ON $4-\gamma_c-$ CRITICAL GRAPHS WITH CUT-VERTICES

By

Pawaton Kaemawichanurat

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

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กราฟ $4-\gamma_c-$critical ที่มีจุดตัด

โดย
นายภวธน เขมะวิชานุรัตน์

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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The Graduate School, Silpakorn University has approved and accredited the Thesis title of "On $4-\gamma_c-$ Critical Graphs with Cut-Vertices" submitted by Mr. Pawaton Kaemawichanurat as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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PAWATON KAEMAWICHANURAT : ON $4-\gamma_c-$CRITICAL GRAPHS WITH CUT-VERTICES . THESIS ADVISOR : ASSOC. PROF.NAWARAT ANANCHUEN, Ph.D. 38 pp.

Let $\gamma_c(G)$ denote the connected domination number of a graph G . G is said to be $k-\gamma_c-$ critical if $\gamma_c(G)=k$ and for each pair of non-adjacent vertices u and v of G, $\gamma_c(G+uv)<k$.

In this thesis, we show that $4-\gamma_c-$ critical graphs of connectivity one contain at most two cut-vertices. A characterization of $4-\gamma_c-$ critical graphs containing two cut-vertices is given. We also establish that a $4-\gamma_c-$ critical graph of connectivity one contains a perfect matching if it is of even order and a near perfect matching if it is of odd order. Moreover, we establish that a $2-$connected $4-\gamma_c-$ critical $K_{1,4}-$free graph of even order contains a perfect matching but it need not be bicritical and a $2-$connected $4-\gamma_c-$ critical $K_{1,4}-$free graph of odd order contains a near perfect matching but it need not be factor critical. Finally, we provide some examples to show that $2-$connected $4-\gamma_c-$ critical $K_{1,n}-$free graphs of even order need not be bicritical when n is 3 and 4 and $3-$connected $4-\gamma_c-$ critical $K_{1,n}-$free graphs of even order need not be bicritical when n is 4 and 5.

Department of Mathematics Graduate School, Silpakorn University Academic Year 2010

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คำสำคัญ : จำนวนควบคุมที่เชื่อมโยง / วิกฤติ / การจับคู่สมบูรณ์ / การจับคู่เกือบสมบูรณ์ / แฟคเตอร์ คริติคัล / ไบคริติคัล

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กำหนดให้ $\gamma_c(G)$ แทนขนาดของเซตควบคุมที่เชื่อมโยงที่เล็กที่สุดของกราฟ G เราจะเรียกกราฟ G ว่า $k-\gamma_c-$critical เมื่อ $\gamma_c(G)=k$ และสำหรับจุด u และ v ใน G ที่ไม่ประชิดกันแล้ว $\gamma_c(G+uv)<k$

ในวิทยานิพนธ์นี้เราแสดงว่ากราฟที่เป็น $4-\gamma_c-$ critical ซึ่งมีค่าความเชื่อมโยงเท่ากับ 1 มีจุดตัดได้ไม่เกิน 2 จุด เราได้ให้ลักษณะเฉพาะเจาะจงของกราฟ $4-\gamma_c-$ critical ที่มีจุดตัด 2 จุด และแสดงว่ากราฟ $4-\gamma_c-$ critical ที่มีค่าความเชื่อมโยงเท่ากับ 1 มีการจับคู่สมบูรณ์เมื่อมีอันดับคู่และมีการจับคู่เกือบสมบูรณ์เมื่อมีอันดับคี่ ยิ่งไปกว่านั้นเราแสดงว่ากราฟ $4-\gamma_c-$ critical $K_{1,4}-$free ที่มีค่าความเชื่อมโยงเท่ากับ 2 และมีอันดับคู่ มีการจับคู่สัมบูรณ์ แต่ไม่จำเป็นต้องเป็นกราฟไบคริติคัล และกราฟ $4-\gamma_c-$ critical $K_{1,4}-$free ที่มีค่าความเชื่อมโยงเท่ากับ 2 และมีอันดับคี่ มีการจับคู่เกือบสมบูรณ์ แต่ไม่จำเป็นต้องเป็นกราฟแฟคเตอร์ คริติคัล และสุดท้ายเราให้ตัวอย่างในการแสดงว่ากราฟ $4-\gamma_c-$ critical $K_{1,n}-$free ที่มีค่าความเชื่อมโยงเท่ากับ 2 อันดับคู่ไม่จำเป็นต้องเป็นไบคริติคัลเมื่อ n มีค่าเป็น 3 และ 4 และ กราฟ $4-\gamma_c-$ critical $K_{1,n}-$free ที่มีค่าความเชื่อมโยงเท่ากับ 3 อันดับคู่ไม่จำเป็นต้องเป็นไบคริติคัลเมื่อ n มีค่าเป็น 4 และ 5

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ลายมือชื่อนักศึกษา..

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Chapter 1

Introduction

Graph theory has emerged as an important and rapidly growing branch of mathematics that is rich in theory and application. The reason for this growth is that the very simple structure of a graph, a collection of points called vertices connected by lines called edges, makes it a very useful tool in mathematical modeling.

A graph G is an order triple $(V(G), E(G), \psi_G)$ (see Bondy and Murty [5]) consisting of a non-empty set $V(G)$ of vertices and a set $E(G)$, disjoint from $V(G)$, of edges and an incidence function ψ_G that assosiates with each edge of G an unordered pair of (not necessarily distinct) vertices of G. If u and v are vertices of a graph G identified with an edge e, that is $\psi_G(e) = uv$, then e is said to join u and v and we write $e = uv$. The vertices u and v are called the ends of e. We also say that the vertices u and v are incident with the edge e, and vice versa. Further, u and v are adjacent.

A graph G is said to be finite if both $V(G)$ and $E(G)$ are finite sets. The order of a graph G is the number of vertices of G. An edge with identical ends is called a loop. Two or more edges joining the same pair of vertices are called multiple edges. A graph with no loops and multiple edges is called a simple graph. All graphs considered in this thesis are finite and simple.

A complete graph is a graph in which every pair of vertices are adjacent. A complete graph of order n is denote by K_n. A graph G is bipartite if $V(G)$ can be partitioned into two(non-empty) subset V_1 and V_2 such that every edge of G joins a vertex of V_1 and a vertex of V_2. Moreover, if every vertex of V_1 is joined to every vertex of V_2 in such graph, then G is called a complete bipartite graph. A complete bipartite graph which the cardinality of at least one of partitioned vertex set equals to one is called a star, denoted by $K_{1,n}$. A graph $H = (V(H), E(H), \psi_H)$ is a subgraph of $G = (V(G), E(G), \psi_G)$ if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. For a non-empty set S of $V(G)$, the subgraph of G induced by S, denoted by $G[S]$, is a graph with vertex set S and $E(G[S]) = \{uv \in E(G) | u, v \in S\}$. For a graph H, a graph G is called $H-$free if G does not contain H as an induced subgraph.

A walk in a graph G is a finite, non-empty alternating sequence $W = v_0 e_1$

$v_1e_2...e_nv_n$ of vertices and edges such that for $1 \leq i \leq n$, the ends of edge e_i are v_{i-1} and v_i. W is said to be a walk from v_o to v_n. A path is a walk with distinct vertices. Two vertices u and v of G are connected if there is a path from u to v. A graph G is connected if every pair of vertices of G are connected otherwise G is disconnected. A maximal connected subgraph of G is called a component of G. It is odd or even depending on its order. We let $\omega(G)$ and $\omega_o(G)$ denote the number of components and odd components of G, respectively.

The neighborhood of a vertex v of G, denote by $N_G(v)$, is $\{u \in V(G)|uv \in E(G)\}$. A degree of v in G, denoted by $deg_G(v)$, is $|N_G(v)|$. $N_S(v)$ denotes either $N_G(v) \cap S$ if S is a subset of $V(G)$ or $N_G(v) \cap V(S)$ if S is a subgraph of G. A vertex v is called an end vertex of G if $deg_G(v) = 1$. We denote the minimum degree of a graph G by $\delta(G)$. The distance between vertices u and v of G, denoted by $d(u, v)$, is the length of a shortest $(u, v)-$path in G. The diameter of G, denoted by diam(G), is the maximum distance between two vertices of G.

A vertex v of G is called a cut-vertex if the number of components of $G - v$ is more than the number of components of G. A block of a graph G is a maximal connected subgraph with no cut-vertices. An end block of G is a block of G containing exactly one cut-vertex of G. A set $S \subseteq V(G)$ is called a cutset if the number of components of $G - S$ is more than the number of components of G. The connectivity $\kappa(G)$ of graph G is the cardinality of minimum cutset of G. A graph G is $n-$connected if $\kappa(G) \geq n$.

For $M \subseteq E(G)$, M is a matching in G if no two edges of M have a common end vertex. A matching M in G is maximum if G has no matching M' with $|M'| > |M|$. For a matching M in G, a vertex v of G is called $M-$saturated if v is incident with some edge of M; otherwise, v is $M-$unsaturated. If every vertex of G is $M-$saturated, then M is called a perfect matching. A graph G has a near perfect matching if there exists a vertex $x \in V(G)$ such that $G - x$ contains a perfect matching. A graph G is $k-$factor critical if, for every set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. For more specially, we say that G is factor critical if $k = 1$ and G is bicritical if $k = 2$.

A subset S of $V(G)$ is a dominating set of G if every vertex of G either belongs to S or is adjacent to a vertex of S. A dominating set S of G is a connected dominating set of G if S dominates G and S is connected. We will write $S \succ_c G$ if S is a connected dominating set of G. Further, if $S = \{u\}$, then we say that u dominates G rather than $\{u\}$ dominates G. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Similarly, the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. We say that S is a γ_c-set of G if S is a connected dominating set of G with $|S| = \gamma_c(G)$. Note that $\gamma(G) \leq \gamma_c(G)$ and only connected graphs can contain a connected dominating set.

Graph G is said to be k-γ-critical if $\gamma(G) = k$ but $\gamma(G + uv) < k$ for each pair of non-adjacent vertices u and v of G. The concept of k-γ-critical graphs was introduced, in 1983, by Sumner and Blitch [15]. They gave a characterization of 2-γ-critical graphs and 3-γ-critical disconnected graphs. They also established some properties of connected 3-γ-critical graphs. One of them is an existence of a perfect matching in 3-γ-critical graphs of even order. Since then, the concept of k-γ-critical graphs have been received considerable attention. Most of the known results concern 3-γ-critical graphs, see [3, 4, 8, 9, 10, 12, 17] for examples. Sumner and Wojcicka [16] asked whether connected k-γ-critical graphs of even order, for $k \geq 4$, contain a perfect matching. To date, this problem is unresolved.

In 2004, Chen et al. [7] introduced the concept of connected domination to k-γ -critical graphs. Graph G is said to be k-γ_c-critical graph if $\gamma_c(G) = k$ but $\gamma_c(G + uv) < k$ for each pair of non-adjacent vertices u and v of G. Chen et al. [7] obtained some results on k-γ_c-critical graphs, most of them are analogous to k-γ-critical graphs. One of them concerns an existence of a perfect matching (see Theorem 2.3). Ananchuen [1] and Ananchuen et al. [2] further studied k-γ_c-critical graphs. Most of the known results concern 3-γ_c-critical graphs. Similar to the problem concerning an existence of a perfect matching in k-γ- critical graphs of even order, for $k \geq 4$, posed by Sumner and Wojcicka [16], we might ask whether k-γ_c-critical graphs of even order, for $k \geq 4$, contain a perfect matching.

We will show in Chapter 3 that a 4-γ_c-critical graph having connectivity one contains a perfect matching if such graph is of even order and a near perfect matching if such graph is of odd order. This partially responds to such a problem. We also establish that 4-γ_c-critical graphs contain at most two cut-vertices. Further, a characterization of 4-γ_c-critical graphs containing exactly two cut-vertices is given. In Chapter 4, we establish that a $2-$connected $4 - \gamma_c-$critical $K_{1,4}-$free graph of even order contains a perfect matching but it need not be bicritical and a $2-$connected $4 - \gamma_c-$critical $K_{1,4}-$free graph of odd order contains a near perfect matching but it need not be factor critical. Finally, we provide some examples to show that $2-$connected $4 - \gamma_c-$critical $K_{1,n}-$free graphs of even order need not be bicritical when n is 3 and 4 and $3-$connected $4 - \gamma_c-$critical $K_{1,n}-$free graphs of even order need not be bicritical when n is 4 and 5.

Chapter 2

Prelimminaries

In this chapter we state a number of results that we make use of in establishing our results.

Theorem 2.1. *(see Chartrand and Oellermann [6] p. 24) Let G be a connected graph with at least one cut-vertex. Then G has at least 2 end blocks.*

The next result, first appeared in [7], follows immediately from the definition of k-γ_c-critical graphs.

Lemma 2.2. *[7] Let G be a k-γ_c-critical graph and let u and v be a pair of non-adjacent vertices of G. Suppose S is a γ_c-set of $G + uv$. Then*

(1) $k - 2 \leq |S| \leq k - 1$,

(2) $S \cap \{u, v\} \neq \emptyset$,

(3) If $u \in S$ and $v \notin S$, then $N_G(v) \cap (S - \{u\}) = \emptyset$.

Chen et al. [7] established a result concerning an existence of a perfect matching in 3-γ_c-critical graphs. More precisely, they proved that :

Theorem 2.3. *[7] Let G be a 3-γ_c-critical graph. If G is a graph of even order, then G has a perfect matching.*

Theorem 2.4. *[2] Let G be a 3-γ_c-critical graph. If G is a graph of odd order, then G has a near perfect matching.*

The two following results provide some properties of k-γ_c-critical graphs with a cut-vertex.

Lemma 2.5. *[1] For $k \geq 3$, let G be a k-γ_c-critical graph with a cut-vertex x. Then*

(1) $G - x$ contains exactly two components.

(2) If C_1 and C_2 are the components of $G - x$, then $G[N_{C_1}(x)]$ and $G[N_{C_2}(x)]$ are complete.

Lemma 2.6. *[1] For $k \geq 3$, let G be a k-γ_c-critical graph with a cut-vertex x and let C_1 and C_2 be the components of $G - x$. Suppose S is a γ_c-set of G. Then*

(1) $x \in S$.

(2) If C is a non-singleton component of $G - x$ with $\gamma_c(C) = k - 1$, then C is $(k-1)$-γ_c-critical.

The next result provides a necessary and sufficient condition for a graph to have a perfect matching.

Theorem 2.7. *(Tutte's Theorem) (see Chartrand and Oellermann [6] p. 188) A nontrivial graph G has a perfect matching if and only if, for every proper subset S of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$.*

Theorem 2.8. *(Whitney's Theorem) (see Chartrand and Oellermann [6] p. 155) For $n \geq 1$, a graph G is $n-$connected if and only if every pair of vertices of G is connected by at least n internally disjoint paths.*

We conclude this chapter by establishing results concerning n-factor critical graphs.

Theorem 2.9. *[11] A graph G is n-factor critical if and only if $\omega_o(G-C) \leq |C|-n$ for every $C \subseteq V(G)$ with $|C| \geq n$.*

Theorem 2.10. *[11] If G is n-factor critical, then G is $(n+1)$-edge-connected.*

Chapter 3

4-γ_c-critical graphs of connectivity 1.

In this chapter, we establish that $4-\gamma_c-$critical graphs contain at most 2 cut-vertices. The characterization of $4-\gamma_c-$critical graphs having 2 cut-vertices is provided. We also show that $4-\gamma_c-$critical graphs of connectivity one contain a perfect matching if such graphs are of even order and contain a near perfect matching if such graphs are of odd order.

Our first result provides an upper bound on the diameter of k-γ_c-critical graphs.

Lemma 3.1. *If G is a k-γ_c-critical graph, then $diam(G) \leq k$.*

Proof. Let G be a k-γ_c-critical graph. Suppose that $diam(G) = m \geq k+1$. Choose $x, y \in V(G)$ such that $d(x, y) = m$. Consider $G + xy$. Let S be a γ_c-set of $G + xy$. By Lemmas 2.2(1) and 2.2(2), $|S| \leq k - 1$ and either $x \in S$ or $y \in S$. We may suppose without loss of generality that $x \in S$. Let $L_i = \{z \in V(G)|d(z, x) = i\}$ for $0 \leq i \leq m$. Clearly, $L_i \neq \emptyset$. Further, $L_0 = \{x\}$ and $y \in L_m$. Let n be a maximum integer in which $S \cap L_i \neq \emptyset$ for each $0 \leq i \leq n$ and $G[\cup_{i=0}^{n}(S \cap L_i)]$ is connected. Since $n+1 \leq |S| \leq k-1$ and $m \geq k+1$, it follows that $n \leq m-3$. Consider L_{n+2}. Clearly, no vertex of $\cup_{i=0}^{n}(S \cap L_i)$ dominates L_{n+2}. Thus, $S \cap (L_{n+2} \cup L_{n+3}) \neq \emptyset$. Consequently, $S \cap L_j \neq \emptyset$ for each $n + 3 \leq j \leq m$. Then $y \in S$ because S is connected. Thus $|\cup_{j=n+3}^{m}(S \cap L_j)| \geq m-(n+3)+1 = m-n-2 \geq k+1-n-2 = k-n-1$. Therefore $|S| = |(\cup_{i=0}^{n}(S \cap L_i)) \cup (\cup_{j=n+3}^{m}(S \cap L_j))| \geq 1+n+(k-n-1) = k$, a contradiction. This completes the proof of our lemma. □

Our next result gives an upper bound on a number of cut-vertices of 4-γ_c-critical graphs.

Theorem 3.2. *Let G be a 4-γ_c-critical graph. Then G has at most two cut-vertices.*

Proof. Let C be a set of all cut-vertices of G. Suppose to the contrary that $|C| \geq 3$. By Theorem 2.1, G has at least two end blocks. Let B_1 be an end block of G with $V(B_1) \cap C = \{c_1\}$ for some $c_1 \in C$. By Lemma 2.5(1), the only components of $G - c_1$ are $G[V(B_1) - \{c_1\}]$ and $G_1 = G[V(G) - V(B_1)]$. Since B_1 is an end block, $C - \{c_1\} \subseteq V(G_1)$. Thus G_1 contains at least two cut-vertices. By Theorem 2.1, G_1 has at least two end blocks. Let B_2 be an end block of G_1 in such a way that B_2 is also an end block of G. Such B_2 exists as otherwise G contains exactly one end block. Put $\{c_2\} = V(B_2) \cap (C - \{c_1\})$. Let $c_3 \in C - \{c_1, c_2\}$. Note that c_3 is not adjacent to any vertex of $(V(B_1) \cup V(B_2)) - \{c_1, c_2\}$. Further, $G - c_3$ contains exactly two components, D_1 and D_2 say. Without loss of generality, we may assume that $V(B_1) \subseteq V(D_1)$. Then $V(B_2) \subseteq V(D_1)$ or $V(B_2) \subseteq V(D_2)$.

Claim : For $1 \leq i \leq 2$, c_i dominates B_i.
Suppose to the contrary that c_1 does not dominate B_1. Then there exists a vertex $x \in V(B_1) - \{c_1\}$ such that $xc_1 \notin E(G)$. Since B_1 is connected, there is an $x - c_1$ path of length at least 2. Thus $|V(B_1) - \{c_1, x\}| \geq 1$. Let $x_1 \in V(B_1) - \{c_1, x\}$. Choose $y \in V(B_2) - \{c_2\}$ and consider $G + xy$. Let S' be a γ_c-set of $G + xy$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since x and y are not adjacent to c_3, $1 \leq |S' - \{x, y\}| \leq 2$. Further, $c_1 \in S'$ or $c_2 \in S'$ because of connectedness of S'. We first suppose that $\{x, y\} \subseteq S'$. Then $S' = \{x, y, c_2\}$ because of connectedness of S'. If $V(B_2) \subseteq V(D_1)$, then no vertex of S' is adjacent to a vertex of D_2, a contradiction. Hence $V(B_2) \subseteq V(D_2)$. But then no vertex of S' is adjacent to c_1 because c_3 is a cut-vertex, again a contradiction. This proves that $|\{x, y\} \cap S'| = 1$. If $\{x, y\} \cap S' = \{x\}$, then $S' = \{x, a, c_1\}$ for some $a \in V(B_1) - \{x, c_1\}$ since $xc_1 \notin E(G)$. But then no vertex of S' is adjacent to a vertex of D_2, a contradiction. Hence, $\{x, y\} \cap S' \neq \{x\}$. By Lemma 2.2(2), $\{x, y\} \cap S' = \{y\}$. Then $c_2 \in S'$ since S' is connected. Because $x_1 \in V(B_1) - \{c_1, x\}$ and $\{y, c_2\} \subseteq V(B_2)$, x_1 is adjacent to neither y nor c_2. It follows that $S' - \{y, c_2\} = \{c_1\}$ since $\{c_1\} = V(B_1) \cap C$. Then $c_1c_2 \in E(G)$ and thus $V(B_2) \subseteq V(D_1)$. But then no vertex of S' is adjacent to a vertex of D_2, a contradiction. So $\gamma_c(G + xy) > 3$, again a contradiction. This proves that c_1 dominates B_1. By similar arguments, c_2 dominates B_2.

Now let $x \in V(B_1) - \{c_1\}$ and $y \in V(B_2) - \{c_2\}$. Clearly, $xy \notin E(G)$. By our claim, $xc_1 \in E(G)$ and $yc_2 \in E(G)$. We now distinguish two cases.

Case 1: $V(B_2) \subseteq V(D_1)$.
Consider $G + xy$. Let S' be a γ_c-set of $G + xy$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since c_3 is a cut-vertex of $G + xy$, $c_3 \in S'$. Because $c_3x \notin E(G)$ and $c_3y \notin E(G)$, it follows that $|S' \cap \{x, y\}| = 1$ since S' is connected. Without loss of generality, we may assume that $x \in S'$. Then $S' = \{x, c_1, c_3\}$. Thus $c_1c_3 \in E(G)$ and c_3 dominates D_2. Further, $\{c_1, c_3\}$ dominates $V(D_1) - \{y\}$ by our claim and the fact that $x \in B_1$ and B_1 is an end block. Since S' is a γ_c-set of $G + xy$ and $xc_2 \notin E(G)$, it follows that $c_2c_1 \in E(G)$ or $c_2c_3 \in E(G)$. But then $\{c_1, c_2, c_3\}$ is a connected dominating set of G, a contradiction. This proves that Case 1 cannot occur.

Case 2 : $V(B_2) \subseteq V(D_2)$.

By Lemma 3.1, $d(x,y) \leq 4$. Since B_1 and B_2 are end blocks, $d(c_1,c_2) \leq 2$. Because $c_1 \in V(D_1)$ and $c_2 \in V(D_2)$, we have $c_1c_3 \in E(G)$ and $c_2c_3 \in E(G)$. Now consider $G + c_1c_2$. Let S' be a γ_c-set of $G + c_1c_2$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since B_i is an end block and $V(B_i) \cap C = \{c_i\}$ for $i = 1,2$, it follows that $\{c_1,c_2\} \subseteq S'$. If $S' = \{c_1,c_2\}$, then $\{c_1,c_2,c_3\}$ is a connected dominating set of G, a contradiction. Hence, $S' \neq \{c_1,c_2\}$. Therefore, $|S' - \{c_1,c_2\}| = 1$ since $|S'| \leq 3$. Let $\{a\} = S' - \{c_1,c_2\}$. Clearly, $a \neq c_3$. Then $a \in V(D_1)$ or $a \in V(D_2)$.

Subcase 2.1 : $a \in V(D_1)$.
Thus c_2 dominates D_2 and there is a vertex $z \in V(D_1)$ such that $zc_1 \notin E(G)$ but $za \in E(G)$. By our claim, $z \notin V(B_1)$. Observe that no vertex of B_1 is adjacent to z since B_1 is an end block. Consider $G + c_1z$. Let S'' be a γ_c-set of $G + c_1z$. By Lemma 2.2(1), $|S''| \leq 3$. Because c_1 is a cut-vertex of $G + c_1z$, $c_1 \in S''$. Since S'' is connected and c_1 and c_3 are not adjacent to any vertex of $B_2 - \{c_2\} \subseteq V(D_2)$, it follows that $S'' = \{c_1,c_2,c_3\}$. Thus $c_3z \notin E(G)$.

Recall that $x \in V(B_1) - \{c_1\}$ and $y \in V(B_2) - \{c_2\}$. Now consider $G + xy$. Let S''' be a γ_c-set of $G + xy$. Since $xc_3 \notin E(G)$ and $yc_3 \notin E(G)$ together with the fact that B_i is an end block with $V(B_i) \cap C = \{c_i\}$ for $i = 1,2$, we have $c_1 \in S'''$ or $c_2 \in S'''$. If $S''' = \{x,y,c_1\}$ or $S''' = \{x,y,c_2\}$, then no vertex of S''' is adjacent to z, a contradiction. Hence, $|S''' \cap \{x,y\}| = 1$. We first suppose that $S''' \cap \{x,y\} = \{x\}$. Then $S''' = \{x,c_1,c_3\}$ since S''' is connected and $x \in B_1$ where B_1 is an end block. But then no vertex of S''' is adjacent to z, a contradiction. Hence $S''' \cap \{x,y\} \neq \{x\}$ and therefore $S''' \cap \{x,y\} = \{y\}$ by Lemma 2.2(2). Clearly, $S''' = \{y,c_2,c_3\}$. But then no vertex of S''' is adjacent to z, again a contradiction. Hence, $\gamma_c(G + xy) > 3$. This contradicts the criticality of G. Thus Subcase 2.1 cannot occur.

Subcase 2.2 : $a \in V(D_2)$.
By similar arguments as in the proof of Subcase 2.1, Subcase 2.2 cannot occur. This proves that Case 2 cannot occur and completes the proof of our theorem. □

In what follows, we shall assume that $|S \cap V(C_2)| \leq |S \cap V(C_1)|$ where S is a γ_c-set of a 4-γ_c-critical graph G with a cut-vertex c and C_1 and C_2 are the only components of $G - c$ (by Lemma 2.5 (1)). Note that, since $c \in S$ by Lemma 2.6(1), either $|S \cap V(C_2)| = 0$ and $|S \cap V(C_1)| = 3$ or $|S \cap V(C_2)| = 1$ and $|S \cap V(C_1)| = 2$.

Our next result provides a structure of such graph with $|S \cap V(C_2)| = 1$ and $|S \cap V(C_1)| = 2$.

Theorem 3.3. *Let G be a 4-γ_c-critical graph with a cut-vertex c and S a γ_c-set of G. Further, let C_1 and C_2 be the two components of $G - c$ and for $1 \leq i \leq 2$, let $X_i = \overline{N}_{C_i}(c)$. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then*

(1) For $1 \leq i \leq 2$, $X_i \neq \emptyset$ and $G[X_i]$ is complete.

(2) For each $x \in N_{C_1}(c)$, $|N_{X_1}(x)| = |X_1| - 1$.

(3) For each $x \in X_1$, there is a vertex $y \in N_{C_1}(c)$ such that $xy \in E(G)$.

(4) $|X_2| = 1$.

(5) If $X_2 = \{b\}$, then $deg_G(b) = |N_{C_2}(c)|$.

Proof. (1) Clearly, for $1 \le i \le 2$, $X_i \ne \emptyset$, as otherwise $\gamma_c(G) < 4$. We now show that $G[X_i]$ is complete. Suppose to the contrary that there exist $u, v \in X_1$ such that $uv \notin E(G)$. Let S' be a γ_c-set of $G + uv$. By Lemmas 2.2(1) and 2.2(2), $|S'| \le 3$ and either $u \in S'$ or $v \in S'$. We may assume without loss of generality that $u \in S'$. Because S' is connected and c is a cut-vertex in $G + uv$, $c \in S'$. Further, since $uc \notin E(G)$, there exists $a \in (S' - \{u, c\}) \cap V(C_1)$ such that $ua, ac \in E(G)$. Thus $S' \cap V(C_2) = \emptyset$. This implies that c dominates C_2. But this contradicts the fact that $X_2 \ne \emptyset$. Hence, $G[X_1]$ is complete. By similar arguments, $G[X_2]$ is complete. This settles (1).

(2) Suppose to the contrary that there exist $a \in N_{C_1}(c)$ and $b, d \in V(C_1) - \{a\}$ such that $ab, ad \notin E(G)$. Lemma 2.5(2) implies that $b, d \in X_1$. Now consider $G + ab$. Let S' be a γ_c-set of $G + ab$. Since c is a cut-vertex of $G + ab$, $c \in S'$. Then $|S' \cap (V(C_1) \cup V(C_2))| \le 2$. If $|S' \cap V(C_2)| = 0$, then c dominates C_2. But then $X_2 = \emptyset$, a contradiction. Hence, $|S' \cap V(C_2)| \ge 1$. By Lemma 2.2(2), $\emptyset \ne S' \cap \{a, b\} \subseteq S' \cap V(C_1)$. It follows that $|S' \cap V(C_1)| = 1$ and $|S' \cap V(C_2)| = 1$. Since $b \in X_1$ and S' is connected, it follows that $S' \cap \{a, b\} = \{a\}$. But then no vertex of S' is adjacent to d, a contradiction. This proves that for each $x \in N_{C_1}(c)$, $|N_{X_1}(x)| \ge |X_1| - 1$. We next suppose that there exists $y \in N_{C_1}(c)$ such that $|N_{X_1}(y)| = |X_1|$. Let $\{z\} = S \cap V(C_2)$. Then $\{y, c, z\}$ is a connected dominating set of size 3 of G, a contradiction. This completes the proof of (2).

(3) Let $x \in X_1$. Suppose to the contrary that x is not adjacent to any vertex of $N_{C_1}(c)$. Then $d(x, c) \ge 3$. Since $X_2 \ne \emptyset$, there exists $z \in X_2$. So $d(c, z) \ge 2$. Because c is a cut-vertex of G, $d(x, z) = d(x, c) + d(c, z) \ge 5$. But this contradicts Lemma 3.1 and completes the proof of (3).

(4) Suppose to the contrary that $|X_2| > 1$. Let $b, d \in X_2$. Now consider $G + cb$. Let S' be a γ_c-set of $G + cb$. Then $|S'| \le 3$. Since c is a cut-vertex of $G + cb$, $c \in S'$. Because $X_1 \ne \emptyset$, $|S' \cap V(C_1)| \ge 1$. If $|S' \cap V(C_1)| = 1$, then the only vertex of $S' \cap V(C_1)$ is adjacent to c and dominates X_1. But this contradicts (2). Hence, $|S' \cap V(C_1)| = 2$ since $|S'| \le 3$. Then no vertex of S' is adjacent to d, a contradiction. Thus $|X_2| = 1$. This settles (4).

(5) Now let $X_2 = \{b\}$. Suppose to the contrary that there exists $d \in N_{C_2}(c)$ such that $db \notin E(G)$. Now consider $G + bd$. Let S' be a γ_c-set of $G + db$. Then $|S'| \le 3$ by Lemma 2.2(1). Since c is a cut-vertex of $G + bd$, $c \in S'$. By similar arguments as in the proof of (4), $|S' \cap V(C_1)| = 2$. Thus $S' \cap \{b, d\} = \emptyset$ since $|S' \cap (\{c\} \cup V(C_1))| = 3$ and $|S'| \le 3$. But this contradicts Lemma 2.2(2) and thus (5) is proved, completing the proof of our theorem. □

Corollary 3.4. *Let G, c, S, C_1, C_2, X_1, X_2 and b be defined as in Theorem 3.3. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then $|X_1| \ge 2$ and $|N_{C_1}(c)| \ge 2$. Further, if G contains an end vertex, then G has exactly one end vertex and the only end vertex of G is b.*

Proof. By Theorems 3.3(2) and 3.3(3) together with the fact that G is connected, $|X_1| \ge 2$ and $|N_{C_1}(c)| \ge 2$. Suppose x is an end vertex of G. Clearly, $x \notin$

$V(C_1) \cup \{c\}$. Thus $x \in V(C_2)$. Since $X_2 = \{b\}$ and $N_G(b) = |N_{C_2}(c)|$, it follows that $x \notin N_{C_2}(c)$. Thus $x \in X_2 = \{b\}$. Therefore, $x = b$ and $|N_{C_2}(c)| = 1$. This proves our corollary. □

Lemma 3.5. *Let G be a 4-γ_c-critical graph with a cut-vertex c and let C_1 and C_2 be the two components of $G - c$. Let S be a γ_c-set of G. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then*

(1) If G is of even order, then G has a perfect matching.

(2) If G is of odd order, then G has a near perfect matching.

Proof. For simplicity, let $X_i = \overline{N}_{C_i}(c)$ for $1 \le i \le 2$. By Theorem 3.3(4), $|X_2| = 1$. Let $|X_1| = n$, $|N_{C_1}(c)| = m$ and $|N_{C_2}(c)| = p$. By Corollary 3.4, $n \ge 2$ and $m \ge 2$. Now let $X_2 = \{b\}$, $X_1 = \{x_i | 1 \le i \le n\}$, $N_{C_1}(c) = \{y_i | 1 \le i \le m\}$ and $N_{C_2}(c) = \{z_i | 1 \le i \le p\}$. By Lemma 2.5(2) and Theorem 3.3(1), $G[X_1]$, $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete. Further, by Theorem 3.3(5), $N_G(b) = N_{C_2}(c)$.

(1) Suppose G is of even order. Since $V(G) = X_1 \cup N_{C_1}(c) \cup \{c\} \cup N_{C_2}(c) \cup X_2$, $n + m + p$ must be even. We distinguish 4 cases.

Case 1 : n, m, p are even.
Clearly, $F_1 = \{x_i x_{i+\frac{n}{2}} | 1 \le i \le \frac{n}{2}\} \cup \{y_i y_{i+\frac{m}{2}} | 1 \le i \le \frac{m}{2}\}$ is a perfect matching in C_1. Since $N_{C_2}(c) = N_G(b)$ and $G[N_{C_2}(c)]$ is complete, it follows that $G[N_{C_2}(c) - \{z_1, z_{1+\frac{p}{2}}\}]$ is complete. Then $F_2 = \{z_i z_{i+\frac{p}{2}} | 2 \le i \le \frac{p}{2}\} \cup \{cz_1, bz_{1+\frac{p}{2}}\}$ is a perfect matching in $G[V(C_2) \cup \{c\}]$. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 1.

Case 2 : n is even, m and p are odd.
Clearly, $F_1 = \{x_i x_{i+\frac{n}{2}} | 1 \le i \le \frac{n}{2}\}$ is a perfect matching in $G[X_1]$. Because m is odd, $F_2 = \{y_i y_{i+\lfloor \frac{m}{2} \rfloor} | 1 \le i \le \lfloor \frac{m}{2} \rfloor\} \cup \{y_m c\}$ is a perfect matching in $G[N_{C_1}(c) \cup \{c\}]$. By a similar argument, $F_3 = \{z_i z_{i+\lfloor \frac{p}{2} \rfloor} | 1 \le i \le \lfloor \frac{p}{2} \rfloor\} \cup \{z_p b\}$ is a perfect matching in $G[N_{C_2}(c) \cup \{b\}]$. Thus $F_1 \cup F_2 \cup F_3$ is a perfect matching in G. This proves Case 2.

Case 3 : n and p are odd, m is even.
By Theorem 3.3(3), there exists a vertex $u \in N_{C_1}(c)$ such that $x_n u \in E(G)$. Without loss of generality, we may assume that $u = y_1$. Since $G[X_1]$ and $G[N_{C_1}(c)]$ are complete, $F_1 = \{x_i x_{i+\lfloor \frac{n}{2} \rfloor} | 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{x_n y_1, cy_{1+\frac{m}{2}}\} \cup \{y_i y_{i+\frac{m}{2}} | 2 \le i \le \frac{m}{2}\}$ is a perfect matching in $G[V(C_1) \cup \{c\}]$. Since $G[N_{C_2}(c)]$ is complete and $N_G(b) = N_{C_2}(c)$, it follows that $F_2 = \{z_i z_{i+\lfloor \frac{p}{2} \rfloor} | 1 \le i \le \lfloor \frac{p}{2} \rfloor\} \cup \{bz_p\}$ is a perfect matching in C_2. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 3.

Case 4 : n and m are odd, p is even.
By Theorem 3.3(3), there exists a vertex $u \in N_{C_1}(c)$ such that $x_n u \in E(G)$. Without loss of generality, we may assume that $u = y_m$. Since $G[X_1]$ and $G[N_{C_1}(c)]$ are complete, $F_1 = \{x_i x_{i+\lfloor \frac{n}{2} \rfloor} | 1 \le i \le \lfloor \frac{n}{2} \rfloor\} \cup \{y_i y_{i+\lfloor \frac{m}{2} \rfloor} | 1 \le i \le \lfloor \frac{m}{2} \rfloor\} \cup \{x_n y_m\}$ is a perfect matching in $G[V(C_1)]$. Since $N_G(b) = N_{C_2}(c)$ and $G[N_{C_2}(c)]$ is complete, it follows that $G[N_{C_2}(c) - \{z_1, z_{1+\frac{p}{2}}\}]$ is complete. Then $F_2 = \{cz_1, bz_{1+\frac{p}{2}}\} \cup$

$\{z_i z_{i+\frac{p}{2}} | 2 \leq i \leq \frac{p}{2}\}$ is a perfect matching in $G[V(C_2) \cup \{c\}]$. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 4 and completes the proof of (1).

(2) We now suppose G is of odd order. By similar arguements, it is not difficult to show that G has a near perfect matching. □

Theorem 3.6. *Let G be a 4-γ_c-critical graph with a cut-vertex c and let C_1 and C_2 be the two components of $G - c$. Let S be a γ_c-set of G. Suppose $|S \cap V(C_1)| = 3$ and $|S \cap V(C_2)| = 0$. Then*

(1) $|V(C_2)| = 1$.

(2) If G is of even order, then G has a perfect matching.

(3) If G is of odd order, then G has a near perfect matching.

Proof. Clearly, c dominates C_2. Thus $V(C_2) \subseteq N_G(c)$. Note that $|V(C_1)| \geq 4$ since $|S \cap V(C_1)| = 3$.

(1) Choose $a \in V(C_2)$ and $d \in N_{C_1}(c)$. Consider $G + ad$. Let S' be a γ_c-set of $G + ad$. By Lemmas 2.2(1) and 2.2(2), $|S'| \leq 3$ and either $a \in S'$ or $d \in S'$. Suppose first that $\{a, d\} \subseteq S'$. Then $|S' \cap V(C_1)| \leq 2$. Clearly, $S' \cap V(C_1) \succ_c C_1$. But then $(S' \cap V(C_1)) \cup \{c\}$ is a connected dominating set of G, a contradiction. Hence, $|S' \cap \{a, d\}| = 1$. Since $\{a, d\} \subseteq N_G(c)$, $c \notin S' - \{a, d\}$ by Lemma 2.2(3). If $S' \cap \{a, d\} = \{a\}$, then $S' \subseteq V(C_2)$ because of connectedness of S'. But then no vertex of S' is adjacent to a vertex of $V(C_1) - \{d\}$, a contradiction. Hence, $S' \cap \{a, d\} = \{d\}$. Because of the connectedness of S' and the fact that $c \notin S' - \{a, d\}$, it follows that $S' \subseteq V(C_1)$. Therefore, $V(C_2) = \{a\}$. This proves (1).

Now let $V(C_2) = \{a\}$. Then $ac \in E(G)$. We first show that $2 \leq \gamma_c(C_1) \leq 3$. Since S is connected and $c \in S$, it follows that $S \cap N_{C_1}(c) \neq \emptyset$. Because $G[N_{C_1}(c)]$ is complete by Lemma 2.5(2), $S \cap V(C_1) \succ_c C_1$. Hence, $\gamma_c(C_1) \leq 3$. Note that for each γ_c-set W of C_1 and for each $x \in N_{C_1}(c)$, $W \cup \{x, c\} \succ_c G$. It follows that $|W| \geq 2$ as otherwise $\gamma_c(G) \leq 3$. Consequently, $\gamma_c(C_1) \geq 2$. Before we establish (2) and (3), we first show that if $\gamma_c(C_1) = 2$, then the following Claims $1 - 5$ hold.

Suppose $\gamma_c(C_1) = 2$. Put $X = \overline{N}_{C_1}(c)$. Let $Y = \{u \in X | ux \in E(G)$ for some $x \in N_{C_1}(c)\}$ and $Z = X - Y$.

Claim 1 : If $Z \neq \emptyset$, then $G[Z]$ is complete.
Suppose that $G[Z]$ is not complete. Then there exist $u, v \in Z$ such that $uv \notin E(G)$. Consider $G + uv$ and let S' be a γ_c-set of $G + uv$. By Lemma 2.2(1), $|S'| \leq 3$. Because c is a cut-vertex in $G + uv$, $c \in S'$. Since $|S'| \leq 3$ and S' is connected, it follows that $(G + uv)[S']$ contains an $u - c$ path or $v - c$ path of length at most 2. But this is not possible since neither u nor v is adjacent to a vertex of $N_{C_1}(c)$. Hence $G[Z]$ is complete. This settles our claim.

Claim 2 : If D is a γ_c-set of C_1, then $D \cap N_{C_1}(c) = \emptyset$.
Suppose $D = \{x, y\}$ where $x \in N_{C_1}(c)$. Then $\{x, y, c\} \succ_c G$, a contradiction. This proves our claim.

Claim 3 : If X has no perfect matching, then there is a maximum matching M in X such that $X - V(M) \subseteq Y$.
Let M_0 be a maximum matching in X and let $B = (X - V(M_0)) \cap Z$. By Claim 1 and the fact that $X - V(M_0)$ is an independent set, it follows that $0 \le |B| \le 1$. If $|B| = 0$, then our claim follows. So we may suppose that $|B| = 1$. Let $\{z\} = B$. Since $d(z, a) \le 4$ by Lemma 3.1, there is a vertex $v_1 \in Y$ such that $zv_1 \in E(G)$. If v_1 is M_0-unsaturated, then $M_0 \cup \{zv_1\}$ is a matching of X with $|M_0 \cup \{zv_1\}| > |M_0|$, contradicting the fact that M_0 is a maximum matching. Hence, v_1 is M_0-saturated. Let $v_2 \in X$ such that $v_1v_2 \in M_0$.
Now let

$$M = \begin{cases} (M_0 - \{v_1v_2\}) \cup \{zv_1\} & if \;\; v_2 \in Y \\ (M_0 - \{v_1v_2\}) \cup \{zv_2\} & if \;\; v_2 \in Z. \end{cases}$$

Clearly, M is a matching in X with $X - V(M) \subseteq Y$. This proves our claim.

Claim 4 : Let M be a maximum matching in X such that $X - V(M) \subseteq Y$. If $A = X - V(M) \neq \emptyset$ and $|A| \ge 2$, then there exists a vertex $z \in V(M)$ such that $\{z\} \cup A$ is an independent set.
Clearly, A is independent. Let D be a γ_c-set of C_1. Then $|D \cap A| \le 1$ because of connectedness of $G[D]$. By Claim 2, $D \subseteq X = V(M) \cup A$. Thus $1 \le |D \cap V(M)| \le 2$.

Case 4.1 : $|D \cap A| = 1$ and $|D \cap V(M)| = 1$.
Let $\{x\} = D \cap A$ and $\{y\} = D \cap V(M)$. Note that $xy \in E(G)$. Since A is independent, y dominates A. Let $w \in V(M)$ where $yw \in M$. If $wu \in E(G)$ for some $u \in A$, then $M_1 = (M - \{yw\}) \cup \{wu, yv\}$ where $v \in A - \{u\}$ is a matching in X with $|M_1| > |M|$, a contradiction. Hence, $wu \notin E(G)$ for all $u \in A$. Thus $\{w\} \cup A$ is an independent set as required. This settles Case 2.1.

Case 4.2 : $|D \cap V(M)| = 2$.
Let $\{x_1, x_2\} = D \cap V(M)$, Clearly, $x_1x_2 \in E(G)$ and $A \subseteq N_G(x_1) \cup N_G(x_2)$. Put $A_1 = \{a \in A | ax_1 \in E(G)\}$ and $A_2 = \{a \in A | ax_2 \in E(G)\}$. Note that $A_1 \cup A_2 = A$ and either $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$ since $|A| \ge 2$. We may suppose that $A_1 \neq \emptyset$.

We first suppose that $x_1x_2 \in M$. If $A_2 \neq \emptyset$, then, since $|A| \ge 2$, there is a matching $M_2 = (M - \{x_1x_2\}) \cup \{a_1x_1, a_2x_2\}$, where $a_i \in A_i$ for $1 \le i \le 2$, in X with $|M_2| \ge |M|$, a contradiction. Hence, $A_2 = \emptyset$. Consequently, $A_1 = A$ and $\{x_2\} \cup A$ is an independent set as required.

We may now suppose that $x_1x_2 \notin M$. Let $y_1, y_2 \in V(M)$ where $x_1y_1, x_2y_2 \in M$. If $A_1 = A$, then y_1 is not adjacent to any vertex of A_1 as otherwise X contains a matching M' with $|M'| > |M|$ since $|A| \ge 2$. Consequently, $A \cup \{y_1\}$ is an independent set. Similarly, if $A_2 = A$, then $A \cup \{y_2\}$ is an independent set. So we may now suppose that $A_1 \neq A$ and $A_2 \neq A$. Since $A_1 \neq A$, $A_2 \neq \emptyset$. Observe that $y_1u \notin E(G)$ for all $u \in A_2$ and $y_2v \notin V(G)$ for all $v \in A_1$. Further, if $|A_1| \ge 2$, then $y_1u \notin E(G)$ for all $u \in A_1$ as otherwise X contains a matching M'' with $|M''| > |M|$. Similarly, if $|A_2| \ge 2$, then $y_2u \notin E(G)$ for all $u \in A_2$. We now distinguish two subcases.

Subcase 4.2.1 : $|A| \ge 3$.
Then $|A_1| \ge 2$ or $|A_2| \ge 2$. If $|A_1| \ge 2$, then $A \cup \{y_1\}$ is an independent set and

if $|A_2| \geq 2$, then $A \cup \{y_2\}$ is an independent set. This settles Subcase 2.2.1.

Subcase 4.2.2 : $|A| = 2$.
Recall that, for $1 \leq i \leq 2$, $A_i \neq A$ and $A_i \neq \emptyset$. Thus $|A_1| = 1$ and $|A_2| = 1$. For $1 \leq i \leq 2$, let $\{a_i\} = A_i$. If $a_1y_1 \in E(G)$ and $a_2y_2 \in E(G)$, then $M_3 = (M - \{x_1y_1, x_2y_2\}) \cup \{a_1y_1, x_1x_2, a_2y_2\}$ is a matching in X with $|M_3| > |M|$, a contradiction. Hence, either $a_1y_1 \notin E(G)$ or $a_2y_2 \notin E(G)$. Then either $A \cup \{y_1\}$ or $A \cup \{y_2\}$ is an independent set. This proves Subcase 2.2.2 and then Case 2, completing the proof of our Claim.

Claim 5 : Let B be an independent subset of X where $|B| = n \geq 3$. Then the vertices of B can be ordered as $b_1, ..., b_n$ in such a way that, for $1 \leq i \leq n-1$, there exist vertices $u_1, ..., u_{n-1}$ of $N_{C_1}(c)$ where $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$ and, for $1 \leq i \neq j \leq n-1$, $u_i \neq u_j$.

Let u, v be a pair of non-adjacent vertices of B. Let S' be a γ_c-set of $G + uv$. So $|S'| \leq 3$. Because c is a cut-vertex of $G + uv$, $c \in S'$. By Lemma 2.2(2), $\{u, v\} \cap S' \neq \emptyset$. Since $\{u, v\} \cap N_{C_1}(c) = \emptyset$ and S' is connected, it follows that there exists $x \in N_{C_1}(c) \cap S'$. Then $|\{u, v\} \cap S'| = 1$ and $|S'| = 3$. Hence, $\{u, x, c\} \succ_c G + uv$ or $\{v, x, c\} \succ_c G + uv$. Because $G[N_{C_1}(c)]$ is complete by Lemma 2.5(2) and c is not adjacent to any vertex of X, $\{x, u\} \succ_c C_1 - v$ or $\{x, v\} \succ_c C_1 - u$. Now consider $\overline{G}[B]$. Clearly, $\overline{G}[B]$ is complete. For a pair of vertices u and v of B, we orient u to v in $\overline{G}[B]$ if there exists a vertex $x \in N_{C_1}(c)$ such that $\{x, u\} \succ_c C_1 - v$. Let G^* be a spanning subdigraph of $\overline{G}[B]$ where G^* is a tournament. Then G^* contains P as a spanning directed path. Such a path exists since every tournament contains a spanning directed path. We now label the vertices of B as $b_1, ..., b_n$ where (b_i, b_{i+1}) is an arc of P for $i = 1, ..., n-1$. Hence, there exists u_i such that $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$ for $i = 1, ..., n-1$. Since $\{b_1, ..., b_n\}$ is independent, $u_jb_k \in E(G)$ for $1 \leq j \leq n-1$, and $1 \leq k \neq j+1 \leq n$. Thus $u_j \neq u_i$ for $1 \leq i \neq j \leq n-1$. This settles our claim.

We are now ready to prove (2) and (3).

(2) Suppose G is of even order. Recall that $V(C_2) - \{a\}$ and $ac \in E(G)$. Then $|V(C_1)| = |V(G) - \{a, c\}|$ is even. If $\gamma_c(C_1) = 3$, then C_1 is $3 - \gamma_c$-critical by Lemma 2.6(2). Then C contains M as a perfect matching by Theorem 2.3. So $M \cup \{ac\}$ is a perfect matching in G.

We now assume that $\gamma_c(C_1) = 2$. Note that $G[N_{C_1}(c)]$ is complete by Lemma 2.5(2). If X contains M as a perfect matching, then, $M \cup F \cup \{ac\}$ is a perfect matching in G where F is a perfect matching in $G[N_{C_1}(c)]$. We may now suppose that X does not contain a perfect matching. By Claim 3, there is a maximum matching M_1 in X such that $X - V(M_1) \subseteq Y$. Put $A = X - V(M_1)$. Clearly, $|A| \geq 1$. We first suppose that $|A| = 1$. Let $A = \{u\}$. Since $A \subseteq Y$, there exists $v \in N_{C_1}(c)$ such that $uv \in E(G)$. Then $G[N_{C_1}(c) - \{v\}]$ contains F_1 as a perfect matching since $G[N_{C_1}(c)]$ is complete and $|V(G)|$ is even. Thus $M_1 \cup F_1 \cup \{uv, ac\}$ is a perfect matching in G.

We now consider $|A| \geq 2$. By Claim 4, there exists a vertex $y \in V(M_1)$ such that $\{y\} \cup A$ is an independent set. Put $|\{y\} \cup A| = n$. Note that $n \geq 3$. By

Claim 5, the vertices of $\{y\} \cup A$ can be ordered as $b_1, ..., b_n$ in such a way that for $1 \le i \le n-1$ there exists $u_i \in N_{C_1}(c)$ where $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$. By Lemma 2.2(3), $u_i b_{i+1} \notin E(G)$ for $1 \le i \le n-1$. Since $\{b_1, ..., b_n\}$ is independent and for $1 \le i \le n-1$, $G[\{b_i, u_i\}]$ is connected, it follows that $u_i b_j \in E(G)$ for $1 \le i \le n-1$ and $1 \le j \ne i+1 \le n$. We may suppose that $y = b_k$ for some $1 \le k \le n$. Now let F_2 be a matching in C_1 where

$$F_2 = \begin{cases} \{u_i b_i | 1 \le i \ne k \le n-1\} \cup \{u_k b_n\} & if \;\; 1 \le k \le n-2, \\ \{u_i b_i | 2 \le i \le n-2\} \cup \{u_1 b_n, u_{n-1} b_1\} & if \;\; k = n-1, \\ \{u_i b_i | 1 \le i \le n-1\} & if \;\; k = n. \end{cases}$$

Note that $X \subseteq V(M_1 \cup F_2)$. Since $G[N_{C_1}(c)]$ is complete, $G[N_{C_1}(c) - \{u_1, ..., u_{n-1}\}]$ contains F_3 as a perfect matching. Then $M_1 \cup F_2 \cup F_3 \cup \{ac\}$ is a perfect matching in G. This proves (2).

(3) We now suppose G is of odd order. By similar arguements as in (2) together with Theorem 2.4, it is not difficult to show that G has a near perfect matching. □

Theorem 3.7. *Let G be a 4-γ_c-critical graph of connectivity one.*

(1) If G is of even order, then G has a perfect matching.

(2) If G is of odd order, then G has a near perfect matching.

Proof. Let c be a cut-vertex of G and let C_1 and C_2 be the two components of $G - c$. Further, let S be a γ_c-set of G. By Lemma 2.6(1), $c \in S$. Then $|V(C_1) \cap S| + |V(C_2) \cap S| = 3$. We may suppose without loss of generality that $|V(C_1) \cap S| \ge |V(C_2) \cap S|$.

(1) If $|V(C_1) \cap S| = 3$ and $|V(C_2) \cap S| = 0$, then G has a perfect matching by Theorem 3.6. Further, if $|V(C_1) \cap S| = 2$ and $|V(C_2) \cap S| = 1$, then G has a perfect matching by Lemma 4.3. This proves (1).

(2) We now suppose G is of odd order. By similar arguements, it is easy to see that G has a near perfect matching. □

Our next result provides a construction of 4-γ_c-critical graphs having two cut-vertices.

Lemma 3.8. *Let H_1 and H_2 be complete graphs of order $n \ge 2$ and $m \ge 2$, respectively. Let H be a graph with $n + m + 3$ distinct vertices where $V(H) = V(H_1) \cup V(H_2) \cup \{x, y, z\}$, $E(H[V(H_1) \cup \{x, y, z\}]) = E(H_1) \cup \{xy, yz\} \cup \{xa | a \in V(H_1)\}$, $E(H[V(H_2)]) = E(H_2)$. Further $H[V(H_1) \cup V(H_2)]$ has the following properties*
(i) For all $u \in V(H_1), |N_{H_2}(u)| = |V(H_2)| - 1$.
(ii) For all $u \in V(H_2)$, there exists $v \in V(H_1)$ such that $uv \in E(H)$.
Then H is a 4-γ_c-critical graph containing x and y as cut-vertices.

Proof. Clearly, x and y are cut-vertices of H. We first show that $\gamma_c(H) = 4$. Since x, y are cut-vertices of H, for each γ_c-set S of H, $\{x, y\} \subseteq S$. Note that $\{x, y\}$ dominates $V(H_1) \cup \{x, y, z\}$. Suppose there exists $a \in V(H_1) \cup V(H_2)$ such that $\{x, y, a\} \succ_c H$. Then $a \in V(H_1)$ and a dominates H_2. But this contradicts hypothesis (i). Thus $\gamma_c(H) > 3$. Choose $p \in V(H_1)$ and $q \in N_{H_2}(p)$. Then $\{p, q\} \succ_c H[V(H_1) \cup V(H_2)]$. So $\{p, q, x, y\} \succ_c H$. Hence, $\gamma_c(H) = 4$.

We next show that H is $4{-}\gamma_c$-critical. Let u and v be a pair of non-adjacent vertices of H. Consider $H + uv$. We distinguish 4 cases.

Case 1 : $z \in \{u, v\}$.
We may assume that $z = u$. Then $v \in \{x\} \cup V(H_1) \cup V(H_2)$. We first suppose that $v = x$. Choose $a \in V(H_1)$ and $b \in N_{H_2}(a)$. Then $\{v, a, b\} \succ_c G + uv$. Now suppose $v \in V(H_1)$. Choose $c \in N_{H_2}(v)$. Then $\{v, c, x\} \succ_c G + uv$. Finally, suppose $v \in V(H_2)$. By hypothesis (ii), there exists a vertex $d \in V(H_1)$ such that $vd \in E(H)$. Then $\{v, d, x\} \succ_c G + uv$. This proves Case 1.

Case 2 : $y \in \{u, v\}$.
We may assume that $y = u$. Then $v \in V(H_1) \cup V(H_2)$. We first suppose that $v \in V(H_1)$. Choose $a \in N_{H_2}(v)$. Then $\{u, v, a\} \succ_c G + uv$. Now suppose $v \in V(H_2)$. By hypothesis (ii), there exists a vertex $b \in V(H_1)$ such that $vb \in E(G)$. Then $\{u, v, b\} \succ_c G + uv$. This proves Case 2.

Case 3 : $x \in \{u, v\}$.
We may assume that $x = u$. By Case 1, we need only to consider when $v \in V(H_2)$. Clearly, $\{u, v, y\} \succ_c G + uv$. This proves Case 3.

Case 4 : $\{u, v\} \subseteq V(H_1) \cup V(H_2)$.
Since H_1 and H_2 are complete, we may assume that $u \in V(H_1)$ and $v \in V(H_2)$. By hypothesis (i), $\{u, x, y\} \succ_c G + uv$. This proves Case 4 and completes the proof of our theorem. □

We next establish a characterization of 4-γ_c-critical graphs with 2 cut-vertices.

Theorem 3.9. *Let G be a 4-γ_c-critical graph with 2 cut-vertices c_1 and c_2. Then G is isomorphic to the graph H defined in Lemma 3.8.*

Proof. We first show that G has exactly 3 blocks. By Lemma 2.5(1), $G - c_1$ contains exactly 2 components. Let A_1 and A_2 be the two components of $G - c_1$. We may suppose without loss of generality that $c_2 \in V(A_2)$. Then, by Theorem 3.2, $B_1 = G[V(A_1) \cup \{c_1\}]$ is a block of G. Similarly, $G - c_2$ contains exactly 2 components, say A'_1, A'_2. We may suppose without loss of generality that $c_1 \in V(A'_1)$. Hence, $B_2 = G[V(A'_2) \cup \{c_2\}]$ is a block of G. Since G has only 2 cut-vertices, it follows that $B_3 = G[V(G) - (V(A_1) \cup V(A'_2))]$ is a block of G. Therefore, G contains exactly 3 blocks B_1, B_2 and B_3.

Let S be a γ_c-set of G. Since c_1 and c_2 are cut-vertices of G, $\{c_1, c_2\} \subseteq S$ by Lemma 2.6(1).

Claim 1 : $|(S-\{c_1,c_2\})\cap V(B_1)|=2$ or $|(S-\{c_1,c_2\})\cap V(B_2)|=2$. Suppose to the contrary that $|(S-\{c_1,c_2\})\cap V(B_1)|\neq 2$ and $|(S-\{c_1,c_2\})\cap V(B_2)|\neq 2$. We distinguish 4 cases.

Case 1 : $|(S-\{c_1,c_2\})\cap V(B_1)|=1$ and $|(S-\{c_1,c_2\})\cap V(B_2)|=1$. Let $\{a\}=(S-\{c_1,c_2\})\cap V(B_1)$ and $\{b\}=(S-\{c_1,c_2\})\cap V(B_2)$. Note that $ac_1, bc_2, c_1c_2\in E(G)$ since S is connected. Since $\gamma_c(G)=4$, there exists $x\in V(B_1)-\{a,c_1\}$ such that $xa\in E(G)$ but $xc_1\notin E(G)$. Similarly, there exists $y\in V(B_2)-\{b,c_2\}$ such that $yb\in E(G)$ but $yc_2\notin E(G)$. So $d(x,y)=d(x,c_1)+d(c_1,c_2)+d(c_2,y)\geq 5$ since c_1 and c_2 are cut-vertices, contradicting Lemma 3.1. Then Case 1 cannot occur.

Case 2 : $|(S-\{c_1,c_2\})\cap V(B_3)|=2$. Then c_1 dominates B_1 and c_2 dominates B_2. Thus $|V(B_3)-\{c_1,c_2\}|\geq 2$. Choose $x\in V(B_1)-\{c_1\}$ and $y\in V(B_2)-\{c_2\}$. Consider $G+xy$. Let S' be a γ_c-set of $G+xy$. By Lemmas 2.2(1) and 2.2(2), $|S'|\leq 3$ and either $x\in S'$ or $y\in S'$. We may suppose without loss of generality that $x\in S'$. We first show that $y\in S'$. Suppose $y\notin S'$. Then $c_2\notin S'$ by Lemma 2.2(3). By a connectedness of S', $c_1\in S'$. Since $|(S-\{c_1,c_2\})\cap V(B_3)|=2$, $\{x,c_1\}$ does not dominate B_3. Thus $|S'-\{x,c_1\}|=1$. Let $\{d\}=S'-\{x,c_1\}$. Then $d\in V(B_3)-\{c_1,c_2\}$ and $\{c_1,d\}\succ_c (B_2\cup B_3)-\{y\}$. Consequently, $\{c_1,d,c_2\}\succ_c G$, a contradiction. Hence, $y\in S'$ and therefore $\{x,y\}\subseteq S'$. Since $|S'|\leq 3$, $|S'-\{x,y\}|\leq 1$. Because $|V(B_3)-\{c_1,c_2\}|\geq 2$ and no vertex of $\{x,y\}$ is adjacent to a vertex of $V(B_3)-\{c_1,c_2\}$, it follows that either $S'-\{x,y\}=\{c_1\}$ or $S'-\{x,y\}=\{c_2\}$ since S' is connected. We may assume that $S'-\{x,y\}=\{c_1\}$. So $c_1\succ_c V(B_3)-\{c_2\}$. Since B_3 is a block, these exists a vertex $w\in V(B_3)-\{c_2\}$ such that $wc_2\in E(G)$. Then $\{c_1,w,c_2\}\succ_c G$ since $c_1\succ_c (V(B_1)\cup V(B_3))-\{c_2\}$ and $c_2\succ_c B_2$. But this contradicts the fact that $\gamma_c(G)=4$. Hence, Case 2 cannot occur.

Case 3 : $|(S-\{c_1,c_2\})\cap V(B_1)|=1$ and $|(S-\{c_1,c_2\})\cap V(B_3)|=1$. Let $\{a\}=(S-\{c_1,c_2\})\cap V(B_1)$ and $\{b\}=(S-\{c_1,c_2\})\cap V(B_3)$. Clearly, $c_2\succ_c B_2$ and $ac_1\in E(G)$. Further, $bc_1\in E(G)$ or $bc_2\in E(G)$. Since $S\cap V(B_1)=\{a,c_1\}$, there is a vertex $x\in V(B_1)-\{a,c_1\}$ such that $xc_1\notin E(G)$ but $xa\in E(G)$. Choose a vertex $y\in V(B_2)-\{c_2\}$. Since c_1 and c_2 are cut-vertices of G, $d(x,y)=d(x,c_1)+d(c_1,c_2)+d(c_2,y)\leq 4$ by Lemma 3.1. It follows that $c_1c_2\in E(G)$. Because $\gamma_c(G)=4$, there exists $d\in V(B_3)-\{b,c_1,c_2\}$ such that $dc_1, dc_2\notin E(G)$ but $bd\in E(G)$. Now consider $G+dy$. Let S' be a γ_c-set of $G+dy$. Note that c_1 is a cut-vertex of $G+dy$. Then $c_1\in S'$. Since $c_1d\notin E(G)$ and $c_1y\notin E(G)$, there exists a vertex, w say, of $S'-\{c_1,d,y\}$ such that either $\{c_1,w,d\}\succ_c G+dy$ or $\{c_1,w,y\}\succ_c G+dy$. In either case $w\notin V(B_1)$. Thus $c_1\succ_c B_1$. But this contradicts the fact that $xc_1\notin E(G)$. Hence, Case 3 cannot occur.

Case 4 : $|(S-\{c_1,c_2\})\cap V(B_2)|=1$, $|(S-\{c_1,c_2\})\cap V(B_3)|=1$. By similar arguments as in the proof of Case 3, Case 4 follows.

Claim 2 : If $|(S-\{c_1,c_2\})\cap V(B_1)|=2$, then $|V(B_2)|=2$. Since no vertex of $(S-\{c_1,c_2\})\cap V(B_1)$ is adjacent to a vertex of $(V(B_2)\cup V(B_3))-\{c_1\}$, $\{c_1,c_2\}\succ_c B_3$ and $c_2\succ_c B_2$. It follows that for each $z\in N_{B_1}(c_1)$, there is a vertex $z'\in V(B_1)-\{c_1,z\}$ such that $z'z\notin E(G)$ and $z'c_1\notin E(G)$.

Hence, if D is a connected dominating set of size at most 2 of B_1, then $c_1 \notin D$. Choose $x \in V(B_2) - \{c_2\}$ and consider $G + xc_1$. Let S' be a γ_c-set for $G + xc_1$. By Lemma 2.2(1), $|S'| \leq 3$. Since c_1 is a cut-vertex of $G + xc_1$, $c_1 \in S'$. Then $|S' \cap V(B_1)| \geq 3$ because $S' \cap V(B_1) \succ_c B_1$. Hence, $|S' \cap V(B_1)| = 3$ and thus $x \notin S'$. Consequently, $V(B_2) - \{c_2, x\} = \emptyset$. This settles our claim.

Claim 3 : If $|(S - \{c_1, c_2\}) \cap V(B_2)| = 2$, then $|V(B_1)| = 2$.
By similar arguments as in the proof of Claim 2, our claim follows.

We may now assume that $|(S - \{c_1, c_2\}) \cap V(B_1)| = 2$. Thus $\{c_1, c_2\}$ dominates $B_2 \cup B_3$. By Claim 2, $|V(B_2)| = 2$. Let $\{x\} = V(B_2) - \{c_2\}$. Then $xc_2 \in E(G)$.

Claim 4 : $V(B_3) = \{c_1, c_2\}$.
Suppose to the contrary that there exists $b \in V(B_3) - \{c_1, c_2\}$. Consider $G + bx$. Let S' be a γ_c-set of $G + bx$. By Lemma 2.2(1), $|S'| \leq 3$. Since c_1 is a cut-vertex of $G + bx$, $c_1 \in S'$. If $|(S' \cap V(B_1)) - \{c_1\}| < 2$, then $((S' \cap V(B_1)) - \{c_1\}) \cup \{c_1, c_2\}$ is a connected dominating set of size at most 3 of G, a contradiction. Hence, $|(S' \cap V(B_1)) - \{c_1\}| \geq 2$ and thus $|(S' \cap V(B_1)) - \{c_1\}| = 2$ since $|S'| \leq 3$. It follows that $|S' \cap V(B_1)| = 3$. Because S' is a γ_c-set of $G + bx$, $S' \cap \{b, x\} \neq \emptyset$. Then $|S'| \geq 4$, a contradiction. This proves our claim.

Recall that $B_1 = G[V(A_1) \cup \{c_1\}]$. By Lemma 2.5(2), $G[N_{B_1}(c_1)] = G[N_{A_1}(c_1)]$ is complete. Since $|S \cap V(A_1)| = |(S - \{c_1, c_2\}) \cap V(B_1)| = 2$, $G[\overline{N}_{B_1}(c_1)] = G[\overline{N}_{A_1}(c_1)]$ is complete by Theorem 3.3. Further,
(i) For all $x \in N_{B_1}(c_1)$, $|N_{\overline{N}_{B_1}(c_1)}(x)| = |\overline{N}_{B_1}(c_1)| - 1$.
(ii) For all $y \in \overline{N}_{B_1}(c_1)$, there exists $x \in N_{B_1}(c_1)$ such that $xy \in E(G)$.
Therefore, G is isomorphic to the graph H defined in Lemma 3.8 as required. This completes the proof of our theorem. □

We conclude this chapter by providing a necessary and sufficient for $4-\gamma_c-$critical graphs of odd order having connectivity one to be factor critical.

Theorem 3.10. *Let G be a $4-\gamma_c-$critical graph of odd order with a cut-vertex c. Let S be a γ_c-set of $G - c$. Further, let C_1 and C_2 be the two components of $G - c$. Suppose $|S \cap V(C_2)| \leq |S \cap V(C_1)|$. Then G is factor critical if and only if $|S \cap V(C_1)| = 2$ and $|V(C_2)| \geq 4$ is even.*

Proof. Assume that G is factor critical. Since $c \in S$, $|S \cap V(C_1)| = 3$ or $|S \cap V(C_1)| = 2$. We first show that $|S \cap V(C_1)| = 2$. Suppose to the contrary that $|S \cap V(C_1)| = 3$. By Theorem 3.6(1), $\delta(G) = 1$. But this contradicts Theorem 2.10. Hence, $|S \cap V(C_1)| \neq 3$. Then $|S \cap V(C_1)| = 2$. If $|V(C_2)|$ is odd, then $\omega_o(G - c) = 2$ by Lemma 2.5(1), contradicting Theorem 2.9. Hence, $|V(C_2)|$ is even. Further, it follows by Theorems 2.9 and 3.3(4) that $|V(C_2)| \geq 4$. This proves the necessity.

We now establish the sufficiency. Suppose $|S \cap V(C_1)| = 2$ and $|V(C_2)|$ is even. Then $|V(C_1)|$ is even.

Claim 1 : $G[V(C_1)]$ contains a perfect matching.
By Lemma 2.5(2) and Theorem 3.3(1), $G[N_{C_1}(c)]$ and $G[\overline{N}_{C_1}(c)]$ are complete.

Then $G[\overline{N}_{C_1}(c)]$ has a perfect matching or has a maximum matching M such that $|\overline{N}_{C_1}(c) - V(M)| = 1$. If $G[\overline{N}_{C_1}(c)]$ has a perfect matching, then $G[N_{C_1}(c)]$ has a perfect matching. So we now suppose that $|\overline{N}_{C_1}(c) - V(M)| = 1$. Let x be an $M-$unsaturated. Then, by Theorem 3.3(3), there is $y \in N_{C_1}(c)$ such that $xy \in E(G)$. Because $|V(C_1)|$ is even, $|V(C_1)-(V(M)\cup\{x,y\})| = |N_{C_1}(c)-\{y\}|$ is even. So $G[N_{C_1}(c)-\{y\}]$ contains M_1 as a perfect matching. Thus $G[C_1]$ contains $M \cup M_1 \cup \{xy\}$ as a perfect matching.

Claim 2 : For each $u \in V(C_1)$, $G[(V(C_1) - \{u\}) \cup \{c\}]$ contains a perfect matching.
Note that $G[N_{C_1}(c) \cup \{c\}]$ is complete. By similar arguements as in the proof of Claim 1, it is not difficult to show that $G[(V(C_1) - \{u\}) \cup \{c\}]$ has a perfect matching.

Let $v \in V(G)$. We now show that $G - v$ has a perfect matching.

Case 1 : $v \in V(C_2) \cup \{c\}$.
By Lemma 2.5(2) and Theorem 3.3(5), $G[V(C_2)]$ is complete. Then $G[(V(C_2) - \{v\}) \cup \{c\}]$ is of order at least 4 and has a perfect matching. By Claim 1, $G[V(C_1)]$ has a perfect matching. Thus $G - v$ has a perfect matching. This proves Case 1.

Case 2 : $v \in V(C_1)$.
By Claim 2, $G[(V(C_1) - \{v\}) \cup \{c\}]$ has a perfect matching. By Lemma 2.5(2) and Theorem 3.3(5), $G[V(C_2)]$ is complete. Since $|V(C_2)|$ is even and at least 4, $G[V(C_2)]$ has a perfect matching. Then $G - v$ contains a perfect matching. This proves Case 2 and completes the proof of our Theorem. $\square$

Chapter 4

2-connected 4-γ_c-critical graphs

In this chapter, we establish sufficient conditions for 2-connected 4-γ_c-critical graphs of even order to contain a perfect matching and of odd order to contain a near perfect matching in terms of $K_{1,n}$-free. We also show that 2-connected 4-γ_c-critical $K_{1,n}$-free graphs of even order need not be bicritical when n is 3 and 4 and 3-connected 4-γ_c-critical $K_{1,n}$-free graphs of even order need not be bicritical when n is 4 and 5.

We do not know whether $2-$connected $4-\gamma_c-$critical graphs of even order contain a perfect matching. Sumner [14] proved that, if G is an $n-$connected $K_{1,n+1}-$free graph of even order, then G has a perfect matching. Then a $2-$ connected $K_{1,3}-$free graph has a perfect matching. Note that $2-$connected $K_{1,4}-$free graphs need not contain a perfect matching. The graph in Figure 1 is a counter example. We first show that a $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free graph contains a perfect matching if it is of even order and contains a near perfect matching if it is of odd order. We begin with the following lemmas.

Lemma 4.1. *Let G be a $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free graph with a cutset C. Let $H_1, H_2, ..., H_n$ be the components of $G-C$. Suppose S is a γ_c-set of $G+u_iu_j$ where $u_i \in V(H_i)$, $u_j \in V(H_j)$, $1 \le i \ne j \le n$. If $n \ge 5$, then $|S \cap C| = 2$.*

Proof. By Lemma 2.2(2), $S \cap \{u_i, u_j\} \ne \emptyset$. Then $|S \cap C| \le 2$. Because S is connected and $n \ge 5$, it follows that $|S \cap C| \ge 1$. Suppose $|S \cap C| = 1$. Let $\{c\} = S \cap C$. So c is adjacent to some vertex in at least 4 different components of $G-C$, contradicting the fact that G is $K_{1,4}-$free. Then $|S \cap C| = 2$. This proves our lemma. □

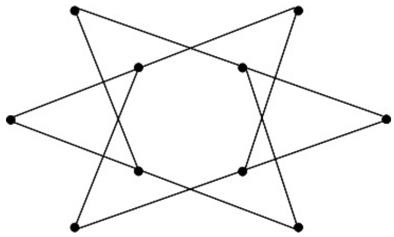

Figure 1 : A 2−connected $K_{1,4}$−free graph of even order without a perfect matching.

Lemma 4.2. *Let G be a 2−connected $4-\gamma_c$−critical $K_{1,4}$−free graph. Let C be a cutset of G. Then $\omega(G-C) \le 6$.*

Proof. Let $H_1, H_2, ..., H_n$ be the components of $G-C$. Suppose to the contrary that $n \ge 7$. Let $u_1 \in V(H_1)$ and $u_2 \in V(H_2)$. Consider $G+u_1u_2$. Let S be a γ_c−set of $G+u_1u_2$. By Lemma 4.1, $|S \cap C| = 2$. Let $\{c_1, c_2\} = S \cap C$. So $|\{u_1, u_2\} \cap S| = 1$ by Lemma 2.2(2). We may suppose without loss of generlity that $u_1 \in S$. Then $\{c_1, c_2\} \succ \cup_{i=3}^{n} H_i$. Because $n \ge 7$, c_1 together with c_2 is adjacent to at least 5 components of $\cup_{i=3}^{n} H_i$. By a connectedness of S, at least one vertex of $\{c_1, c_2\}$ is adjacent to u_1, c_1 say. Since G is $K_{1,4}$−free, it follows that c_1 and c_2 are adjacent to at most 6 different components. Because $\{c_1, c_2\}$ dominates $\cup_{i=3}^{n} V(H_i) \cup \{u_1\}$, we have that $n \le 7$. So $n = 7$. Without loss of generality we may assume that c_2 dominates H_5, H_6, H_7 and c_1 dominates H_3, H_4 and $\{u_1\}$. So c_2 is not adjacent to any vertex in H_3, H_4 and $\{u_1\}$. By the connectedness of S, $c_1c_2 \in E(G)$. Then G contains $K_{1,4}$ centered at c_1 as an induced subgraph, a contradiction. Then $\omega(G-C) \le 6$, completing the proof of our lemma. □

Lemma 4.3. *If G is 2−connected $4-\gamma_c$−critical $K_{1,4}$−free graph, then $\omega_o(G-C) \le |C|+1$.*

Proof. Let $\omega_o(G-C) = n$ and $|C| = m$. Suppose to the contrary that $\omega_o(G-C) \ge |C|+2$. Then, by Lemma 4.2, $m+2 \le n \le 6$. Then $2 \le m \le 4$ since G is 2−connected. Let $H_1, H_2, ..., H_n$ denote the odd components of $G-C$. We distinguish 3 cases according to m.

Case 1 : $m = 2$.

Then C is a minimum cutset and $n \ge 4$. Thus for each vertex $c \in C$, $N_{H_i}(c) \ne \emptyset$ for all $1 \le i \le n$. This implies that G contains $K_{1,4}$ as an induced subgraph, a contradiction. This settles Case 1.

Case 2 : $m = 3$.

Then $n \ge 5$. Because G is 2 connected for $1 \le i \le n$, there are at least two edges joining vertices of H_i to two different vertices of C. Since $n \ge 5$, there are at least 10 edges joining vertices of $\cup_{i=1}^{n} V(H_i)$ to vertices of C. Because $|C| = 3$, there is a vertex $c \in C$ such that c is adjacent to some vertex in at least 4 different components of $G-C$. Thus G contains $K_{1,4}$ centered at c as an induced subgraph, a contradiction. This settles Case 2.

Case 3 : $m = 4$.
By Lemma 4.2, $n = 6$ and thus $G - C$ has no even components. We first establish the following claim.

Claim For each $x \in C$, $N_C(x) \neq \emptyset$.
Suppose c is a vertex of C such that $N_C(c) = \emptyset$. Because G is $K_{1,4}-$free and $n = 6$, there are components H_k and H_l with $N_{H_k}(c) = \emptyset$ and $N_{H_l}(c) = \emptyset$, for some $1 \leq k \neq l \leq 6$. Let $x_k \in V(H_k)$ and $x_l \in V(H_l)$. Since G is $2-$connected, by Theorem 2.8 there are at least two internally disjoint paths between x_k and x_l. Because $N_{H_k}(c) = \emptyset$ and $N_{H_l}(c) = \emptyset$, c cannot be an internally vertex of such paths. Since $|C| = 4$, there is a vertex of C, c_1 say, such that $c_1u_k \in E(G)$ and $c_1u_l \in E(G)$ for some $u_k \in V(H_k)$ and $u_l \in V(H_l)$. Consider $G + u_ku_l$. Let S denote the γ_c-set of $G+u_ku_l$. It follows by Lemma 2.2(3) that $c_1 \notin S$. By Lemma 4.1, $|S \cap C| = 2$. Thus by Lemma 2.2(2), $|S \cap \{u_k, u_l\}| = 1$. We may suppose without loss of generality that $S \cap \{u_k, u_l\} = \{u_k\}$. Because $N_G(c) \cap (V(H_k) \cup V(H_l) \cup (C - \{c\})) = \emptyset$, if $c \notin S$, then no vertex of S is adjacent to c, a contradiction. Hence, $c \in S$. But then S is disconnected since c is adjacent to neither u_k nor a vertex of $C - \{c\}$. This settles our claim.

Because G is $2-$connected, for $1 \leq i \leq n$, there are at least two edges joining vertices of H_i to two different vertices of C. Then there are at least 12 edges joining vertices of $\cup_{i=1}^{6} V(H_i)$ to vertices of C. Because G is $K_{1,4}-$free, each vertex of C is adjacent to some vertex in exactly 3 components of $G - C$. Let $c_1 \in C$. We may suppose without loss of generality that c_1 is adjacent to a vertex of H_1, H_2 and H_3. Then c_1 is not adjacent to any vertex of $\cup_{i=4}^{6} H_i$. Let $u_5 \in V(H_5)$ and $u_6 \in V(H_6)$. Consider $G + u_5u_6$. Let S be a γ_c-set of $G + u_5u_6$. By Lemmas 4.1 and 2.2(2), $|S \cap C| = 2$ and thus $|S \cap \{u_5, u_6\}| = 1$. Let $S \cap C = \{x, y\}$. Without loss of generality, we may assume that $S \cap \{u_5, u_6\} = \{u_5\}$ and $xu_5 \in E(G)$. Note that $x \neq c_1$. We distinguish 2 subcases.

Subcase 3.1 : $|V(H_6)| > 1$.
Then $\{x, y\}$ dominates $\cup_{i=1, i\neq 5}^{6} V(H_i) - \{u_6\}$. Thus x together with y is adjacent to some vertex in 6 different components. So each vertex of $\{x, y\}$ is adjacent to some vertex in exactly 3 different components of $G - C$ by $K_{1,4}-$free. Suppose y dominates $(H_j \cup H_k \cup H_l) - \{u_6\}$ where $1 \leq j, k, l \leq 6$. Clearly, $5 \notin \{j, k, l\}$. Since G is $K_{1,4}-$free and $n = 6$, x is not adjacent to any vertex of $H_j \cup H_k \cup H_l$. Similarly, y is not adjacent to any vertex of $N_G(x) \cap (\cup_{i=1}^{6} V(H_i))$. Then $yu_5 \notin E(G)$. Because S is connected, $xy \in E(G)$. But then y is a center of $K_{1,4}$, a contradiction. This completes the proof of our subcase.

Subcase 3.2 : $|V(H_6)| = 1$.
Then $V(H_6) = \{u_6\}$. By Lemma 2.2(2), $xu_6 \notin E(G)$ and $yu_6 \notin E(G)$. Since G is $2-$connected, u_6 is adjacent to every vertex of $C - \{x, y\}$. It follows that $y = c_1$. Thus $S \cap C = \{c_1, x\}$. Since $c_1u_5 \notin E(G)$, by the connectedness of S, $c_1x \in E(G)$. Now $\{c_1, x\} \succ_c \cup_{i=1}^{4} V(H_i) \cup \{u_5\}$. Because c_1 is not adjacent to any vertex of H_4, x is adjacent to every vertex of H_4. By $K_{1,4}-$free of G, x is adjacent to some vertex in at most one component of H_1, H_2, H_3. We may suppose without loss of generality that x is not adjacent to any vertex of $V(H_1) \cup V(H_2)$. Let

$v_1 \in N_{H_1}(c_1), v_2 \in N_{H_2}(c_1)$. Consider $G+v_1v_2$. Let S' be a γ_c-set of $G+v_1v_2$. By Lemma 4.1, $|S' \cap C| = 2$. Because $c_1v_1, c_1v_2 \in E(G)$, $c_1 \notin S'$, by Lemma 2.2(3). We first show that $x \notin S' \cap C$. Suppose $S' \cap C = \{x, c_2\}$ where $c_2 \in C - \{c_1, x\}$. By the connectedness of S and $xv_1, xv_2 \notin E(G)$, $c_2x \in E(G)$. So $\{c_2, x\} \succ_c \cup_{i=3}^{6} H_i$. Then $\{c_1, c_2, x\} \succ_c \cup_{i=1}^{6} H_i$. Since $G - C$ has no even components and by our claim, $\{c_1, c_2, x\}$ is a γ_c-set of size 3 of G, a contradiction. Hence, $x \notin S' \cap C$. Then $S' \cap C = C - \{c_1, x\}$. Now let $\{c_3, c_4\} = S' \cap C$. Then $S' \cap \{v_1, v_2\} = \{v_1\}$ or $S' \cap \{v_1, v_2\} = \{v_2\}$. We first show that $S' \cap \{v_1, v_2\} = \{v_1\}$. Suppose to the contrary that $S' \cap \{v_1, v_2\} = \{v_2\}$. Then $\{c_3, c_4\} \succ \cup_{i=3}^{6} H_i$. Because c_3 and c_4 are adjacent to u_6. By $K_{1,4}-$free, each vertex of $\{c_3, c_4\}$ is adjacent to some vertex in exactly 2 components of H_2, H_3, H_4 and H_5. So c_3 and c_4 is not adjacent to any vertex of H_1. Because x is not adjacent to any vertex of H_1, c_1 is the only one vertex of C which is adjacent to vertex of H_1. So c_1 is a cut-vertex, a contradiction. Then $S' \cap \{v_1, v_2\} = \{v_1\}$. By similar arguements, $S' \cap \{v_1, v_2\} \neq \{v_1\}$. This contradicts Lemma 2.2(2), completing the proof of Subcase 3.2 and the Case 3. This completes the proof of our lemma. □

Theorem 4.4. *Let G be $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free graph. Then*

(1) If G is of even order, then G has a perfect matching.

(2) If G is of odd order, then G has a near perfect matching.

Proof. (1) Suppose to the contrary that G has no perfect matching. Then, by Theorem 2.7, there exists a cutset C such that $\omega_o(G-C) > |C|$. Let $|C| = m$ and $\omega_o(G-C) = n$. By Lemma 4.2 and parity, $m+2 \leq n \leq 6$. Thus by Lemma 4.3, $n \leq m+1 < m+2 \leq n$, a contradiction. Hence, G has a perfect matching. This complete the prove of (1).

(2) Let G_1 be a graph obtained by adding a new vertex x to G and joining x to each vertex of G. That is $V(G_1) = V(G) \cup \{x\}$ and $E(G_1) = E(G) \cup \{xu | u \in V(G)\}$. Clearly, G_1 is of even order. It is not difficult to show that G_1 has a perfect matching if and only if G has a near perfect matching. So we only need to show that G_1 has a perfect matching. Suppose to the contrary that G_1 has no perfect matching. Then, by Theorem 2.7, there exists a cutset C_1 such that $\omega_o(G_1 - C_1) > |C_1|$. By parity, $|C_1| + 2 \leq \omega_o(G_1 - C_1)$. Since $xu \in E(G_1)$ for all $u \in V(G_1) - \{x\}$, $x \in C_1$. Thus $C = C_1 - \{x\}$ is a cutset of G. Then $|C|+1 = |C_1|$ and $\omega_o(G-C) = \omega_o(G_1 - C_1)$. Thus $(|C|+1)+2 = |C_1|+2 \leq \omega_o(G_1 - C_1) = \omega_o(G-C)$. Because $G_1 - x = G$ is $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free, it follows by Lemma 4.3 that $|C|+3 \leq \omega_o(G-C) \leq |C|+1$, a contradiction. Thus G_1 has a perfect matching and Hence, G has a near perfect matching. This completes the proof of (2) and our theorem. □

We might expect that a $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free graph G is factor critical if order of G is odd and bicritical if order of G is even. But these are not the cases. The graphs in Figure 2 and 3 are $2-$connected $4-\gamma_c-$critical $K_{1,4}-$free. Observe that the graph in Figure 2 is of order 9 which is not factor critical and the graph in Figure 3 is of order $2(m+n)+6$, for some positive integer m,n which is not bicritical. Note that in our diagram a *double line* denotes the join between corresponding graphs. Although we strengthen the condition $K_{1,4}-$free of G to $K_{1,3}-$free, G need not be bicritical. One of them is shown in Figure 4.

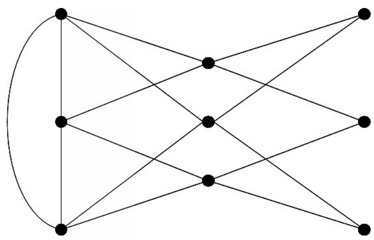

Figure 2 : A 2− connected $4-\gamma_c-$critical $K_{1,4}-$free graph of odd order which is not factor critical.

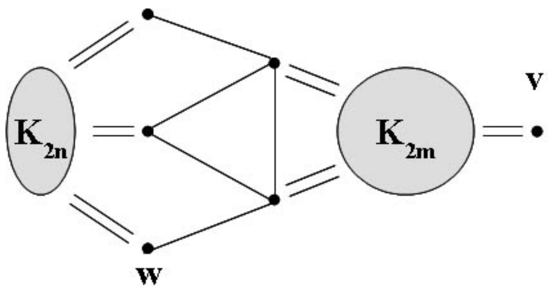

Figure 3 : A 2−connected $4-\gamma_c-$critical $K_{1,4}-$free graph of even order with $\delta(G)=4$ which is not bicritical.

We conclude this chapter by establishing that 3-connected 4-γ_c-critical graphs of even order need not be bicritical. Some examples are graphs in Figures 5 and 6. Note that the graph in Figure 5 is $K_{1,5}$-free while the graph in Figure 6 is $K_{1,4}$-free. In fact, the graph in Figure 6, can be expand to the one of larger order as follow. Choose a white vertex w_i, for some $1 \leq i \leq 4$. Then replace the vertex w_i with a complete graph $K_n(w_i)$ where $n \geq 2$ and join every vertex of $K_n(w_i)$ to every vertex in the neighberhood of w_i. The resulting graph is 3-connected 4-γ_c-critical $K_{1,4}$-free graph. If we replace every vertex of $\{w_1, ..., w_4\}$ with $K_{n_1}(w_1)$, $K_{n_2}(w_2)$, $K_{n_3}(w_3)$ and $K_{n_4}(w_4)$ where $n_i \geq 3$ is odd for $1 \geq i \geq 4$, then the minimum degree of the resulting graph is $min\{n_i{+}2|1 \leq i \leq 4\}$. Note also that this graph is not bicritical. Further, Plummer [13] proved that 3-connected $K_{1,3}$-free graphs are bicritical. So it is not possible to provide suffcient conditions for 3-connected 4-γ_c-critical $K_{1,n}$-free graphs, for $n \geq 4$ to be bicritical.

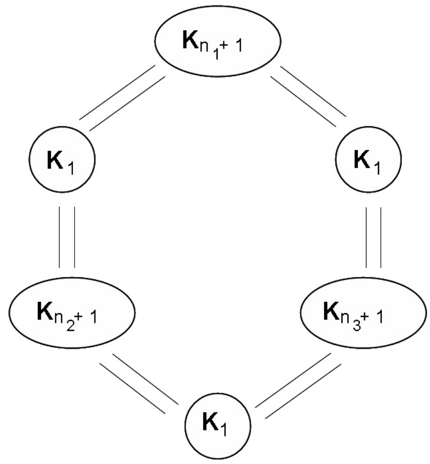

Figure 4 : A 2−connected $4 - \gamma_c$−critical $K_{1,3}$−free graph with $\delta(G) = 4$ which is not bicritical.

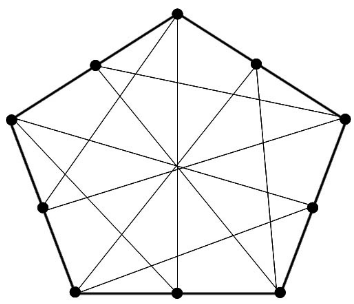

Figure 5 : A 3−connected $4 - \gamma_c$−critical $K_{1,5}$−free graph of order 10 which is not bicritical.

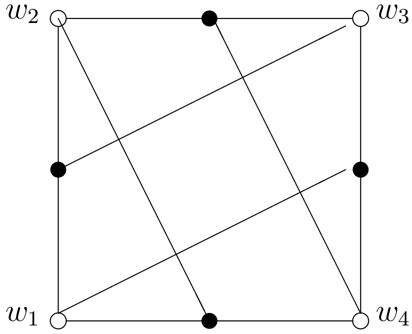

Figure 6 : A 3−connected $4-\gamma_c$−critical $K_{1,4}$−free graph of order 8 which is not bicritical.

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Appendix

or 2, respectively), $s = 2$ and $\gamma_a(T) = 3 > (3n - \ell - s + 4)/8$. Thus assume that $n' \geq 2$. Then $S - \{u, v\}$ is a gda of T' and by induction on T', and the fact that $n' = n-k-3$, $\ell' \leq \ell-k+1$, $s' \leq s$, we obtain $\gamma_a(T) > (3n - \ell - s + 4)/8$. Thus assume that $w \in S$. If $d(w) = 2$, then let $T' = T - T_v$. We obtain $\gamma_a(T) > (3n - \ell - s + 4)/8$. Finally if $d(w) \geq 3$, then seeing the previous cases, we may assume that all children of w are in S except some leaves (if any). If w is a support vertex, then let $T' = T - T_u$ and if w is not a support vertex, then let $T' = T - T_v$. In each case $S \cap V(T')$ is a gda and it can be seen that $\gamma_a(T) > (3n - \ell - s + 4)/8$. □

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On 4-γ_c-critical graphs with cut-vertices

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Abstract

Let $\gamma_c(G)$ denote the connected domination number of a graph G. G is said to be k-γ_c-critical if $\gamma_c(G) = k$ and for each pair of non-adjacent vertices u and v of G, $\gamma_c(G + uv) < k$. In this paper, we show that 4-γ_c-critical graphs contain at most two cut-vertices. A characterization of 4-γ_c-critical graphs containing exactly two cut-vertices is given. We also establish that a 4-γ_c-critical graph of even order having connectivity one contains a perfect matching.

1 Introduction

Let G denote a finite undirected simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq V(G)$, $G[S]$ denotes the induced subgraph of G by S. The neighborhood of a vertex v of G is denoted by $N_G(v)$ while $N_S(v)$ denotes either $N_G(v) \cap S$ if S is a subset of $V(G)$ or $N_G(v) \cap V(S)$ if S is a subgraph of G. A degree of v in G, denoted by $deg_G(v)$, is $|N_G(v)|$. A vertex v is called an end vertex of G if $deg_G(v) = 1$. The distance between vertices u and v of G, denoted by $d(u, v)$, is the length of a shortest (u, v)−path in G. The diameter of G, denoted by diam(G), is the maximum distance between two vertices of G.

A vertex v of G is called a cut-vertex if the number of components of $G - v$ is more than the number of components of G. A block of a graph G is a maximal connected subgraph with no cut-vertices. An end block of G is a block of G containing exactly one cut-vertex of G.

For $M \subseteq E(G)$, M is a matching in G if no two edges of M have a common end vertex. A matching M in G is maximum if G has no match-

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ing M' with $|M'| > |M|$. For a matching M in G, a vertex v of G is called M−saturated if v is incident with some edge of M; otherwise, v is M−unsaturated. If every vertex of G is M−saturated, then M is called a perfect matching.

A subset S of $V(G)$ is a dominating set of G if every vertex of G either belongs to S or is adjacent to a vertex of S. A dominating set S of G is a connected dominating set of G if S dominates G and $G[S]$ is connected. We will write $S \succ_c G$ if S is a connected dominating set of G. Further, if $S = \{u\}$, then we say that u dominates G rather than $\{u\}$ dominates G. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by $\gamma(G)$. Similarly, the minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$. We say that S is a γ_c-set of G if S is a connected dominating set of G with $|S| = \gamma_c(G)$. Note that $\gamma(G) \leq \gamma_c(G)$ and only connected graphs can contain a connected dominating set. In what follows, we shall assume that all graphs are connected.

Graph G is said to be k-γ-critical if $\gamma(G) = k$ but $\gamma(G + uv) < k$ for each pair of non-adjacent vertices u and v of G. The concept of k-γ-critical graphs was introduced, in 1983, by Sumner and Blitch [11]. They gave a characterization of 2-γ-critical graphs and 3-γ-critical disconnected graphs. They also established some properties of connected 3-γ-critical graphs. One of them is an existence of a perfect matching in 3-γ-critical graph of even order. Since then, the concept of k-γ-critical graphs have been received considerable attention. Most of the known results concern 3-γ-critical graphs, see [3, 4, 7, 8, 9, 10, 13] for examples. Sumner and Wojcicka [12] asked whether connected k-γ-critical graphs of even order, for $k \geq 4$, contain a perfect matching. To date, this problem is unresolved.

In 2004, Chen et al. [6] introduced the concept of connected domination to k-γ -critical graphs. Graph G is said to be k-γ_c-critical graph if $\gamma_c(G) = k$ but $\gamma_c(G+uv) < k$ for each pair of non-adjacent vertices u and v of G. Chen et al. [6] obtained some results on k-γ_c-critical graphs, most of them are analogous to k-γ-critical graphs. One of them concerns an existence of a perfect matching (see Theorem 2.3). Ananchuen [1] and Ananchuen et al. [2] further studied k-γ_c-critical graphs. Most of the known results concern 3-γ_c-critical graphs. Similar to the problem concerning an existence of a perfect matching in k-γ- critical graphs of even order, for $k \geq 4$, posed by Sumner and Wojcicka [12], we might ask whether k-γ_c-critical graphs of even order, for $k \geq 4$, contain a perfect matching. We will show in Section 3 that a 4-γ_c-critical graph of even order having connectivity one contains a perfect matching. This partially responds to such a problem. Further, we establish that 4-γ_c-critical graphs contain at most two cut-vertices. A characterization of 4-γ_c-critical graphs containing exactly two cut-vertices is also given.

2 Prelimminaries

In this section we state a number of results that we make use of in establishing our results.

Theorem 2.1. *(see Ghartrand and Oellermann [5] p. 24) Let G be a connected graph with at least one cut-vertex. Then G has at least 2 end blocks.*

Our next result, first appeared in [6], follows immediately from the definition of k-γ_c-critical graphs.

Lemma 2.2. *[6] Let G be a k-γ_c-critical graph and let u and v be a pair of non-adjacent vertices of G. Suppose S is a γ_c-set of $G + uv$. Then*

(1) $k - 2 \leq |S| \leq k - 1$,

(2) $S \cap \{u, v\} \neq \emptyset$,

(3) If $u \in S$ and $v \notin S$, then $N_G(v) \cap (S - \{u\}) = \emptyset$.

Chen et al. [6] established a result concerning an existence of a perfect matching in 3-γ_c-critical graphs. More precisely, they proved that :

Theorem 2.3. *[6] Let G be a 3-γ_c-critical graph. If C is a graph of even order, then G has a perfect matching.*

The last two results provide some properties of k-γ_c-critical graphs with a cut-vertex.

Lemma 2.4. *[1] For $k \geq 3$, let G be a k-γ_c-critical graph with a cut-vertex x. Then*

(1) $C - x$ contains exactly two components.

(2) If C_1 and C_2 are the components of $G-x$, then $G[N_{C_1}(x)]$ and $G[N_{C_2}(x)]$ are complete.

Lemma 2.5. *[1] For $k \geq 3$, let G be a k-γ_c-critical graph with a cut-vertex x and let C_1 and C_2 be the components of $G - x$. Suppose S is a γ_c-set of G. Then*

(1) $x \in S$.

(2) If C is a non-singleton component of $G - x$ with $\gamma_c(C) = k - 1$, then C is $(k - 1)$-γ_c-critical.

3 Main results

Our first result provides an upper bound on the diameter of k-γ_c-critical graphs.

Lemma 3.1. *If G is a k-γ_c-critical graph, then $diam(G) \leq k$.*

Proof. Let G be a k-γ_c-critical graph. Suppose that $diam(G) = m \geq k+1$. Choose $x, y \in V(G)$ such that $d(x,y) = m$. Consider $G + xy$. Let S be a γ_c-set of $G + xy$. By Lemmas 2.2(1) and 2.2(2), $|S| \leq k-1$ and either $x \in S$ or $y \in S$. We may suppose without loss of generality that $x \in S$. Let $L_i = \{z \in V(G) | d(z,x) = i\}$ for $0 \leq i \leq m$. Clearly, $L_i \neq \emptyset$. Further, $L_0 = \{x\}$ and $y \in L_m$. Let n be a maximum integer in which $S \cap L_i \neq \emptyset$ for each $0 \leq i \leq n$ and $G[\cup_{i=0}^{n}(S \cap L_i)]$ is connected. Since $n+1 \leq |S| \leq k-1$ and $m \geq k+1$, it follows that $n \leq m-3$. Consider L_{n+2}. Clearly, no vertex of $\cup_{i=0}^{n}(S \cap L_i)$ dominates L_{n+2}. Thus, $S \cap (L_{n+2} \cup L_{n+3}) \neq \emptyset$. Consequently, $S \cap L_j \neq \emptyset$ for each $n+3 \leq j \leq m$. Then $y \in S$ because S is connected. Thus $|\cup_{j=n+3}^{m}(S \cap L_j)| \geq m-(n+3)+1 = m-n-2 \geq k+1-n-2 = k-n-1$. Therefore $|S| = |(\cup_{i=0}^{n}(S \cap L_i)) \cup (\cup_{j=n+3}^{m}(S \cap L_j))| \geq 1+n+(k-n-1) = k$, a contradiction. This completes the proof of our lemma. □

Our next result gives an upper bound on a number of cut-vertices of 4-γ_c-critical graphs.

Theorem 3.2. *Let G be a 4-γ_c-critical graph. Then G has at most two cut-vertices.*

Proof. Let C be a set of all cut-vertices of G. Suppose to the contrary that $|C| \geq 3$. By Theorem 2.1, C has at least two end blocks. Let B_1 be an end block of G with $V(B_1) \cap C = \{c_1\}$ for some $c_1 \in C$. By Lemma 2.4(1), the only components of $G - c_1$ are $G[V(B_1) - \{c_1\}]$ and $G_1 = G[V(G) - V(B_1)]$. Since B_1 is an end block, $C - \{c_1\} \subseteq V(C_1)$. Thus G_1 contains at least two cut-vertices. By Theorem 2.1, G_1 has at least two end blocks. Let B_2 be an end block of G_1 in such a way that B_2 is also an end block of G. Such B_2 exists as otherwise G contains exactly one end block. Put $\{c_2\} = V(B_2) \cap (C - \{c_1\})$. Let $c_3 \in C - \{c_1, c_2\}$. Note that c_3 is not adjacent to any vertex of $(V(B_1) \cup V(B_2)) - \{c_1, c_2\}$. Further, $G - c_3$ contains exactly two components, D_1 and D_2 say. Without loss of generality, we may assume that $V(B_1) \subseteq V(D_1)$. Then $V(B_2) \subseteq V(D_1)$ or $V(B_2) \subseteq V(D_2)$.

Claim : For $1 \leq i \leq 2$, c_i dominates B_i.
Suppose to the contrary that c_1 does not dominate B_1. Then there exists a vertex $x \in V(B_1) - \{c_1\}$ such that $xc_1 \notin E(G)$. Since B_1 is connected, there is an $x - c_1$ path of length at least 2. Thus $|V(B_1) - \{c_1, x\}| \geq 1$. Let $x_1 \in V(B_1) - \{c_1, x\}$.

Choose $y \in V(B_2) - \{c_2\}$ and consider $G + xy$. Let S' be a γ_c-set of $G + xy$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since x and y are not adjacent to c_3, $1 \leq |S' - \{x, y\}| \leq 2$. Further, $c_1 \in S'$ or $c_2 \in S'$ because of connectedness of S'. We first suppose that $\{x, y\} \subseteq S'$. Then $S' = \{x, y, c_2\}$ because of connectedness of S'. If $V(B_2) \subseteq V(D_1)$, then no vertex of S' is adjacent to a vertex of D_2, a contradiction. Hence $V(B_2) \subseteq V(D_2)$. But then no vertex of S' is adjacent to c_1 because c_3 is a cut-vertex, again a contradiction. This proves that $|\{x, y\} \cap S'| = 1$. If $\{x, y\} \cap S' = \{x\}$, then $S' = \{x, a, c_1\}$ for some $a \in V(B_1) - \{x, c_1\}$ since $xc_1 \notin E(G)$. But then no vertex of S' is adjacent to a vertex of D_2, a contradiction. Hence, $\{x, y\} \cap S' \neq \{x\}$. By Lemma 2.2(2), $\{x, y\} \cap S' = \{y\}$. Then $c_2 \in S'$ since S' is connected. Because $x_1 \in V(B_1) - \{c_1, x\}$ and $\{y, c_2\} \subseteq V(B_2)$, x_1 is adjacent to neither y nor c_2. It follows that $S' - \{y, c_2\} = \{c_1\}$ since $\{c_1\} = V(B_1) \cap C$. Then $c_1c_2 \in E(G)$ and thus $V(B_2) \subseteq V(D_1)$. But then no vertex of S' is adjacent to a vertex of D_2, a contradiction. So $\gamma_c(G + xy) > 3$, again a contradiction. This proves that c_1 dominates B_1. By similar arguments, c_2 dominates B_2.

Now let $x \in V(B_1) - \{c_1\}$ and $y \in V(B_2) - \{c_2\}$. Clearly, $xy \notin E(G)$. By our claim, $xc_1 \in E(G)$ and $yc_2 \in E(G)$. We now distinguish two cases.

Case 1: $V(B_2) \subseteq V(D_1)$.
Consider $G + xy$. Let S' be a γ_c-set of $G + xy$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since c_3 is a cut-vertex of $G + xy$, $c_3 \in S'$. Because $c_3x \notin E(G)$ and $c_3y \notin E(G)$, it follows that $|S' \cap \{x, y\}| = 1$ since S' is connected. Without loss of generality, we may assume that $x \in S'$. Then $S' = \{x, c_1, c_3\}$. Thus $c_1c_3 \in E(G)$ and c_3 dominates D_2. Further, $\{c_1, c_3\}$ dominates $V(D_1) - \{y\}$ by our claim and the fact that $x \in B_1$ and B_1 is an end block. Since S' is a γ_c-set of $G + xy$ and $xc_2 \notin E(G)$, it follows that $c_2c_1 \in E(G)$ or $c_2c_3 \in E(G)$. But then $\{c_1, c_2, c_3\}$ is a connected dominating set of G, a contradiction. This proves that Case 1 cannot occur.

Case 2 : $V(B_2) \subseteq V(D_2)$.
By Lemma 3.1, $d(x, y) \leq 4$. Since B_1 and B_2 are end blocks, $d(c_1, c_2) \leq 2$. Because $c_1 \in V(D_1)$ and $c_2 \in V(D_2)$, we have $c_1c_3 \in E(G)$ and $c_2c_3 \in E(G)$. Now consider $G + c_1c_2$. Let S' be a γ_c-set of $G + c_1c_2$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since B_i is an end block and $V(B_i) \cap C = \{c_i\}$ for $i = 1, 2$, it follows that $\{c_1, c_2\} \subseteq S'$. If $S' = \{c_1, c_2\}$, then $\{c_1, c_2, c_3\}$ is a connected dominating set of G, a contradiction. Hence, $S' \neq \{c_1, c_2\}$. Therefore, $|S' - \{c_1, c_2\}| = 1$ since $|S'| \leq 3$. Let $\{a\} = S' - \{c_1, c_2\}$. Clearly, $a \neq c_3$. Then $a \in V(D_1)$ or $a \in V(D_2)$.

Subcase 2.1 : $a \in V(D_1)$.
Thus c_2 dominates D_2 and there is a vertex $z \in V(D_1)$ such that $zc_1 \notin E(G)$ but $za \in E(G)$. By our claim, $z \notin V(B_1)$. Observe that no vertex of B_1 is adjacent to z since B_1 is an end block. Consider $G + c_1z$. Let S'' be a γ_c-set of $G + c_1z$. By Lemma 2.2(1), $|S''| \leq 3$. Because c_1 is a cut-vertex

of $G + c_1z$, $c_1 \in S''$. Since S'' is connected and c_1 and c_3 are not adjacent to any vertex of $B_2 - \{c_2\} \subseteq V(D_2)$, it follows that $S'' = \{c_1, c_2, c_3\}$. Thus $c_3z \notin E(G)$.

Recall that $x \in V(B_1) - \{c_1\}$ and $y \in V(B_2) - \{c_2\}$. Now consider $G + xy$. Let S''' be a γ_c-set of $G + xy$. Since $xc_3 \notin E(G)$ and $yc_3 \notin E(G)$ together with the fact that B_i is an end block with $V(B_i) \cap C = \{c_i\}$ for $i = 1, 2$, we have $c_1 \in S'''$ or $c_2 \in S'''$. If $S''' = \{x, y, c_1\}$ or $S''' = \{x, y, c_2\}$, then no vertex of S''' is adjacent to z, a contradiction. Hence, $|S''' \cap \{x, y\}| = 1$. We first suppose that $S''' \cap \{x, y\} = \{x\}$. Then $S''' = \{x, c_1, c_3\}$ since S''' is connected and $x \in B_1$ where B_1 is an end block. But then no vertex of S''' is adjacent to z, a contradiction. Hence $S''' \cap \{x, y\} \neq \{x\}$ and therefore $S''' \cap \{x, y\} = \{y\}$ by Lemma 2.2(2). Clearly, $S''' = \{y, c_2, c_3\}$. But then no vertex of S''' is adjacent to z, again a contradiction. Hence, $\gamma_c(G + xy) > 3$. This contradicts the criticality of G. Thus Subcase 2.1 cannot occur.

Subcase 2.2 : $a \in V(D_2)$.
By similar arguments as in the proof of Subcase 2.1, Subcase 2.2 cannot occur. This proves that Case 2 cannot occur and completes the proof of our theorem. □

In what follows, we shall assume that $|S \cap V(C_2)| \leq |S \cap V(C_1)|$ where S is a γ_c-set of a 4-γ_c-critical graph G with a cut-vertex c and C_1 and C_2 are the only components of $G - c$ (by Lemma 2.4 (1)). Note that, since $c \in S$ by Lemma 2.5(1), either $|S \cap V(C_2)| = 0$ and $|S \cap V(C_1)| = 3$ or $|S \cap V(C_2)| = 1$ and $|S \cap V(C_1)| = 2$.

Our next result provides a structure of such graph with $|S \cap V(C_2)| = 1$ and $|S \cap V(C_1)| = 2$.

Theorem 3.3. *Let G be a 4-γ_c-critical graph with a cut-vertex c and S a γ_c-set of G. Further, let C_1 and C_2 be the two components of $G - c$ and for $1 \leq i \leq 2$, let $X_i = \overline{N}_{C_i}(c)$. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then*
(1) For $1 \leq i \leq 2$, $X_i \neq \emptyset$ and $G[X_i]$ is complete.
(2) For each $x \in N_{C_1}(c)$, $|N_{X_1}(x)| = |X_1| - 1$.
(3) For each $x \in X_1$, there is a vertex $y \in N_{C_1}(c)$ such that $xy \in E(G)$.
(4) $|X_2| = 1$.
(5) If $X_2 = \{b\}$, then $deg_G(b) = |N_{C_2}(c)|$.

Proof. (1) Clearly, for $1 \leq i \leq 2$, $X_i \neq \emptyset$, as otherwise $\gamma_c(G) < 4$. We now show that $G[X_i]$ is complete. Suppose to the contrary that there exist $u, v \in X_1$ such that $uv \notin E(G)$. Let S' be a γ_c-set of $G + uv$. By Lemmas 2.2(1) and 2.2(2), $|S'| \leq 3$ and either $u \in S'$ or $v \in S'$. We may assume without loss of generality that $u \in S'$. Because S' is connected and c is a cut-vertex in $G + uv$, $c \in S'$. Further, since $uc \notin E(G)$, there exists

$a \in (S' - \{u, c\}) \cap V(C_1)$ such that $ua, ac \in E(G)$. Thus $S' \cap V(C_2) = \emptyset$. This implies that c dominates C_2. But this contradicts the fact that $X_2 \neq \emptyset$. Hence, $G[X_1]$ is complete. By similar arguments, $G[X_2]$ is complete. This settles (1).

(2) Suppose to the contrary that there exist $a \in N_{C_1}(c)$ and $b, d \in V(C_1) - \{a\}$ such that $ab, ad \notin E(G)$. Lemma 2.4(2) implies that $b, d \in X_1$. Now consider $G + ab$. Let S' be a γ_c-set of $G + ab$. Since c is a cut-vertex of $G + ab$, $c \in S'$. Then $|S' \cap (V(C_1) \cup V(C_2))| \leq 2$. If $|S' \cap V(C_2)| = 0$, then c dominates C_2. But then $X_2 = \emptyset$, a contradiction. Hence, $|S' \cap V(C_2)| \geq 1$. By Lemma 2.2(2), $\emptyset \neq S' \cap \{a, b\} \subseteq S' \cap V(C_1)$. It follows that $|S' \cap V(C_1)| = 1$ and $|S' \cap V(C_2)| = 1$. Since $b \in X_1$ and S' is connected, it follows that $S' \cap \{a, b\} = \{a\}$. But then no vertex of S' is adjacent to d, a contradiction. This proves that for each $x \in N_{C_1}(c)$, $|N_{X_1}(x)| \geq |X_1| - 1$. We next suppose that there exists $y \in N_{C_1}(c)$ such that $|N_{X_1}(y)| = |X_1|$. Let $\{z\} = S \cap V(C_2)$. Then $\{y, c, z\}$ is a connected dominating set of size 3 of G, a contradiction. This completes the proof of (2).

(3) Let $x \in X_1$. Suppose to the contrary that x is not adjacent to any vertex of $N_{C_1}(c)$. Then $d(x, c) \geq 3$. Since $X_2 \neq \emptyset$, there exists $z \in X_2$. So $d(c, z) \geq 2$. Because c is a cut-vertex of G, $d(x, z) = d(x, c) + d(c, z) \geq 5$. But this contradicts Lemma 3.1 and completes the proof of (3).

(4) Suppose to the contrary that $|X_2| > 1$. Let $b, d \in X_2$. Now consider $G + cb$. Let S' be a γ_c-set of $G + cb$. Then $|S'| \leq 3$. Since c is a cut-vertex of $G + cb$, $c \in S'$. Because $X_1 \neq \emptyset$, $|S' \cap V(C_1)| \geq 1$. If $|S' \cap V(C_1)| = 1$, then the only vertex of $S' \cap V(C_1)$ is adjacent to c and dominates X_1. But this contradicts (2). Hence, $|S' \cap V(C_1)| = 2$ since $|S'| \leq 3$. Then no vertex of S' is adjacent to d, a contradiction. Thus $|X_2| = 1$. This settles (4).

(5) Now let $X_2 = \{b\}$. Suppose to the contrary that there exists $d \in N_{C_2}(c)$ such that $db \notin E(G)$. Now consider $G + bd$. Let S' be a γ_c-set of $G + db$. Then $|S'| \leq 3$ by Lemma 2.2(1). Since c is a cut-vertex of $G + bd$, $c \in S'$. By similar arguments as in the proof of (4), $|S' \cap V(C_1)| = 2$. Thus $S' \cap \{b, d\} = \emptyset$ since $|S' \cap (\{c\} \cup V(C_1))| = 3$ and $|S'| \leq 3$. But this contradicts Lemma 2.2(2) and thus (5) is proved, completing the proof of our theorem. □

Corollary 3.4. *Let C, c, S, C_1, C_2, X_1, X_2 and b be defined as in Theorem 3.3. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then $|X_1| \geq 2$ and $|N_{C_1}(c)| \geq 2$. Further, if G contains an end vertex, then G has exactly one end vertex and the only end vertex of G is b.*

Proof. By Theorems 3.3(2) and 3.3(3) together with the fact that G is connected, $|X_1| \geq 2$ and $|N_{C_1}(c)| \geq 2$. Suppose x is an end vertex of G. Clearly, $x \notin V(C_1) \cup \{c\}$. Thus $x \in V(C_2)$. Since $X_2 = \{b\}$ and $N_G(b) = |N_{C_2}(c)|$, it follows that $x \notin N_{C_2}(c)$. Thus $x \in X_2 = \{b\}$. Therefore, $x = b$ and $|N_{C_2}(c)| = 1$. This proves our corollary. □

Lemma 3.5. *Let G be a 4-γ_c-critical graph of even order with a cut-vertex c and let C_1 and C_2 be the two components of $G - c$. Let S be a γ_c-set of G. Suppose $|S \cap V(C_1)| = 2$ and $|S \cap V(C_2)| = 1$. Then G has a perfect matching.*

Proof. For simplicity, let $X_i = \overline{N}_{C_i}(c)$ for $1 \le i \le 2$. By Theorem 3.3(4), $|X_2| = 1$. Let $|X_1| = n$, $|N_{C_1}(c)| = m$ and $|N_{C_2}(c)| = p$. By Corollary 3.4, $n \ge 2$ and $m \ge 2$. Now let $X_2 = \{b\}$, $X_1 = \{x_i | 1 \le i \le n\}$, $N_{C_1}(c) = \{y_i | 1 \le i \le m\}$ and $N_{C_2}(c) = \{z_i | 1 \le i \le p\}$. By Lemma 2.4(2) and Theorem 3.3(1), $G[X_1]$, $G[N_{C_1}(c)]$ and $G[N_{C_2}(c)]$ are complete. Further, by Theorem 3.3(5), $N_G(b) = N_{C_2}(c)$. Since $|V(G)|$ is even and $V(G) = X_1 \cup N_{C_1}(c) \cup \{c\} \cup N_{C_2}(c) \cup X_2$, it follows that $n + m + p$ must be even. We distinguish 4 cases.

Case 1 : n, m, p are even.
Clearly, $F_1 = \{x_i x_{i+\frac{n}{2}} | 1 \le i \le \frac{n}{2}\} \cup \{y_i y_{i+\frac{m}{2}} | 1 \le i \le \frac{m}{2}\}$ is a perfect matching in C_1. Since $N_{C_2}(c) = N_G(b)$ and $G[N_{C_2}(c)]$ is complete, it follows that $G[N_{C_2}(c) - \{z_1, z_{1+\frac{p}{2}}\}]$ is complete. Then $F_2 = \{z_i z_{i+\frac{p}{2}} | 2 \le i \le \frac{p}{2}\} \cup \{cz_1, bz_{1+\frac{p}{2}}\}$ is a perfect matching in $G[V(C_2) \cup \{c\}]$. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 1.

Case 2 : n is even, m and p are odd.
Clearly, $F_1 = \{x_i x_{i+\frac{n}{2}} | 1 \le i \le \frac{n}{2}\}$ is a perfect matching in $G[X_1]$. Because m is odd, $F_2 = \{y_i y_{i+\lfloor\frac{m}{2}\rfloor} | 1 \le i \le \lfloor\frac{m}{2}\rfloor\} \cup \{y_m c\}$ is a perfect matching in $G[N_{C_1}(c) \cup \{c\}]$. By a similar argument, $F_3 = \{z_i z_{i+\lfloor\frac{p}{2}\rfloor} | 1 \le i \le \lfloor\frac{p}{2}\rfloor\} \cup \{z_p b\}$ is a perfect matching in $G[N_{C_2}(c) \cup \{b\}]$. Thus $F_1 \cup F_2 \cup F_3$ is a perfect matching in G. This proves Case 2.

Case 3 : n and p are odd, m is even.
By Theorem 3.3(3), there exists a vertex $u \in N_{C_1}(c)$ such that $x_n u \in E(G)$. Without loss of generality, we may assume that $u = y_1$. Since $G[X_1]$ and $G[N_{C_1}(c)]$ are complete, $F_1 = \{x_i x_{i+\lfloor\frac{n}{2}\rfloor} | 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{x_n y_1, cy_{1+\frac{m}{2}}\} \cup \{y_i y_{i+\frac{m}{2}} | 2 \le i \le \frac{m}{2}\}$ is a perfect matching in $G[V(C_1) \cup \{c\}]$. Since $G[N_{C_2}(c)]$ is complete and $N_G(b) = N_{C_2}(c)$, it follows that $F_2 = \{z_i z_{i+\lfloor\frac{p}{2}\rfloor} | 1 \le i \le \lfloor\frac{p}{2}\rfloor\} \cup \{bz_p\}$ is a perfect matching in C_2. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 3.

Case 4 : n and m are odd, p is even.
By Theorem 3.3(3), there exists a vertex $u \in N_{C_1}(c)$ such that $x_n u \in E(G)$. Without loss of generality, we may assume that $u = y_m$. Since $G[X_1]$ and $G[N_{C_1}(c)]$ are complete, $F_1 = \{x_i x_{i+\lfloor\frac{n}{2}\rfloor} | 1 \le i \le \lfloor\frac{n}{2}\rfloor\} \cup \{y_i y_{i+\lfloor\frac{m}{2}\rfloor} | 1 \le i \le \lfloor\frac{m}{2}\rfloor\} \cup \{x_n y_m\}$ is a perfect matching in $G[V(C_1)]$. Since $N_G(b) = N_{C_2}(c)$ and $G[N_{C_2}(c)]$ is complete, it follows that $G[N_{C_2}(c) - \{z_1, z_{1+\frac{p}{2}}\}]$ is complete. Then $F_2 = \{cz_1, bz_{1+\frac{p}{2}}\} \cup \{z_i z_{i+\frac{p}{2}} | 2 \le i \le \frac{p}{2}\}$ is a perfect matching in $G[V(C_2) \cup \{c\}]$. Thus $F_1 \cup F_2$ is a perfect matching in G. This proves Case 4 and completes the proof of our lemma. □

Theorem 3.6. *Let G be a 4-γ_c-critical graph with a cut-vertex c and let C_1 and C_2 be the two components of $G - c$. Let S be a γ_c-set of G. Suppose $|S \cap V(C_1)| = 3$ and $|S \cap V(C_2)| = 0$. Then*

(1) $|V(C_2)| = 1$.

(2) If G is of even order, then G has a perfect matching.

Proof. Clearly, c dominates C_2. Thus $V(C_2) \subseteq N_G(c)$. Note that $|V(C_1)| \ge 4$ since $|S \cap V(C_1)| = 3$.

(1) Choose $a \in V(C_2)$ and $d \in N_{C_1}(c)$. Consider $G + ad$. Let S' be a γ_c-set of $G + ad$. By Lemmas 2.2(1) and 2.2(2), $|S'| \le 3$ and either $a \in S'$ or $d \in S'$. Suppose first that $\{a, d\} \subseteq S'$. Then $|S' \cap V(C_1)| \le 2$. Clearly, $S' \cap V(C_1) \succ_c C_1$. But then $(S' \cap V(C_1)) \cup \{c\}$ is a connected dominating set of G, a contradiction. Hence, $|S' \cap \{a, d\}| = 1$. Since $\{a, d\} \subseteq N_G(c)$, $c \notin S' - \{a, d\}$ by Lemma 2.2(3). If $S' \cap \{a, d\} = \{a\}$, then $S' \subseteq V(C_2)$ because of connectedness of S'. But then no vertex of S' is adjacent to a vertex of $V(C_1) - \{d\}$, a contradiction. Hence, $S' \cap \{a, d\} = \{d\}$. Because of the connectedness of S' and the fact that $c \notin S' - \{a, d\}$, it follows that $S' \subseteq V(C_1)$. Therefore, $V(C_2) = \{a\}$. This proves (1).

(2) Now let $V(C_2) = \{a\}$. Then $ac \in E(G)$. Since G is of even order, $|V(C_1)| = |V(G) - \{a, c\}|$ is even. We first show that $2 \le \gamma_c(C_1) \le 3$. Since $G[S]$ is connected and $c \in S$, it follows that $S \cap N_{C_1}(c) \ne \emptyset$. Because $G[N_{C_1}(c)]$ is complete by Lemma 2.4(2), $S \cap V(C_1) \succ_c C_1$. Hence, $\gamma_c(C_1) \le 3$. Note that for each γ_c-set W of C_1 and for each $x \in N_{C_1}(c)$, $W \cup \{x, c\} \succ_c G$. It follows that $|W| \ge 2$ as otherwise $\gamma_c(G) \le 3$. Consequently, $\gamma_c(C_1) \ge 2$. We now distinguish two cases.

Case 1: $\gamma_c(C_1) = 3$.
By Lemma 2.5(2), C_1 is 3-γ_c-critical. Then C_1 contains M as a perfect matching by Theorem 2.3 and the fact that $|V(C_1)|$ is even. So $M \cup \{ac\}$ is a perfect matching in G. This proves Case 1.

Case 2: $\gamma_c(C_1) = 2$.
Put $X = \overline{N}_{C_1}(c)$. Let $Y = \{u \in X | ux \in E(G)$ for some $x \in N_{C_1}(c)\}$ and $Z = X - Y$.

Claim 1 : If $Z \ne \emptyset$, then $G[Z]$ is complete.
Suppose that $G[Z]$ is not complete. Then there exist $u, v \in Z$ such that $uv \notin E(G)$. Consider $G + uv$ and let S' be a γ_c-set of $G + uv$. By Lemma 2.2(1), $|S'| \le 3$. Because c is a cut-vertex in $G + uv$, $c \in S'$. Since $|S'| \le 3$ and S' is connected, it follows that $(G + uv)[S']$ contains an $u - c$ path or

$v-c$ path of length at most 2. But this is not possible since neither u nor v is adjacent to a vertex of $N_{C_1}(c)$. Hence $G[Z]$ is complete. This settles our claim.

Claim 2 : If D is a γ_c-set of C_1, then $D \cap N_{C_1}(c) = \emptyset$.
Suppose $D = \{x, y\}$ where $x \in N_{G_1}(c)$. Then $\{x, y, c\} \succ_c G$, a contradiction. This proves our claim.

Claim 3 : If X has no perfect matching, then there is a maximum matching M in X such that $X - V(M) \subseteq Y$.
Let M_0 be a maximum matching in X and let $B = (X - V(M_0)) \cap Z$. By Claim 1 and the fact that $X - V(M_0)$ is an independent set, it follows that $0 \le |B| \le 1$. If $|B| = 0$, then our claim follows. So we may suppose that $|B| = 1$. Let $\{z\} = B$. Since $d(z, a) \le 4$ by Lemma 3.1, there is a vertex $v_1 \in Y$ such that $zv_1 \in E(G)$. If v_1 is M_0-unsaturated, then $M_0 \cup \{zv_1\}$ is a matching of X with $|M_0 \cup \{zv_1\}| > |M_0|$, contradicting the fact that M_0 is a maximum matching. Hence, v_1 is M_0-saturated. Let $v_2 \in X$ such that $v_1 v_2 \in M_0$.
Now let

$$M = \begin{cases} (M_0 - \{v_1 v_2\}) \cup \{zv_1\} & if \;\; v_2 \in Y \\ (M_0 - \{v_1 v_2\}) \cup \{zv_2\} & if \;\; v_2 \in Z. \end{cases}$$

Clearly, M is a matching in X with $X - V(M) \subseteq Y$. This proves our claim.

Claim 4 : Let M be a maximum matching in X such that $X - V(M) \subseteq Y$. If $A = X - V(M) \neq \emptyset$ and $|A| \ge 2$, then there exists a vertex $z \in V(M)$ such that $\{z\} \cup A$ is an independent set.
Clearly, A is independent. Let D be a γ_c-set of C_1. Then $|D \cap A| \le 1$ because of connectedness of $G[D]$. By Claim 2, $D \subseteq X = V(M) \cup A$. Thus $1 \le |D \cap V(M)| \le 2$.

Case 2.1 : $|D \cap A| = 1$ and $|D \cap V(M)| = 1$.
Let $\{x\} = D \cap A$ and $\{y\} = D \cap V(M)$. Note that $xy \in E(G)$. Since A is independent, y dominates A. Let $w \in V(M)$ where $yw \in M$. If $wu \in E(G)$ for some $u \in A$, then $M_1 = (M - \{yw\}) \cup \{wu, yv\}$ where $v \in A - \{u\}$ is a matching in X with $|M_1| > |M|$, a contradiction. Hence, $wu \notin E(G)$ for all $u \in A$. Thus $\{w\} \cup A$ is an independent set as required. This settles Case 2.1.

Case 2.2 : $|D \cap V(M)| = 2$.
Let $\{x_1, x_2\} = D \cap V(M)$, Clearly, $x_1 x_2 \in E(G)$ and $A \subseteq N_G(x_1) \cup N_G(x_2)$. Put $A_1 = \{a \in A | ax_1 \in E(G)\}$ and $A_2 = \{a \in A | ax_2 \in E(G)\}$. Note that $A_1 \cup A_2 = A$ and either $A_1 \neq \emptyset$ or $A_2 \neq \emptyset$ since $|A| \ge 2$. We may suppose that $A_1 \neq \emptyset$.
We first suppose that $x_1 x_2 \in M$. If $A_2 \neq \emptyset$, then, since $|A| \ge 2$, there is a matching $M_2 = (M - \{x_1 x_2\}) \cup \{a_1 x_1, a_2 x_2\}$, where $a_i \in A_i$ for $1 \le i \le 2$, in X with $|M_2| \ge |M|$, a contradiction. Hence, $A_2 = \emptyset$. Consequently, $A_1 = A$ and $\{x_2\} \cup A$ is an independent set as required.

We may now suppose that $x_1 x_2 \notin M$. Let $y_1, y_2 \in V(M)$ where $x_1 y_1$, $x_2 y_2 \in M$. If $A_1 = A$, then y_1 is not adjacent to any vertex of A_1 as otherwise X contains a matching M' with $|M'| > |M|$ since $|A| \ge 2$. Consequently, $A \cup \{y_1\}$ is an independent set. Similarly, if $A_2 = A$, then $A \cup \{y_2\}$ is an independent set. So we may now suppose that $A_1 \neq A$ and $A_2 \neq A$. Since $A_1 \neq A$, $A_2 \neq \emptyset$. Observe that $y_1 u \notin E(G)$ for all $u \in A_2$ and $y_2 v \notin V(G)$ for all $v \in A_1$. Further, if $|A_1| \ge 2$, then $y_1 u \notin E(G)$ for all $u \in A_1$ as otherwise X contains a matching M'' with $|M''| > |M|$. Similarly, if $|A_2| \ge 2$, then $y_2 u \notin E(G)$ for all $u \in A_2$. We now distinguish two subcases.

Subcase 2.2.1 : $|A| \ge 3$.
Then $|A_1| \ge 2$ or $|A_2| \ge 2$. If $|A_1| \ge 2$, then $A \cup \{y_1\}$ is an independent set and if $|A_2| \ge 2$, then $A \cup \{y_2\}$ is an independent set. This settles Subcase 2.2.1.

Subcase 2.2.2 : $|A| = 2$.
Recall that, for $1 \le i \le 2$, $A_i \neq A$ and $A_i \neq \emptyset$. Thus $|A_1| = 1$ and $|A_2| = 1$. For $1 \le i \le 2$, let $\{a_i\} = A_i$. If $a_1 y_1 \in E(G)$ and $a_2 y_2 \in E(G)$, then $M_3 = (M - \{x_1 y_1, x_2 y_2\}) \cup \{a_1 y_1, x_1 x_2, a_2 y_2\}$ is a matching in X with $|M_3| > |M|$, a contradiction. Hence, either $a_1 y_1 \notin E(G)$ or $a_2 y_2 \notin E(G)$. Then either $A \cup \{y_1\}$ or $A \cup \{y_2\}$ is an independent set. This proves Subcase 2.2.2 and then Case 2, completing the proof of our Claim.

Claim 5 : Let B be an independent subset of X where $|B| = n \ge 3$. Then the vertices of B can be ordered as $b_1, ..., b_n$ in such a way that, for $1 \le i \le n-1$, there exist vertices $u_1, ..., u_{n-1}$ of $N_{C_1}(c)$ where $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$ and, for $1 \le i \neq j \le n-1$, $u_i \neq u_j$.

Let u, v be a pair of non-adjacent vertices of B. Let S' be a γ_c-set of $G + uv$. So $|S'| \le 3$. Because c is a cut-vertex of $G + uv$, $c \in S'$. By Lemma 2.2(2), $\{u, v\} \cap S' \neq \emptyset$. Since $\{u, v\} \cap N_{C_1}(c) = \emptyset$ and S' is connected, it follows that there exists $x \in N_{C_1}(c) \cap S'$. Then $|\{u, v\} \cap S'| = 1$ and $|S'| = 3$. Hence, $\{u, x, c\} \succ_c G + uv$ or $\{v, x, c\} \succ_c G + uv$. Because $G[N_{C_1}(c)]$ is complete by Lemma 2.4(2) and c is not adjacent to any vertex of X, $\{x, u\} \succ_c C_1 - v$ or $\{x, v\} \succ_c C_1 - u$. Now consider $\overline{G}[B]$. Clearly, $\overline{G}[B]$ is complete. For a pair of vertices u and v of B, we orient u to v in $\overline{G}[B]$ if there exists a vertex $x \in N_{C_1}(c)$ such that $\{x, u\} \succ_c C_1 - v$. Let G^* be a spanning subdigraph of $\overline{G}[B]$ where G^* is a tournament. Then G^* contains P as a spanning directed path. Such a path exists since every tournament contains a spanning directed path. We now label the vertices of B as $b_1, ..., b_n$ where (b_i, b_{i+1}) is an arc of P for $i = 1, ..., n-1$. Hence, there exists u_i such that $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$ for $i = 1, ..., n-1$. Since $\{b_1, ..., b_n\}$ is independent, $u_j b_k \in E(G)$ for $1 \le j \le n-1$, and $1 \le k \neq j+1 \le n$. Thus $u_j \neq u_i$ for $1 \le i \neq j \le n-1$. This settles our claim.

We are now ready to show that G contains a perfect matching. Note that $G[N_{C_1}(c)]$ is complete by Lemma 2.4(2). If X contains M as a perfect matching, then, $M \cup F \cup \{ac\}$ is a perfect matching in G where F is a perfect matching in $C[N_{C_1}(c)]$. We may now suppose that X does not contain a perfect matching. By Claim 3, there is a maximum matching M_1 in X such that $X - V(M_1) \subseteq Y$. Put $A = X - V(M_1)$. Clearly, $|A| \geq 1$. We first suppose that $|A| = 1$. Let $A = \{u\}$. Since $A \subseteq Y$, there exists $v \in N_{C_1}(c)$ such that $uv \in E(G)$. Then $G[N_{C_1}(c) - \{v\}]$ contains F_1 as a perfect matching since $G[N_{C_1}(c)]$ is complete and $|V(G)|$ is even. Thus $M_1 \cup F_1 \cup \{uv, ac\}$ is a perfect matching in G.

We now consider $|A| \geq 2$. By Claim 4, there exists a vertex $y \in V(M_1)$ such that $\{y\} \cup A$ is an independent set. Put $|\{y\} \cup A| = n$. Note that $n \geq 3$. By Claim 5, the vertices of $\{y\} \cup A$ can be ordered as $b_1, ..., b_n$ in such a way that for $1 \leq i \leq n-1$ there exists $u_i \in N_{C_1}(c)$ where $\{b_i, u_i\} \succ_c C_1 - b_{i+1}$. By Lemma 2.2(3), $u_i b_{i+1} \notin E(G)$ for $1 \leq i \leq n-1$. Since $\{b_1, ..., b_n\}$ is independent and for $1 \leq i \leq n-1$, $G[\{b_i, u_i\}]$ is connected, it follows that $u_i b_j \in E(G)$ for $1 \leq i \leq n-1$ and $1 \leq j \neq i+1 \leq n$. We may suppose that $y = b_k$ for some $1 \leq k \leq n$. Now let F_2 be a matching in C_1 where

$$F_2 = \begin{cases} \{u_i b_i | 1 \leq i \neq k \leq n-1\} \cup \{u_k b_n\} & if \;\; 1 \leq k \leq n-2, \\ \{u_i b_i | 2 \leq i \leq n-2\} \cup \{u_1 b_n, u_{n-1} b_1\} & if \;\; k = n-1, \\ \{u_i b_i | 1 \leq i \leq n-1\} & if \;\; k = n. \end{cases}$$

Note that $X \subseteq V(M_1 \cup F_2)$. Since $G[N_{C_1}(c)]$ is complete, $G[N_{C_1}(c) - \{u_1, ..., u_{n-1}\}]$ contains F_3 as a perfect matching. Then $M_1 \cup F_2 \cup F_3 \cup \{ac\}$ is a perfect matching in G. This proves Case 2 and completes the proof of our theorem. □

Theorem 3.7. *Let G be a 4-γ_c-critical graph of connectivity one. If G is of even order, then G has a perfect matching.*

Proof. Let c be a cut-vertex of G and let C_1 and C_2 be the two components of $G - c$. Further, let S be a γ_c-set of G. By Lemma 2.5(1), $c \in S$. Then $|V(C_1) \cap S| + |V(C_2) \cap S| = 3$. We may suppose without loss of generality that $|V(C_1) \cap S| \geq |V(C_2) \cap S|$. If $|V(C_1) \cap S| = 3$ and $|V(C_2) \cap S| = 0$, then G has a perfect matching by Theorem 3.6. Now we may assume that $|V(C_1) \cap S| = 2$ and $|V(C_2) \cap S| = 1$. Then G has a perfect matching by Lemma 3.5. This completes the proof of our theorem. □

Our next result provides a construction of 4-γ_c-critical graphs having two cut-vertices.

Lemma 3.8. *Let H_1 and H_2 be complete graphs of order $n \geq 2$ and $m \geq 2$, respectively. Let H be a graph with $n+m+3$ distinct vertices where $V(H) = V(H_1) \cup V(H_2) \cup \{x, y, z\}$, $E(H[V(H_1) \cup \{x, y, z\}]) = E(H_1) \cup \{xy, yz\} \cup \{xa | a \in V(H_1)\}$, $E(H[V(H_2)]) = E(H_2)$. Further $H[V(H_1) \cup V(H_2)]$ has the following properties*
(i) For all $u \in V(H_1)$, $|N_{H_2}(u)| = |V(H_2)| - 1$.
(ii) For all $u \in V(H_2)$, there exists $v \in V(H_1)$ such that $uv \in E(H)$.
Then H is a 4-γ_c-critical graph containing x and y as cut-vertices.

Proof. Clearly, x and y are cut-vertices of H. We first show that $\gamma_c(H) = 4$. Since x, y are cut-vertices of H, for each γ_c-set S of H, $\{x, y\} \subseteq S$. Note that $\{x, y\}$ dominates $V(H_1) \cup \{x, y, z\}$. Suppose there exists $a \in V(H_1) \cup V(H_2)$ such that $\{x, y, a\} \succ_c H$. Then $a \in V(H_1)$ and a dominates H_2. But this contradicts hypothesis (i). Thus $\gamma_c(H) > 3$. Choose $p \in V(H_1)$ and $q \in N_{H_2}(p)$. Then $\{p, q\} \succ_c H[V(H_1) \cup V(H_2)]$. So $\{p, q, x, y\} \succ_c H$. Hence, $\gamma_c(H) = 4$.

We next show that H is $4 - \gamma_c$-critical. Let u and v be a pair of non-adjacent vertices of H. Consider $H + uv$. We distinguish 4 cases.

Case 1 : $z \in \{u, v\}$.
We may assume that $z = u$. Then $v \in \{x\} \cup V(H_1) \cup V(H_2)$. We first suppose that $v = x$. Choose $a \in V(H_1)$ and $b \in N_{H_2}(a)$. Then $\{v, a, b\} \succ_c G + uv$. Now suppose $v \in V(H_1)$. Choose $c \in N_{H_2}(v)$. Then $\{v, c, x\} \succ_c G + uv$. Finally, suppose $v \in V(H_2)$. By hypothesis (ii), there exists a vertex $d \in V(H_1)$ such that $vd \in E(H)$. Then $\{v, d, x\} \succ_c G + uv$. This proves Case 1.

Case 2 : $y \in \{u, v\}$.
We may assume that $y = u$. Then $v \in V(H_1) \cup V(H_2)$. We first suppose that $v \in V(H_1)$. Choose $a \in N_{H_2}(v)$. Then $\{u, v, a\} \succ_c G + uv$. Now suppose $v \in V(H_2)$. By hypothesis (ii), there exists a vertex $b \in V(H_1)$ such that $vb \in E(G)$. Then $\{u, v, b\} \succ_c G + uv$. This proves Case 2.

Case 3 : $x \in \{u, v\}$.
We may assume that $x = u$. By Case 1, we need only to consider when $v \in V(H_2)$. Clearly, $\{u, v, y\} \succ_c G + uv$. This proves Case 3.

Case 4 : $\{u, v\} \subseteq V(H_1) \cup V(H_2)$.
Since H_1 and H_2 are complete, we may assume that $u \in V(H_1)$ and $v \in V(H_2)$. By hypothesis (i), $\{u, x, y\} \succ_c G + uv$. This proves Case 4 and completes the proof of our theorem. □

We conclude our paper by establishing a characterization of 4-γ_c-critical graphs with 2 cut-vertices.

Theorem 3.9. *Let G be a 4-γ_c-critical graph with 2 cut-vertices c_1 and c_2. Then G is isomorphic to the graph H defined in Lemma 3.8.*

Proof. We first show that G has exactly 3 blocks. By Lemma 2.4(1), $G - c_1$ contains exactly 2 components. Let A_1 and A_2 be the two components of

$G - c_1$. We may suppose without loss of generality that $c_2 \in V(A_2)$. Then, by Theorem 3.2, $B_1 = G[V(A_1) \cup \{c_1\}]$ is a block of G. Similarly, $G - c_2$ contains exactly 2 components, say A'_1, A'_2. We may suppose without loss of generality that $c_1 \in V(A'_1)$. Hence, $B_2 = G[V(A'_2) \cup \{c_2\}]$ is a block of G. Since G has only 2 cut-vertices, it follows that $B_3 = G[V(G) - (V(A_1) \cup V(A'_2))]$ is a block of G. Therefore, C contains exactly 3 blocks B_1, B_2 and B_3.

Let S be a γ_c-set of G. Since c_1 and c_2 are cut-vertices of G, $\{c_1, c_2\} \subseteq S$ by Lemma 2.5(1).

Claim 1 : $|(S - \{c_1, c_2\}) \cap V(B_1)| = 2$ or $|(S - \{c_1, c_2\}) \cap V(B_2)| = 2$.
Suppose to the contrary that $|(S - \{c_1, c_2\}) \cap V(B_1)| \neq 2$ and $|(S - \{c_1, c_2\}) \cap V(B_2)| \neq 2$. We distinguish 4 cases.

Case 1 : $|(S - \{c_1, c_2\}) \cap V(B_1)| = 1$ and $|(S - \{c_1, c_2\}) \cap V(B_2)| = 1$.
Let $\{a\} = (S - \{c_1, c_2\}) \cap V(B_1)$ and $\{b\} = (S - \{c_1, c_2\}) \cap V(B_2)$. Note that $ac_1, bc_2, c_1c_2 \in E(G)$ since S is connected. Since $\gamma_c(G) = 4$, there exists $x \in V(B_1) - \{a, c_1\}$ such that $xa \in E(G)$ but $xc_1 \notin E(G)$. Similarly, there exists $y \in V(B_2) - \{b, c_2\}$ such that $yb \in E(G)$ but $yc_2 \notin E(G)$. So $d(x, y) = d(x, c_1) + d(c_1, c_2) + d(c_2, y) \geq 5$ since c_1 and c_2 are cut-vertices, contradicting Lemma 3.1. Then Case 1 cannot occur.

Case 2 : $|(S - \{c_1, c_2\}) \cap V(B_3)| = 2$.
Then c_1 dominates B_1 and c_2 dominates B_2. Thus $|V(B_3) - \{c_1, c_2\}| \geq 2$. Choose $x \in V(B_1) - \{c_1\}$ and $y \in V(B_2) - \{c_2\}$. Consider $G + xy$. Let S' be a γ_c-set of $G + xy$. By Lemmas 2.2(1) and 2.2(2), $|S'| \leq 3$ and either $x \in S'$ or $y \in S'$. We may suppose without loss of generality that $x \in S'$. We first show that $y \in S'$. Suppose $y \notin S'$. Then $c_2 \notin S'$ by Lemma 2.2(3). By a connectedness of S', $c_1 \in S'$. Since $|(S - \{c_1, c_2\}) \cap V(B_3)| = 2$, $\{x, c_1\}$ does not dominate B_3. Thus $|S' - \{x, c_1\}| = 1$. Let $\{d\} = S' - \{x, c_1\}$. Then $d \in V(B_3) - \{c_1, c_2\}$ and $\{c_1, d\} \succ_c (B_2 \cup B_3) - \{y\}$. Consequently, $\{c_1, d, c_2\} \succ_c G$, a contradiction. Hence, $y \in S'$ and therefore $\{x, y\} \subseteq S'$. Since $|S'| \leq 3$, $|S' - \{x, y\}| \leq 1$. Because $|V(B_3) - \{c_1, c_2\}| \geq 2$ and no vertex of $\{x, y\}$ is adjacent to a vertex of $V(B_3) - \{c_1, c_2\}$, it follows that either $S' - \{x, y\} = \{c_1\}$ or $S' - \{x, y\} = \{c_2\}$ since S' is connected. We may assume that $S' - \{x, y\} = \{c_1\}$. So $c_1 \succ_c V(B_3) - \{c_2\}$. Since B_3 is a block, these exists a vertex $w \in V(B_3) - \{c_2\}$ such that $wc_2 \in E(G)$. Then $\{c_1, w, c_2\} \succ_c G$ since $c_1 \succ_c (V(B_1) \cup V(B_3)) - \{c_2\}$ and $c_2 \succ_c B_2$. But this contradicts the fact that $\gamma_c(G) = 4$. Hence, Case 2 cannot occur.

Case 3 : $|(S - \{c_1, c_2\}) \cap V(B_1)| = 1$ and $|(S - \{c_1, c_2\}) \cap V(B_3)| = 1$.
Let $\{a\} = (S - \{c_1, c_2\}) \cap V(B_1)$ and $\{b\} = (S - \{c_1, c_2\}) \cap V(B_3)$. Clearly, $c_2 \succ_c B_2$ and $ac_1 \in E(G)$. Further, $bc_1 \in E(G)$ or $bc_2 \in E(G)$. Since $S \cap V(B_1) = \{a, c_1\}$, there is a vertex $x \in V(B_1) - \{a, c_1\}$ such that $xc_1 \notin E(G)$ but $xa \in E(G)$. Choose a vertex $y \in V(B_2) - \{c_2\}$. Since c_1 and c_2 are cut-vertices of G, $d(x, y) = d(x, c_1) + d(c_1, c_2) + d(c_2, y) \leq 4$ by Lemma 3.1. It follows that $c_1c_2 \in E(G)$. Because $\gamma_c(G) = 4$, there exists $d \in V(B_3) - \{b, c_1, c_2\}$ such that $dc_1, dc_2 \notin E(G)$ but $bd \in E(G)$. Now consider $G + dy$. Let S' be a γ_c-set of $G + dy$. Note that c_1 is a cut-vertex of $G + dy$. Then $c_1 \in S'$. Since $c_1d \notin E(G)$ and $c_1y \notin E(G)$, there exists a vertex, w say, of $S' - \{c_1, d, y\}$ such that either $\{c_1, w, d\} \succ_c G + dy$ or $\{c_1, w, y\} \succ_c G + dy$. In either case $w \notin V(B_1)$. Thus $c_1 \succ_c B_1$. But this contradicts the fact that $xc_1 \notin E(G)$. Hence, Case 3 cannot occur.

Case 4 : $|(S - \{c_1, c_2\}) \cap V(B_2)| = 1$, $|(S - \{c_1, c_2\}) \cap V(B_3)| = 1$.
By similar arguments as in the proof of Case 3, Case 4 follows.

Claim 2 : If $|(S - \{c_1, c_2\}) \cap V(B_1)| = 2$, then $|V(B_2)| = 2$.
Since no vertex of $(S - \{c_1, c_2\}) \cap V(B_1)$ is adjacent to a vertex of $(V(B_2) \cup V(B_3)) - \{c_1\}$, $\{c_1, c_2\} \succ_c B_3$ and $c_2 \succ_c B_2$. It follows that for each $z \in N_{B_1}(c_1)$, there is a vertex $z' \in V(B_1) - \{c_1, z\}$ such that $z'z \notin E(G)$ and $z'c_1 \notin E(G)$. Hence, if D is a connected dominating set of size at most 2 of B_1, then $c_1 \notin D$. Choose $x \in V(B_2) - \{c_2\}$ and consider $G + xc_1$. Let S' be a γ_c-set for $G + xc_1$. By Lemma 2.2(1), $|S'| \leq 3$. Since c_1 is a cut-vertex of $G + xc_1$, $c_1 \in S'$. Then $|S' \cap V(B_1)| \geq 3$ because $S' \cap V(B_1) \succ_c B_1$. Hence, $|S' \cap V(B_1)| = 3$ and thus $x \notin S'$. Consequently, $V(B_2) - \{c_2, x\} = \emptyset$. This settles our claim.

Claim 3 : If $|(S - \{c_1, c_2\}) \cap V(B_2)| = 2$, then $|V(B_1)| = 2$.
By similar arguments as in the proof of Claim 2, our claim follows.

We may now assume that $|(S - \{c_1, c_2\}) \cap V(B_1)| = 2$. Thus $\{c_1, c_2\}$ dominates $B_2 \cup B_3$. By Claim 2, $|V(B_2)| = 2$. Let $\{x\} = V(B_2) - \{c_2\}$. Then $xc_2 \in E(G)$.

Claim 4 : $V(B_3) = \{c_1, c_2\}$.
Suppose to the contrary that there exists $b \in V(B_3) - \{c_1, c_2\}$. Consider $G + bx$. Let S' be a γ_c-set of $G + bx$. By Lemma 2.2(1), $|S'| \leq 3$. Since c_1 is a cut-vertex of $G + bx$, $c_1 \in S'$. If $|(S' \cap V(B_1)) - \{c_1\}| < 2$, then $((S' \cap V(B_1)) - \{c_1\}) \cup \{c_1, c_2\}$ is a connected dominating set of size at most 3 of G, a contradiction. Hence, $|(S' \cap V(B_1)) - \{c_1\}| \geq 2$ and thus $|(S' \cap V(B_1)) - \{c_1\}| = 2$ since $|S'| \leq 3$. It follows that $|S' \cap V(B_1)| = 3$. Because S' is a γ_c-set of $G + bx$, $S' \cap \{b, x\} \neq \emptyset$. Then $|S'| \geq 4$, a contradiction. This proves our claim.

Recall that $B_1 = G[V(A_1) \cup \{c_1\}]$. By Theorem 2.4(2), $G[N_{B_1}(c_1)] = G[N_{A_1}(c_1)]$ is complete. Since $|S \cap V(A_1)| = |(S - \{c_1, c_2\}) \cap V(B_1)| = 2$, $G[\overline{N}_{B_1}(c_1)] = G[\overline{N}_{A_1}(c_1)]$ is complete by Theorem 3.3. Further,
(i) For all $x \in N_{B_1}(c_1)$, $|N_{\overline{N}_{B_1}(c_1)}(x)| = |\overline{N}_{B_1}(c_1)| - 1$.
(ii) For all $y \in \overline{N}_{B_1}(c_1)$, there exists $x \in N_{B_1}(c_1)$ such that $xy \in E(G)$.
Therefore, G is isomorphic to the graph H defined in Lemma 3.8 as required. This completes the proof of our theorem. □

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Some Results for Generalized Cauchy Numbers[1]

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Abstract

In this paper, we study the properties of the generalized Cauchy numbers. Using the method of coefficient, we establish some identities related to the generalized Cauchy numbers and extend some recurrence relations for Cauchy numbers. We further give some asymptotic expansions of certain sums involving these numbers.

1. Introduction

Given an integer r, the generalized Cauchy number $c_n^{[r]}$ is defined in [9] by

$$\sum_{n=0}^{\infty} c_n^{[r]} \frac{t^n}{n!} = \frac{t(1+t)^{1-r}}{\ln(1+t)},$$

which is clearly a generalization of the well-known first and second kinds of Cauchy numbers [4]. Initial values of some numbers are as follows:

n	0	1	2	3	4	5
$c_n^{[1]}$	1	1/2	−1/6	1/4	−19/30	9/4
$c_n^{[2]}$	1	−1/2	5/6	−9/4	251/30	−475/12
$c_n^{[3]}$	1	−3/2	23/6	−55/4	1901/30	4277/42

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