1 Introduction

Given $a, b, c \in \mathbb{Z}$ with $a \neq 0$ the quadratic equations of the form

$$p(x) = ax^2 + bx + c = 0 \tag{1}$$

have different types of roots which are easily determined by the quadratic formula, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. For example, $p_1(x) = x^2 + 2x + 4 = 0$ has complex roots, while $p_2(x) = x^2 + 2x - 4 = 0$ has real roots but not rational. $p_3(x) = x^2 + 2x - 3 = 0$ and $p_4(x) = 66c^2 + 68x - 70 = 0$ have rational roots. Surprisingly,($|1|, |2|, |-3|$) and ($|66|, |68|, |-70|$), are in arithmetic progression with the common difference 1 and 2, respectively. I suddenly wondered if there are other quadratic equations in (1) such that the roots are rational and ($|a|, |b|, |c|$) is in arithmetic progression with the common difference $d \in \mathbb{Z}$. Before searching for the answer, let us notice that the quadratic equation (1) can be spread out into 8 different equations. Assuming $a, b, c \in \mathbb{N} \cup \{0\}$ with $a \neq 0$, one has

$$\begin{aligned} p_1 &:= ax^2 + bx + c = 0 & p_2 &:= ax^2 + bx - c = 0 \\ p_3 &:= ax^2 - bx + c = 0 & p_4 &:= ax^2 - bx - c = 0 \\ p_5 &:= -ax^2 + bx + c = 0 & p_6 &:= -ax^2 + bx - c = 0 \\ p_7 &= -ax^2 - bx + c = 0 & p_8 &:= -ax^2 - bx - c = 0 \end{aligned} \tag{2}$$

Note that $p_1 = -p_8$, $p_2 = -p_7$, $p_3 = -p_6$, and $p_4 = -p_5$ and also if r is a root of p_1 and s is a root of p_2, then $-r$ and $-s$ are roots of p_3 and p_4, respectively. Moreover, for $p_1^* := cx^2 + bx + a = 0$ and $p_2^* := -cx^2 + bx + a = 0$, if the two nonzero rationals r and s are the respective roots of p_1 and p_2 then it is easy to show that $\frac{1}{r}$ and $\frac{1}{s}$ are a root of p_1^* and p_2^*, respectively. Now, let $\mathcal{A}$ be the set of all arithmetic progression triples (a, b, c) with $a \in \mathbb{N}$ and the common difference $d \in \mathbb{N}$.

Thus, in this paper, it suffices to consider only the equations of the forms

$$\begin{aligned} p_1 &:= ax^2 + bx + c = 0 \\ p_2 &:= ax^2 + bx - c = 0 \end{aligned} \tag{3}$$

where $(a, b, c) \in \mathcal{A}$. In the case of the common difference $d = 1$, Schwartzman [4] has already shown that there are infinitely many quadratic equations of the form p_2 such that the roots are rational. In his paper, Schwartzman did not mention the form p_1 because for this particular case, $d = 1$, all the roots are complex. Also, he claimed without proof that the coefficients b of p_2 are necessarily elements of the Fibonacci numbers. Otherwise, $p_2 = 0$ could not have rational roots. However, it is easy to see that $2x^2 + 3x - 4 = 0$ has no rational root but 3 is an element of the Fibonacci numbers.

In this paper, based on Schwartzman's conjecture, we proved it in a more general result for the quadratic equation of the form p_2. We also showed that under some conditions on d the quadratic equation of the form p_1 can possibly have rational roots. In fact, we proved that once the common difference d is given, there are only a finite number of elements $(a, b, c) \in \mathcal{A}$ such that the quadratic equation of the form p_1 has rational roots. In other words, the set

$$P_d = \{(a, b, c) \in \mathcal{A} \mid q(x) = ax^2 + bx + c = 0 \text{ has rational roots }\}$$

is finite. In addition, once the common difference d is given, the upper bound of the number $n(P_d)$ can be found. Finally, the algorithms based on the *scilab* code are given for generating all possible quadratic equations of the form p_1 and p_2 for a given d. In particular, in the case of p_1 the least upper bound of $n(P_d)$ is also confirmed by the algorithms.

2 Main Results

2.1 Rational Roots of $p_2 = 0$

Since $(a, b, c) \in \mathcal{A}$, the quadratic equation $p_2 = 0$ can be written as

$$p_2 := nx^2 + (n+d)x - (n+2d) = 0. \tag{4}$$

The roots of p_2 are rational if and only if the discriminant of p_2 is a perfect square. That is, for some $M \in \mathbb{N}$, one must have

$$(n+d)^2 + 4n(n+2d) = M^2 \text{ or } 5n^2 + 10nd + (d^2 - M^2) = 0 \tag{5}$$

Solving equation (5) for n yields

$$n = -d \pm \sqrt{\frac{4d^2 + M^2}{5}}. \tag{6}$$

In orter to get an integer radicand in equation (6), it is necessary to have

$$\begin{aligned} 4d^2 + M^2 &\equiv 0 \mod 5 \\ M^2 &\equiv d^2 \mod 5 \\ M &\equiv \pm d \mod 5. \end{aligned} \tag{7}$$

Unfortunately, the condition (7) on M is necessary to have an integer radicand but not sufficient to guarantee whether the radicand is perfect. Now, for each d, let us consider the set

$$\mathcal{M}_d = \left\{ M \in \mathbb{N} \mid M \equiv \pm d \mod 5 \ \& \ \frac{4d^2 + M^2}{5} \text{ is a perfect square} \right\}.$$

We claim that $\mathcal{M}_d$ is an infinite set. To prove the claim, we first need the following lemma.

Lemma 2.1. *For each n, let F_n and L_n be a Fibonacci number and a Lucas number, respectively. Then we have*

$$L_n^2 = 5F_n^2 + 4(-1)^n, \tag{8}$$

and hence for each $d \in \mathbb{N}$ the formula $(dL_n)^2 = 5(dF_n)^2 + 4d^2(-1)^n$ holds.

Proof. A proof is straightforward via using the fundamental identities

$$F_n^2 - F_{n-1}^2 = F_{n-1}F_n - (-1)^n \text{ and } L_n = F_{n+1} + F_{n-1}.$$

□

Then we have the following Lemma.

Lemma 2.2. *For each $k \in \mathbb{N}$ and $d \in \mathbb{Z}$, we have $dL_{2k+1} \in \mathcal{M}_d$ and hence $\mathcal{M}_d$ is an infinite set.*

Proof. By Lemma 2.1, we have

$$(dL_{2k+1})^2 = 5(dF_{2k+1})^2 + 4d^2(-1)^{2k+1} = 5(dF_{2k+1})^2 - 4d^2.$$

Let $M = dL_{2k+1}$. Then $(dF_{2k+1})^2 = \dfrac{M^2 + 4d^2}{5}$ which implies that $M \in \mathcal{M}_d$. □

To prove the main theorem we need two lemmas.

Lemma 2.3. *Let $x_1 + y_1\sqrt{D}$ be the fundamental solution of the Pell's equation $x^2 - Dy^2 = -4$ where D is a squarefree. Then all positive integer solutions of the equation $x^2 - Dy^2 = -4$ are given by*

$$x_n + y_n\sqrt{D} = \frac{\left(x_1 + y_1\sqrt{D}\right)^{2n-1}}{2^{2n-2}} \tag{9}$$

for $n \in \mathbb{N}$.

Proof. The proof can be found in [1], [3]. □

Lemma 2.4. *Given $d \in \mathbb{N}$, if $5x^2 - 4d^2$ is a perfect square, x is divisible by d, and x is positive, then $x = \pm dF_{2k+1}$ for some k.*

Proof. Let $d \in \mathbb{N}$. Assume that $5x^2 - 4d^2$ is a perfect square, x is divisible by d, and x is positive. Then there exists an integer l such that $x = dl$. Thus,

$$5x^2 - 4d^2 = 5(dl)^2 - 4d^2 = d^2(5l^2 - 4).$$

Since $d^2(5l^2 - 4)$ is a perfect square, it forces that $5l^2 - 4$ must also be perfect. So that $5l^2 - 4 = r^2$ for some positive integer r.

Next we will show that $l = F_{2k-1}$ for some k. Consider the Pell's equation

$$x^2 - 5y^2 = -4. \tag{10}$$

We can see that the fundamental solution to the equation 10 is $1 + \sqrt{5}$. By the Lemma 2.3, all positive solutions to the equation (2.3) are

$$x_n + y_n\sqrt{5} = \frac{(1 + \sqrt{5})^{2n-1}}{2^{2n-2}}. \tag{11}$$

Equating and expanding the equation (11), we have

$$y_n = \frac{1}{(\sqrt{5})2^{2n-2}} \sum_{t \text{ odd}} \binom{2n-1}{t} (\sqrt{5})^t.$$

We also know that the sequence F_n of Fibonacci numbers is defined by the recurrence relation

$$F_n = F_{n-1} + F_{n-2}$$

where $F_0 = 0$ and $F_1 = 1$ and the mth element in the Fibonaci sequence is

$$F_m = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^m - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^m.$$

For $m = 2n - 1$, we can see that

$$\begin{aligned}
F_{2n-1} &= \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n-1} \\
&= \frac{1}{(\sqrt{5})2^{2n-1}}\left[\left(1+\sqrt{5}\right)^{2n-1} - \left(1-\sqrt{5}\right)^{2n-1}\right] \\
&= \frac{2}{(\sqrt{5})2^{2n-1}} \sum_{t \text{ odd}} \binom{2n-1}{t} (\sqrt{5})^t \\
&= \frac{1}{(\sqrt{5})2^{2n-2}} \sum_{t \text{ odd}} \binom{2n-1}{t} (\sqrt{5})^t \\
&= y_n
\end{aligned}$$

Since $5l^2-4=r^2$, it follows that (r,l) is a positive solution of the equation (10) and hence $l=y_k=F_{2k-1}$ for some k. This implies that $x=dF_{2k-1}$ for some k. □

For the Lemma 2.4, if x is not divisible by d, then it is not necessary that $x=dF_{2k+1}$. For example if $d=11$ and $x=13$, then it is obvious that $5(13)^2-4(11)^2=19^2$ which is a perfect square but $x\neq 11F_{2k-1}$ for all k. Now we are ready to prove the main theorem.

Theorem 2.5. *Let $d\in\mathbb{N}$ and $(a,b,c)\in\mathcal{A}$ be such that $\gcd(a,b,c)$ is divisible by d. Then $p_2(x)=ax^2+bx-c=0$ has rational roots if and only if $b=dF_{2k-1}$ for some k where F_{2k-1} is a $2k-1$th Fibonacci number.*

Proof. The converse part is true via Corollary 2.2. Now assume that $p_2(x)=ax^2+bx-c=0$ has rational roots. Then the discriminant D is a perfect square . That is, $D=b^2+4ac=5n^2+10dn+d^2=M^2$ for some $M,n\in\mathbb{N}$. From the equation (6), we have $5b^2-4d^2=M^2$, a perfect square. Since $\gcd(a,b,c)$ is divisible by d, so is b. Thus by Lemma 2.4, $b=dF_{2k-1}$ for some k. □

2.2 Rational Roots of $p_1=0$

Similarly, the roots of the $p_1=0$ are rational if and only if the discriminant of p_1 is a perfect square. In fact,

$$D=(n+d)^2-4n(n+2d)=N^2 \text{ or } 3n^2+6nd-d^2+N^2=0 \tag{12}$$

for some $N\in\mathbb{N}$. Observe that the discriminant D depending on n can be negative. Hence, first we have to figure out the possible value of n for a given d such that D is positive. In this case, it is easy to show that

$$\lfloor -d-\frac{2}{\sqrt{3}}|d|\rfloor+1\leq n\leq \lfloor -d+\frac{2}{\sqrt{3}}|d|\rfloor.$$

Solving equation (12) for n yields

$$n=-d\pm\sqrt{\frac{4d^2-N^2}{3}}. \tag{13}$$

In other to get an integer radicand in equation (13), it is necessary to have

$$\begin{aligned}4d^2-N^2&\equiv 0 \mod 3\\ N^2&\equiv d^2 \mod 3\\ N&\equiv \pm d \mod 3.\end{aligned} \tag{14}$$

Unfortunately, the condition (14) on N is necessary to have an integer radicand but not sufficient to guarantee whether the radicand is perfect. Now, for each d, let us consider the sets

$$\mathcal{N}_d=\left\{N\in\mathbb{N}\mid N\equiv\pm d \mod 3\ \&\ \frac{4d^2-N^2}{3}\text{ is a perfect square}\right\}$$

and

$$P_d=\{(a,b,c)\in\mathcal{A}\mid q(x)=ax^2+bx+c=0\text{ has rational roots}\}.$$

Note that $\mathcal{N}_d\neq\emptyset$ if and only if $P_d\neq\emptyset$. The next theorem shows that for each d, the set P_d is always finite.

Theorem 2.6. *Let a,b,c,d be positive integers such that $(a,b,c)\in\mathcal{A}$. If $p_1: ax^2+bx+c=0$ has rational roots, then $\mathcal{N}_d$ and P_d are finite sets.*

Proof. Let $a = n$ for some $n \in \mathbb{N}$. Since $p_1 : ax^2 + bx + c = 0$ has rational roots, it is straightforward to check that

$$\lfloor -d - \frac{2}{\sqrt{3}}|d| \rfloor + 1 \leq n \leq \lfloor -d + \frac{2}{\sqrt{3}}|d| \rfloor.$$

Thus it follows that $\mathcal{N}_d < \infty$ and $n(P_d) < \infty$. $\square$

However, for a given d, it is not necessarily that $P_d \neq \emptyset$. For example, $P_d = \emptyset$ if $d = 1, 2, 3, 4, 5, 6$. So, it is quite interesting to find the integer d such that P_d is always non-empty. The following lemma gives such integers d but not all.

Lemma 2.7. *Let $N \in \mathbb{N}$. For each $k \in \mathbb{N}$, define d_k and n_k as follow*

$$d_k = \frac{N}{4}\left((2+\sqrt{3})(7+4\sqrt{3})^k + (2-\sqrt{3})(7-4\sqrt{3})^k\right)$$
$$n_k = \frac{N}{4\sqrt{3}}\left((7+4\sqrt{3})^k - (7-4\sqrt{3})^k\right).$$

Then $p(x) = n_k x^2 + (n_k + d_k)x + (n_k + 2d_k) = 0$ has rational roots.

Proof. It suffices to show that for each k, d_k and n_k are integers and satisfied

$$3n_k^2 + 6n_k d_k - d_k^2 + N^2 = 0.$$

Note that the pair of

$$\frac{d_k - 3n_k}{N} = \frac{(7+4\sqrt{3})^k + (7-4\sqrt{3})^k}{2}$$

and

$$\frac{n_k}{N} = \frac{(7+4\sqrt{3})^k - (7-4\sqrt{3})^k}{4\sqrt{3}}$$

are all integer solutions of the Pell's equation $x^2 - 12y^2 = 1$, [2]. That is, $(d_k - 3n_k)^2 - 12n_k^2 = N^2$ and so it implies that $3n_k^2 + 6n_k d_k - d_k^2 + N^2 = 0$. $\square$

2.3 Interesting Properties

Lemma 2.8. *If α and β are the roots of p_2, then $(\alpha - 2)(\beta - 2) = 5$. Consequently, if α and β are integers, then $(\alpha, \beta) \in \{(3,7), (7,3), (1,-3), (-3,1)\}$.*

Proof. Assume that α and β are the roots of $p_2(x) = nx^2 + (n+d)x - (n+2d) = 0$ for some d and n. Then $\alpha + \beta = -(\frac{n+d}{n})$ and $\alpha\beta = -\frac{n+2d}{n}$. And so,

$$\begin{aligned}(\alpha - 2)(\beta - 2) &= \alpha\beta - 2(\alpha + \beta) + 4 \\ &= -(\frac{n+2d}{n}) - 2(-\frac{n+d}{n}) + 4 \\ &= 5.\end{aligned}$$

Furthermore, if α and β are integers, then both $\alpha - 2$ and $\beta - 2$ are in $\{\pm 1, \pm 5\}$ and hence $(\alpha, \beta) \in \{(3,7), (7,3), (1,-3), (-3,1)\}$. $\square$

Lemma 2.9. *If α and β are the roots of p_1, then $(\alpha + 2)(\beta + 2) = 3$. Consequently, if α and β are integers, then $(\alpha, \beta) \in \{(-1,1), (1,-1), (-3,-5), (-5,-3)\}$.*

Proof. Assume that α and β are the roots of $p_2(x) = nx^2 + (n+d)x + (n+2d) = 0$ for some d and n. Then $\alpha + \beta = -(\frac{n+d}{n})$ and $\alpha\beta = \frac{n+2d}{n}$. And so,

$$\begin{aligned}(\alpha+2)(\beta+2) &= \alpha\beta + 2(\alpha+\beta) + 4 \\ &= \frac{n+2d}{n} + 2(-\frac{n+d}{n}) + 4 \\ &= 3.\end{aligned}$$

Furthermore, if α and β are integers, then both $\alpha + 2$ and $\beta + 2$ are in $\{\pm 1, \pm 3\}$ and hence $(\alpha, \beta) \in \{(-1,1),(1,-1),(-3,-5),(-5,-3)\}$. □

Lemma 2.10. *Let $d \in \mathbb{Z}$. For each $i \in \mathbb{N}$, let n_i be a positive integer such that $p_2^i(x) = n_i x^2 + (n_i + d)x - (n_i + 2d) = 0$ has rational roots. If α_i and β_i are such rational roots of $p_2(x) = 0$ with $\alpha_i < \beta_i$ then $\{\alpha_i\}$ converges to $\frac{-1-\sqrt{5}}{2}$ and $\{\beta_i\}$ converges to $\frac{-1+\sqrt{5}}{2}$.*

Proof. Note that $\alpha_i + \beta_i = -\frac{n_i+d}{n_i}$ and $\alpha_i\beta_i = -\frac{n_i+2d}{n_i}$ and hence as n_i approaches to infinity, both $\alpha_i + \beta_i$ and $\alpha_i\beta_i$ converge to -1. Moreover, α_i and β_i are the two solutions of the quadratic equation $x^2 - (\alpha_i + \beta_i)x + \alpha_i\beta_i = 0$. Since $\alpha_i < \beta_i$, it follows that

$$\beta_i = \frac{(\alpha_i+\beta_i) + \sqrt{(\alpha_i+\beta_i)^2 - 4(\alpha_i\beta_i)}}{2} \to \frac{-1+\sqrt{5}}{2}$$

and

$$\alpha_i = \frac{(\alpha_i+\beta_i) - \sqrt{(\alpha_i+\beta_i)^2 - 4(\alpha_i\beta_i)}}{2} \to \frac{-1-\sqrt{5}}{2}.$$

□

3 Algorithms

Finally, we complete this study by giving algorithms in *scilab* codes. A simple yet important reason to choose scilab because it is a freeware. Anyone can download it from https://www.scilab.org/.

Let us first consider the quadratic equation $p_2(x) = 0$. Recall that once d is chosen we want to find M such that

$$n = -d \pm \sqrt{\frac{4d^2 + M^2}{5}}$$

is integer and hence the quadratic equation $p_2(x) = nx^2 + (n+d)x - (n+2d) = 0$ has rational roots. For a given d, the function "findm" is looking for those M such that $5|(4d^2 + M^2)$. After that the function "getperfect" will screen out these M for only $\frac{4d^2+M^2}{5}$ is perfect. Recall that $\mathcal{M}_d$ is an infinite set so that in the input argument of the function "findm" d represents the common difference while q restricts the number of M we desire. Furthermore, the vector m in the output argument collects all the q number of M. For example, for $d = 1$ and the first 1500 numbers of M ($q = 1500$) there are only seven of M which is the output $md = (4,\ 11,\ 29,\ 76,\ 199,\ 521,\ 1364)$ so that $\frac{4d^2+M^2}{5}$ is perfect. Now, in this case, with $d = 1$ and $M = (4,\ 11,\ 29,\ 76,\ 199,\ 521,\ 1364)$, we obtain the corresponding leading coefficient $n = (1,\ 4,\ 12,\ 33,\ 88,\ 232,\ 609)$ so that $p_2(x) = 0$ has rational roots. For examples, for 232 and 609, one has $232x^2 + 233x - 234 = (8x-13)(29x+18)$ and $609x^2 + 610x - 611 = (21x-13)(29x+47)$, respectively.

```
function [m]=findm(d, q),
i = 1;
if d <> 5 then
    k = 1;
    while i < q,
        m(i) = 5 * k - d;
        m(i + 1) = 5 * k + d;
        k = k + 1;
        i = i + 2;
    end
else for k = 1 : q,
        m(k) = 5 * k;
    end
end
endfunction
```

```
function [md]=getperfect(m, d)
n = max(size(m));
i = 1;
md = 0;
for k = 1 : n,
    c = (4*d∧2 + m(k)∧2)/5;
    if (c == (floor(sqrt(c))∧2) ) then
        md(i) = m(k);
        i = i + 1;
    end
end
endfunction
```

It would be useful to note here that for $d > 1$ the coefficient we get are actually the multiple of the case $d = 1$. For example with $d = 2$, the first several leading coefficients n are 2, 8, 24, 66, 176 and 464. For examples, for 176 and 464, one has $176x^2 + 178x - 180 = 2(88x^2 + 89x - 90) = 2(8x + 5)(11x - 18)$ and $464x^2 + 466x - 468 = 2(232x^2 + 233x - 234) = 2(8x - 13)(29x + 8)$, respectively.

Next, let us turn to the quadratic equation $p_1(x)$ of the form $nx^2 + (n + d)x + (n + 2d) = 0$. The first difference made by the condition on n:

$$\lfloor -d - \frac{2}{\sqrt{3}}|d| \rfloor + 1 \leq n \leq \lfloor -d + \frac{2}{\sqrt{3}}|d| \rfloor.$$

In this study, only the positive leading coefficients are of interest. So that the above condition forces d to be greater than 7. With the quite similar algorithms we can obtain the integer N such that $\frac{4d^2 - N^2}{3}$ is integer and perfect. For example, the case $d = 7$ all the possible leading coefficients are shown in Table 1. They are actually within the interval [-16,1] confirmed by

n	$p_1(x)$	n	$p_1(x)$
1	$x^2 + 8x + 15 = (x + 3)(x + 5)$	-10	$-10x^2 - 3x + 4 = (5x + 4)(-2x + 1)$
-2	$-2x^2 + 5x + 112 = (2x + 3)(-x + 4)$	-12	$-12x^2 - 5x + 2 = (3x + 2)(-4x + 1)$
-4	$-4x^2 + 3x + 10 = (4x + 5)(-x + 2)$	-15	$-15x^2 - 8x - 1 = -(3x + 1)(5x + 1)$
-7	$-7x^2 + 7 = -7(x + 1)(x - 1)$		

Table 1: All possible leading coefficients for $d = 7$ so that $p_1(x) = 0$ has rational roots.

However, since our interest is restricted only on the positive leading coefficient, so that in this case, $d = 7$, the eligible leading coefficient is only $n = 1$. In particular, we have $x^2 + 8x + 15 = (x + 3)(x + 5)$.

4 Conclusions

In this study, it was shown that the quadratic equation of the form $p_2(x) = ax^2 + bx - c = 0$ has infinitely many $(a, b, c) \in \mathcal{A}$ such that $p_2(x) = 0$ has rational roots. While for the quadratic equation of the form $p_1(x) = ax^2 + bx + c = 0$ has only a finite number of $(a, b, c) \in \mathcal{A}$ such that $p_1(x) = 0$ has rational roots. Furthermore, for $0 < d < 7$ and $n > 0$, we found that there is no rational solution of $p_1(x) = 0$. However