = p_0$

X_{t-i+1} is the binomial observations at the $t-i+1^{th}$; $t = 2, 3, \ldots$

q is a weighted parameter $(0 \le q < 1)$ and $q = 1-\lambda$

w is an adjustment smoothing constant $(0 < w \le 1)$.

The mean and variance of the GWMA statistic are $E(Y_t) = \alpha_0$ and $Var(Y_t) = \sigma^2_{Y_t} = Q_t\sigma^2$, respectively. The upper and lower control limits of GWMA chart are

$$UCL = p_0 + L\sigma\sqrt{Q_t} = h_U \tag{8}$$

$$LCL = p_0 - L\sigma\sqrt{Q_t} = h_L \tag{9}$$

where $Q_t = \sum_{i=1}^{t}(q^{(i-1)^w} - q^{i^w})^2$, L is the width of a control limit and let $LCL = h_L = 0.$

The corresponding stopping time for the GWMA procedure is given by

$$\tau = \inf\{t > 0 : Y_t > UCL \text{ or } Y_t < LCL\}.$$

3. Approximation of ARL using the MCA

The MCA is one of the most effective methods to study the characteristics of a control chart. This approach has been discussed by many authors (e.g. Brook and Evans, 1972; Lucas and Saccucci, 1990). Lucas and Saccucci (1990) introduced the MCA to approximate an ARL *t* state in an in-control process assuming the observation x_j; $j = 1, 2, ..., \mathrm{N}$ is an in-control state and $j = N+1$ is an out-of-control state. The transition probability (P_{ij}) is the probability of moving from state i to state j in one step and is given by

$$P_{ij} = (X_{ij} = x_j \,|\, X_t = x_i). \tag{10}$$

The transition probability matrix (**P**) and element of matrix (P_{ij}) can be rewritten as

$$\mathbf{P} = \begin{bmatrix} P_{11} & \cdots & P_{1N} & | & P_{1,\mathrm{N}+1} \\ \vdots & \ddots & \vdots & | & \vdots \\ P_{N1} & \cdots & P_{NN} & | & P_{\mathrm{N},\mathrm{N}+1} \\ --- & --- & --- & | & --- \\ 0 & \cdots & 0 & | & 1 \end{bmatrix} \text{ or } \mathbf{P} = \begin{bmatrix} P_{11} & \cdots & P_{1(\mathrm{N}+1)} \\ \vdots & \ddots & \vdots \\ P_{(\mathrm{N}+1)1} & \cdots & P_{(\mathrm{N}+1)(\mathrm{N}+1)} \end{bmatrix} \text{ or } \mathbf{P} = \begin{bmatrix} \mathbf{R} & (\mathbf{I}_N - \mathbf{R})\mathbf{1}_N \\ \mathbf{0}_N^T & \mathbf{1} \end{bmatrix}, \tag{11}$$

where **R** is the $N \times N$ transition probability matrix among the in-control states, $\mathbf{I}_N$ is the $N \times N$ identity matrix, $\mathbf{1}_N$ is the $N \times 1$ column vector of ones, **0** is the $1 \times N$ row vector of zeros and **1** is the scalar of one.

The *k* stage transition probability matrix $\mathbf{P}_k$ is useful for evaluating ARL because it contains the probability that the chain goes from one state to another state in *k* steps. This matrix is

$$\mathbf{P}_k = \begin{bmatrix} \mathbf{R}^k & (\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N \\ \mathbf{0}_N^T & \mathrm{I}_N \end{bmatrix}.$$

The vector $(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N$ is the vector of transition probabilities from state $i < N+1$ to the state $N+1$ in k steps. Hence

$$P(\tau_i \le \mathrm{k} \mid X_0 = x_i) = \mathrm{element}[(\mathbf{I} - \mathbf{R}^k)\mathbf{1}_N](\mathrm{i})$$

$$= \mathbf{P}_N^{(\mathrm{i})\mathrm{T}}(\mathbf{I}_N - \mathbf{R}^k)\mathbf{1}_N$$

where $\mathbf{P}_N^{(\mathrm{i})\mathrm{T}}$ is the initial probability vector with 1 at i^{th} position and 0 otherwise. Then,

$$P(\tau_i = \mathrm{k} \mid X_0 = x_i) = P(\tau_i \le \mathrm{k} \mid X_0 = x_i) - P(\tau_i \le \mathrm{k}-1 \mid X_0 = x_i)$$

$$= \mathbf{P}_N^{(\mathrm{i})\mathrm{T}}(\mathbf{R}^{k-1} - \mathbf{R}^k)\mathbf{1}_N. \tag{12}$$

Using Equation (12), the ARL can be rewritten as

$$ARL(\mathrm{N}) = \sum_{k=1}^{\infty} k\mathbf{P}_N^{(i)T}(\mathbf{R}^{k-1} - \mathbf{R}^k)\mathbf{1}_N$$

$$= \sum_{k=1}^{\infty} \mathbf{P}_N^{(i)T}\mathbf{R}^{k-1}\mathbf{1}_N$$

$$= \mathbf{P}_N^{(i)T}(\mathbf{I}_N - \mathbf{R})^{-1}\mathbf{1}_N \tag{13}$$

where $\mathbf{P}_N^{(i)T}$ is a vector with initial probability vector $[0, \ \ldots, \ 0, \ 1, \ 0, \ \ldots, \ 0]_{1\times N}$, $\mathbf{I}$ is the identity matrix, $\mathbf{1}$ is a unit vector.

An approximation of the ARL using the MCA to detect the mean changes of a process is in the interval of the lower control limit and upper control limit. The region of the in-control state is divided into n subintervals.

The j^{th} subinterval of the upper control limit (U_j), j^{th} subinterval of the lower control limit (L_j) and the i^{th} subinterval of the midpoint (m_i) are given by

$$U_j = h_L + \frac{j(h_U - h_L)}{n},$$

$$L_j = h_L + \frac{(j-1)(h_U - h_L)}{n}$$

and $$m_i = h_L + \frac{(2i-1)(h_U - h_L)}{2n}.$$

Consequently, the transition probability equation (P_{ij}) can be rewritten as

$$P_{ij} = P(L_j \le Z_t \le U_j \mid Z_{t-1} = m_i) \tag{14}$$

and we can substitute the GWMA statistic (Y_t), L_j, U_j, and m_i into Equation (14). This transition probability equation is

$$P_{ij} = P(L_j < \frac{(1-q)(q-1)-(q-1)q(1-q)}{(q-1)(1-q)} X_{t-i+1} + q^w Y_{t-1} < U_j \mid Y_{t-1} = m_i)$$

$$= P\left(L_j < \frac{(1-q)(q-1)-(q-1)q(1-q)}{(q-1)(1-q)} X_{t-i+1} + q^w m_i < U_j \right)$$

$$= P\left(\begin{array}{l} \dfrac{[2nh_L + 2(j-1)(h_U - h_L) - 2nq^w h_L - q^w(2i-1)(h_U - h_L)](q-1)(1-q)}{2n[(1-q)(q-1)-(q-1)q(1-q)]} < X_{t-i+1} \\ < \dfrac{[2nh_L + 2j(h_U - h_L) - 2nq^w h_L - q^w(2i-1)(h_U - h_L)](q-1)(1-q)}{2n[(1-q)(q-1)-(q-1)q(1-q)]} \end{array} \right). \tag{15}$$

4. Numerical Results

In this section, the numerical results of the ARL approximation of the GWMA chart using the MCA and MC approaches are presented and the performance is compared in terms of monitoring the change in the process mean of the binomial GWMA and EWMA charts. Tables 1–4 shows the accuracy of the numerical results of the ARL for GWMA chart obtained from the MCA and MC when the observations are from a binomial distribution. We assume that the ARL_0 values are 370 and 500, the in-control process mean is $p_0 \in \{0.01,\ 0.05\}$, the weighted factors are $\lambda \in \{0.05,\ 0.1\}$ and $q \in \{0.9,\ 0.95\}$ and the magnitudes of the change in the process mean $\delta \in \{0.01, 0.05, 0.10, 0.25, 0.50\}$. The results show that the numerical results obtained by the MCA are close to the results obtained from the MC. In addition, we compared the results of ARL_0 and ARL_1 obtained from the MCA for the GWMA and EWMA charts and the results obtained from the MC verified the accuracy of the closed-form formulae. The results in Tables 1–4 show that the ARL_1 values obtained from the MCA method for the GWMA chart are less than the values obtained from the EWMA chart for small changes. For larger changes, the EWMA chart gives a smaller value of the ARL_1 than the GWMA chart. Note that calculations with the MCA clearly take much less computational time than the MC where the CPU was the Core i7-4700MQ processor 4th Generation.

Table 1. ARLs for the GWMA and EWMA charts given p_0 = 0.01, λ =0.05, q = 0.95 and ARL_0 =370.

δ		GWMA					EWMA
		w =0.1 UCL=4.27	w =0.3 UCL=1.884	w =0.5 UCL=1.267	w = 0.7 UCL=.9845	w = 0.9 UCL=.8214	w =1 UCL=.7630
0.00	MCA	370.958 (40.479*)	370.891 (41.762)	370.513 (41.34)	370.189 (41.465)	369.528 (41.808)	370.281 (41.574)
	MC	367.55 ±0.281** (524.241*)	371.646 ±0.778 (528.406)	369.723 ±0.3033 (527.581)	370.561 ±1.058 (528.282)	369.037 ±1.122 (519.624)	368.072 ±1.149 (516.832)
0.01	MCA	359.989 (40.888)	345.495 (43.555)	341.278 (41.497)	340.024 (42.385)	339.249 (42.448)	339.921 (41.871)
	MC	356.594 ±0.268 (527.19)	344.928 ±0.701 (503.216)	340.493 ±0.8722 (498.345)	339.746 ±0.9629 (494.133)	339.157 ±1.028 (492.76)	336.574 ±1.052 (486.317)
0.05	MCA	322.663 (39.936)	269.654 (41.777)	254.238 (41.98)	249.289 (42.776)	247.228 (42.682)	247.254 (42.183)
	MC	320.584 ±0.222 (475.969)	270.29 ±0.4932 (407.49)	253.649 ±0.6078 (369.551)	249.06 ±0.682 (362.921)	247.503 ±0.737 (357.664)	246.28 ±0.755 (356.572)

Table 1. Continued.

δ		GWMA					EWMA
		w =0.1 UCL=4.27	w =0.3 UCL=1.884	w =0.5 UCL=1.267	w = 0.7 UCL=.9845	w = 0.9 UCL=.8214	w =1 UCL=.7630
0.10	MCA	286.718	210.628	187.706	179.243	175.259	174.359
		(36.983)	(42.526)	(42.401)	(42.713)	(42.526)	(41.948)
	MC	285.334	211.167	187.569	179.754	174.875	173.592
		$\pm$0.1819	$\pm$0.3428	$\pm$0.4136	$\pm$0.468	$\pm$0.504	$\pm$0.520
		(420.969)	(311.503)	(275.638)	(263.251)	(254.827)	(250.522)
0.25	MCA	217.5	127.798	100.09	87.865	80.981	78.598
		(40.28)	(42.136)	(42.245)	(42.136)	(42.011)	(41.98)
	MC	216.975	127.968	100.645	87.935	81.267	78.653
		$\pm$0.119	$\pm$0.1573	$\pm$0.1789	$\pm$0.194	$\pm$0.211	$\pm$0.218
		(318.57)	(188.324)	(152.569)	(128.405)	(128.576)	(114.177)
0.50	MCA	157.336	78.888	55.241	44.065	37.335	34.828
		(40.155)	(41.965)	(43.056)	(42.776)	(42.136)	(42.027)
	MC	157.379	78.866	55.149	41.159	37.328	35.116
		$\pm$0.0759	$\pm$0.0756	$\pm$0.076	$\pm$0.078	$\pm$0.082	$\pm$0.055
		(233.517)	(116.22)	(80.263)	(64.537)	(54.913)	(51.355)

* CPU Times (second)
** standard deviation of ARL

Table 2. ARLs for the GWMA and EWMA charts given p_0 = 0.01, λ =0.05, q = 0.95 and ARL$_0$ =500.

δ		GWMA					EWMA
		w =0.1 UCL=4.617	w =0.3 UCL=1.9505	w =0.5 UCL=1.304	w = 0.7 UCL=1.0121	w = 0.9 UCL=.8435	w =1 UCL=.7835
0.00	MCA	500.585	500.279	500.171	500.921	500.9	501.14
		(43.93)	(44.351)	(45.583)	(47.455)	(49.733)	(46.457)
	MC	484.517	504.054	499.248	498.917	495.803	495.833
		$\pm$0.142	$\pm$0.366	$\pm$0.428	$\pm$0.46	$\pm$0.48	$\pm$0.489
		(6898.91)	(7170.6)	(7774.9)	(7091.18)	(7513.16)	(7036.03)
0.01	MCA	480.257	459.139	**455.531**	455.993	456.528	455.636
		(40.186)	(41.574)	(41.886)	(41.668)	(41.871)	(40.998)
	MC	465.889	463.691	456.77	454.012	450	453.538
		$\pm$0.417	$\pm$1.035	$\pm$1.222	$\pm$1.313	$\pm$1.368	$\pm$1.41
		(706.794)	(683.44)	(725.389)	(663.738)	(724.796)	(663.987)
0.05	MCA	415.05	340.997	325.96	**323.458**	323.77	323.525
		(39.64)	(42.042)	(42.526)	(41.839)	(45.116)	(41.918)
	MC	406.194	343.415	325.535	321.929	319.343	322.213
		$\pm$0.324	$\pm$0.692	$\pm$0.821	$\pm$0.899	$\pm$0.959	$\pm$0.992
		(594.878)	(503.181)	(516.192)	(473.369)	(518.251)	(470.546)
0.10	MCA	357.578	254.817	231.192	224.589	222.776	**221.873**
		(39.749)	(42.183)	(60.918)	(42.183)	(43.851)	(71.339)
	MC	352.117	256.268	231.752	225.548	222.104	222.927
		$\pm$0.251	$\pm$0.459	$\pm$0.53	$\pm$0.603	$\pm$0.645	$\pm$0.676
		(515.037)	(374.621)	(362.765)	(332.016)	(357.195)	(324.622)
0.25	MCA	258.342	144.493	114.738	102.658	96.502	**94.158**
		(40.295)	(42.151)	(70.107)	(42.026)	(42.121)	(99.185)
	MC	256.435	144.758	114.807	103.231	96.536	94.70
		$\pm$0.147	$\pm$0.188	$\pm$0.212	$\pm$0.234	$\pm$0.254	$\pm$0.265
		(376.134)	(212.052)	(170.322)	(160.509)	(165.174)	(138.482)
0.50	MCA	181.017	85.924	60.516	48.925	42.18	**39.582**
		(40.357)	(42.26)	(42.167)	(41.824)	(41.949)	(45.912)
	MC	180.63	86.088	60.683	49.184	42.304	39.655
		$\pm$0.088	$\pm$0.84	$\pm$0.085	$\pm$0.089	$\pm$0.004	$\pm$0.005
		(265.326)	(126.423)	(89.825)	(72.307)	(68.468)	(57.986)

Table 3. ARLs for the GWMA and EWMA charts given p_0 = 0.05, λ =0.1, q = 0.90 and ARL_0=370.

δ		GWMA					EWMA
		w =0.1 UCL=23.95	w =0.3 UCL=9.155	w =0.5 UCL=5.925	w = 0.7 UCL=4.499	w = 0.9 UCL=3.6901	w =1 UCL=3.403
0.00	MCA	370.313 (39.655)	370.465 (44.414)	369.999 (42.557)	370.459 (43.056)	370.165 (42.182)	371.702 (42.973)
	MC	368.302 ±0.383 (532.524)	369.206 ±0.888 (524.928)	363.448 ±0.999 (519.28)	359.886 ±1.055 (516.754)	354.556 ±1.082 (504.211)	354.379 ±1.104 (520.685)
0.01	MCA	349.616 (35.272)	314.73 (40.217)	**313.657** (43.01)	314.821 (43.103)	315.94 (43.01)	317.594 (42.619)
	MC	340.333 ±0.326 (498.881)	315.988 ±0.727 (461.7)	309.232 ±0.837 (452.481)	305.441 ±0.883 (453.588)	305.357 ±0.933 (449.844)	304.479 ±0.951 (459.828)
0.05	MCA	269.066 (35.334)	191.312 (40.264)	179.899 (41.917)	**178.349** (42.932)	178.968 (43.337)	179.917 (42.744)
	MC	268.151 ±0.196 (392.061)	192.433 ±0.364 (282.222)	178.299 ±0.430 (259.882)	175.097 ±0.477 (257.62)	176.561 ±0.519 (261.177)	179.74 ±0.533 (262.285)
0.10	MCA	217.767 (34.96)	126.145 (40.592)	108.29 (42.635)	102.681 (42.9)	100.479 (42.588)	**99.973** (43.15)
	MC	217.404 ±0.125 (318.398)	126.485 ±0.188 (185.532)	108.249 ±0.430 (157.67)	101.948 ±0.253 (150.011)	99.908 ±0.279 (146.719)	99.052 ±0.293 (149.418)
0.25	MCA	146.342 (35.053)	65.041 (39.624)	46.233 (42.729)	37.895 (43.072)	32.909 (42.963)	**30.967** (42.791)
	MC	146.332 ±0.058 (215.594)	65.084 ±0.058 (95.16)	46.36 ±0.063 (67.936)	37.934 ±0.0682 (55.973)	33.173 ±0.045 (49.187)	31.022 ±0.07833 (46.8)
0.50	MCA	98.902 (35.303)	38.385 (39.936)	24.203 (42.526)	17.565 (42.729)	13.426 (42.978)	**11.805** (43.244)
	MC	98.885 ±0.0306 (144.987)	38.448 ±0.024 (56.285)	24.234 ±0.023 (35.506)	17.604 ±0.0223 (26.442)	33.447 ±0.022 (20.124)	11.893 ±0.021 (18.112)

Table 4. ARLs for the GWMA and EWMA charts given p_0 = 0.05, λ =0.1, q = 0.90 and ARL_0=500.

δ		GWMA					EWMA
		w =0.1 UCL=24.625	w =0.3 UCL=9.298	w =0.5 UCL=6.016	w = 0.7 UCL=4.568	w = 0.9 UCL=3.746	w =1 UCL=3.4553
0.00	MCA	500.338 (35.677)	499.877 (40.155)	500.458 (42.23)	499.678 (42.464)	499.813 (42.884)	499.343 (42.885)
	MC	487.472 ±0.659 (686.607)	493.502 ±1.266 (693.846)	493.635 ±1.41 (694.127)	479.749 ±1.436 (686.17)	473.286 ±1.461 (675.172)	473.175 ±1.476 (663.223)
0.01	MCA	446.531 (37.222)	**415.818** (40.42)	416.54 (42.339)	418.788 (43.103)	421.732 (42.931)	422.503 (42.822)
	MC	435.22 ±0.533 (633.24)	409.326 ±1.01 (592.71)	408.279 ±1.145 (593.677)	405.963 ±1.203 (596.563)	402.301 ±1.229 (590.089)	402.922 ±1.253 (581.322)
0.05	MCA	320.63 (35.1)	232.435 (39.577)	**224.01** (42.432)	225.495 (43.165)	229.25 (43.29)	230.709 (42.713)
	MC	317.326 ±0.278 (460.655)	231.708 ±0.471 (336.249)	222.184 ±0.563 (322.532)	220.841 ±0.619 (323.92)	222.181 ±0.664 (328.585)	223.219 ±0.685 (324.154)

Table 4. Continued.

δ		GWMA					EWMA
		w =0.1 UCL=24.625	w =0.3 UCL=9.298	w =0.5 UCL=6.016	w = 0.7 UCL=4.568	w = 0.9 UCL=3.746	w =1 UCL=3.4553
0.10	MCA	246.01 (34.944)	143.889 (39.952)	127.144 (42.448)	123.311 (42.573)	**123.165** (42.947)	123.182 (42.979)
	MC	244.916 $\pm$0.158 (355.292)	143.383 $\pm$0.225 (207.638)	126.61 $\pm$0.278 (192.567)	122.558 $\pm$0.316 (181.476)	121.246 $\pm$0.345 (178.387)	121.312 $\pm$0.36 (176.562)
0.25	MCA	157.827 (35.428)	69.381 40.139)	50.074 (42.744)	41.824 (42.62)	37.150 (43.072)	**35.283** (42.557)
	MC	157.669 $\pm$0.065 (229.244)	69.285 $\pm$0.065 (100.461)	50.237 $\pm$0.071 (73.243)	41.9321 $\pm$0.078 (61.777)	37.035 $\pm$0.085 (54.616)	35.464 $\pm$0.90 (51.824)
0.50	MCA	104.532 (34.991)	40.001 (39.937)	25.386 (41.964)	18.619 (43.025)	14.460 (43.493)	**12.814** (42.619)
	MC	104.513 $\pm$0.033 (152.787)	39.971 $\pm$0.025 (58.485)	25.438 $\pm$0.024 (37.191)	18.668 $\pm$0.024 (27.815)	14.474 $\pm$0.0242 (21.544)	12.871 $\pm$0.025 (18.86)

5. Discussion

An approximation of the ARL using the MCA for a GWMA was presented when observations are from a binomial distribution. The results showed that the numerical results obtained from the MCA are in good agreement with the results obtained from the MC. Additionally, we compared the effectiveness of the GWMA and EWMA procedures to detect changes in binomial distributions. The comparison of control charts is based on ARL_0 and ARL_1 criteria. We demonstrated that the performance of the GWMA chart is superior to the EWMA chart for small changes; otherwise, the performance of the EWMA chart is superior to the GWMA chart. However, the performance of the GWMA chart could be improved by modification of a pair of the weighted parameters (w, q).

6. Conclusions

The GWMA chart has memory-less properties and the ability to detect small shifts where $\delta \leq 0.05$. Without loss of generality, this chart can be relaxed due to its feasibility with two parameters as a time weighted parameter (q) and adjustment smoothing constant (w). Furthermore, the GWMA chart performs better as the values of w increase for small shifts.

Acknowledgements

The author would like to express her gratitude to King Mongkut's University of Technology North Bangkok, Thailand for providing research grant No: KMUTNB-GEN-59-11. Also, thanks go out to the Department of Applied Statistics for supporting materials and a high performance computer.

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