CHAPTER II

RELATED THEORIES

In this chapter, an overview of the fundamental theory used in modeling and designing PWA control systems is given.

2.1 Bicycle Properties

In this section, we describe the important properties of the bicycle that affect the stability of the bicycle.

2.1.1 Nature of the Bicycle

The bicycle is naturally unstable. When it stays upright, by no holding force, it will roll down left or right. However, it is not too hard to learn riding a bicycle by human. We turn the steering to the right when the bicycle seems to roll to the right side. It behaves the same manner for the left hand side. That is a mean of dynamic control of the bicycle.

2.1.2 The Trail

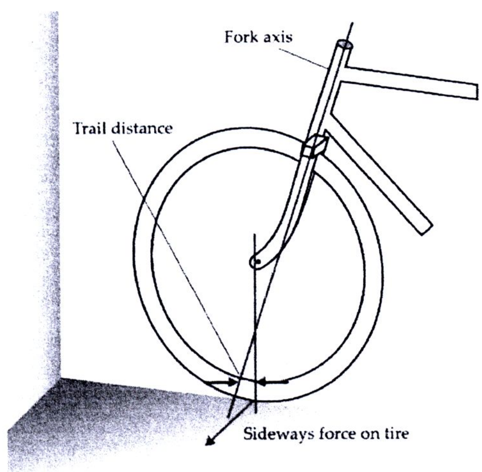

Figure 2.1: The position of the trail distance of the bicycle [48].

At the beginning, the bicycle has no trail or front fork. That means the handle bar axis is perpendicular to the ground. This type of bicycle has no effect of trail to the rolling angle when we steer the handlebar. The non-zero trail distance produce a major impact to the dynamics of the bicycle. D.E.H. Jones [48] studied this effect by constructing the bicycle with different kinds of trail distance. The interesting case is the positive trail which we are always familiar with. Positive trail provides a torque about the steering axis that counteracts angular momentum when the bike body leans to the left or the right. This counteracting torque causes the front wheel to turn in the direction opposite to the direction of lean, and thus enhances the stability of the bike. This torque does not appear only when the bicycle is moving but it is also generated when the bicycle is tilting.

2.1.3 Self-stability

Imagine when the bicycle is running forward along a road with non-zero speed and no maneuvering. We know that the bicycle is an unstable system. However, it was proved that the bicycle that has the positive front fork trail is itself stable in an interval of speed [15, 16, 21]. David E.H. Jones [48] emphasized that the steering geometry dramatically influences the stability. When the bicycle is tilting, its center of gravity is lower. Then, the front wheel steers to the tilting direction to minimize its gravitational potential energy. This will not occur if the trail is zero. In addition, Åström presented in another perspective. The ground reaction force exerts a torque on the front fork assembly to made the front fork steer. The analysis through the simple second order linearized model is discussed in [15].

It is essential to understand the self-stabilization behavior of the bicycle. To control the bicycle upright and running on a straight path, we do not need any inputs to stabilize the bicycle in a particular speed range unless we have a curvature path. Here, we show our analysis using the experimental bicycle parameters that we measure ourselves. The moment of inertias are retrieved via CATIA CAD-software. The analysis is based on the linear 4th-order equation (The Whipple model) in [21]. The equation of motion is

$$\mathbf{M}\begin{bmatrix}\ddot{\varphi}\\ \ddot{\beta}\end{bmatrix} + v\mathbf{C_1}\begin{bmatrix}\dot{\varphi}\\ \dot{\beta}\end{bmatrix} + (g\mathbf{K_0} + v^2\mathbf{K_2})\begin{bmatrix}\varphi\\ \beta\end{bmatrix} = \begin{bmatrix}T_\varphi\\ T_\beta\end{bmatrix} \tag{2.1}$$

With our real measured and CAD-program calculated parameters in Table 2.1, we have

$$\begin{aligned}\mathbf{M} &= \begin{bmatrix}8.6551 & 0.9466\\ 0.9466 & 0.3165\end{bmatrix} & \mathbf{C_1} &= \begin{bmatrix}0 & 9.3019\\ -0.7057 & 1.4885\end{bmatrix}\\ \mathbf{K_0} &= \begin{bmatrix}-14.7837 & -2.0400\\ -2.040 & -0.7164\end{bmatrix} & \mathbf{K_2} &= \begin{bmatrix}0 & 14.0139\\ 0 & 1.9743\end{bmatrix}\end{aligned} \tag{2.2}$$

The result is that the self-stable speed range is $3.60 < v < 10.26$ m/s. Note that the range is wider than a bicycle with the rider which has more weight.

We will take this advantage of self-stability to leave the steering bar move freely when we want the bicycle to run on a straight path at that particular speed range. Also, it is not necessary to control the roll angle by precessing the gyroscopic flywheel when the bicycle is self-stabilized. The explanation about how to control the bicycle roll angle will be discuss in the section 4.

2.1.4 Gyroscopic Effect at the Front Wheel

The gyroscopic action at the front wheel affect the stability of the bicycle. In Figure 2.3, we assume the bicycle is running with forward speed. According to our earth fixed coordinate frame, the spinning axis is perpendicular to the direction of the bicycle and have a positive ω_{speed}. To say, it points to the same direction as y-axis. Next, when we steer the handlebar to the left, ω_{steer} vector points vertically with the z-axis. This will result to the bicycle to roll to the right side. The rolling direction can be found mathematically by $\tau_{\text{roll}} = (I_{s2}\omega_{\text{speed}}\mathbf{e}_2) \times \omega_{\text{steer}}\mathbf{e}_3 = \omega_{\text{roll}}\mathbf{e}_1$ where I_{s2} is the moment of inertia of the steering handlebar with respect to the principal axis $\mathbf{e}_2$.

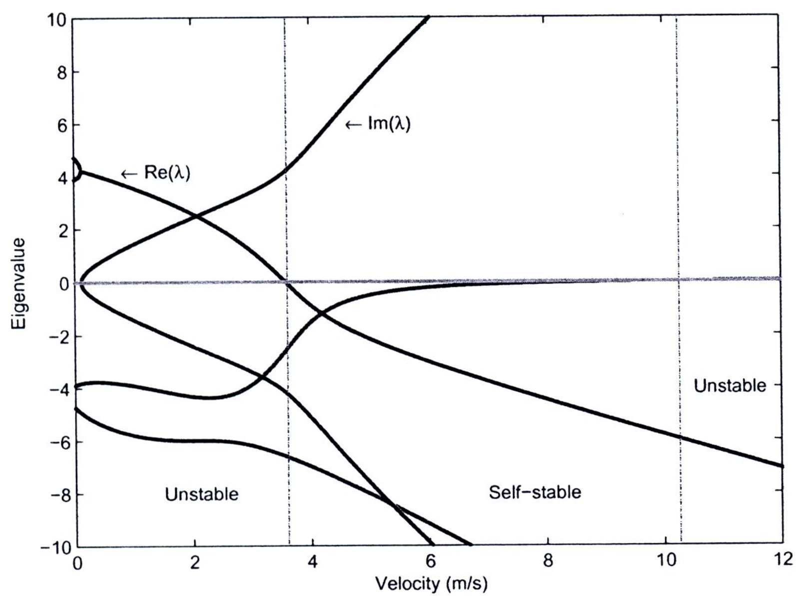

Figure 2.2: Eigenvalues from the linearized self-stability analysis.

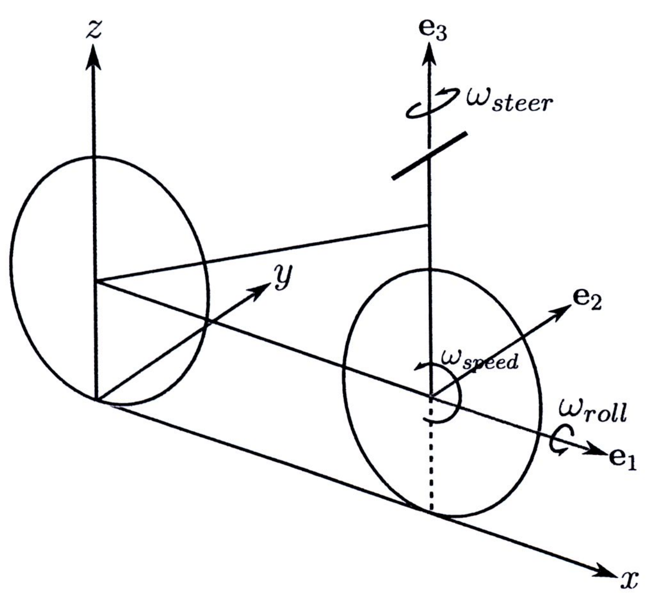

Figure 2.3: Gyroscopic effect at the front wheel coordinate and notation.

Table 2.1: Parameters of the Experimental Bicycle for Self-stability Analysis.

Parameter	**Symbol**	**Value**	**Unit**
Wheelbase	w	1.07	m
Trail	c	0.12	m
Front fork angle	λ	20.56	degree
Front wheel			
Mass	m_F	2.234	kg
Mass center	(x_F, y_F, z_F)	(1.07,0,0.32)	m
Radius	r_F	0.32	m
Moment of inertia	$\begin{bmatrix} I_{Fxx} & I_{Fxy} & I_{Fxz} \\ I_{Fyx} & I_{Fyy} & I_{Fyz} \\ I_{Fzx} & I_{Fzy} & I_{Fzz} \end{bmatrix}$	$\begin{bmatrix} 0.099 & 0 & 0 \\ 0 & 0.197 & 0 \\ 0 & 0 & 0.099 \end{bmatrix}$	kg· m^2
Rear wheel			
Mass	m_R	2.234	kg
Mass center	(x_R, y_R, z_R)	(0,0,0.32)	m
Radius	r_R	0.32	m
Moment of inertia	$\begin{bmatrix} I_{Rxx} & I_{Rxy} & I_{Rxz} \\ I_{Ryx} & I_{Ryy} & I_{Ryz} \\ I_{Rzx} & I_{Rzy} & I_{Rzz} \end{bmatrix}$	$\begin{bmatrix} 0.099 & 0 & 0 \\ 0 & 0.197 & 0 \\ 0 & 0 & 0.099 \end{bmatrix}$	kg· m^2
Body (with battery)			
Mass	m_B	30	kg
Mass center	(x_B, y_B, z_B)	(0.49,0,0.39)	m
Moment of inertia	$\begin{bmatrix} I_{Bxx} & I_{Bxy} & I_{Bxz} \\ I_{Byx} & I_{Byy} & I_{Byz} \\ I_{Bzx} & I_{Bzy} & I_{Bzz} \end{bmatrix}$	$\begin{bmatrix} 1.995 & 0 & -0.053 \\ 0 & 2.347 & 0 \\ -0.053 & 0 & 0.487 \end{bmatrix}$	kg· m^2
Front fork & Handlebar			
Mass	m_H	2.148	kg
Mass center	(x_H, y_H, z_H)	(0.92,0,0.77)	m
Moment of inertia	$\begin{bmatrix} I_{Hxx} & I_{Hxy} & I_{Hxz} \\ I_{Hyx} & I_{Hyy} & I_{Hyz} \\ I_{Hzx} & I_{Hzy} & I_{Hzz} \end{bmatrix}$	$\begin{bmatrix} 0.168 & 0 & 0.023 \\ 0 & 0.132 & 0 \\ 0.023 & 0 & 0.047 \end{bmatrix}$	kg· m^2
Gyroscopic Flywheel			
Mass	m_G	9.0329	kg
Mass center	(x_G, y_G, z_G)	(0.49,0,0.88)	m
Moment of inertia	$\begin{bmatrix} I_{Gxx} & I_{Gxy} & I_{Gxz} \\ I_{Gyx} & I_{Gyy} & I_{Gyz} \\ I_{Gzx} & I_{Gzy} & I_{Gzz} \end{bmatrix}$	$\begin{bmatrix} 0.138 & 0 & 0 \\ 0 & 0.138 & 0 \\ 0 & 0 & 0.274 \end{bmatrix}$	kg· m^2

Nevertheless, the effect on the bicycle is very small compared to the gravitational torque and gyroscopic flywheel unless the wheel spinning speed is very high. Our model therefore neglects this effect.

2.2 Lagrangian Mechanics

Lagrangian mechanics is a re-formulation of classical mechanics that combines conservation of momentum with conservation of energy [49]. The Lagrangian is an efficient method to derive the equation of motion through the energy aspect. The Lagrangian function $\mathcal{L}$ is defined as

$$\mathcal{L}(q, \dot{q}) \equiv T(q, \dot{q}) - V(q)$$

where T is the Kinetic energy, V is the Potential energy, and q is the generalized coordinate. According to the derivation in [50], the result Lagrangian's equations is then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{L}}{\partial q_k} = Q_k^{(nc)}; \qquad k = 1, 2, \ldots, M \tag{2.3}$$

where Q_k^{nc} is the nonconservative generalized forces.

The Kinetic energy of the rigid body can be calculated by

$$T = \frac{1}{2} m \mathbf{v}_c \cdot \mathbf{v}_c + \frac{1}{2} \mathbf{H}_c \cdot \omega \tag{2.4}$$

or in the matrix form

$$T = \frac{1}{2} m \mathbf{v}_c^T \mathbf{v}_c + \frac{1}{2} \omega^T \mathbf{I}_c \omega \tag{2.5}$$

where $\mathbf{v}_c$ is the linear velocity of the rigid body, ω is the angular velocity about its mass center, $\mathbf{H}_c$ is the angular momentun about the mass center, and $\mathbf{I}_c$ is the moment of inertia of the rigid body.

The Potential energy may be caused by gravitational force, elastic spring force, elastic force between two charges, etc. It can be represented as

$$\mathbf{F} = -\nabla V(\mathrm{r}) \tag{2.6}$$

In this thesis, the instrumental force for the potential energy is from the gravity near the Earth's surface. It is given by

$$\mathbf{F} = -mg\mathbf{e}_z, \qquad V = mgz$$

2.3 Piecewise Affine System

The Piecewise Affine system is a kind of nonlinear system which is linear in each local cell/partition where each partition has its own dynamics. The fascinating advantage of this type of control is that it is linear, however in a region, but provides more accuracy than a linearized model and the controller synthesis based on Piecewise Quadratic Lyapunov function is global. One time solving a batch LMIs problem, the obtained gain can be used to stabilize the system in overall operating point.

2.3.1 Model Representation

Consider piecewise affine systems on the form

$$\begin{cases} \dot{x}(t) = A_i x(t) + a_i + B_i u(t) \\ y(t) = C_i x(t) + c_i + D_i u(t) \end{cases} \quad x \in X_i, i \in I \tag{2.7}$$

Here, $x(t)$ is the continuous state vector, $u(t)$ is an exogenous signal (control or disturbance, depending on the context), $\{X_i\}_{i \in I} \subseteq \mathbf{R}^n$ is a partition of the state space into a number of closed polyhedral cells and I is the set of cell indices. Assume that the cells have disjoint interior (so that any two cells can only share a common boundary) and that they form a partition of some compact subspace $X = \cup_{i \in I} X_i$ of $\mathbf{R}^n$. Let $x(t)$ be a continuous piecewise function on the time interval $[0, T]$. We say that $x(t)$ is a trajectory of (2.7), if for every $t \in [0, T]$ such that the derivative $\dot{x}(t)$ is defined, the equation $\dot{x}(t) = A_i x(t) + a_i + B_i u(t)$ holds for all i with $x(t) \in X_i$. Note that for a given system there may be initial values such that a corresponding trajectory only exists for small T.

Focus on properties of the equilibrium $x = 0$, and let $I_0 \subseteq I$ be the index set for cells that contain the origin, let $I_1 = I \setminus I_0$, and assume that $a_i = 0$, $c_i = 0$ for $i \in I_0$. For convenient, we use the notation $\bar{x} = \begin{bmatrix} x & 1 \end{bmatrix}^T$,

$$\bar{A}_i = \begin{bmatrix} A_i & a_i \\ 0 & 0 \end{bmatrix}, \ \bar{B}_i = \begin{bmatrix} B_i \\ 0 \end{bmatrix}, \ \bar{C}_i = \begin{bmatrix} C_i & c_i \end{bmatrix}$$

and re-write (2.7) as

$$\begin{aligned} \dot{\bar{x}}(t) &= \bar{A}_i \bar{x}(t) + \bar{B}_i u(t) \\ y(t) &= \bar{C}_i \bar{x}(t) + D_i u(t) \end{aligned} \tag{2.8}$$

Each polyhedral cell of the system (2.8) is partitioned by K hyperplanes

$$\partial \mathcal{H}_k = \{x \mid H_k x + h_k = 0\} \qquad \forall h_k \leq 0, \quad k = 1, \ldots, K \tag{2.9}$$

For convenient, all hyperplanes are represented as a *hyperplane matrix*

$$\bar{H} = \begin{bmatrix} H_k & h_k \end{bmatrix} \tag{2.10}$$

The polyhedral cells are represented on the form

$$X_i = \{x \mid G_i x + g_i \succeq 0\} \tag{2.11}$$

where $\succeq$ denotes elementwise inequality. To made it more compact, we construct matrices

$$\bar{G}_i = \begin{bmatrix} G_i & g_i \end{bmatrix}$$

where $\bar{G}_i$ is called a *cell identifier*.

2.3.2 Quadratic Stability

The term *quadratic stability* refers to stability that can be established using a quadratic Lyapunov function. It is possible to prove stability of piecewise linear systems using a globally quadratic Lyapunov function $V(x) = x^T P x$. In particular, if $a_i = 0 \ \forall i \in I$ and there exists $P > 0$ such that

$$A_i^T P + P A_i < 0 \qquad \forall i \in I \tag{2.12}$$

Then every trajectory of (2.7) tends to zero exponentially. The stability of a family of linear system depends on each cell partition. The equation (2.12) are linear matrix inequalities in P which can be solved as a convex optimization problem.

To verify that there exists no matrix P satisfying (2.12), it is a dual problem to find a positive definite matrices R_i , $i \in I$ such that

$$\sum_{i \in I} A_i^T R_i + R_i A_i > 0 \tag{2.13}$$

If the condition (2.13) is satisfied, then the Lyapunov function P in (2.12) will not be admitted.

2.3.3 Piecewise Quadratic Stability

We consider functions that are continuous and piecewise quadratic. This condition must be satisfied with all cell X_i, so it is sufficient to require that

$$x^T(A_i^T P + PA_i)x < 0, \quad \text{for } x \in X_i \tag{2.14}$$

To obtain a relaxed conditions for quadratic stability, one applies the $\mathcal{S}$-procedure and construct positive definite matrices S_i, $i \in I$ such that

$$A_i^T P + PA_i + S_i < 0 \tag{2.15}$$

Matrices S_i in $\mathcal{S}$-procedure can be construct from the system description, in this case are cell bounding matrices E_i and $\bar{E}_i$. With nonnegative entries matrices U_i, we have

$$\begin{array}{ll} x^T E_i^T U_i E_i x \geq 0 \ , & x \in X_i, i \in I_0 \\ \bar{x}^T \bar{E}_i^T U_i \bar{E}_i \bar{x} \geq 0 \ , & x \in X_i, i \in I_1 \end{array} \tag{2.16}$$

The cell boundings are important parameters from the partition information to enforce the positivity of the quadratic Lyapunov functions for all $x \in X_i$. The *polyhedral cell bounding* matrices can be defined as

$$\bar{E}_i = \begin{bmatrix} E_i & e_i \end{bmatrix} \quad \text{and} \quad \bar{E}_i \bar{x} \succeq 0, \quad x(t) \in X_i$$

The next step is to make the quadratic Lyapunov functions to be valid in all regions and continuous across cell boundaries. Let

$$\begin{array}{ll} P_i &= F_i^T T F_i, \quad i \in I_0 \\ \bar{P}_i &= \bar{F}_i^T T \bar{F}_i, \quad i \in I_1 \end{array} \tag{2.17}$$

where F_i and $\bar{F}_i$ are called the *continuity matrices* with their properties

$$\bar{F}_i = \begin{bmatrix} F_i & f_i \end{bmatrix} \quad \text{and} \quad \bar{F}_i \bar{x}(t) = \bar{F}_j \bar{x}(t) \quad \text{for } x(t) \in X_i \cap X_j$$

Since the expression for P_i is linear in a symmetric matrix T, it will be possible to state the search for a piecewise quadratic Lyapunov function as a set of linear matrix inequalities. The constructed Lyapunov function will in general have the form

$$V(x) = \begin{cases} x^T P_i x, & x \in X_i, \quad i \in I_0 \\ \bar{x}^T \bar{P}_i \bar{x}, & x \in X_i, \quad i \in I_1 \end{cases} \tag{2.18}$$

Next, we formulate LMIs for finding an existence of piecewise quadratic Lyapunov function of the system (2.7).

Theorem 2.1 (Piecewise Quadratic Stability). *[3]*
Consider symmetrics T, U_i and W_i have nonnegative entries, while $P_i = F_i^T T F_i, i \in I_0$ and $\bar{P}_i = \bar{F}_i^T T \bar{F}_i, i \in I_1$

$$\begin{cases} 0 > A_i^T P_i + P_i A_i + E_i^T U_i E_i \\ 0 < P_i - E_i^T W_i E_i \end{cases} \quad i \in I_0 \tag{2.19}$$

$$\begin{cases} 0 > \bar{A}_i^T \bar{P}_i + \bar{P}_i \bar{A}_i + \bar{E}_i^T U_i \bar{E}_i \\ 0 < \bar{P}_i - \bar{E}_i^T W_i \bar{E}_i \end{cases} \quad i \in I_1 \tag{2.20}$$

then every trajectory $x(t)$ of (2.7) with $u \equiv 0$ for $t \geq 0$ tends to zero exponentially.

2.3.4 Piecewise Quadratic Stabilization of PWA system

This section will show how to obtain the globally linear state feedback that stabilizes a PWA system. This can be cast as a convex optimization problem. Let us consider the state feedback

$$u = -Lx$$

which results in the closed loop system

$$\dot{x}(t) = (A_i - B_i L)x(t) + a_i \qquad x \in X_i \quad i \in I. \tag{2.21}$$

to be asymptotically stable for all region.

For the quadratic stabilization problem, we need to find a gain L that admits a quadratic Lyapunov function $V(x) = x^T P x$. For each cell X_i, we use the ellipsoid cell boundings

$$\|S_i x + s_i\|_2 \leq 1 \qquad \forall x \in X_i \tag{2.22}$$

or

$$1 - (S_i x + s_i)^T (S_i x + s_i) \geq 0 \qquad \forall x \in X_i \tag{2.23}$$

or

$$\begin{bmatrix} x \\ 1 \end{bmatrix}^T \begin{bmatrix} -S_i^T S_i & -S_i^T s_i \\ -s_i^T S_i & 1 - s_i^T s_i \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} \geq 0 \qquad \forall x \in X_i \tag{2.24}$$

and the condition in (2.14) to derive the sufficient condition for PWA system stability via S-procedure. Then, the closed-loop system is quadratically stable if we can find a positive definite matrix $P = P^T \geq 0$ and positive scalars $u_i \geq 0$ such that

$$\begin{cases} 0 > (A_i - B_i L)^T P + P(A_i - B_i L) & i \in I_0 \\ 0 > \begin{bmatrix} (A_i - B_i L)^T P + P(A_i - B_i L) & P a_i \\ a_i^T P & 0 \end{bmatrix} + u_i \begin{bmatrix} -S_i^T S_i & -S_i^T s_i \\ -s_i^T S_i & 1 - s_i^T s_i \end{bmatrix} & i \in I_1 \end{cases} \tag{2.25}$$

The above condition is bilinear in L and P and not efficient to be solved, however the problem can be transformed and resulted in Theorem 2.2.

Theorem 2.2 (Quadratic Stabilization). *[3]*
If there exists a positive definite matrix $Q = Q^T > 0$, positive scalars $v_i \geq 0$ and a matrix Y such that

$$\begin{cases} 0 > QA_j^T + A_jQ - Y^TB_j^T - B_jY \\ 0 < \begin{bmatrix} QA_i^T + A_iQ - Y^TB_i^T - B_iY - v_ia_ia_i^T & QS_i^T - v_ia_is_i^T \\ (QS_i^T - v_ia_is_i^T)^T & v_i(I - s_is_i^T) \end{bmatrix} \end{cases} \tag{2.26}$$

where $j \in I_0$ and $i \in I_1$. Then, the feedback $u = -Lx$ with $L = YQ^{-1}$ renders the piecewise linear system exponentially stable.

In this thesis, we will use this criteria to design the PWA state feedback control laws.