CHAPTER IV

BICYCLE DYNAMIC MODEL

The equation of motion of 3D rigid body can be derived in 3 aspects. Those are the conservation of force (torque), momentum (angular momentum), and energy. The model of a bicycle with gyroscopic stabilization is mostly derived by the Lagrangian method (Energy aspect) because it is easy to obtain the linear and angular velocity while the internal force or any other workless forces can be ignored. From the literature review, we have inspected many types and complicated levels of the bicycle. We end up with the nonlinear dynamic model from Spry [2] and extend the model to PWA model.

We next define the bicycle geometry, the assumption and limitation of the model, the notation of the parameters and lastly the nonlinear model with neglecting relatively small-value terms.

4.1 Bicycle Geometry

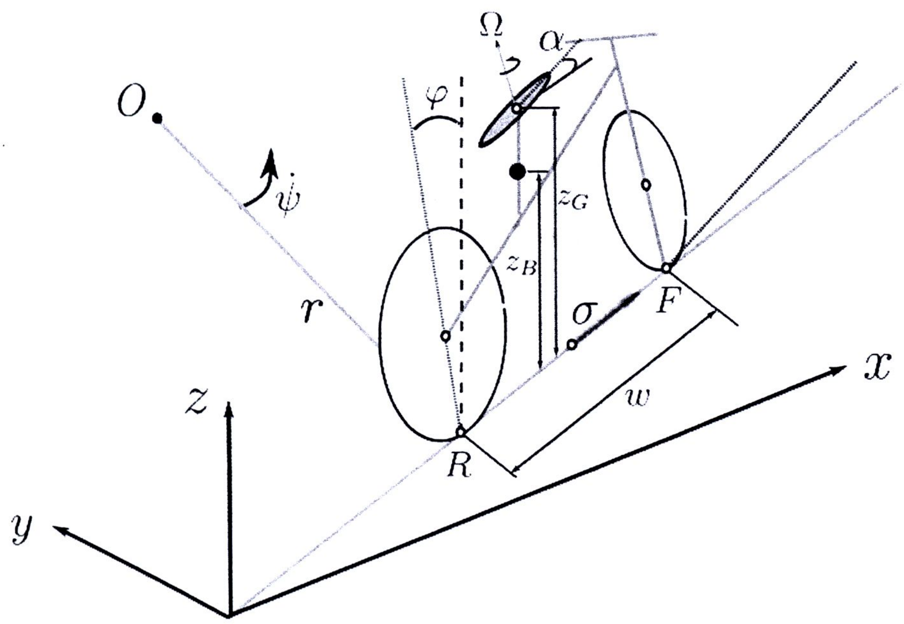

Figure 4.1: The Bicycle Geometry.

Parameter Definition

The parameter notations here are also consistent with the measured parameter in Tables 2.1 and 3.1. We present them separately to emphasize each component; the bicycle dimension, the flywheel (for design calculation), and the bicycle with gyroscopic flywheel model parameters. These are shown in Table 4.1. The constant mass, moment of inertia, and height of the center of mass are obtained via CATIA CAD software. These values in Table 4.1 may differ from Tables 2.1 and 3.1 since we consider the model here in two parts; the body and the flywheel. See more in the bicycle model assumption.

Table 4.1: Parameters for Bicycle Gyroscopic Flywheel Dynamic Model.

Parameter	Symbol	Value	Unit
Bike roll angle	φ	-	rad
Flywheel precession angle	α	-	rad
Bike rotation rate	$\dot{\psi}$	-	rad/s
Flywheel spinning rate	Ω	-	rad/s
Track radius curvature	r	-	m
The midpoint of track segment	s	-	m
The distance between s and wheelbase midpoint	h	-	m
The wheelbase midpoint speed	σ	-	m/s
Disturbance force	F_d	-	N
Bicycle body mass	m_B	30	kg
Flywheel mass	m_G	9	kg
Height of Bicycle center of mass	z_G	0.39	m
Height of Flywheel center of mass	z_B	0.88	m
Bike Moment of inertia	$(I_{Bxx}, I_{Byy}, I_{Bzz})$	(5.947,8.083,2.295)	kg$\cdot$ m^2
Flywheel Moment of inertia	$(I_{Gxx}, I_{Gyy}, I_{Gzz})$	(0.138,0.138,0.274)	kg$\cdot$ m^2

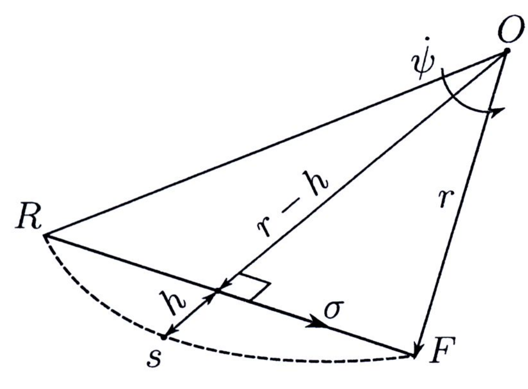

Figure 4.2: The bicycle curvature path.

To explain more about the curvature path of the bicycle, see Figure 4.2. In Figure, F is the front wheel ground contact point, R is the rear wheel ground contact point, and O is the center point of rotation. The distance between R and F is called "wheelbase length " (w). We can find the relation between $\dot{s}$ and $\dot{\psi}$ is $\sigma = \dot{s}(r-h)/r = \dot{\psi}(r-h)$. For straight path running, $r \to \infty$, $h = 0$ and $\sigma = \dot{s}$.

4.2 Model Assumptions

It is much more complex to treat the bicycle model as a 3D rigid body. The simplified model that captured the major effects on the bicycle and is well enough to describe the bicycle dynamics is a better choice. However, we should be careful to define the assumption and its limitation as shown below.

- The steering axis has no trail.
- The bicycle is rolling on a flat plane
- The tires has no width and no deformation.
- The longituditional and lateral slips at the front and rear wheel are neglected.
- The bicycle is considered as a point mass at the center of mass height z_B
- The flywheel is considered as a point mass at the center of mass height z_G
- The mass moment of inertia of the front and rear wheel are neglected.

4.2.1 Nonlinear Dynamic Model

The model derivation is done by Lagrangian method. We follow the derivation in [2] but we combine the load and flywheel cage into the bicycle body in one point mass. The kinetic energy of the system is

$$T = \frac{1}{2}m_B\mathbf{v}_B^T\mathbf{v}_B + \frac{1}{2}\omega_B^T\mathbf{I_B}\omega_B + \frac{1}{2}m_G\mathbf{v}_G^T\mathbf{v}_G + \frac{1}{2}\omega_G^T\mathbf{I_G}\omega_G \quad (4.1)$$

and the potential energy of the system is

$$V = (m_B z_B + m_G z_G)g\sin\varphi \quad (4.2)$$

where

$$\omega_B = \begin{bmatrix} \dot\varphi \\ \dot\psi\sin\varphi \\ \dot\psi\cos\varphi \end{bmatrix} \qquad \mathbf{v}_B = \begin{bmatrix} \sigma + (\dot\psi\sin\varphi)z_B \\ \dot\varphi z_B \\ 0 \end{bmatrix}$$

$$\omega_\mathbf{G} = \begin{bmatrix} \dot\varphi\cos\alpha - \dot\psi\cos\varphi\sin\alpha \\ \dot\psi\sin\varphi + \dot\alpha \\ \dot\varphi\sin\alpha + \dot\psi\cos\varphi\cos\alpha + \Omega \end{bmatrix} \qquad \mathbf{v}_G = \begin{bmatrix} \sigma + (\dot\psi\sin\varphi)z_G \\ \dot\varphi z_G \\ 0 \end{bmatrix}$$

From the Lagrangian $\mathcal{L}(q,\dot q) \equiv T(q,\dot q) - V(q)$, we can derive the Lagrange's equations in the form

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot q_k}\right) - \frac{\partial\mathcal{L}}{\partial q_k} = Q_k \quad (4.3)$$

where the generalized coordinates are

$$\begin{cases} q_1 & : \ \varphi \text{ (Bike roll angle)} \\ q_2 & : \ \alpha \text{ (Flywheel precession angle)} \end{cases}$$

and the generalized forces are

$$\begin{cases} Q_1 & = \ F_d z_B\cos\varphi \\ Q_2 & = \ T_\alpha \end{cases}$$

According to (4.3), the equation of motions are obtained as follow:

Bicycle rolling equation

$$\left.\begin{aligned}
&(k_9 + k_4\cos^2\alpha + k_6\sin^2\alpha)\ddot{\varphi}\\
&-2k_{10}\dot{\varphi}\dot{\alpha}\sin\alpha\cos\alpha\\
&+\dot{\psi}\dot{\alpha}\cos\varphi(k_{10}(\sin^2\alpha-\cos^2\alpha)-k_5)\\
&-\dot{\psi}^2\cos\varphi\sin\varphi(k_{11}-k_4\sin^2\alpha-k_6\cos^2\alpha)\\
&+(\Omega I_{Gzz}\cos\alpha)\dot{\alpha}\\
&+\dot{\psi}\Omega I_{Gzz}\cos\alpha\sin\varphi\\
&-k_7 g\sin\varphi
\end{aligned}\right\} = k_7\sigma\dot{\psi}\cos\varphi + F_d z_B\cos\varphi \tag{4.4}$$

Flywheel precessing equation

$$\left.\begin{aligned}
&k_5\ddot{\alpha}\\
&+k_5\dot{\psi}\dot{\varphi}\cos\varphi\\
&+k_{10}(\dot{\varphi}^2\cos\alpha\sin\alpha-\dot{\psi}^2\cos^2\varphi\cos\alpha\sin\alpha-\dot{\psi}\dot{\varphi}\cos\varphi(\sin^2\alpha-\cos^2\alpha))\\
&+\Omega(\dot{\psi}\cos\varphi\sin\alpha-\dot{\varphi}\cos\alpha)I_{Gzz}
\end{aligned}\right\} = T_\alpha \tag{4.5}$$

where

$$\begin{aligned}
k_1 &= I_{Bxx} & k_2 &= I_{Byy}\\
k_3 &= I_{Bzz} & k_4 &= I_{Gxx}\\
k_5 &= I_{Gyy} & k_6 &= I_{Gzz}\\
k_7 &= m_B z_B + m_G z_G & k_8 &= m_B z_B^2 + m_G z_G^2\\
k_9 &= k_1 + k_8 & k_{10} &= k_4 - k_6\\
k_{11} &= k_8 + k_2 - k_3 + k_5
\end{aligned}$$

4.3 Linearized Dynamic Model

The conventional simple way to deal with the nonlinear system is to linearize the nonlinear system around its equilibrium point. We will use this linearized model for a comparison with our reduced nonlinear in the next section. Let the state vector

$$x = \begin{bmatrix}\varphi & \alpha & \dot{\varphi} & \dot{\alpha}\end{bmatrix}^T$$

Linearize the nonlinear model (4.4) and (4.5) about $\mathbf{x} = 0$, then

$$(k_9 + k_4)\ddot{\varphi} - k_7\sigma\dot{\psi} + \Omega I_{Gzz}\dot{\alpha} + \Omega I_{Gzz}\dot{\psi}\varphi = k_7 g\varphi \tag{4.6}$$

$$k_5\ddot{\alpha} + \Omega(\dot{\psi}\alpha - \dot{\varphi})I_{Gzz} = T_\alpha \tag{4.7}$$

Rewrite the above two equations in a state space form

$$\frac{d}{dt}\begin{bmatrix}\varphi\\ \alpha\\ \dot{\varphi}\\ \dot{\alpha}\end{bmatrix} = \begin{bmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ a_{31} & 0 & 0 & a_{34}\\ 0 & a_{42} & a_{43} & 0\end{bmatrix}\begin{bmatrix}\varphi\\ \alpha\\ \dot{\varphi}\\ \dot{\alpha}\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \sigma\dot{\psi}k_7/(k_9+k_4)\\ T_\alpha/k_5\end{bmatrix} \tag{4.8}$$

where

$$a_{31} = \frac{k_7 g - \dot{\psi}\Omega I_{Gzz}}{k_9 + k_4} \qquad a_{34} = \frac{-\Omega I_{Gzz}}{k_9 + k_4}$$

$$a_{42} = -\frac{1}{k_5}(\dot{\psi}\Omega I_{Gzz}) \qquad a_{43} = \frac{\Omega I_{Gzz}}{k_5}$$