CHAPTER V

PIECEWISE AFFINE MODEL FOR BICYCLE ROBOT

In order to synthesis the controller by PWQ stabilization technique, one needs to model the nonlinear system dynamics to be the PWA model with the error as small as possible. In this chapter, we describe how to obtain the PWA model by a simple trigonometric terms approximation method, least-square-error without boundary constraints, and least-square-error without boundary constraints.

We starts with defining the regions that will be approximated by PWA model. The bicycle roll angle is partitioned into 3 regions, the flywheel precession angle does so. Thus, the operating regions were split into 9 regions or polyhedral cells, see Fig. 5.1. X_5 is considered to be in I_0 or the steady state point region where the state trajectory rest at the point $(0,0,0,0)$ when the system is made stable. The other cells X_i are in the set I_1. Note that these 9 regions is not the best choice to reduce model error. More regions lead to more accurate model but more calculation is needed.

The nonlinear differential equations (4.4) and (4.5) can be approximated by continuous PWA functions into the state-space form (2.7). We define the parameters for our bicycle robot model as

$$x = \begin{bmatrix} \varphi \\ \alpha \\ \dot{\varphi} \\ \dot{\alpha} \end{bmatrix} \qquad u = T_\alpha \qquad A_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ A_{31}^{(i)} & A_{32}^{(i)} & A_{33}^{(i)} & A_{34}^{(i)} \\ A_{41}^{(i)} & A_{42}^{(i)} & A_{43}^{(i)} & A_{44}^{(i)} \end{bmatrix} \qquad a_i = \begin{bmatrix} 0 \\ 0 \\ a_{3a}^{(i)} \\ a_{4a}^{(i)} \end{bmatrix} \qquad B_i = \begin{bmatrix} 0 \\ 0 \\ 0 \\ B_{41}^{(i)} \end{bmatrix}$$

Next, the PWA model approximation methods will be shown from a simple method (fast calculation but roughly accuracy) to the more complex (longer time for calculation but more accuracy) method. All constant terms are taken from Table 4.1.

5.1 Trigonometric Terms Approximation

We approximate the nonlinear terms sin and cos by least square error method in each interval, while the other nonlinear terms are approximated by linearization about the operationg point $(0,0,0,0)$.

- Approximate the nonlinear terms sin and cos by least square error method and use 'θ' to represent φ and α only for this occasion as follow
  - When $\theta \leq -0.1745$, we approximate $\sin\theta \approx m_1(\theta + 0.1745)$ and $\cos\theta \approx m_2(\theta + 0.1745)$,

$$m_1 = \text{argmin} \int_{-1.0472}^{-0.1745} (m_1(\theta + 0.1745) - 0.1745 - \sin\theta)^2 d\theta$$
$$m_2 = \text{argmin} \int_{-1.0472}^{-0.1745} (m_2(\theta + 0.1745) + 1 - \cos\theta)^2 d\theta$$

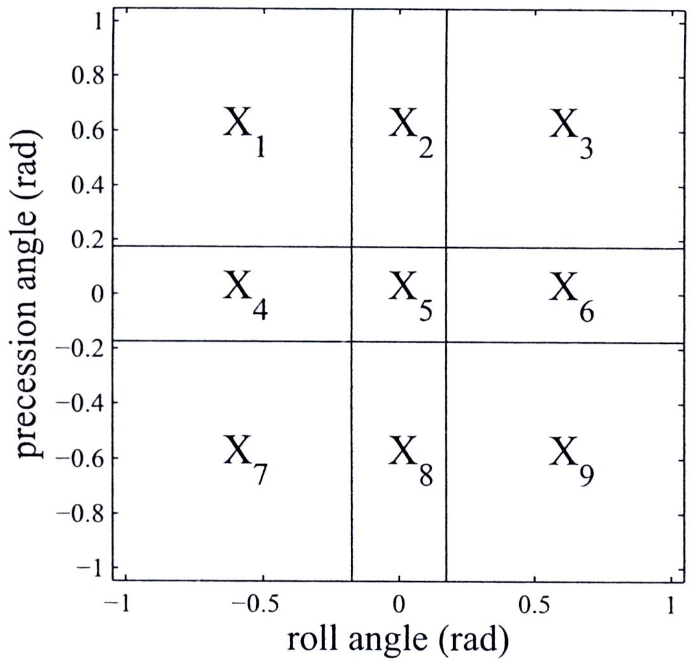

Figure 5.1: Polyhedral partition of the PWA bicycle state space model.

*The calculation is done only in the range $-1.0472 \leq \theta \leq -0.1745$ or $-60° \leq \theta \leq -10°$.

- When $-0.1745 \leq \theta \leq 0.1745$, we linearize $\sin\theta$ and $\cos\theta$ around $\theta = 0$,

$$\sin\theta \approx \theta \qquad\qquad \cos\theta \approx 1$$

- When $\theta \geq 0.1745$, we approximate $\sin\theta \approx m_3(\theta - 0.1745)$ and $\cos\theta \approx m_4(\theta - 0.1745)$,

$$m_3 = \text{argmin} \int_{0.1745}^{1.0472} (m_3(\theta - 0.1745) + 0.1745 - \sin\theta)^2 d\theta$$

$$m_4 = \text{argmin} \int_{0.1745}^{1.0472} (m_4(\theta - 0.1745) + 1 - \cos\theta)^2 d\theta$$

*The calculation is done only in the range $-1.0472 \leq \theta \leq -0.1745$ or $10° \leq \theta \leq 60°$.

Finally,

$$\sin\theta \approx \begin{cases} 0.8558\theta - 0.02516 & \theta \leq -0.1745 \\ \theta & -0.1745 \leq \theta \leq 0.1745 \\ 0.8558\theta + 0.02516 & \theta \geq 0.1745 \end{cases} \tag{5.1}$$

$$\cos\theta \approx \begin{cases} 0.4957\theta + 1.0865 & \theta \leq -0.1745 \\ 1 & -0.1745 \leq \theta \leq 0.1745 \\ -0.4957\theta + 1.0865 & \theta \geq 0.1745 \end{cases} \tag{5.2}$$

- Substitute the approximated functions from (5.1) and (5.2) shown below into (4.4) and (4.5).

$$\sin\alpha \approx a_{1i}\alpha + b_{1i} \qquad \cos\alpha \approx a_{2i}\alpha + b_{2i}$$
$$\sin\varphi \approx a_{3i}\varphi + b_{3i} \qquad \cos\varphi \approx a_{4i}\varphi + b_{4i}$$

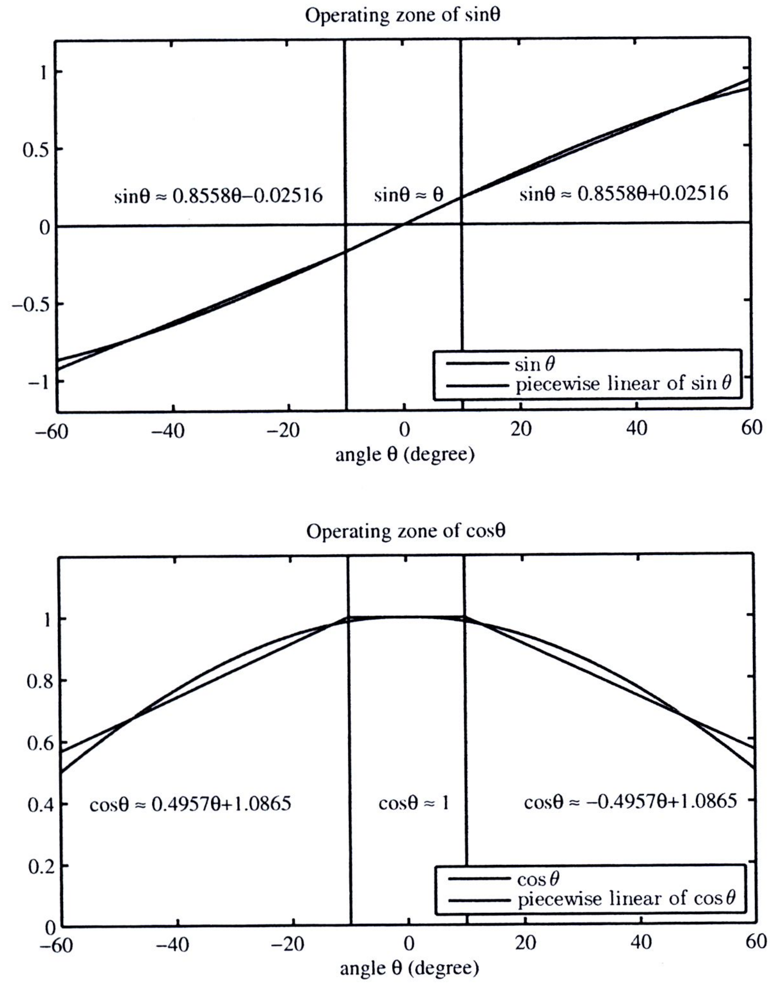

Figure 5.2: Affine approximation of functions sin and cos.

where $i = 1, \ldots, 9$ indicated the region of approximation.

- Approximate the higher order terms and other nonlinear terms of $\dot{\alpha}$ and $\dot{\varphi}$ based on linearization about the operating point $(\alpha, \dot{\alpha}, \varphi, \dot{\varphi}) = (0, 0, 0, 0)$ into the state-space form (2.7) where

$$
\begin{aligned}
A_{31}^{(i)} &= (K_9^{(i)} + K_{11}^{(i)} - K_3^{(i)} - K_7^{(i)})/K_1^{(i)} \\
A_{32}^{(i)} &= -(K_4^{(i)} + K_6^{(i)})/K_1^{(i)} \\
A_{33}^{(i)} &= 0 \\
A_{34}^{(i)} &= -(K_2^{(i)} + K_5^{(i)})/K_1^{(i)} \\
a_{3a}^{(i)} &= (K_{10}^{(i)} - K_8^{(i)} - K_{22}^{(i)})/K_1^{(i)}
\end{aligned}
$$

$$
\begin{aligned}
A_{41}^{(i)} &= -(K_{14}^{(i)} + K_{18}^{(i)})/k_5 \\
A_{42}^{(i)} &= -(K_{21}^{(i)} + K_{15}^{(i)})/k_5 \\
A_{43}^{(i)} &= -(K_{13}^{(i)} + K_{17}^{(i)} + K_{20}^{(i)})/k_5 \\
A_{44}^{(i)} &= 0 \\
a_{4a}^{(i)} &= -(K_{16}^{(i)} + K_{19}^{(i)})/k_5 \\
B_{41}^{(i)} &= 1/k_5
\end{aligned}
$$

Table 5.1: Approximated trigonometric functions in each polyhedral cell.

$\alpha \uparrow$ \ $\varphi \rightarrow$	$\varphi < -10^\circ$	$-10^\circ \le \varphi \le 10^\circ$	$\varphi > 10^\circ$
$\alpha > 10^\circ$	$\sin\alpha \approx 0.8558\alpha + 0.02516$ $\cos\alpha \approx -0.4957\alpha + 1.0865$ $\sin\varphi \approx 0.8558\varphi - 0.02516$ $\cos\varphi \approx 0.4957\varphi + 1.0865$	$\sin\alpha \approx 0.8558\alpha + 0.02516$ $\cos\alpha \approx -0.4957\alpha + 1.0865$ $\sin\varphi \approx \varphi$ $\cos\varphi \approx 1$	$\sin\alpha \approx 0.8558\alpha + 0.02516$ $\cos\alpha \approx -0.4957\alpha + 1.0865$ $\sin\varphi \approx 0.8558\varphi + 0.02516$ $\cos\varphi \approx -0.4957\varphi + 1.0865$
$-10^\circ \le \alpha \le 10^\circ$	$\sin\alpha \approx \alpha$ $\cos\alpha \approx 1$ $\sin\varphi \approx 0.8558\varphi - 0.02516$ $\cos\varphi \approx 0.4957\varphi + 1.0865$	$\sin\alpha \approx \alpha$ $\cos\alpha \approx 1$ $\sin\varphi \approx \varphi$ $\cos\varphi \approx 1$	$\sin\alpha \approx \alpha$ $\cos\alpha \approx 1$ $\sin\varphi \approx 0.8558\varphi + 0.02516$ $\cos\varphi \approx -0.4957\varphi + 1.0865$
$\alpha < -10^\circ$	$\sin\alpha \approx 0.8558\alpha - 0.02516$ $\cos\alpha \approx 0.4957\alpha + 1.0865$ $\sin\varphi \approx 0.8558\varphi - 0.02516$ $\cos\varphi \approx 0.4957\varphi + 1.0865$	$\sin\alpha \approx 0.8558\alpha - 0.02516$ $\cos\alpha \approx 0.4957\alpha + 1.0865$ $\sin\varphi \approx \varphi$ $\cos\varphi \approx 1$	$\sin\alpha \approx 0.8558\alpha - 0.02516$ $\cos\alpha \approx 0.4957\alpha + 1.0865$ $\sin\varphi \approx 0.8558\varphi + 0.02516$ $\cos\varphi \approx -0.4957\varphi + 1.0865$

$$
\begin{aligned}
K_1^{(i)} &= (k_9 + k_4 b_{2i}^2 + k_6 b_{1i}^2) & K_2^{(i)} &= \dot{\psi} b_{4i}(k_{10}(b_{1i}^2 - b_{2i}^2) - k_5) \\
K_3^{(i)} &= -\dot{\psi}^2(a_{3i}b_{4i} + a_{4i}b_{3i})(k_{11} - k_4 b_{1i}^2 - k_6 b_{2i}^2) & K_4^{(i)} &= -\dot{\psi}^2 b_{3i} b_{4i}(-2k_4 a_{1i} b_{1i} - 2k_6 a_{2i} b_{2i}) \\
K_5^{(i)} &= \Omega I_{Gzz} b_{2i} & K_6^{(i)} &= \dot{\psi}\Omega I_{Gzz}(a_{2i}b_{3i}) \\
K_7^{(i)} &= \dot{\psi}\Omega I_{Gzz}(a_{3i}b_{2i}) & K_8^{(i)} &= \dot{\psi}\Omega I_{Gzz}(b_{3i}b_{2i}) \\
K_9^{(i)} &= k_7 g a_{3i} & K_{10}^{(i)} &= k_7 g b_{3i} \\
K_{11}^{(i)} &= k_7 \sigma \dot{\psi} a_{4i} & K_{12}^{(i)} &= k_7 \sigma \dot{\psi} b_{4i} \\
K_{13}^{(i)} &= k_5 \dot{\psi} b_{4i} & K_{14}^{(i)} &= -k_{10}\dot{\psi}^2(2a_{4i}b_{4i}b_{1i}b_{2i}) \\
K_{15}^{(i)} &= -k_{10}\dot{\psi}^2(a_{1i}b_{2i} + a_{2i}b_{1i})b_{4i}^2 & K_{16}^{(i)} &= -k_{10}\dot{\psi}^2 b_{1i}b_{2i}b_{4i}^2 \\
K_{17}^{(i)} &= -k_{10}\dot{\psi} b_4(b_{1i}^2 - b_{2i}^2) & K_{18}^{(i)} &= \Omega\dot{\psi} I_{Gzz}(a_{4i}b_{1i}) \\
K_{19}^{(i)} &= \Omega\dot{\psi} I_{Gzz}(b_{1i}b_{4i}) & K_{20}^{(i)} &= -\Omega I_{Gzz} b_{2i} \\
K_{21}^{(i)} &= \Omega\dot{\psi} I_{Gzz}(a_{1i}b_{4i}) & K_{22}^{(i)} &= -\dot{\psi}^2 b_{3i}b_{4i}(k_{11} - k_4 b_{1i}^2 - k_6 b_{2i}^2)
\end{aligned}
$$

- Substitute the bicycle parameters in the Table 4.1 and get the resulting system matrices

5.2 Least-Square Error Approximation without Boundary Constraints

This approximation method gives a discontinuous model at the cell boundaries since the error is forced to be minimized while nothing concerning with the boundary constraints are taken into account. To approximate the nonlinear terms of $\ddot{\alpha}$ and $\ddot{\varphi}$ into a state-space form, we formulate the least square problem from the proposed approximated linear model :

$$\ddot{y}_{N\times1} = G_{N\times m}\theta_{m\times1} + \mu_{N\times1} \tag{5.3}$$

where

$$
\begin{aligned}
G &= \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & 1 \end{bmatrix} \\
\theta_\varphi &= \begin{bmatrix} A_{31}^{(i)} & A_{32}^{(i)} & A_{33}^{(i)} & A_{34}^{(i)} & a_3^{(i)} \end{bmatrix}^T \text{ or} \\
\theta_\alpha &= \begin{bmatrix} A_{41}^{(i)} & A_{42}^{(i)} & A_{43}^{(i)} & A_{44}^{(i)} & a_4^{(i)} \end{bmatrix}^T
\end{aligned}
$$

$\ddot{y}$ is the exact value of $\ddot{\varphi}$ or $\ddot{\alpha}$ obtained from the bicycle dynamic equation (4.4) and (4.5)
μ is the approximation error
x_k is the k^{th} vector containing N realizations of a uniform random variable in the range $[x_{k_{min}}, x_{k_{max}}]$ in each X_i
N is the number of realization (higher is better)
m is the number the state plus a single affine term

Then we can present the problem as

$$\hat{\theta}^{(i)} = \underset{\theta^{(i)}}{\operatorname{argmin}} \parallel \ddot{y} - G^{(i)}\theta^{(i)} \parallel_2^2 \tag{5.4}$$

Solving (5.4) for each cell, we will get all 9 sets of system matrices of the bicycle PWA model.

5.3 Least-Square Error Approximation with Boundary Constraints

This model is continuous across the boundary. We carefully begin an approximation with the cell $X_5 \in I_0$ in order to made this cell the most accurate. The benefit is that there is no constraint for model continuity at the first approximation in X_5. When the first cell has already been placed, it introduces one more boundary constraint at its attached polyhedral cell. This is in case II and in the same manner for more constraints in case III.

- *Case I: No constraint*

  Formulate the least square problem (5.4) with the same methodology for the operating-point region X_5. The closed form solution is

$$\hat{\theta}^{(5)} = (G^{(5)T}G^{(5)})^{-1}G^{(5)T}\ddot{y} \tag{5.5}$$

- *Case II: One constraint*

  One constraint of the problem is appeared when the approximation is done in the nearby region of X_5 i.e. X_2, X_4, X_6, X_8. Consider an example of X_6, the continuity the model at boundary $x_1 = \gamma$ connecting X_5 to X_6. The solution for $\hat{\theta}^{(6)}$ can be obtained by solving the following problem

$$
\begin{aligned}
&\text{minimize} && \parallel \ddot{y} - G^{(6)}\theta^{(6)} \parallel_2^2 \\
&\text{subject to} && G_\gamma\theta^{(6)} = G_\gamma\theta^{(5)}
\end{aligned} \tag{5.6}
$$

  where $G_\gamma = \begin{bmatrix} \gamma & x_2 & x_3 & x_4 & 1 \end{bmatrix}$

  For the rest of X_5 connected regions X_2, X_4, X_6, X_8, the approximation is applied in the similar fashion.

- *Case III: Two constraints*

  Two constraints are taken into account when an approximation is done in the region X_1, X_3, X_7, X_9. Consider the continuity of the model in X_3 at the boundary $x_1 = \gamma$ that connects the region X_2 and X_3 and the boundary $x_2 = \beta$ that connects to the region X_6 and X_3, the problem can be written in this form

$$\begin{aligned} \text{minimize} \quad & \| \ddot{y} - G^{(3)}\theta^{(3)} \|_2^2 \\ \text{subject to} \quad & G_\gamma \theta^{(3)} = G_\gamma \theta^{(2)} \\ & G_\beta \theta^{(3)} = G_\beta \theta^{(6)} \end{aligned} \tag{5.7}$$

$$\text{where} \quad \begin{aligned} G_\gamma &= \begin{bmatrix} \gamma & x_2 & x_3 & x_4 & 1 \end{bmatrix} \\ G_\beta &= \begin{bmatrix} x_1 & \beta & x_3 & x_4 & 1 \end{bmatrix} \end{aligned}$$

In 5.2 and 5.3, the range $[x1_{min}, x1_{max}]$ and $[x2_{min}, x2_{max}]$ are defined upon the region X_i. For the angular velocities as represented by x_3 and x_4, there is no partition region given. Hence, we assign an operating point $(0, 0)$ for them. The approximation of x_1 and x_2 will be varied in each region but x_3 and x_4 will be fixed at $(0, 0)$ which its resulting models are like the linearisation model around this point.

The example of system matrices in each region after substituting constant parameters are shown in the next pages. They are calculated by the assumption that the bicycle rotating velocity and forward velocity are very small and no disturbance force ($F_d = 0$) in the system. Also, the constant terms $\dot{\psi} = 0.01$ rad/s, $\sigma = 0$ m/s, $\Omega = 3000$ rpm $= 314.16$ rad/s, and other values from Table 4.1 are also included.

Trigonometric terms approximation model

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.291 & -0.0006 & 0 & -5.3011 \\ -0.0778 & -5.8 & 677.723 & 0 \end{bmatrix} \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ -0.2732 \\ -0.1705 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8566 & 0 & 0 & -5.3011 \\ 0 & -5.3383 & 677.7228 & 0 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -0.1569 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.291 & 0.0006 & 0 & -5.3011 \\ 0.0778 & -5.8 & 677.723 & 0 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ 0.2732 \\ -0.1705 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3079 & 0 & 0 & -4.886 \\ 0 & -6.7773 & 623.7653 & 0 \end{bmatrix} \quad a_4 = \begin{bmatrix} 0 \\ 0 \\ -0.2736 \\ 0 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8762 & 0 & 0 & -4.886 \\ 0 & -6.2378 & 623.7654 & 0 \end{bmatrix} \quad a_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3079 & 0 & 0 & -4.886 \\ 0 & -6.7773 & 623.7653 & 0 \end{bmatrix} \quad a_6 = \begin{bmatrix} 0 \\ 0 \\ 0.2736 \\ 0 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.291 & 0.0006 & 0 & -5.3011 \\ 0.0778 & -5.8 & 677.723 & 0 \end{bmatrix} \quad a_7 = \begin{bmatrix} 0 \\ 0 \\ -0.2732 \\ 0.1705 \end{bmatrix} \quad B_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8566 & 0 & 0 & -5.3011 \\ 0 & -5.3383 & 677.7228 & 0 \end{bmatrix} \quad a_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0.1569 \end{bmatrix} \quad B_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.291 & -0.0006 & 0 & -5.3011 \\ -0.0778 & -5.8 & 677.723 & 0 \end{bmatrix} \quad a_9 = \begin{bmatrix} 0 \\ 0 \\ 0.2732 \\ 0.1705 \end{bmatrix} \quad B_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$C_i = 0 \qquad c_i = 0 \qquad D_i = 0$$

$$I_0 = \{5\} \qquad I_1 = \{1, 2, 3, 4, 6, 7, 8, 9\}$$

Least-square error approximation without boundary constraints model – Discontinuous model

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7185 & 0.0357 & -0.0015 & -3.862 \\ -1.6106 & -3.7379 & 494.975 & 0.1775 \end{bmatrix} \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ -0.7247 \\ -1.6888 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8284 & 0.0011 & 0.0053 & -3.87 \\ 0.717 & -5.586 & 495.2182 & -0.671 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ 0.0026 \\ 0.0386 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7356 & -0.019 & -0.0014 & -3.8658 \\ 0.5886 & -3.0051 & 495.1564 & 0.1435 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ 0.7089 \\ -1.2207 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7439 & -0.0007 & -0.0001 & -4.8606 \\ -0.0724 & -4.9344 & 620.5933 & -0.0131 \end{bmatrix} \quad a_4 = \begin{bmatrix} 0 \\ 0 \\ -0.7004 \\ -0.055 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8427 & 0.0001 & 0.0001 & -4.861 \\ -0.0354 & -6.0111 & 620.6116 & -0.0106 \end{bmatrix} \quad a_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7377 & -0.0044 & -0.0004 & -4.8619 \\ -0.0269 & -4.9932 & 620.6132 & 0.0017 \end{bmatrix} \quad a_6 = \begin{bmatrix} 0 \\ 0 \\ 0.7045 \\ 0.0196 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7356 & -0.0484 & 0.002 & -3.8671 \\ 1.4033 & -5.4668 & 494.6352 & -0.2225 \end{bmatrix} \quad a_7 = \begin{bmatrix} 0 \\ 0 \\ -0.7257 \\ 0.435 \end{bmatrix} \quad B_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.7681 & 0.014 & 0.0001 & -3.8668 \\ 6.6803 & -3.2871 & 494.709 & -0.0058 \end{bmatrix} \quad a_8 = \begin{bmatrix} 0 \\ 0 \\ 0.0061 \\ 0.7354 \end{bmatrix} \quad B_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 8.7063 & 0.014 & -0.0018 & -3.8661 \\ -2.721 & -6.1022 & 494.6512 & 0.0637 \end{bmatrix} \quad a_9 = \begin{bmatrix} 0 \\ 0 \\ 0.7192 \\ 0.8622 \end{bmatrix} \quad B_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$C_i = 0$ $\qquad$ $c_i = 0$ $\qquad$ $D_i = 0$

$I_0 = \{5\}$ $\qquad$ $I_1 = \{1, 2, 3, 4, 6, 7, 8, 9\}$

Least-square error approximation with boundary constraints model – Continuous model

$$A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3186 & 0.0104 & 0.0001 & -4.861 \\ 0.011 & -6.2415 & 620.6116 & -0.0106 \end{bmatrix} \quad a_1 = \begin{bmatrix} 0 \\ 0 \\ -0.2678 \\ 0.0483 \end{bmatrix} \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8427 & 0.0104 & 0.0001 & -4.861 \\ -0.0354 & -6.2415 & 620.6116 & -0.0106 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 0 \\ -0.0018 \\ 0.0402 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3166 & 0.0104 & 0.0001 & -4.861 \\ 0.0088 & -6.2415 & 620.6116 & -0.0106 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ 0.2646 \\ 0.0325 \end{bmatrix} \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3186 & 0.0001 & 0.0001 & -4.861 \\ 0.011 & -6.0111 & 620.6116 & -0.0106 \end{bmatrix} \quad a_4 = \begin{bmatrix} 0 \\ 0 \\ -0.266 \\ 0.0081 \end{bmatrix} \quad B_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8427 & 0.0001 & 0.0001 & -4.861 \\ -0.0354 & -6.0111 & 620.6116 & -0.0106 \end{bmatrix} \quad a_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad B_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_6 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3166 & 0.0001 & 0.0001 & -4.861 \\ 0.0088 & -6.0111 & 620.6116 & -0.0106 \end{bmatrix} \quad a_6 = \begin{bmatrix} 0 \\ 0 \\ 0.2664 \\ -0.0077 \end{bmatrix} \quad B_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_7 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3186 & 0.013 & 0.0001 & -4.861 \\ 0.011 & -2.9629 & 620.6116 & -0.0106 \end{bmatrix} \quad a_7 = \begin{bmatrix} 0 \\ 0 \\ -0.2638 \\ 0.5401 \end{bmatrix} \quad B_7 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_8 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 10.8427 & 0.013 & 0.0001 & -4.861 \\ -0.0354 & -2.9629 & 620.6116 & -0.0106 \end{bmatrix} \quad a_8 = \begin{bmatrix} 0 \\ 0 \\ 0.0022 \\ 0.532 \end{bmatrix} \quad B_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$$A_9 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 9.3166 & 0.013 & 0.0001 & -4.861 \\ 0.0088 & -2.9629 & 620.6116 & -0.0106 \end{bmatrix} \quad a_9 = \begin{bmatrix} 0 \\ 0 \\ 0.2686 \\ 0.5243 \end{bmatrix} \quad B_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7.2464 \end{bmatrix}$$

$C_i = 0 \qquad c_i = 0 \qquad D_i = 0$

$I_0 = \{5\} \qquad I_1 = \{1, 2, 3, 4, 6, 7, 8, 9\}$

5.4 Comparison of Model Error

The rms errors are calculated from the 10,000 uniform random points within the respected region. The values are collected in the Table 5.4 and Table 5.4. The error in each partitioned region is shown in three dimensions plot in Figures 5.3, 5.4, 5.5, 5.6, 5.7, 5.8, 5.9, and 5.10. The model which has the highest to the lowest error are linearized model, trigonometric terms approximation model, continuous model, and discontinuous model, respectively. This happens to both the bicycle roll angle and flywheel precessing angle. The approximation yields a good result for partitioning the roll angle at $\pm 10°$. Partitioning for more regions will possibly reduce the error.

Table 5.2: Summary of the root-mean-square error of the approximated PWA model.

Bicycle angle

Model region	Linearized	Continuous	Discontinuous	trig. terms approx.
X_1	0.8168	0.2100	0.1537	0.2116
X_2	0.0026	0.0027	0.0036	0.0054
X_3	0.8147	0.2100	0.1528	0.2077
X_4	0.8037	0.2101	0.1533	0.2123
X_5	0.0015	0.0025	0.0013	0.0013
X_6	0.8036	0.2101	0.1534	0.2124
X_7	0.8175	0.2100	0.1532	0.2071
X_8	0.0026	0.0027	0.0070	0.0067
X_9	0.8169	0.2100	0.1539	0.2123
Average	0.5422	0.1409	0.1036	0.1419

Flywheel angle

Model region	Linearized	Continuous	Discontinuous	trig. terms approx.
X_1	1.4861	1.0376	0.3231	1.1844
X_2	0.6506	0.1398	0.2099	0.4688
X_3	1.4574	1.0376	0.4514	1.1873
X_4	0.1816	0.1880	0.0823	0.1117
X_5	0.0310	0.0053	0.0183	0.0183
X_6	0.1817	0.1880	0.0794	0.1118
X_7	1.4642	1.0376	0.4358	0.6338
X_8	0.6498	0.1398	0.9693	1.2216
X_9	1.4905	1.0376	0.6298	0.6384
Average	0.8437	0.5346	0.3555	0.6196

Table 5.3: Summary of the maximum absolute error of the approximated PWA model.

Bicycle angle

Model region	Linearized	Continuous	Discontinuous	trig. terms approx.
X_1	1.9624	0.6175	0.4319	0.6295
X_2	0.0114	0.0128	0.0117	0.0195
X_3	1.9663	0.6175	0.4512	0.6459
X_4	1.9333	0.6033	0.4396	0.6069
X_5	0.0038	0.0099	0.0041	0.0041
X_6	1.9332	0.6033	0.4379	0.6052
X_7	1.9651	0.6175	0.4476	0.6499
X_8	0.0115	0.0128	0.0138	0.0218
X_9	1.9612	0.6175	0.4359	0.6255
Maximum	1.9663	0.6175	0.4512	0.6499

Flywheel angle

Model region	Linearized	Continuous	Discontinuous	trig. terms approx.
X_1	4.1177	3.4618	1.2155	3.7983
X_2	1.4790	0.4272	0.6162	1.1821
X_3	4.0637	3.4618	1.0550	3.7934
X_4	0.6119	0.6413	0.3404	0.5110
X_5	0.0702	0.0220	0.0346	0.0346
X_6	0.6148	0.6413	0.3385	0.5091
X_7	4.0478	3.4618	1.9893	1.6790
X_8	1.4816	0.4272	2.3126	1.7674
X_9	4.1000	3.4618	1.7019	1.6914
Maximum	4.1177	3.4618	2.3126	3.7983

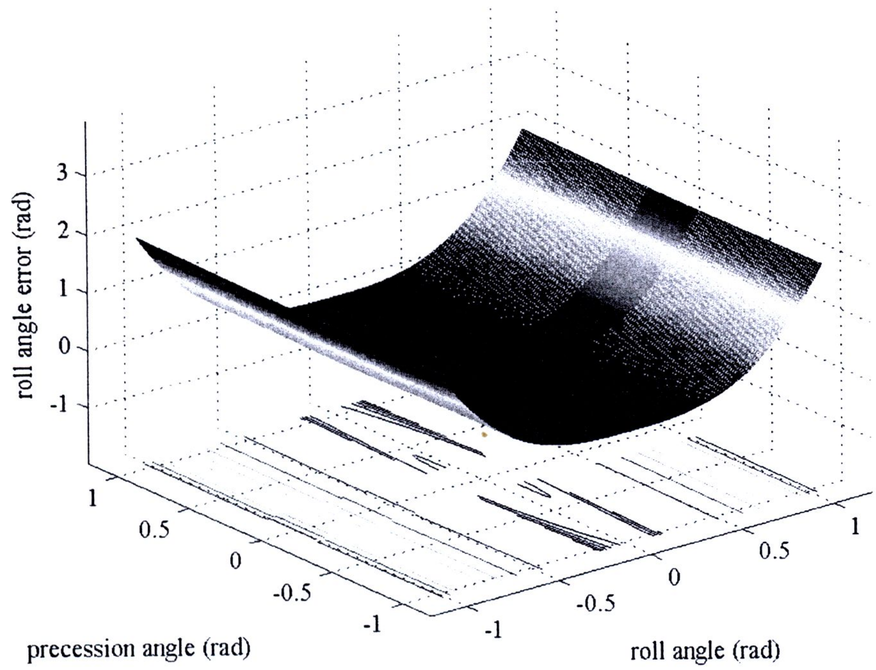

Figure 5.3: The roll angle error plane of the linearized model.

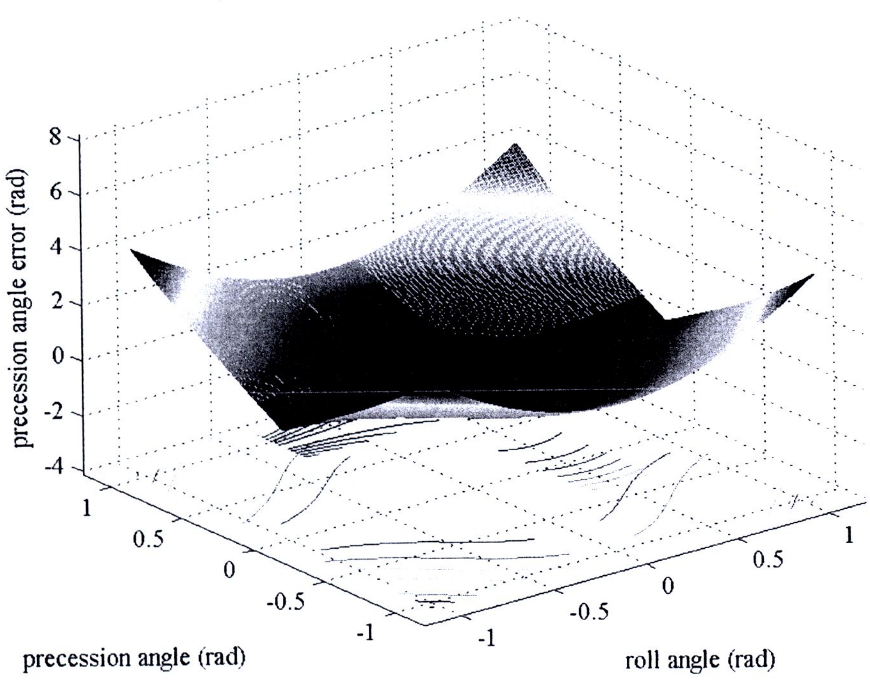

Figure 5.4: The precession angle error plane of the linearized model.

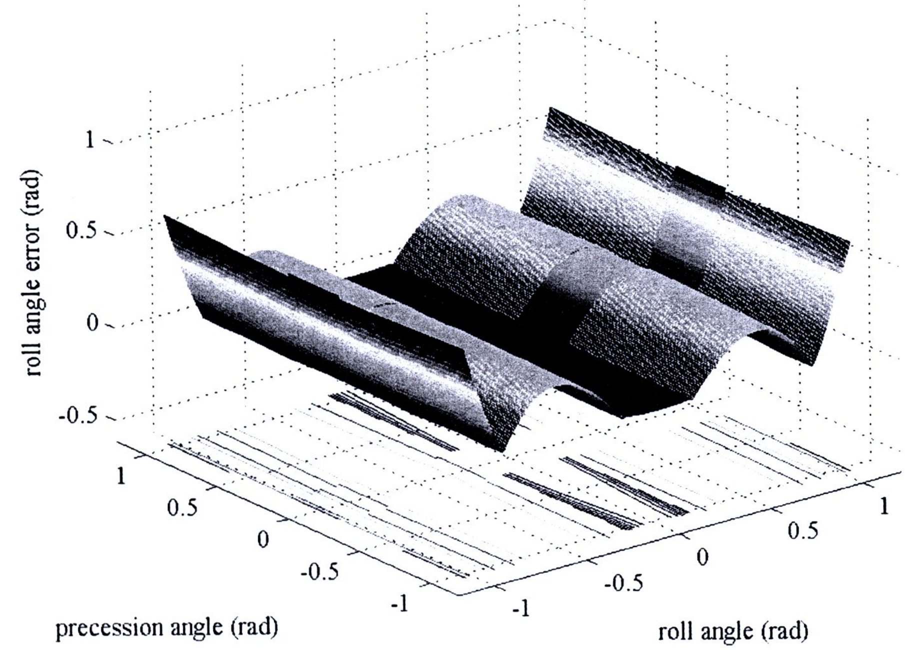

Figure 5.5: The roll angle error plane of the trigonometic terms approximation PWA model.

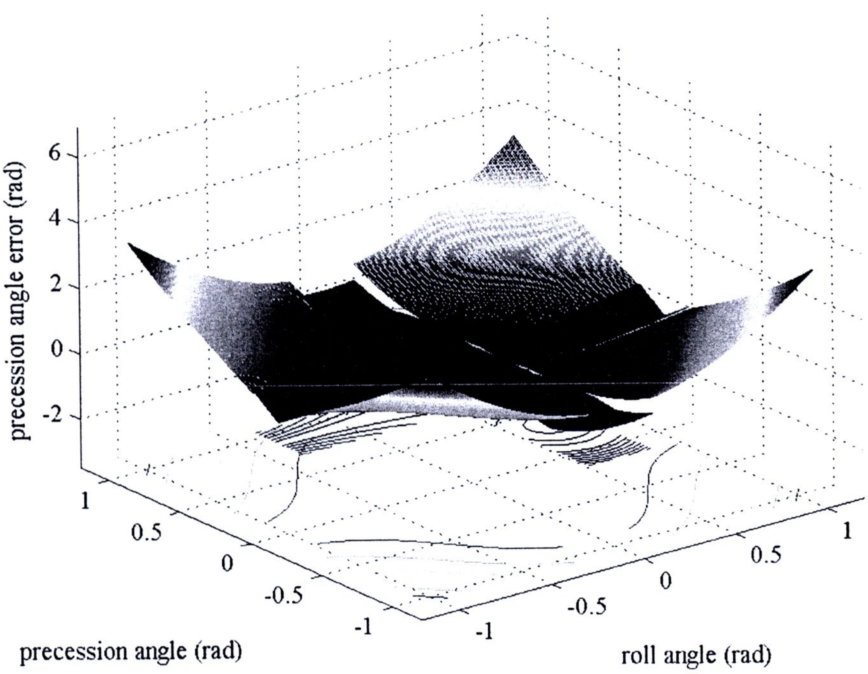

Figure 5.6: The precession angle error plane of the trigonometic terms approximation PWA model.

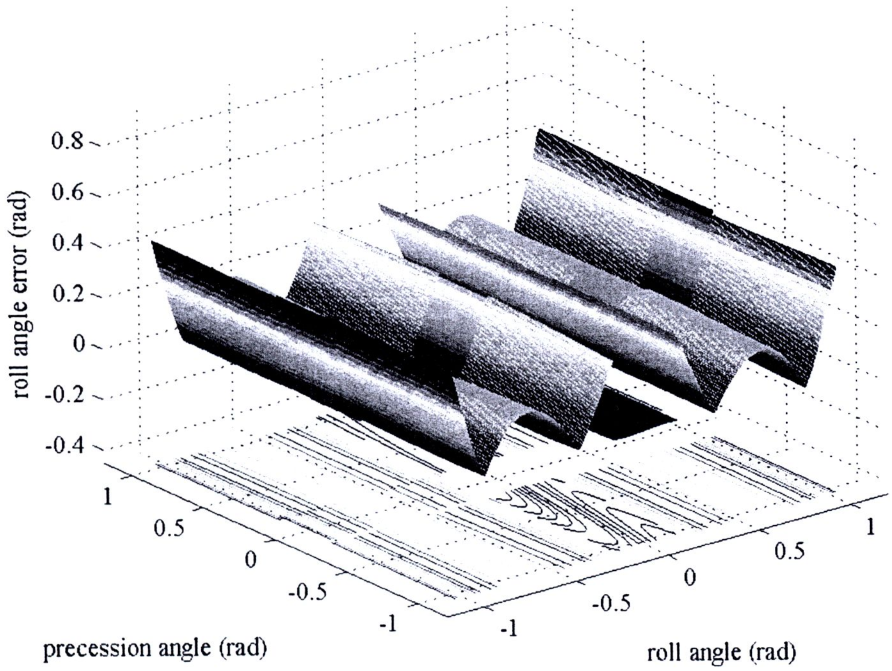

Figure 5.7: The roll angle error plane of the discontinuous PWA model.

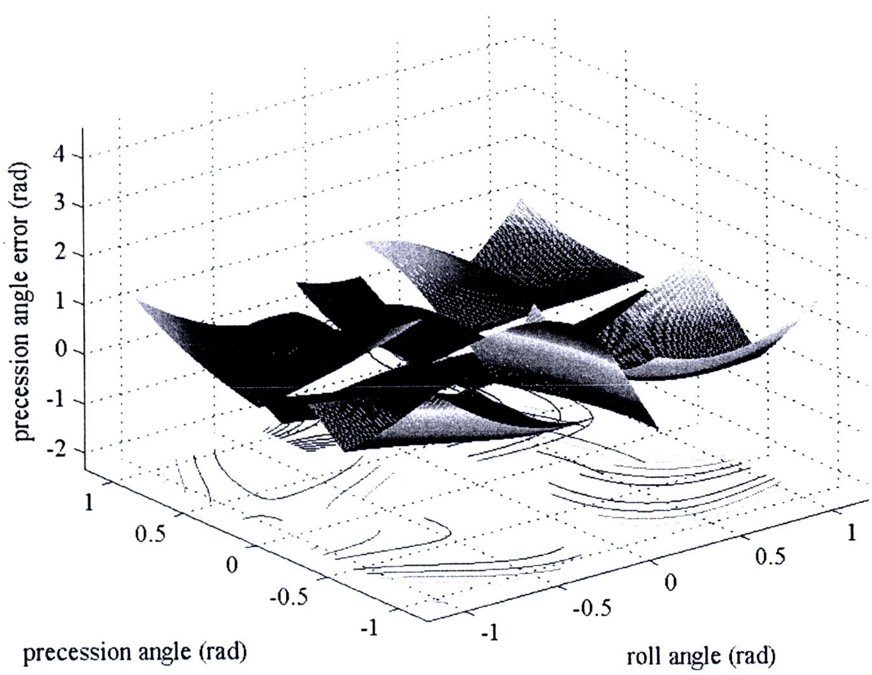

Figure 5.8: The precession angle error plane of the discontinuous PWA model.

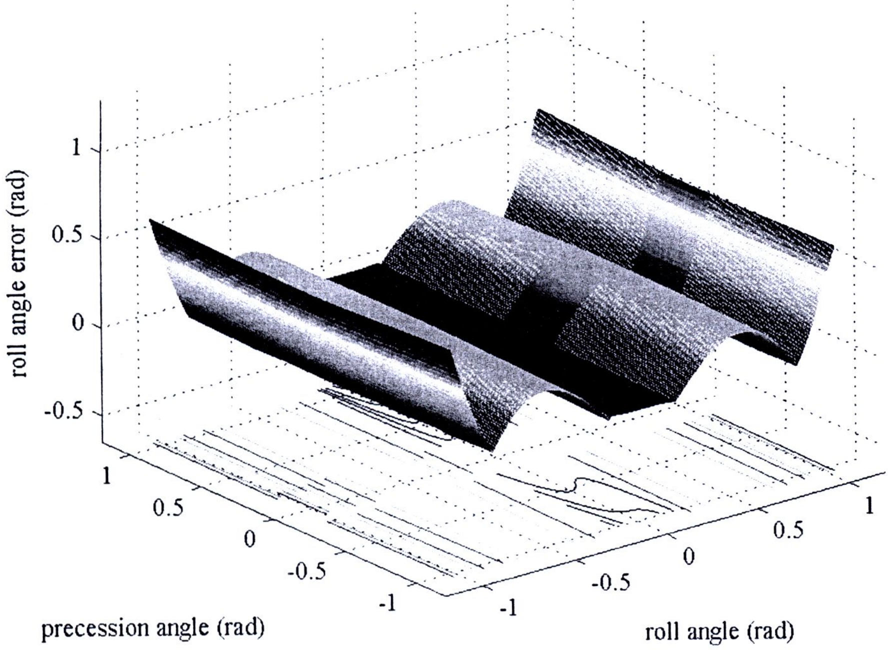

Figure 5.9: The roll angle error plane of the continuous PWA model.

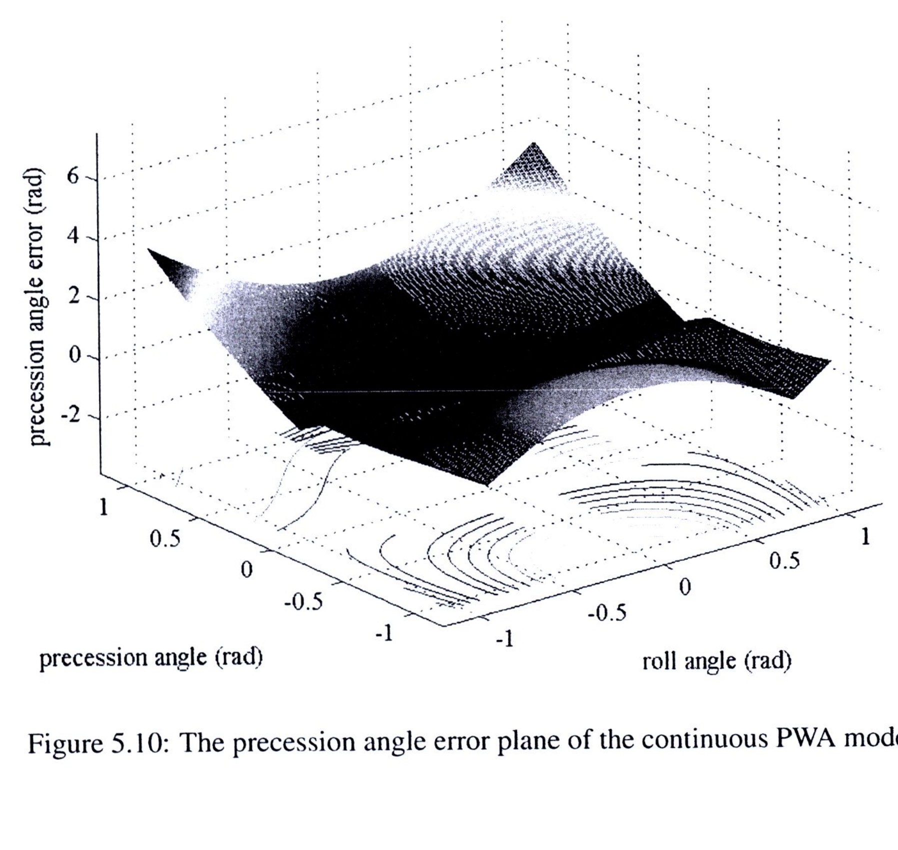

Figure 5.10: The precession angle error plane of the continuous PWA model.