SOME PROPERTIES OF 3-i-CRITICAL GRAPHS

By
Watcharintorn Ruksasakchai

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
MASTER OF SCIENCE
Department of Mathematics
Graduate School
SILPAKORN UNIVERSITY
2010

SOME PROPERTIES OF 3-i-CRITICAL GRAPHS

By
Watcharintorn Ruksasakchai

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree
MASTER OF SCIENCE
Department of Mathematics
Graduate School
SILPAKORN UNIVERSITY
2010

สมบัติของกราฟ 3-i-CRITICAL

โดย
นางสาววัชรินทร รักษาศักดิ์ชัย

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
สาขาวิชาคณิตศาสตร์
ภาควิชาคณิตศาสตร์
บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร
ปีการศึกษา 2553

The Graduate School, Silpakorn University has approved and accredited the Thesis title of "Some properties of 3-i-critical graphs" submitted by Miss Watcharintorn Ruksasakchai as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

………...

(Assistant Professor Panjai Tantatsanawong, Ph.D.)

Dean of Graduate School

........../.................../..........

The Thesis Advisor

Associate Professor Nawarat Ananchuen, Ph.D.

The Thesis Examination Committee

.... Chairman

(Jitti Rakbud, Ph.D.)

............/......................./.............

.... Member

(Associate Professor Watcharaphong Ananchuen, Ph.D.)

............/......................./.............

.... Member

(Associate Professor Nawarat Ananchuen, Ph.D.)

............/......................./.............

51304201 : MAJOR : MATHEMATICS
KEY WORDS : INDEPENDENT DOMINATION / CRITICAL / NEAR-PERFECT MATCHING / DEFECT K-FACTOR-CRITICL / FACTOR-CRITICAL / BICRITICAL
WATCHARINTORN RUKSASAKCHAI : SOME PROPERTIES OF 3-i-CRITICAL GRAPHS. THESIS ADVISOR : ASSOC. PROF.NAWARAT ANANCHUEN, Ph.D. 39 pp.

Let $\gamma(G)$ denote the domination number of a graph G. A graph G is said to be k-edge-critical if $\gamma(G) = k$ and for each pair of non-adjacent vertices u and v of G, $\gamma(G+uv) < k$.

Let $i(G)$ denote the independent domination number of a graph G. A graph G is said to be k-edge-i-critical if $i(G) = k$ and for each pair of non-adjacent vertices u and v of G, $i(G+uv) < k$.

A graph G is said to be defect k-factor-critical if for any subset S of $V(G)$ with $|S| = k$, the graph G-S contains a near-perfect matching.

In this thesis, we provide some properties of defect k-factor-critical graphs. We prove that a graph G with $|V(G)| \equiv (k+1)(mod 2)$ is defect k-factor-critical if and only if $c_0(G-S) \leq |S|-k+1$ for every subset S of $V(G)$ such that $|S| \geq k$ where $c_0(G-S)$ denotes the n umber of odd components of G-S. We also show that connected 3-edge-critical graphs of even order are defect 1-factor-critical. Moreover, we establish some sufficient conditions for connected 3-edge-critical graphs to be defect k-factor-critical for $2 \leq k \leq 4$. In addition, we study some properties of 3-edge-i-critical graphs. We also concentrate on sufficient conditions for connected 3-edge-i-critical graphs to be defect k-factor-critical for $1 \leq k \leq 4$. Finally, we provide some sufficient conditions for 3-edge-i-critical graphs to be factor-critical and bicritical.

Department of Mathematics Graduate School, Silpakorn University Academic Year 2010
Student's signature ....
Thesis Advisor's signature ....

51304201 : สาขาวิชาคณิตศาสตร์
คำสำคัญ : จำนวนควบคุมที่เป็นอิสระ / วิกฤติ / การจับคู่เกือบสมบูรณ์ / defect k-factor-critical / แฟคเตอร์คริติคัล / ไบคริติคัล

วัชรินทร รักษาศักดิ์ชัย : สมบัติของกราฟ 3-i-CRITICAL. อาจารย์ที่ปรึกษาวิทยานิพนธ์ : รศ.ดร.นวรัตน์ อนันต์ชื่น. 39 หน้า.

กำหนดให้ $\gamma(G)$ แทนขนาดของเซตควบคุมที่เล็กที่สุดของกราฟ G เราจะเรียกกราฟ G ว่า k-edge-critical เมื่อ $\gamma(G) = k$ และสำหรับจุด u และ v ใน G ที่ไม่ประชิดกันแล้ว $\gamma(G+uv) < k$

กำหนดให้ $i(G)$ แทนขนาดของเซตควบคุมที่เป็นอิสระที่เล็กที่สุดของกราฟ G เราจะเรียกกราฟ G ว่า k-edge-i-critical เมื่อ $i(G) = k$ และสำหรับจุด u และ v ใน G ที่ไม่ประชิดกันแล้ว $i(G+uv) < k$

กราฟ G จะถูกเรียกว่าเป็น defect k-factor-critical ถ้า G-S มีการจับคู่เกือบสมบูรณ์สำหรับทุกๆสับเซต S ของเซตของจุด ซึ่งมีขนาดเท่ากับ k

ในวิทยานิพนธ์นี้เราได้ศึกษาคุณสมบัติของกราฟ defect k-factor-critical เราได้แสดงว่ากราฟซึ่งมีจำนวนจุดคอนกรูเอ็นซ์ k+1 มอดุโล 2 จะเป็น defect k-factor-critical ก็ต่อเมื่อ $c_0(G-S) \leq |S|-k+1$ สำหรับทุกๆสับเซต S ของจุดใน G ซึ่ง $|S| \geq k$ และเรายังได้แสดงด้วยว่า สำหรับกราฟ 3-edge-critical ซึ่งเป็นกราฟไม่ขาดตอนและมีจำนวนจุดเป็นจำนวนคู่นั้นจะเป็น defect 1-factor-critical ยิ่งไปกว่านั้นเราได้ให้เงื่อนไขที่เพียงพอสำหรับการเป็น defect k-factor-critical ของกราฟ 3-edge-critical ซึ่งเป็นกราฟไม่ขาดตอน สำหรับ $2 \leq k \leq 4$ นอกจากนี้เราได้ศึกษาคุณสมบัติของกราฟ 3-edge-i-critical และเราได้ให้เงื่อนไขที่เพียงพอสำหรับการเป็น defect k-factor-critical ของกราฟ 3-edge-i-critical ซึ่งเป็นกราฟไม่ขาดตอนสำหรับ $1 \leq k \leq 4$ สุดท้ายเรายังได้ให้เงื่อนไขที่เพียงพอสำหรับการเป็นแฟคเตอร์คริติคัลและไบคริติคัลของกราฟ 3-edge-i-critical

ภาควิชาคณิตศาสตร์ บัณฑิตวิทยาลัย มหาวิทยาลัยศิลปากร ปีการศึกษา 2553

ลายมือชื่อนักศึกษา..

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

ACKNOWLEDGEMENTS

This thesis has been completed by the involvement of people about whom I would like to mention here.

I would like to thank Assoc. Prof. Dr. Nawarat Ananchuen, my advisor for her valuable suggestions and excellent advices throughout the study with great attention.

I would like to thank Dr. Jitti Rakbud and Assoc. Prof. Dr. Watcharaphong Ananchuen, Chairman and Member of the thesis Committee, for their valuable comments and suggestions.

Finally, I would like to express my gratitude to my family and my friends for their understanding, encouragement and moral support during the study.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

Introduction

In this chapter, we introduce some definitions and notations used in this thesis.

A **graph** is an order pair $G = (V(G), E(G))$, where $V(G)$ is a finite (possibly empty) set of vertices and $E(G)$ is a set of unordered pairs of distinct vertices. The elements of $E(G)$ are called **edges**. If $\{u, v\} \in E(G)$, we denote $\{u, v\}$ by uv and we say that u and v are **adjacent**. For any $uv \in E(G)$, the vertices u and v are called **end vertices** of uv. Two or more edges that join the same pair of vertices are called **parallel edges**. An edge that joins itself is a **loop**. A graph G is **simple** if G has no loops and parallel edges.

A graph H is a **subgraph** of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. H is an **induced subgraph** of G, denoted by $G[H]$, if, for every pair of $u, v \in V(H)$, $uv \in E(H)$ if and only if $uv \in E(G)$. A graph G is called **H-free** if G does not contain H as an induced subgraph. The **neighborhood** of a vertex v in a graph G denoted by $N_G(v)$ is $\{u \in V(G) | \ uv \in E(G)\}$. A **degree** of v in G, denoted by $d(v)$, is $|N_G(v)|$. The **minimum degree** of all vertices in G is denoted by $\delta(G)$. A vertex of degree 0 is called an **isolated vertex** and a vertex of degree 1 is called an **end vertex**.

A **complete graph** is a graph in which every pair of vertices are adjacent. A complete graph of order n is denoted by K_n. For $k \geq 2$, a graph G is a **k-partite graph** if $V(G)$ can be partitioned into k nonempty subsets $V_1, V_2, ..., V_k$ such that no edge of G joins vertices in the same set. The sets $V_1, V_2, ..., V_k$ are called the **partite sets** of G. If G is a k-partite graph having partite sets $V_1, V_2, ..., V_k$ such that every vertex of V_i is joined to every vertex of V_j, where $1 \leq i < j \leq n$, then G is called a **complete k-partite graph**. If $|V_i| = p_i$, for $i = 1, 2, ..., k$, then we denote G by $K_{p_1,p_2,...,p_k}$. Further, for $1 \leq i \leq k$, if $|V_i| = m$, we denote $K_{p_1,p_2,...,p_k}$ by $K_{k,mk}$. If G is a k-partite (complete k-partite) graph where $k = 2$, then G is called **bipartite** (respectively, **complete bipartite**). A complete bipartite graph which the cardinality of at least one of partitioned vertex set equals to one is called a **star**, denoted by $K_{1,n}$. $K_{1,3}$ is called a **claw**.

A **walk** in a graph G is a finite, non-empty alternating sequence $W = v_0e_1v_1e_2...e_nv_n$ of vertices and edges such that for $1 \leq i \leq n$, the ends of edge e_i are v_{i-1} and v_i. W is said to be a walk from v_o to v_n. A **path** is a walk with distinct vertices. A **cycle** is a walk $v_0, v_1, ..., v_n$ in which $n \geq 3$, $v_0 = v_n$ and the n vertices $v_1, v_2, ..., v_n$ are distinct. A path that contains every vertex of G is called a **hamiltonian path** of G; similarly, a **hamiltonian cycle** of G is a cycle that contains every vertex of G. A graph is **hamiltonian** if it contains a hamiltonian cycle. Two vertices u and v of G are **connected** if there is a path from u to v. A graph G is **connected** if every pair of vertices of G are connected otherwise G is disconnected. A maximal connected subgraph of G is called a **component** of G. We denote the number of components (odd components) of graph G by $c(G)$ (respectively, $c_0(G)$).

A graph that can be drawn in the plane without any of its edges intersecting is called a **planar graph**.

The **distance** between two vertices u, v, denoted by $d(u, v)$, is the length of a shortest (u, v)-path in G. The **diameter** of G, denoted by $diam(G)$, is the maximum distance between two vertices of G.

A vertex v of G is called a **cutvertex** if the number of components of $G - v$ is more than the number of components of G. A set $S \subseteq V(G)$ is called a **vertex cutset** if the number of components of $G - S$ is more than the number of components of G. The **connectivity** $\kappa(G)$ of a graph G is the minimum number of vertices whose deletion from G produces a disconnected or trivial graph. The **edge-connectivity** $\lambda(G)$ of a graph G is the minimum cardinality of a set E of edges of G such that $G - E$ is disconnected.

For $M \subseteq E(G)$, M is a **matching** in G if no two edges of M have common end vertex. A **perfect matching** of a graph G is a matching covering all vertices in G. A **near-perfect matching** of a graph G is a matching covering all but one vertex in G. A graph G is k**-factor-critical** if for any set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a perfect matching. Note that if G is k-factor-critical, then $|V(G)| \equiv k(mod2)$. For more specifically, we say that G is **factor-critical** if

$k = 1$ and G is **bicritical** if $k = 2$.

A graph G is said to be **defect k-factor-critical** if for any set $S \subseteq V(G)$ with $|S| = k$, the graph $G - S$ contains a near-perfect matching. Note that if G is defect k-factor-critical, then $|V(G)| \equiv (k+1)(mod 2)$.

A subset D of $V(G)$ is said to **dominate** G if every vertex of G either belongs to D or is adjacent to a vertex of D. Such a set D is called a **dominating set** for G. The smallest cardinality of a dominating set for G is the **domination number** of G, denoted by $\gamma(G)$.

A subset S of $V(G)$ is said to be an **independent set** if no two vertices in S are adjacent and the **independence number** $\alpha(G)$ is the maximum cardinality of an independent set. A dominating set D of G is called an **independent dominating set** if D is independent and the **independent domination number** denoted by $i(G)$ is the smallest cardinality of an independent dominating set for G.

A graph G is said to be **k-edge-critical** if $\gamma(G) = k$, but $\gamma(G+e) < k$ for each edge $e \notin E(G)$.

Let G be a 3-edge-critical graph and let u and v be a pair of non-adjacent vertices of G. Then $\gamma(G + uv) = 2$. It is easy to see that exactly one of $\{u, v\}$ belongs to a dominating set for $G+uv$. Then there exists a vertex $w \in V(G)-\{u, v\}$ such that either $\{u, w\}$ or $\{v, w\}$ is a dominating set for $G+uv$. Clearly, if $\{u, w\}$ is a dominating set for $G + uv$, then $\{u, w\}$ dominates $G - v$. In this case, we write $[u, w] \rightarrow v$. Similarly, if $\{v, w\}$ is a dominating set for $G + uv$, then $\{v, w\}$ dominates $G - u$. In this case, we write $[v, w] \rightarrow u$.

A graph G is said to be **k-edge-i-critical** if $i(G) = k$, but $i(G+e) < k$ for each edge $e \notin E(G)$.

Let G be a 3-edge-i-critical graph and let u and v be a pair of non-adjacent vertices of G. Then $i(G + uv) = 2$. It is easy to see that exactly one of $\{u, v\}$ belongs to an independent dominating set for $G + uv$. Then there exists a vertex $w \in V(G)-\{u, v\}$ such that either $\{u, w\}$ or $\{v, w\}$ is an independent dominating set for $G+uv$. In either case, $\{u, v, w\}$ is an independent set. Clearly, if $\{u, w\}$ is an independent dominating set for $G + uv$, then $\{u, w\}$ dominates $G - v$. In this case, we write $[u, w]\underset{i}{\rightarrow} v$. Similarly, if $\{v, w\}$ is an independent dominating set for $G + uv$, then $\{v, w\}$ dominates $G - u$. In this case, we write $[v, w]\underset{i}{\rightarrow} u$. Furthermore, there is a natural orientation induced on the edges of $\overline{G}$, the complement of G. If $uv \in E(\overline{G})$ and there exists $w \in V(G)$ such that $[u, w]\underset{i}{\rightarrow} v$, we orient u to v

and if there exists $w \in V(G)$ such that $[v, w] \underset{i}{\rightarrow} u$, we orient v to u. In particular, we allow both (u, v) and (v, u) to be arcs in the orientation of $\overline{G}$.

In what follows, our graphs are simple and finite.

Our next chapter provides some basic background and preliminaries concerning our work. The main results are in Chapter 3 and Chapter 4.

Chapter 2

Literature Reviews

In this chapter we provide some background and results related to our work. The first section concerns results on a perfect matching and k-factor-critical graphs. Results on k-edge-critical graphs and k-edge-i-critical graphs are given in Section 2.2 and Section 2.3, respectively.

2.1 A perfect matching and k-factor-critical graphs

The most well known and usuful result on a perfect matching provided by Tutte.

Theorem 2.1.1. *(Tutte's Theorem)(see Page 76 in [5]) A nontrivial graph G has a perfect matching if and only if, for every proper subset S of $V(G)$, the number of odd components of $G - S$ does not exceed $|S|$.*

This result was also known as Tutte's theorem on a perfect matching.

The concept of k-factor-critical graphs was introduced by Favaron [8]. In [8], she gave several properties of k-factor-critical graphs. We state some of them which we used in establishing our results.

Theorem 2.1.2. *[8] A graph G is k-factor-critical if and only if $c_0(G - S) \leq |S| - k$, for every $S \subseteq V(G)$ and $|S| \geq k$.*

In addition, she also established the following results.

Theorem 2.1.3. *[8] For $k \geq 2$, any k-factor-critical graph of order $n > k$ is $(k-2)$-factor-critical.*

Theorem 2.1.4. *[8] Every k-factor-critical graph of order $n > k$ is k-connected and this result is sharp.*

Theorem 2.1.5. *[8] For $k \geq 1$, every k-factor-critical graph G of order $n > k$ is $(k+1)$-edge-connected.*

2.2 k-edge-critical graphs

The concept of k-edge-critical graphs was first introduced in 1983 by Sumner and Blitch [13]. In [13], the notation $[u, w] \to v$ introduced in Chapter 1 was first used and they provided the following lemma for $n \geq 4$. The case $n = 2$ and $n = 3$ were proved by Flandrin etc. in [10].

Lemma 2.2.1. *[13,10] Let G be a 3-edge-critical graph and let S be an independent set of $n \geq 2$ vertices in $V(G)$.*
(a) Then the vertices of S can be ordered as $a_1, a_2, ..., a_n$ in such a way that there exists a sequence of distinct vertices $x_1, x_2, ..., x_{n-1}$ so that $[a_i, x_i] \to a_{i+1}$ for $i = 1, 2, ..., n-1$.
(b) If, in addition, $n \geq 4$, then the x_i's can be chosen so that $x_1, x_2, ..., x_n$ is a path and $S \cap \{x_1, x_2, ..., x_{n-1}\} = \phi$.

This lemma becomes a very usuful tool in establishing other results on 3-edge-critical graphs.

Sumner and Blitch [13] also gave a characterization of 2-edge-critical graphs and disconnected 3-edge-critical graphs. They proved that:

Theorem 2.2.2. *G is 2-edge-critical if and only if $\overline{G} = \bigcup_{i=1}^{n} K_{1,n_i}$ where $n \geq 1$.*

Theorem 2.2.3. *If G is 3-edge-critical and is not connected, then G is the disjoint union of a 2-edge-critical graph and a complete graph.*

Further, they established some properties of connected 3-edge-critical graphs as follow.

Theorem 2.2.4. *[13] Let G be a connected 3-edge-critical graph. Then if S is a vertex cutset in G, then $G - S$ has at most $|S| + 1$ components.*

Theorem 2.2.5. *[13] The diameter of 3-edge-critical graph is at most 3.*

Most results on k-edge-critical graphs concern the case $k = 3$. Besides the properies established by Sumner and Blitch, there are results related to hamiltonian paths and hamiltonian cycles. Wojcicka [17] showed that:

Theorem 2.2.6. *[17] Every connected* 3*-edge-critical graph on more than six vertices has a hamiltonian path.*

Moreover, she conjectured in [17] that every connected 3-edge-critical graph with $\delta(G) \geq 2$ has a hamiltonian cycle.

Xue and Chen [18] first attempted to prove this conjecture. They established the following theorem.

Theorem 2.2.7. *[18] If G is a connected* 3*-edge-critical graph with $\delta(G) = 1$, then $G - A$ has a hamiltonian cycle, where $A = \{v \in V(G) |\, d(v) = 1\}$.*

Hanson [11] also tried to solve this problem in sence of closure operator.

Theorem 2.2.8. *[11] Let v be a vertex with $d(v) \geq 3$ in a* 2*-connected* 3*-edge-critical graph G and let a and b be the non-adjacent vertices of G such that $[a, b] \to v$. Then $G + ab$ has a hamiltonian cycle if and only if G has a hamiltonian cycle.*

In addiion, there were several researchers (see [7,9,10,14]) who attempted to solve Wojcicka's conjecture which finally was proved as we can see in the following theorem.

Theorem 2.2.9. *[7] Let G be a connected* 3*-edge-critical graph with $\delta(G) \geq 2$. If $\alpha(G) = \kappa(G) + 1$, then G has a hamiltonian cycle.*

Theorem 2.2.10. *[7] Let G be a connected* 3*-edge-critical graph with $\delta(G) \geq 2$. If $\alpha(G) = \kappa(G) + 2$, then G has a hamiltonian cycle.*

Theorem 2.2.11. *[9] If G is a connected* 3*-edge-critical graph, then $\alpha(G) \leq \delta(G) + 2$.*

Theorem 2.2.12. *[9] Let G be a connected* 3*-edge-critical graph with $\delta(G) \geq 2$. If $\alpha(G) \leq \delta(G) + 1$,then G has a hamiltonian cycle.*

Theorem 2.2.13. *[14] Let G be a connected* 3*-edge-critical graph with $\delta(G) \geq 2$. If $\alpha(G) \leq \delta(G) + 2$,then G has a hamiltonian cycle.*

One of interesting properties on 3-edge-critical graphs is an existence of a perfect matching proved by Sumner and Blitch [13].

Theorem 2.2.14. *[13] If G is a connected* 3*-edge-critical graph of even order, then G has a perfect matching.*

Note that this result is a consequence of Theorem 2.2.4 and 2.1.1.

Related to an existence of a perfect matching is an being k-factor-critical. Most results provide sufficient conditions for 3-edge-critical graphs to be k-factor-critical for small k, see [2,3]. One of the usuful tools in establishing these results is the following result proved by Ananchuen and Plummer [1].

Theorem 2.2.15. *[1] Let G be a connected 3-edge-critical graph and let S be a vertex cutset in G. Then*
(a) if $|S| \geq 4$, $G - S$ has at most $|S| - 1$ components,
(b) if $|S| = 3$, then $G - S$ contains at most $|S|$ components, and if $G - S$ has exactly three components, then each component is complete and at least one is a singleton,
(c) if $|S| = 2$, then $G - S$ has at most three components and if $G - S$ has exactly three components, then G must have the structure shown below in Figure 1,
(d) if $|S| = 1$, then $G - S$ has two components, exactly one of which is a singleton. Furthermore, G has exactly one or two cutvertices and if it has two, G is isomorphic to a graph of the type shown in Figure 1.

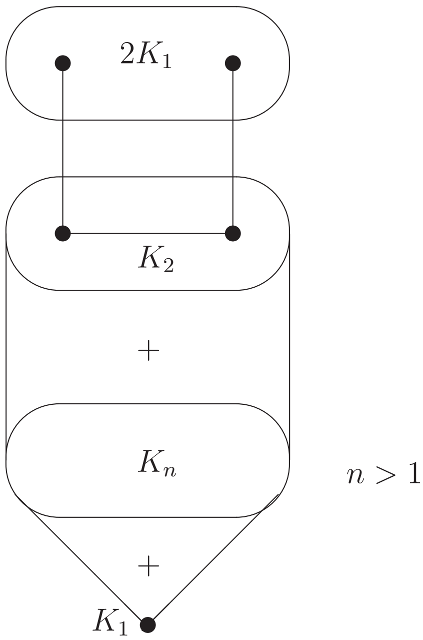

Figure 1: A connected 3-edge-critical graph containing exactly two cutvertices.

Note that the above result is an extension of Theorem 2.2.4.

Ananchuen and Plummer [2,3] proved the following theorems.

Theorem 2.2.16. *[2] Let G be a 3-edge-critical 2-connected graph having odd order. Then G is factor-critical.*

Theorem 2.2.17. *[2] If G is a 3-edge-critical 3-connected graph with minimum degree at least 4 and having even order, then G is bicritical.*

Theorem 2.2.18. *[2] Let G be a 3-edge-critical 2-connected claw-free graph of even order. Then if $\delta(G) \geq 3$, G is bicritical.*

Theorem 2.2.19. *[3] Let G be a 3-edge-critical 3-connected claw-free graph of odd order. Then if $\delta(G) \geq 4$, G is 3-factor-critical.*

2.3 k-edge-i-critical graphs

In 1994, Ao [4] introduced the concept of k-edge-i-critical graphs. He/She gave a characterization of 2-edge-i-critical graphs. He/She also provided a characterization of disconneced 3-edge-i-critical graphs and some structural properties of connected 3-edge-i-critical graphs as following:

Theorem 2.3.1. *A graph G is 2-edge-i-critical if and only if $\overline{G} = \bigcup_{i=1}^{n} K_{1,n_i}$ where $n \geq 1$.*

Theorem 2.3.2. *if G is 3-edge-i-critical, then G has at most 3 components. Furthermore,*
(a) if G has exactly 3 components, then $G = K_n \cup 2K_1$ for some $n \geq 1$;
(b) if G has exactly 2 components, then either $G = K_1 \cup H$, where H is 2-edge-i-critical or $G = K_n \cup K_{m,2m}$ where $n \geq 1$ and $m \geq 2$.

Theorem 2.3.3. *[4] Let G be a connected 3-edge-i-critical graph. If S is a vertex cutset in G, then $G - S$ has at most $|S| + 1$ components.*

Theorem 2.3.4. *[4] The diameter of a connected 3-edge-i-critical graph is at most 3.*

Moreover, he/she also studied the hamiltonian properties of k-edge-i-critical graphs.

Theorem 2.3.5. *[4] If G is connected 3-edge-i-critical graph with $|V(G)| > 6$, then G has a hamiltonian path.*

Theorem 2.3.6. *[4] If G is 2-connected and 3-edge-i-critical, then G is hamiltonian.*

We conclude this chapter by pointing out that the concept of defect k-factor-critical graphs has not yet been studied. In fact, it is first introduced in this thesis. Further, the concept of k-factor-critical graphs was studied only in 3-edge-critical graphs. There are no results on k-factor-critical in 3-edge-i-critical graphs. We provide the investigation of defect k-factor-critical graphs in Chapter 3. Chapter 4 contains sufficient conditions for 3-edge-i-critical graphs to be k-factor-critical for $k = 1$ and 2.

Chapter 3

Defect k-factor-critical in domination critical graphs

In this chapter, we present some properties of defect k-factor-critical graphs. In addition, we provide some sufficient conditions of 3-edge-critical graphs and 3-edge-i-critical graphs to be defect k-factor-critical for small k.

3.1 Fundamental results on defect k-factor-critical

It is the purpose of this section to give some basic properties of defect k-factor-critical graphs. We begin with the following lemma.

Lemma 3.1.1. *A graph G has a near-perfect matching if and only if $c_0(G-S) \leq |S|+1$ for every $S \subseteq V(G)$.*

Proof. Suppose that G has a near-perfect matching. Form a new graph G' from G by adding a new vertex x and joining x to every vertex of G. By our assumption G' has a perfect matching. Let $S \subseteq V(G)$ and put $S' = S \cup \{x\}$. By Theorem 2.1.1, $c_0(G'-S') \leq |S'|$. Hence, $c_0(G-S) = c_0(G'-S') \leq |S'| = |S|+1$.

We now suppose to the contrary that G has no near-perfect matching. Form a new graph G' from G by adding a new vertex x and joining x to every vertex of G. Then G' does not contain a perfect matching. So, by Theorem 2.1.1, there is a set S' in G' such that $c_0(G'-S') \geq |S'|+1$. Since x is adjacent to every vertex of G', $x \in S'$. Put $S = S' - \{x\}$. Thus $c_0(G-S) = c_0(G'-S') \geq |S'|+1 = |S|+2$, a contradiction. Hence, G has a near-perfect matching. This completes the proof of our lemma.

□

Theorem 3.1.2. *Let k be a non-negative integer and let G be a graph with $|V(G)| \equiv (k+1)(mod2)$. Then G is defect k-factor-critical if and only if $c_0(G-S) \leq |S|-k+1$ for every $S \subseteq V(G)$ such that $|S| \geq k$.*

Proof. Suppose G is a defect k-factor-critical graph. Let $S \subseteq V(G)$ such that $|S| \geq k$. Let $T \subseteq S$ with $|T| = k$. Put $S' = S - T$ and $V' = V(G) - T$. Let $G' = G[V']$. Then $c_0(G-S) = c_0(G'-S')$. By our definition, G' has a near-perfect matching. Hence, by Lemma 3.1.1, $c_0(G'-S') \leq |S'|+1$. So $c_0(G-S) = c_0(G'-S') \leq |S'|+1 = |S|-k+1$ as required.

We now establish the sufficient condition. Let $S \subseteq V(G)$ such that $|S| = k$. Suppose to the contrary that $G-S$ has no near-perfect matching. By Lemma 3.1.1, there exists $S' \subseteq V(G-S)$ such that $c_0((G-S)-S') \geq |S'|+2$. Then $c_0(G-(S \cup S')) \geq |S'|+2 = |S \cup S'|-k+2$, contradicting our assumption. So $G-S$ contains a near-perfect matching. This completes the proof of our theorem.

□

Lemma 3.1.3. *For $k \geq 2$, a defect k-factor-critical graph of order $n \geq k+3$ is defect $(k-2)$-factor-critical.*

Proof. Let G be a defect k-factor-critical graph of order $n \geq k+3$. Let $S \subseteq V(G)$ such that $|S| = k-2$. Then $|V(G-S)| = n-(k-2) \geq 5$. Since G is defect k-factor-critical , $G-(S \cup \{x,y\})$ has a near-perfect matching for every pair of vertices $x, y \in V(G-S)$. Hence, $G-S$ contains an edge since $|V(G-S)| \geq 5$ and $G-(S \cup \{x,y\})$ has a near-perfect matching for $x, y \in V(G-S)$. Let $ab \in E(G-S)$. Since $G-(S \cup \{a,b\})$ has a near-perfect matching, say F, then $F \cup \{ab\}$ is a near-perfect matching of $G-S$. This completes the proof of our lemma.

□

Lemma 3.1.4. *If a graph G is defect k-factor-critical, then $G-Y$ is defect $(k-p)$-factor-critical for every integer p with $0 \leq p \leq k$, $p \equiv (k+1)(mod2)$ and every set Y of p vertices of G.*

Proof. Let G be a defect k-factor-critical graph. Let $Y \subseteq V(G)$ such that $|Y| = p$ where $0 \leq p \leq k$. Put $S \subseteq V(G-Y)$ with $|S| = k-p$. Then $X = Y \cup S$ is a set of k vertices of G. Since G is defect k-factor-critical, $G-X = (G-Y)-S$ has a near-perfect matching. Hence, $G-Y$ is defect $(k-p)$-factor-critical. This completes the proof of our lemma.

□

Our next result provides a property of defect 2-factor-critical graphs having a cutvertex.

Theorem 3.1.5. *Let G be a connected defect 2-factor-critical graph and let v be a cutvertex of G.*
(1) If H is an even component of $G-v$, H has a perfect matching.
(2) If H is an odd component of $G-v$, H has a near-perfect matching.
(3) If $G-v$ has an odd component, $c(G-v)=c_0(G-v)=2$.

Proof. (1) Suppose that H has no perfect matching. Then by Theorem 2.1.1, there is a set $S \subseteq V(H)$ such that $c_0(H-S) \geq |S|+1$. Note that $S \neq \phi$. Let $S' = S \cup \{v\}$. Then $c_0(G-S') = c_0(H-S)+c_0(G-v) \geq |S|+1+c_0(G-v) = |S'|+c_0(G-v)$. It follows that $|S'|+c_0(G-v) \leq c_0(G-S') \leq |S'|-1$ by Theorem 3.1.2, a contradiction. Hence, H has a prefect matching.

(2) Suppose that H has no near-perfect matching. Then by Lemma 3.1.1, there is a set $S \subseteq V(H)$ such that $c_0(H-S) \geq |S|+2$. Note that $S \neq \phi$. Let $S' = S \cup \{v\}$. Then $c_0(G-S') = c_0(H-S)+c_0(G-v)-1 \geq |S|+2+c_0(G-v)-1 = |S|+1+c_0(G-v) = |S'|+c_0(G-v)$. It follows that $|S'|+c_0(G-v) \leq c_0(G-S') \leq |S'|-1$ by Theorem 3.1.2, a contradiction. Thus H has a near-prefect matching.

(3) Suppose $G-v$ has an odd component.

Claim 1: $G-v$ has no even component.

Suppose this is not the case. Let C be an even component of $G-v$. Since $|V(G)|$ is odd, $c_0(G-v)$ is even. Let H_1, H_2 be two odd components of $G-v$. Choose $a \in V(C)$. Then H_1, H_2 together with $C-a$ are odd components of $G-\{a,v\}$. But this contradicts Theorem 3.1.2. Hence, $G-v$ has no even component.

We next show that $c(G-v)=2$. Suppose that $c(G-v) \geq 3$. Since $|V(G)|$ is odd, $c(G-v)=c_0(G-v)$ is even by Claim 1. Since $c(G-v) \geq 3, c_0(G-v) \geq 4$. Let H_1, H_2, H_3 be odd components of $G-v$. Choose $a \in V(H_1)$. Then H_2 and H_3 are odd components of $G-\{a,v\}$. But this contradicts Theorem 3.1.2. Hence, $c(G-v)=2$ and so by Claim 1 we have $c_0(G-v)=2$.

□

Let G and H be disjoint graphs and let $u \in V(G)$ and $v \in V(H)$. The **coalescense** of G and H with respect to u and v is the graph $G \cdot_{uv} H$ defined to have the vertex set

$$V(G \cdot_{uv} H) = (V(G) - \{u\}) \cup (V(H) - \{v\}) \cup \{w\},$$

where $w \notin V(G) \cup V(H)$, and edge set

$$E(G \cdot_{uv} H) = E(G - u) \cup E(H - v) \cup \{wx \mid ux \in E(G) \text{ or } vx \in E(H)\}.$$

Informally, $G \cdot_{uv} H$ is the graph obtained from $G \cup H$ by identifying u and v. If the context is clear, or if the vertices u and v are not important, we write $G \cdot H$ instead of $G \cdot_{uv} H$.

We define $G_1 \cdot G_2 \cdot \ldots \cdot G_n$ to be $(G_1 \cdot G_2 \cdot \ldots \cdot G_{n-1}) \cdot G_n$.

For $1 \leq i \leq n+1$, let $G_i \cong K_3$ and let $V(G_i) = \{u_i, v_i, w_i\}$. Let $G = G_{w_1u_2} \cdot G_{w_2u_3} \cdot G_{w_3u_4} \cdot \ldots \cdot G_{w_nu_{n+1}} \cdot G_{n+1}$. Figure 2 illustrates our graph G. It is easy to see that G is defect 2-factor-critical with the vertex cutset $\{w_1u_2, w_2u_3, ..., w_nu_{n+1}\}$. Hence, the size of vertex cutset of defect 2-factor-critical graph is unbounded.

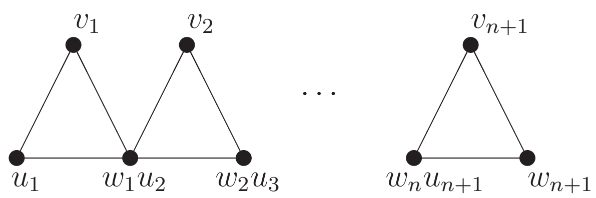

Figure 2: A defect 2-factor-critical graph with a cut vertex.

3.2 3-edge-critical graphs and defect k-factor-critical

In this section, we show that connected 3-edge-critical graphs having even order are defect 1-factor-critical. Further, we also provide some sufficient conditions of connected 3-edge-critical graphs to be defect k-factor-critical for $2 \leq k \leq 4$.

Theorem 3.2.1. *Let G be a connected 3-edge-critical graph. If G has even order, then G is defect 1-factor-critical.*

Proof. Let $x \in V(G)$. Suppose that $G - x$ has no near-perfect matching. Then, by Lemma 3.1.1, there is a set $S \subseteq V(G-x)$ such that $c_0((G - \{x\}) - S) \geq |S| + 2 = |S \cup \{x\}| + 1$. Since $|V(G)|)$ is even, by parity, $c_0(G - (S \cup \{x\})) \geq |S \cup \{x\}| + 2$ which contradicts Theorem 2.2.4. Hence, $G - x$ has a near-perfect matching.

□

Theorem 3.2.2. *Let G be a connected 3-edge-critical graph of odd order. If $\delta(G) \geq 2$, then G is defect 2-factor-critical.*

Proof. Let $S \subseteq V(G)$ such that $|S| \geq 2$. By Theorem 3.1.2, it suffices to show that $c_0(G - S) \leq |S| - 1$. By Theorem 2.2.15.(a), $c_0(G - S) \leq c(G - S) \leq |S| - 1$ if $|S| \geq 4$. So we only need to consider $2 \leq |S| \leq 3$.

Case 1: $|S| = 3$.

Then $G - S$ contains at most $|S|$ components, by Theorem 2.2.15.(b). Then $c_0(G - S) \leq |S|$. If $c_0(G - S) = |S|$, then $|V(G)|$ is even, a contradiction. Hence, $c_0(G - S) \leq |S| - 1$, as required.

Case 2: $|S| = 2$.

Suppose to the contrary that $c_0(G - S) \geq 2$. By Theorem 2.2.4, $G - S$ contains at most $|S| + 1 = 2 + 1 = 3$ components. Then either $c_0(G - S) = 2$ or $c_0(G - S) = 3$. If $c_0(G - S) = 2 = |S|$, then $|V(G)|$ is even, a contradiction. Hence $c_0(G - S) = 3$. By Theorem 2.2.15.(c), G has the structure shown in Figure 2.1. Thus $\delta(G) = 1$, a contradiction. Hence, $c_0(G - S) \leq 1$. In fact, $c_0(G - S) = 1$ since $|V(G)|$ is odd. Therefore, $c_0(G - S) \leq |S| - 1$ as required. This completes the proof of case 2 and thus our theorem.

□

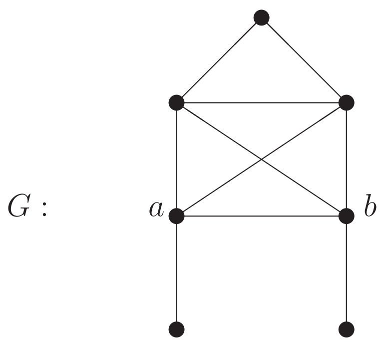

Figure 3: A 3-edge-critical graph of odd order with $\delta(G) = 1$ which is not defcet 2-factor-critical.

The bound on the minimum degree stated in the hypotheses of Theorem 3.2.2 is best possible since there is a 3-edge-critical graph with minimum degree 1 and having odd order, but is not defect 2-factor-critical. Such a graph G is shown in Figure 3. Note that $G-\{a,b\}$ has no near-perfect matching.

Theorem 3.2.3. *Let G be a connected 3-edge-critical graph of even order. If $\delta(G) \geq 4$, then G is defect 3-factor-critical.*

Proof. Let $S \subseteq V(G)$ such that $|S| \geq 3$. By Theorem 3.1.2, it suffices to show that $c_0(G-S) \leq |S|-3+1 = |S|-2$. Suppose to the contrary that $c_0(G-S) \geq |S|-1$. If $|S| \geq 4$, then $c_0(G-S) = c(G-S) = |S|-1$ by Theorem 2.2.15.(a). But then $|V(G)|$ is odd, a contradiction. Hence, $|S| \leq 3$. It follows that $|S| = 3$ and $c_0(G-S) = 2$ or $c_0(G-S) = 3$ by Theorem 2.2.15.(b) . If $c_0(G-S) = 2$, then $|V(G)|$ is odd, a contradiction. Thus $c_0(G-S) = 3$. Let C_1, C_2, C_3 be the odd components of $G-S$. Then, by Theorem 2.2.15.(b), at least one of them is singleton. Without loss of generality, we may assume that $V(C_1) = \{x_1\}$. Since $|S| = 3, d(x_1) \leq 3$, a contradiction. Hence, $c_0(G-S) \leq |S|-2$ for every $S \subseteq V(G)$ with $|S| \geq 3$. So, by Theorem 3.1.2, G is defect 3-factor-critical, as required. This completes the proof of our theorem.

□

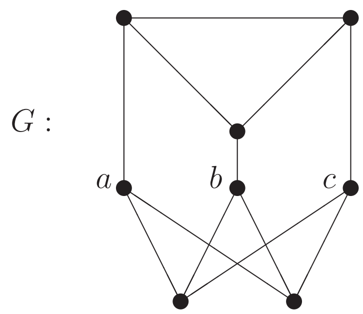

Figure 4: A 3-edge-critical graph of even order with $\delta(G) = 3$ which is not defcet 3-factor-critical.

The bound on the minimum degree stated in the hypotheses of Theorem 3.2.3 is best possible since there is a 3-edge-critical graph with minimum degree 3 having even order which is not defect 3-factor-critical. Such a graph G is shown in Figure 4. Note that $G-\{a,b,c\}$ has no near-perfect matching.

The following theorem shows that the minimum degree in the hypotheses of Theorem 3.2.3 can be decreased by 1 when the assumption claw-free is added.

Theorem 3.2.4. *Let G be a connected claw-free 3-edge-critical graph of even order. If $\delta(G) \geq 3$, then G is defect 3-factor-critical.*

Proof. Suppose that G is not defect 3-factor-critical. Then, by Theorem 3.1.2, there is a set $S \subseteq V(G)$ such that $|S| \geq 3$ and $c_0(G-S) \geq |S|-3+2 = |S|-1$. By Theorem 2.2.15.(a), if $|S| \geq 4$ then $G-S$ has at most $|S|-1$ components and so $c_0(G-S) = |S|-1$ which implies that $|V(G)|$ is odd, a contradiction. Hence, $|S| = 3$. Thus $c_0(G-S) \leq |S|$ by Theorem 2.2.15.(b). Therefore, $2 = |S|-1 \leq c_0(G-S) \leq |S| = 3$. If $c_0(G-S) = 2$ then $|V(G)|$ is odd, a contradiction. Hence, $c_0(G-S) = 3$.

Let C_1, C_2, C_3 be the odd components of $G-S$. For $1 \leq i \leq 3$, choose $w_i \in V(C_i)$. Clearly $W = \{w_1, w_2, w_3\}$ in an independent set. By Lemma 2.2.1.(a), the vertices in W may be ordered as u_1, u_2, u_3 in such a way that there exists a sequence of distinct vertices x_1, x_2 such that $[u_i, x_i] \rightarrow u_{i+1}$ for $i = 1, 2$. Without loss of generality, we may renumber the components of $G-S$ in such a way that $u_i \in V(C_i)$ for $1 \leq i \leq 3$. Since $|S| = 3$ and $G-S$ contains exactly three odd components, it follows that each component is complete and at least one component is singleton by Theorem 2.2.15.(b).

Suppose that $|V(C_1)| = 1$. Then $V(C_1) = \{u_1\}$. So u_1 is adjacent to every vertex in S because $\delta(G) \geq 3$. Since $[u_1, x_1] \rightarrow u_2$, $x_1 \in V(C_3) \cup S$. Suppose that $x_1 \in V(C_3)$. Then $V(C_2) = \{u_2\}$. Thus u_2 is adjacent to every vertex in S because $\delta(G) \geq 3$. Since G is connected, there is a vertex $v \in V(C_3)$ such that v is adjacent to some vertex in S. Hence, there is a claw, a contradiction. Thus $x_1 \in S$ and so x_1 dominates $(V(C_2) \cup V(C_3)) - \{u_2\}$. Since $d(u_2) \geq 3$ and $x_1u_2 \notin E(G)$, it follows that $|V(C_2)| \geq 3$. Hence, there is a claw centered at x_1, again a contradiction. Therefore $|V(C_1)| \geq 3$. By similar arguments, $|V(C_2)| \geq 3$. Thus $|V(C_3)| = 1$. Then $V(C_3) = \{u_3\}$. Thus u_3 is adjacent to every vertex in S because $\delta(G) \geq 3$. Since $[u_1, x_1] \rightarrow u_2$ and $|V(C_2)| \geq 3$, it follows that $x_1 \in S$ and so x_1 dominates $(V(C_2) \cup \{u_3\}) - \{u_2\}$. Because $[u_2, x_2] \rightarrow u_3$, $x_2 \notin S$. So $x_2 \in V(C_1)$ and x_2 dominates $V(C_1)$. Further, $x_1x_2 \in E(G)$ because $x_1u_2 \notin E(G)$. Hence, there is a claw centered at x_1, a contradiction. This completes the proof of our theorem.

□

Theorem 3.2.5. *Let G be a connected claw-free 3-edge-critical graph of odd order. If $\delta(G) \geq 4$, then G is defect 4-factor-critical.*

Proof. Suppose that G is not defect 4-factor-critical. Then, by Theorem 3.1.2, there is a set $S \subseteq V(G)$ such that $|S| \geq 4$ and $c_0(G-S) \geq |S|-4+2 = |S|-2$. By Theorem 2.2.15.(a), $c_0(G-S) \leq |S|-1$. Hence, $|S|-2 \leq c_0(G-S) \leq |S|-1$. If $c_0(G-S) = |S|-2$, then $|V(G)|$ is even, a contradiction. Thus $c_0(G-S) = |S|-1$. Suppose that $|S| = n$ where $n \geq 4$ and let $C_1, C_2, ..., C_{n-1}$ be odd components of $G-S$. For $1 \leq i \leq n-1$, choose $w_i \in V(C_i)$. Clearly, $W = \{w_1, w_2, ..., w_{n-1}\}$ is an independent set.

Case 1: $n \geq 5$.

By Lemma 2.2.1.(b), the vertices in W may be ordered as $a_1, a_2, ..., a_{n-1}$ in such a way that there exists a path $x_1x_2...x_{n-2}$ in $G-W$ such that $[a_i, x_i] \to a_{i+1}$ for $1 \leq i \leq n-2$. Without loss of generality, we may renumber the components of $G-S$ in such a way that $a_i \in V(C_i)$ for $1 \leq i \leq n-1$. Since for $1 \leq i \leq n-1$, a_i does not dominate vertices in C_j for $i \neq j$, it follows that x_i must belong to S. Hence, $|S - \{x_1, x_2, ..., x_{n-2}\}| = 2$. Let $\{w, z\} = S - \{x_1, x_2, ..., x_{n-2}\}$. If $n \geq 6$ then $G[\{x_1, a_3, a_4, a_5\}]$ is a claw centered at x_1, a contradiction. Hence $n = 5$.

Since $[a_1, x_1] \to a_2$, x_1 dominates $(V(C_2) \cup V(C_3) \cup V(C_4)) - \{a_2\}$. Because G is claw-free, $V(C_2) = \{a_2\}$. Hence, $a_2x_2, a_2x_3, a_2w, a_2z \in E(G)$ because $\delta(G) \geq 4$. By similar arguments, $V(C_3) = \{a_3\}$ and $V(C_4) = \{a_4\}$ since $[a_i, x_i] \to a_{i+1}$ for $1 \leq i \leq 3$. Furthermore, $wa_3 \in E(G)$ and $wa_4 \in E(G)$. But then $G[\{w, a_2, a_3, a_4\}]$ is a claw centered at w, a contradiction.

Case 2: $n = 4$.

By Lemma 2.2.1.(a), the vertices in W may be ordered as a_1, a_2, a_3 in such a way that there exists a sequence of distinct vertices x_1, x_2 such that $[a_i, x_i] \to a_{i+1}$ for $1 \leq i \leq 2$. Clearly, $x_ia_{i+1} \notin E(G)$. Without loss of generality, we may renumber the components of $G-S$ in such a way that $a_i \in V(C_i)$ for $1 \leq i \leq 3$.

Claim 1: $|V(C_i)| \geq 3$ for $1 \leq i \leq 3$.

Suppose to the contrary that there is $1 \leq i \leq 3$ such that $|V(C_i)| = 1$. Suppose that $i = 1$. Then a_1 is adjacent to every vertex in S. Since $[a_1, x_1] \to a_2$, $x_1 \in V(C_3) \cup S$. Suppose that $x_1 \in V(C_3)$. Then $V(C_2) = \{a_2\}$. Thus a_2 is adjacent to every vertex in S because $\delta(G) \geq 4$. Since G is connected, there is a vertex $v \in V(C_3)$ such that v is adjacent to some vertex in S. So there is a claw, a contradiction. Hence, $x_1 \in S$. Therefore x_1 dominates $(V(C_2) \cup V(C_3)) - \{a_2\}$. Since $\delta(G) \geq 4$, $|V(C_2)| \geq 3$. Thus there is a claw centered at x_1, a contradiction. Hence, $i \neq 1$. By similar arguments, $i \neq 2$. Hence, $i = 3$ and thus $V(C_3) = \{a_3\}$. Then a_3 is adjacent to every vertex in S because $\delta(G) \geq 4$. Since $[a_1, x_1] \to a_2$ and $|V(C_2)| \geq 3$, it follows that $x_1 \in S$ and so x_1 dominates $(V(C_2) \cup \{a_3\}) - \{a_2\}$. Because $[a_2, x_2] \to a_3$, $x_2 \notin S$. So $x_2 \in V(C_1)$ and x_2 dominates $V(C_1)$. Further, $x_1x_2 \in E(G)$ because $x_1a_2 \notin E(G)$. Hence, there is a claw centered at x_1, a contradiction. This proves our claim .

By Claim 1 and the fact that for $1 \leq i \leq 2$, $[a_i, x_i] \to a_{i+1}$, it follows that x_i must belong to S. Further, x_1 dominates $(V(C_2) \cup V(C_3)) - \{a_2\}$. Because G is claw-free, x_1 is not adjacent to any vertex of $V(C_1)$. In addition, $G[V(C_2) - \{a_2\}]$ and $G[V(C_3)]$ are complete. Since $[a_2, x_2] \to a_3$, x_2 dominates $(V(C_1) \cup V(C_3)) - \{a_3\}$. Thus x_2 is not adjacent to any vertex of $V(C_2)$ and $G[V(C_1)]$ is complete because G is claw-free.

We now consider $G + a_1a_3$. Since $\gamma(G + a_1a_3) = 2$, there is $u \in V(G) - \{a_1, a_3\}$ such that either $[a_1, u] \to a_3$ or $[a_3, u] \to a_1$. Clearly, in either case, $u \in S$ and u dominates $V(C_2)$. Hence, $u \notin \{x_1, x_2\}$.

Subcase 2.1: $[a_1, u] \to a_3$.

Hence, $u \in \{w, z\}$. Without loss of generality, we may assume that $u = w$. Then $[a_1, w] \to a_3$. Thus w dominates $(V(C_2) \cup V(C_3) \cup \{x_1\}) - \{a_3\}$. Since G is claw-free, w is not adjacent to any vertex of $V(C_1)$. Let $a_3' \in V(C_3) - \{a_3\}$. We next consider $G + a_2a_3'$. Since $\gamma(G + a_2a_3') = 2$, there is $y \in V(G) - \{a_2, a_3'\}$ such that either $[a_2, y] \to a_3'$ or $[a_3', y] \to a_2$. In either case $y \in S$. Suppose that $[a_2, y] \to a_3'$. Since $x_1a_3', x_2a_3', wa_3' \in E(G)$, $y \notin \{x_1, x_2, w\}$. Hence, $y = z$. Then $[a_2, z] \to a_3'$ which implies that z dominates $(V(C_1) \cup V(C_3) \cup \{x_1, x_2\}) - \{a_3'\}$. Thus $\{w, z\}$ dominates G, a contradiction. Therefore $[a_3', y] \to a_2$. Since $wa_2 \in E(G)$, $y \neq w$. Because $\{a_3', x_1\}$ does not dominate $V(C_1)$, $y \neq x_1$. Since $\{a_3', x_2\}$ does not dominate $V(C_2) - \{a_2\}$, $y \neq x_2$. Thus $y = z$. Hence, $[a_3', z] \to a_2$ which implies that z dominates $(V(C_1) \cup V(C_2)) - \{a_2\}$. Since G is claw-free, z is not adjacent to any vertex of $V(C_3)$. We now consider $G + a_1a_3'$. Since $\gamma(G + a_1a_3') = 2$, there is $v \in V(G) - \{a_1, a_3'\}$ such that either $[a_1, v] \to a_3'$ or $[a_3', v] \to a_1$. In either case $v \in S$. Suppose that $[a_1, v] \to a_3'$. Since $x_1a_3', x_2a_3', wa_3' \in E(G)$, $v \notin \{x_1, x_2, w\}$.

Because $\{a_1, z\}$ does not dominate $V(C_3)$, $v \neq z$. Hence, $v \notin S$, a contradiction. Thus $[a_3', v] \to a_1$. Since $x_2a_1, za_1 \in E(G)$, $v \notin \{x_2, z\}$. Because $\{a_3', x_1\}$ does not dominate $V(C_1) - \{a_1\}$, $v \neq x_1$. Furthermore, $v \neq w$ because $\{a_3', w\}$ does not dominate $V(C_1)$. Hence, $v \notin S$, again a contradiction.

Subcase 2.2: $[a_3, x] \to a_1$.

By similar arguments as in the proof of Subcase 2.1, we get a contradiction.

□

3.3 3-edge-i-critical graphs and defect k-factor-critical

In this section, we focus on the study of 3-edge-i-critical graphs. We establish some properties of connected 3-edge-i-critical graphs and show that some properties in [1] for connected 3-edge-critical graphs are also true for connected 3-edge-i-critical graphs. After that, we concentrate on sufficient conditions of connected 3-edge-i-critical graphs to be defect k-factor-critical for $1 \leq k \leq 4$. We begin with an infintely family of 3-edge-i-critical graphs J_{k,n_1,n_2} as follow.

For positive integers k, n_1 and n_2, define a graph J_{k,n_1,n_2} of order $2k + n_1 + n_2 + 2$ as follows. Put $V(J_{k,n_1,n_2}) = U \cup Y \cup W \cup \{a, b\}$ where $U = \{u_1, u_2, ..., u_k, v_1, v_2, ..., v_k\}$. The edges of J_{k,n_1,n_2} are defined as follows. $J_{k,n_1,n_2}[U] = \{u_iv_j \mid i \neq j\}$, $J_{k,n_1,n_2}[Y \cup W] = K_{n_1+n_2}$. Further, join each vertex of $Y \cup \{a, b\}$ to every vertex of U. Figure 5 illustrates our construction. It is not difficult to show that J_{k,n_1,n_2} is 3-edge-i-critical graph.

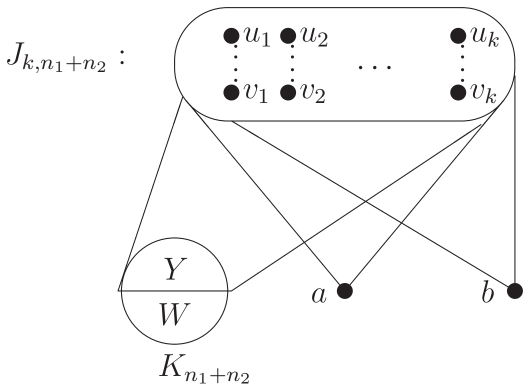

Figure 5: A graph J_{k,n_1,n_2}.

Lemma 3.3.1. *Let G be a connected 3-edge-i-critical graph and let u and v be non-adjacent vertices of G. If x is a vertex of G such that $[u, x]\underset{i}{\rightarrow} v$ or $[v, x]\underset{i}{\rightarrow} u$ then $\{u, v, x\}$ is independent.*

Proof. Assume that $[u, x]\underset{i}{\rightarrow} v$. Suppose that $xv \in E(G)$. Then u and x dominate G, contradicting the assumption that $i(G) = 3$. Similarly, if $[v, x]\underset{i}{\rightarrow} u$ then $xu \notin E(G)$. This completes the proof of our lemma.

□

Lemma 3.3.2. *Let G be a connected 3-edge-i-critical graph and let S be an independent set of $n \geq 3$ vertices in $V(G)$. Then the vertices of S can be ordered as $a_1, a_2, ..., a_n$ in such a way that there exist distinct vertices $x_1, x_2 ..., x_{n-1}$ so that $[a_i, x_i]\underset{i}{\rightarrow} a_{i+1}$ for $i = 1, 2, ..., n-1$. If $\{a_1, a_2, ..., a_n\} \cap \{x_1, x_2, ..., x_{n-1}\} = \phi$, then $x_j x_{j+1} \in E(G)$ for $1 \leq j \leq n-2$.*

Proof. The result was proved by Ao [4] for $n \geq 4$. So we only consider $n = 3$. Since S is independent in G, $\overline{G}[S]$ is a complete graph. By canonical orientation, we have a tournament $\overline{G}[S]$. Since every tournament has a Hamiltonian path, we may label vertices in S as a_1, a_2, a_3 such that for each $i = 1, 2$, there is an arc $a_i a_{i+1}$ in $\overline{G}$. Hence, for each $i = 1, 2$, there is $x_i \in V(G)$ such that $[a_i, x_i]\underset{i}{\rightarrow} a_{i+1}$. Thus $x_1 a_3 \in E(G)$ but $x_2 a_3 \notin E(G)$. So $x_1 \neq x_2$, as required. We now may assume that $\{a_1, a_2, a_3\} \cap \{x_1, x_2\} = \phi$. Since $[a_2, x_2]\underset{i}{\rightarrow} a_3$ and $a_2 x_1 \notin E(G)$, it follows that $x_1 x_2 \in E(G)$. This completes the proof of our lemma.

□

Remark:

1. If $n \geq 4$ in Lemma 3.3.2, then it is not difficult to show that $\{a_1, a_2, ..., a_n\} \cap \{x_1, x_2, ..., x_{n-1}\} = \phi$. Consequently, $x_1 x_2 ... x_{n-1}$ is a path.
2. If $n = 3$, then $\{a_1, a_2, a_3\} \cap \{x_1, x_2\}$ need not be empty. Consider a graph $J_{1,1,2}$ in Figure 6. By our construction, $J_{1,1,2}$ is 3-edge-i-critical graph.

$J_{1,1,2}$:

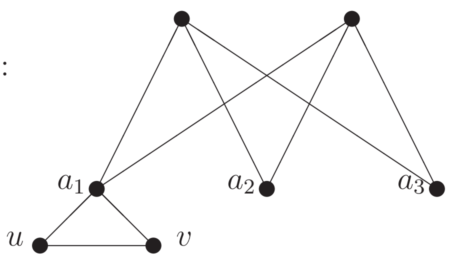

Figure 6: A graph $J_{1,1,2}$.

Clearly, if $[a_1, x_1] \underset{i}{\rightarrow} a_2$ and $[a_2, x_2] \underset{i}{\rightarrow} a_3$, then $x_1 = a_3$ and $x_2 \in \{a_1\} \cup \{u, v\}$. Thus $x_1 x_2 \notin E(G)$.

Lemma 3.3.3. *If G is a connected 3-edge-i-critical graph of even order, then G has a perfect matching.*

Proof. Suppose that G does not contain a perfect matching. By Theorem 2.1.1, there is a set $S \subseteq V(G)$ such that $c_0(G-S) > |S|$. Since $|V(G)|$ is even, $c_0(G-S) \geq |S| + 2$ which contradicts Theorem 2.3.3. Hence, G has a perfect matching.

□

The following two results provide the number of components of 3-edge-i-critical graphs having vertex cutset of size at least three.

Theorem 3.3.4. *Let G be a connected 3-edge-i-critical graph and let S be a vertex cutset in G. If $|S| \geq 4$, then $G - S$ has at most $|S| - 1$ components.*

Proof. Suppose $G - S$ contains $n \geq |S|$ components. Let $C_1, C_2, ..., C_n$ be components of $G - S$. For $1 \leq i \leq n$, choose $a_i \in C_i$. Then $I = \{a_1, a_2, ..., a_n\}$ is an independent set. By Lemma 3.3.2, the vertices in I may be ordered as $u_1, u_2, ..., u_n$ in such a way that there exist distinct vertices $x_1, x_2, ..., x_{n-1}$ such that $[u_i, x_i] \underset{i}{\rightarrow} u_{i+1}$ for $i = 1, 2, ..., n-1$. Without loss of generality, we may renumber the components of $G - S$ in such a way that $u_i \in V(C_i)$ for $i = 1, ..., n$. Note that, for $1 \leq i \leq n-1$, $\{u_i, x_i, u_{i+1}\}$ is independent. Since $n \geq 4$, $x_i \in S$ for $i = 1, 2, ..., n-1$ and so $|S| \geq n-1$. Thus $n-1 \leq |S| \leq n$. Suppose that $|S| = n-1$. Consider $G + u_1 u_{n-1}$. Since $i(G + u_1 u_{n-1}) = 2$, there is $v \in V(G) - \{u_1, u_{n-1}\}$ such that $[u_1, v] \underset{i}{\rightarrow} u_{n-1}$ or $[u_{n-1}, v] \underset{i}{\rightarrow} u_1$. In either case $v \in S$ and $\{u_1, u_{n-1}, v\}$ is independent. Since $u_1 x_i \in E(G)$ for $2 \leq i \leq n-1$, $v \neq x_i$ for $2 \leq i \leq n-1$. Further, $v \neq x_1$ because $u_{n-1} x_1 \in E(G)$. Therefore $v \notin S$ a contradiction. Hence, $|S| = n$.

Since $[u_i, x_i] \underset{i}{\rightarrow} u_{i+1}$ and I is independent, it follows that x_i is adjacent to every vertex of $\bigcup_{j=1}^{n} V(C_j) - (V(C_i) \cup \{u_{i+1}\})$ for $i = 1, 2, ..., n-1$. Let $\{y\} = S - \{x_1, x_2, ..., x_{n-1}\}$.

We now consider $G + u_1 u_n$. Since $i(G + u_1 u_n) = 2$, there is $z \in V(G) - \{u_1, u_n\}$ such that $[u_1, z] \underset{i}{\rightarrow} u_n$ or $[u_n, z] \underset{i}{\rightarrow} u_1$. In either case, $z \in S$ and $\{u_1, u_n, z\}$ is independent. Further, z dominates $\bigcup_{j=1}^{n} V(C_j) - (V(C_1) \cup V(C_n))$. Since $x_i u_i \notin E(G)$ for $2 \leq i \leq n-1$, $z \notin \{x_2, x_3, ..., x_{n-1}\}$. Because $x_1 u_n \in E(G)$, $z \neq x_1$. Thus $z = y$.

Case 1: $[u_1, y] \xrightarrow[i]{} u_n$.

Then y dominates $\bigcup_{j=2}^{n} V(C_j) - \{u_n\}$. Since $u_1x_1 \notin E(G)$, $yx_1 \in E(G)$. Now consider $G + u_nu_{n-2}$. Since $i\,(G + u_nu_{n-2}) = 2$, there is $w \in V\,(G) - \{u_n, u_{n-2}\}$ such that $[u_n, w] \xrightarrow[i]{} u_{n-2}$ or $[u_{n-2}, w] \xrightarrow[i]{} u_n$. In either case, $w \in S$ and $\{u_{n-2}, u_n, w\}$ is independent. Further, w dominates $\bigcup_{j=1}^{n} V(C_j) - (V(C_{n-2}) \cup V(C_n))$. Thus $w \notin \{x_1, ..., x_{n-3}, x_{n-1}, y\}$. It follows that $w = x_{n-2}$. But this contradicts Lemma 3.3.1 since $x_{n-2}u_n \in E(G)$. Hence, Case 1 cannot occur.

Case 2: $[u_n, y] \xrightarrow[i]{} u_1$.

Then y dominates $\bigcup_{j=1}^{n-1} V(C_j) - \{u_1\}$. Since $u_nu_{n-1} \notin E(G)$, $yu_{n-1} \in E(G)$. Now consider $G + u_1u_{n-1}$. Since $i\,(G + u_1u_{n-1}) = 2$, there is $w \in V\,(G) - \{u_1, u_{n-1}\}$ such that $[u_1, w] \xrightarrow[i]{} u_{n-1}$ or $[u_{n-1}, w] \xrightarrow[i]{} u_1$. In either case $w \in S$ and $\{u_1, u_{n-1}, w\}$ is independent. Further, w dominates $\bigcup_{j=1}^{n} V(C_j) - (V(C_1) \cup V(C_{n-1}))$. Thus $w \notin \{x_2, ..., x_{n-2}\}$ since $x_iu_i \notin E(G)$ for $2 \leq i \leq n-2$. Because $x_1u_{n-1}, x_{n-1}u_1, yu_{n-1} \in E(G)$, $w \notin \{x_1, x_{n-1}, y\}$. Hence, $w \notin S$, a contradiction. So Case 2 cannot occur either. This completes the proof of our theorem.

□

Theorem 3.3.5. *Let G be a connected 3-edge-i-critical graph and let S be a vertex cutset in G. If $|S| = 3$, then $G - S$ contains at most 3 components, and if $G - S$ has exactly 3 components, then at least one component is singleton.*

Proof. For $1 \leq i \leq n$, let H_i denote the component of $G - S$. Suppose to the contrary that $n \geq |S| + 1 = 4$. By Theorem 2.3.3, $n = 4$. For $1 \leq i \leq 4$, choose $w_i \in V\,(H_i)$. Clearly, $W = \{w_1, w_2, w_3, w_4\}$ is an independent set. By Lemma 3.3.2, the vertices in W may be ordered as a_1, a_2, a_3, a_4 in such a way that there exist distinct vertices x_1, x_2, x_3 such that $[a_i, x_i] \xrightarrow[i]{} a_{i+1}$ for $1 \leq i \leq 3$. Note that $x_i \in S$ for $1 \leq i \leq 3$ because $G - S$ has 4 components. Further, $x_i \neq x_j$ for $1 \leq i \neq j \leq 3$. Thus $\{x_1, x_2, x_3\} = S$.

We now consider $G + a_1a_3$. Since $i(G + a_1a_3) = 2$, there exists a vertex $y \in V(G) - \{a_1, a_3\}$ such that either $[a_1, y] \xrightarrow[i]{} a_3$ or $[a_3, y] \xrightarrow[i]{} a_1$. Clearly, in either case, $y \in S$. Further, since $\{a_1, a_2, a_3, a_4\}$ is independent, y is adjacent to a_2 and a_4. Then $y \neq x_1$ and $y \neq x_2$ because $x_1a_2 \notin E(G)$ and $x_2a_2 \notin E(G)$. Further, $y \neq x_3$ since $x_3a_4 \notin E(G)$. Thus $y \notin S$, a contradiction. Hence $G - S$ contains at most three components as required.

We now suppose that $G - S$ contains exactly three components. That is, $n = 3$. Suppose by a way of contradiction that each component of $G - S$ has at least two vertices. For $1 \le i \le 3$, choose $w_i \in V(H_i)$. Clearly, $W = \{w_1, w_2, w_3\}$ is an independent set. By Lemma 3.3.2, the vertices in W may be ordered as a_1, a_2, a_3 in such a way that there exists a path x_1x_2 such that $[a_i, x_i] \underset{i}{\rightarrow} a_{i+1}$ for $i = 1, 2$. Without loss of generality, we may renumber the components of $G - S$ in such a way that $a_i \in V(H_i)$ for $i = 1, 2, 3$. Since each component of $G - S$ has at least two vertices, x_1 and x_2 must belong to S. Clearly, $x_1 \neq x_2$. Let $\{w\} = S - \{x_1, x_2\}$. Choose $a_1' \in V(H_1) - \{a_1\}$ and $a_3' \in V(H_3) - \{a_3\}$.

Consider $G + a_1'a_3'$. Since $i(G + a_1'a_3') = 2$, there exists a vertex $z \in V(G) - \{a_1', a_3'\}$ such that $[a_1', z] \underset{i}{\rightarrow} a_3'$ or $[a_3', z] \underset{i}{\rightarrow} a_1'$. Clearly, in either case $z \in S$ and $\{a_1', a_3', z\}$ is independent. Further, z dominates $V(H_2)$. Then $z \notin \{x_1, x_2\}$ since $x_1a_2, x_2a_2 \notin E(G)$. Thus $z = w$.

Case 1: $[a_1', w] \underset{i}{\rightarrow} a_3'$.

Clearly, w dominates $(V(H_2) \cup V(H_3)) - \{a_3'\}$. Now consider $G + a_1'a_3$. Since $i(G + a_1'a_3) = 2$, there is a vertex $z_1 \in G - \{a_1', a_3\}$ such that $[a_1', z_1] \underset{i}{\rightarrow} a_3$ or $[a_3, z_1] \underset{i}{\rightarrow} a_1'$. In either case $z_1 \in S$ and $\{a_1', z_1, a_3\}$ is independent. Further, z_1 dominates $V(H_2)$. Since $x_1a_2, x_2a_2 \notin E(G)$, $z_1 \notin \{x_1, x_2\}$. It then follows that $z_1 = w$. But this contradicts Lemma 3.3.1 since w dominates $V(H_3) - \{a_3'\}$. Hence, Case 1 cannot occur.

Case 2: $[a_3', z] \underset{i}{\rightarrow} a_1'$.

By similar arguments as in the proof of Case 1, Case 2 cannot occur. This completes the proof of our theorem.

□

The following lemma shows that 3-edge-i-critical graphs having odd order have a near-perfect matching.

Lemma 3.3.6. *Let G be a connected 3-edge-i-critical graph. If $|V(G)|$ is odd, then G contains a near-perfect matching.*

Proof. Suppose that G does not contain a near-perfect matching. By Lemma 3.1.1, there is a set $S \subseteq V(G)$ such that $c_0(G-S) \ge |S|+2$. But this contradics Theorem 2.3.3. Thus G contains a near-perfect matching, as required. This completes the proof of our lemma.

□

Theorem 3.3.7. *Let G be a connected 3-edge-i-critical graph. If G has even order, then G is defect 1-factor-critical.*

Proof. By similar arguments as in the proof of Theorem 3.2.1 together with the fact if $[x, y] \underset{i}{\rightarrow} z$, then $\{x, y, z\}$ is independent, our theorem follows.

□

Recall that J_{k,n_1,n_2} is the graph defined in figure 5.

Theorem 3.3.8. *Let $G \neq J_{1,n_1,n_2}$ where $n_1+n_2 \geq 3$ be a connected 3-edge-i-critical graph of odd order. If $\delta(G) \geq 2$, then G is defect 2-factor-critical.*

Proof. Suppose that G is not defect 2-factor-critical. By Theorem 3.1.2, there is a set $S \subseteq V(G)$ such that $|S| \geq 2$ and $c_o(G-S) \geq |S|-2+2 = |S|$. It follows by Theorem 3.3.4 that $2 \leq |S| \leq 3$. So, by Theorem 3.3.5, if $|S| = 3$ then $G - S$ contains at most 3 components and thus $c_0(G - S) = 3 = |S|$ which implies that $|V(G)|$ is even, a contradiction. Therefore, $|S| = 2$. Thus $2 = |S| \leq c_0(G-S) \leq |S|+1 = 3$ by Theorem 2.3.3. If $c_0(G - S) = 2 = |S|$, then $|V(G)|$ is even, a contradiction. So $c_0(G - S) = 3$. Let C_1, C_2, C_3 be odd components of $G - S$. For $1 \leq i \leq 3$, choose $w_i \in V(C_i)$. Clearly, $W = \{w_1, w_2, w_3\}$ is an independent set. By Lemma 3.3.2, the vertices in W may be ordered as a_1, a_2, a_3 in such a way that there exist distinct vertices x_1, x_2 such that $[a_i, x_i] \underset{i}{\rightarrow} a_{i+1}$ for $i = 1, 2$. Without loss of generality, we may renumber the components of $G - S$ in such a way that $a_i \in V(C_i)$ for $i = 1, 2, 3$. By Lemma 3.3.1, $\{a_i, x_i, a_{i+1}\}$ is independent for $i = 1, 2$. Put $S = \{x, y\}$.

Claim 1: $|V(C_1)| \geq 3$.

Suppose to the contrary that $|V(C_1)| = 1$. Then a_1 is adjacent to every vertex in S because $\delta(G) \geq 2$. Since $[a_1, x_1] \underset{i}{\rightarrow} a_2$, $x_1 \in V(C_3)$. Hence, $V(C_2) = \{a_2\}$. Thus a_2 is adjacent to every vertex in S because $\delta(G) \geq 2$. Since $[a_2, x_2] \underset{i}{\rightarrow} a_3$, $x_2 = a_1$. Therefore $V(C_3) = \{a_3\}$ and so a_3 is adjacent to every vertex in S. It follows that $i(G) < 3$, a contradiction. This proves claim 1.

Claim 2: If $|V(C_2)| = 1$, then $G \cong J_{1,n_1,n_2}$.

Suppose that $|V(C_2)| = 1$. Then a_2 is adjacent to every vertex in S because $\delta(G) \geq 2$. Since $[a_2, x_2] \underset{i}{\rightarrow} a_3$, $x_2 \in V(C_1)$ and so $V(C_3) = \{a_3\}$. Hence, a_3 is adjacent to every vertex in S.

We next show that $G[V(C_1)]$ is complete. Suppose to the contrary that there are $u, v \in V(C_1)$ such that $uv \notin E(G)$. We now consider $G + ua_2$. Since $i(G + ua_2) = 2$, there is $a \in V(G) - \{u, a_2\}$ such that $[u, a] \underset{i}{\rightarrow} a_2$ or $[a_2, a] \underset{i}{\rightarrow} u$. By Lemma 3.3.1, $\{u, a_2, a\}$ is independent. Further, in either case, $a \notin S$ because $a_2x, a_2y \in E(G)$. If $[u, a] \underset{i}{\rightarrow} a_2$, then $a = a_3$ but $\{u, a_3\}$ cannot dominate v, a contradiction. Hence, $[a_2, a] \underset{i}{\rightarrow} u$. Then $a \neq a_3$ because $\{a_2, a_3\}$ cannot dominate $V(C_1)$. So $a \in V(C_1)$. But then $\{a_2, a\}$ cannot dominate a_3, again a contradiction. Therefore $G[V(C_1)]$ is complete, as required.

Note that there is at least one vertex $z \in V(C_1)$ such that $zx, zy \notin E(G)$ otherwise $i(G) \leq 2$. Consider $G + xz$. Since $i(G + xz) = 2$, there is $w \in V(G) - \{x, z\}$ such that $[x, w] \underset{i}{\rightarrow} z$ or $[z, w] \underset{i}{\rightarrow} x$. In either case, $w = y$ and so $xy \notin E(G)$.

We next show that $N_{C_1}(x) = N_{C_1}(y)$. Let $b \in V(C_1)$ such that $bx \in E(G)$. Suppose that $by \notin E(G)$. Then $\{b, y\}$ dominates G and so $i(G) = 2$, a contradiction. Hence, $by \in E(G)$. Therefore $N_{C_1}(x) \subseteq N_{C_1}(y)$. By similar arguments, $N_{C_1}(y) \subseteq N_{C_1}(x)$. Hence, $N_{C_1}(x) = N_{C_1}(y)$ as required. Because G is connected, $N_{C_1}(x) = N_{C_1}(y) \neq \phi$. Further, $V(C_1) - N_{C_1}(x) \neq \phi$, since $i(G) = 3$.

Therefore $G \cong J_{1,n_1,n_2}$ where $n_1 = |N_{C_1}(x)|$, $n_2 = |V(C_1)| - |N_{C_1}(x)|$. This completes the proof of claim 2.

Claim 3: $|V(C_3)| \geq 3$.

Suppose to the contrary that $|V(C_3)| = 1$. Then a_3 is adjacent to every vertex in S because $\delta(G) \geq 2$. Since $[a_2, x_2] \underset{i}{\rightarrow} a_3$, $x_2 \in V(C_1)$. Since $[a_1, x_1] \underset{i}{\rightarrow} a_2$, $x_1 \in S \cup \{a_3\}$. If $x_1 = a_3$, then $V(C_2) = \{a_2\}$ and thus by Claim 2, $G \cong J_{1,n_1,n_2}$, a contradiction. Hence, $x_1 \in S$. Without loss of generality, we may assume that $x_1 = x$. Then x dominates $V(C_2) - \{a_2\}$. Since $a_2x \notin E(G)$ and $\delta(G) \geq 2$, it follows that $|V(C_2)| \geq 3$. Choose $a_2' \in V(C_2) - \{a_2\}$. We now consider $G + x_2a_2'$. Since $i(G + x_2a_2') = 2$, there is $w \in V(G) - \{x_2, a_2'\}$ such that $[x_2, w] \underset{i}{\rightarrow} a_2'$ or $[a_2', w] \underset{i}{\rightarrow} x_2$. In either case, $w \in S$. Further, $w \neq x$ because $xa_2' \in E(G)$. Hence, $w = y$. Suppose first that $[x_2, y] \underset{i}{\rightarrow} a_2'$. Then y dominates $V(C_2) - \{a_2'\}$. Choose $a_2'' \in V(C_2) - \{a_2, a_2'\}$. Consider $G + x_2a_2''$. Since $i(G + x_2a_2'') = 2$, there is $z \in V(G) - \{x_2, a_2''\}$ such that $[x_2, z] \underset{i}{\rightarrow} a_2''$ or $[a_2'', z] \underset{i}{\rightarrow} x_2$. In either case, $z \in S$ and $\{x_2, z, a_2''\}$ is independent by Lemma 3.3.1. Since $a_2''y \in E(G)$, $z \neq y$. Further, $z \neq x$ because $xa_2'' \in E(G)$. Thus $z \notin S$, a contradiction. Therefore $[a_2', y] \underset{i}{\rightarrow} x_2$. Then y dominates $V(C_1) - \{x_2\}$. Consider $G + a_1a_2'$. Since $i(G + a_1a_2') = 2$, there is $a \in V(G) - \{a_1, a_2'\}$ such that $[a_1, a] \underset{i}{\rightarrow} a_2'$ or $[a_2', a] \underset{i}{\rightarrow} a_1$. In either case, $a \in S$

and $\{a_1, a, a_2'\}$ is independent by Lemma 3.3.1. Since y dominates $V(C_1) - \{x_2\}$, $ya_1 \in E(G)$. Thus $a \neq y$. Further, $a \neq x$ because $xa_2' \in E(G)$. Hence, $a \notin S$, again a contradiction. This completes the proof of claim 3.

By Claims 1- 3, $|V(C_i)| \geq 3$ for $1 \leq i \leq 3$. From the fact that for $1 \leq i \leq 2$, $[a_i, x_i] \underset{i}{\rightarrow} a_{i+1}$, it follows that x_i must belong to S. Note that $x_1a_2 \notin E(G)$ and $x_2a_2 \notin E(G)$. Consider $G + a_1a_3$. Then there is $z \in V(G) - \{a_1, a_3\}$ such that $[a_1, z] \underset{i}{\rightarrow} a_3$ or $[a_3, z] \underset{i}{\rightarrow} a_1$. In either case, z dominates $V(C_2)$ and $\{a_1, a_3, z\}$ is independent by Lemma 3.3.1. Consequently, $z \notin S$. But z must dominate $(V(C_1) \cup V(C_2)) - \{a_1\}$ or $(V(C_2) \cup V(C_3)) - \{a_3\}$ which is not possible. Hence, G is defect 2-factor-critical as required. This completes the proof of our theorem.

□

Theorem 3.3.9. *Let G be a connected 3-edge-i-critical graph of even order. If $\delta(G) \geq 4$, then G is defect 3-factor-critical.*

Proof. By similar arguments as in the proof of Theorem 3.2.3 together with the fact if $[x, y] \underset{i}{\rightarrow} z$ then $\{x, y, z\}$ is independent, our theorem follows.

□

Theorem 3.3.10. *Let G be a connected 3-edge-i-critical graph of odd order. If $\delta(G) \geq 5$, then G is defect 4-factor-critical.*

Proof. Let $S \subseteq V(G)$ such that $|S| = n \geq 4$. Suppose to the contrary that $c_0(G - S) \geq |S| - 2$. Since $|S| \geq 4$, by Theorem 3.3.4, $G - S$ has at most $|S| - 1$ components. Thus $c_0(G - S) = |S| - 1$ or $c_0(G - S) = |S| - 2$. If $c_0(G - S) = |S| - 2$, then $|V(G)|$ is even, a contradiction. So, $c_0(G - S) = |S| - 1 = n - 1$ and $G - S$ has no even components. Let $C_1, C_2, ..., C_{n-1}$ be the odd components of $G - S$. For each $1 \leq i \leq n - 1$, choose $u_i \in V(C_i)$. Then $I = \{u_1, u_2, ..., u_{n-1}\}$ is independent. By Lemma 3.3.2, the vertices in I may be ordered as $a_1, a_2, ..., a_{n-1}$ in such a way that there exists a sequence of distinct vertices $x_1, x_2, ..., x_{n-2}$ such that $[a_i, x_i] \underset{i}{\rightarrow} a_{i+1}$ for $i = 1, 2, ..., n-2$. Without loss of generality, we may renumber the components of $G - S$ in such a way that $a_i \in V(C_i)$ for $i = 1, 2, ..., n - 2$.

Case 1: $n \geq 5$.

Clearly, x_i must belong to S. Hence, $|S - \{x_1, x_2, ..., x_{n-2}\}| = 2$. Let $\{w, z\} = S - \{x_1, x_2, ..., x_{n-2}\}$. Consider $G + a_1a_{n-1}$. Since $i(G + a_1a_{n-1}) = 2$, there is $x \in V(G) - \{a_1, a_{n-1}\}$ such that $[a_1, x] \underset{i}{\rightarrow} a_{n-1}$ or $[a_{n-1}, x] \underset{i}{\rightarrow} a_1$. In either case $x \in S$ and $\{a_1, a_{n-1}, x\}$ is independent. Since $a_1x_2, a_1x_3, ..., a_1x_{n-2}, a_{n-1}x_1 \in E(G)$, $x \notin \{x_1, x_2, ..., x_{n-2}\}$. Thus $x \in \{w, z\}$. Without loss of generality we may assume that $x = w$. Then either $[a_1, w]\underset{i}{\rightarrow} a_{n-1}$ or $[a_{n-1}, w]\underset{i}{\rightarrow} a_1$.

Subcase 1.1: $[a_1, w]\underset{i}{\rightarrow} a_{n-1}$.

Thus w dominates $(V(C_2) \cup V(C_3) \cup ... \cup V(C_{n-1}) \cup \{x_1\}) - \{a_{n-1}\}$. Now consider $G + a_1a_{n-2}$. Then there exists $y \in V(G) - \{a_1, a_{n-2}\}$ such that $[a_1, y]\underset{i}{\rightarrow} a_{n-2}$ or $[a_{n-2}, y]\underset{i}{\rightarrow} a_1$. Note that $\{a_1, a_{n-2}, y\}$ is independent. Since $a_1x_2, a_1x_3, ..., a_1x_{n-2}, a_{n-2}x_1, a_{n-2}w \in E(G)$, $y \notin \{x_1, x_2, ..., x_{n-2}, w\}$ and so $y = z$.

Subcase 1.1.1: $[a_1, z]\underset{i}{\rightarrow} a_{n-2}$.

Then z dominates $(V(C_2) \cup V(C_3) \cup ... \cup V(C_{n-1}) \cup \{x_1, w\}) - \{a_{n-2}\}$. We first suppose that $n \geq 6$. Let consider $G + a_1a_{n-3}$. Then there is $u \in S$ such that $[a_1, u]\underset{i}{\rightarrow} a_{n-3}$ or $[a_{n-3}, u]\underset{i}{\rightarrow} a_1$ and $\{a_1, a_{n-3}, u\}$ is independent. Since $a_1x_2, a_1x_3, ..., a_1x_{n-2}, a_{n-3}x_1, a_{n-3}w, a_{n-3}z \in E(G)$, $u \notin S$, a contradiction. Hence, $n = 5$. Since $\delta(G) \geq 5$, $a_1x_1, a_1w, a_1z \notin E(G)$ and $a_3x_2, a_3x_3, a_3z \notin E(G)$, it follows that there are $a_1' \in V(C_1)$ and $a_3' \in V(C_3)$ such that $a_1a_1', a_3a_3' \in E(G)$. We now consider $G + a_1'a_3'$. Then there exists $x \in V(G) - \{a_1', a_3'\}$ such that $[a_1', x]\underset{i}{\rightarrow} a_3'$ or $[a_3', x]\underset{i}{\rightarrow} a_1'$. Note that $\{a_1', x, a_3'\}$ is independent and in either case $x \in S$. Since $a_1'x_2, a_1'x_3, a_3'x_1, a_3'w, a_3'z \in E(G)$, $x \notin S$, a contradiction.

Subcase 1.1.2: $[a_{n-2}, z]\underset{i}{\rightarrow} a_1$.

Then z dominates $(V(C_1) \cup V(C_2) \cup ... \cup V(C_{n-3}) \cup V(C_{n-1}) \cup \{x_{n-3}, x_{n-2}\}) - \{a_1\}$. We first suppose that $n \geq 6$. Let consider $G + a_2a_{n-2}$. Then there exists $u \in S$ such that $[a_2, u]\underset{i}{\rightarrow} a_{n-2}$ or $[a_{n-2}, u]\underset{i}{\rightarrow} a_2$ and $\{a_2, a_{n-2}, u\}$ is independent. Since $a_2x_3, a_2x_4, ..., a_2x_{n-2}, a_2w, a_2z, a_{n-2}x_1, a_{n-2}x_2 \in E(G)$, $u \notin S$, a contradiction. Hence, $n = 5$. Since $\delta(G) \geq 5$, $a_1x_1, a_1w, a_1z \notin E(G)$ and $a_3x_2, a_3x_3, a_3z \notin E(G)$, it follows that there are $a_1' \in V(C_1)$ and $a_3' \in V(C_3)$ such that $a_1a_1', a_3a_3' \in E(G)$. We now consider $G + a_1'a_3'$. Then there exists $x \in V(G) - \{a_1', a_3'\}$ such that $[a_1', x]\underset{i}{\rightarrow} a_3'$ or $[a_3', x]\underset{i}{\rightarrow} a_1'$. Note that $\{a_1', x, a_3'\}$ is independent and in either case $x \in S$. Since $a_1'x_2, a_1'x_3, a_1'z, a_3'x_1, a_3'w \in E(G)$, $x \notin S$, a contradiction.

Subcase 1.2: $[a_{n-1}, w]\underset{i}{\rightarrow} a_1$.

By similar arguments as in the proof of subcase 1.1, we get a contradiction. Hence, Case 1 cannot occur either.

Case 2: $n = 4$.

Since $\delta(G) \geq 5$, $|V(C_i)| \geq 3$ for all $1 \leq i \leq 3$. From the fact that for $1 \leq i \leq 3$, $[a_i, x_i] \underset{i}{\rightarrow} a_{i+1}$, it follows that x_i must belong to S. Hence, $|S - \{x_1, x_2\}| = 2$. Let $\{w, z\} = S - \{x_1, x_2\}$. Consider $G + a_1a_3$. Since $i\,(G + a_1a_3) = 2$, there is $x \in V\,(G) - \{a_1, a_3\}$ such that $[a_1, x] \underset{i}{\rightarrow} a_3$ or $[a_3, x] \underset{i}{\rightarrow} a_1$. In either case $x \in S$ and $\{a_1, a_3, x\}$ is independent. Since $a_1x_2, a_3x_1 \in E\,(G)$, $x \notin \{x_1, x_2\}$. Thus $x \in \{w, z\}$. Without loss of generality we may assume that $x = w$. Then either $[a_1, w] \underset{i}{\rightarrow} a_3$ or $[a_3, w] \underset{i}{\rightarrow} a_1$ and $\{a_1, a_3, w\}$ is independent. Since $|V(C_2)| \geq 3$, let $a_2' \in V(C_2) - \{a_2\}$.

Subcase 2.1: $[a_1, w] \underset{i}{\rightarrow} a_3$.

Then w dominates $(V\,(C_2) \cup V\,(C_3) \cup \{x_1\}) - \{a_3\}$. We now consider $G + a_1a_2'$. Since $i\,(G + a_1a_2') = 2$, there is $y \in V\,(G) - \{a_1, a_2'\}$ such that $[a_1, y] \underset{i}{\rightarrow} a_2'$ or $[a_2', y] \underset{i}{\rightarrow} a_1$. In either case $y \in S$. Since $a_1x_2, a_2'x_1, a_2'w \in E\,(G)\,, y \notin \{x_1, x_2, w\}$. So $y = z$. Hence, either $[a_1, z] \underset{i}{\rightarrow} a_2'$ or $[a_2', z] \underset{i}{\rightarrow} a_1$ and $\{a_1, a_2', z\}$ is independent. Note that in either case z dominates $V(C_3)$. Let $a_3' \in V(C_3) - \{a_3\}$. We next consider $G + a_2'a_3'$. Since $i\,(G + a_2'a_3') = 2$, there exists $u \in V\,(G) - \{a_2', a_3'\}$ such that $[a_2', u] \underset{i}{\rightarrow} a_3'$ or $[a_3', u] \underset{i}{\rightarrow} a_2'$. In either case $u \in S$. Since $a_2'x_1, a_2'w, a_3'z, a_3'x_2 \in E\,(G)$, $u \notin S$, a contradiction.

Subcase 2.2: $[a_3, w] \underset{i}{\rightarrow} a_1$.

By similar arguments as in the proof of Subcase 2.1, we get a contradiction. Hence, Case 2 cannot occur.

$\square$

The bound on the minimum degree stated in the hypotheses of Theorem 3.3.10 is best possible since there is a 3-edge-i-critical graph with minimum degree 4 having odd order which is not defect 4-factor-critical. Such a graph G is shown in Figure 7. Note that $G - \{a, b, c, d\}$ has no near-perfect matching.

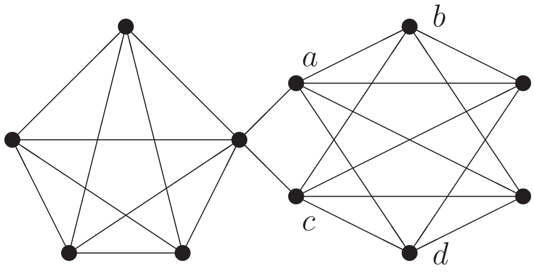

Figure 7: A 3-edge-i-critical graph of odd order with $\delta(G) = 4$ which is not defcet 4-factor-critical.

Chapter 4

k-factor-critical in independent domination critical graphs

In this chapter, we establish sufficient conditions for 3-edge-i-critical graphs to be factor-critical and bicritical.

Theorem 4.1. *Let G be a 2-connected 3-edge-i-critical graph having odd order. Then G is factor-critical.*

Proof. The proof follows by Theorem 2.3.6.

□

The next theorem is a well-known result on a planar graph. Before stating this theorem, we need a new definition.

A subdivision of a graph G is a graph obtained by inserting vertices (of degree 2) into the edges of G.

Theorem 4.2. *(Kuratowski's Theorem)(see Page 254 in [6]) A graph is planar if and only if it contains no subgraph that is isomorphic to or is a subdivision of K_5 or $K_{3,3}$.*

We are now ready to state our next result.

Theorem 4.3. *Let G be a 3-connected 3-edge-i-critical graph of even order. Then G is bicritical if either $\delta(G) \geq 4$ or G is planar.*

Proof. Suppose to the contrary that G is not bicritical. By Theorem 2.1.2, there exists $S \subseteq V(G)$ where $|S| \geq 2$ and $c_0(G-S) > |S|-2$. By Lemma 3.3.3 together with the fact that G is of even order, $c_0(G-S) = |S|$. This implies that $|S| \leq 3$ by Theorem 3.3.4. Then, by the hypothesis that G is 3-connected, $|S| = 3$ and thus S is a minimum vertex cutset. It follows that each vertex of S is joined to a vertex in each component of $G - S$. Consequently, G contains a subdivision of $K_{3,3}$ and a vertex of degree 3 by Theorem 3.3.5. But this contradicts our hypotheses and completes the proof of our theorem.

□

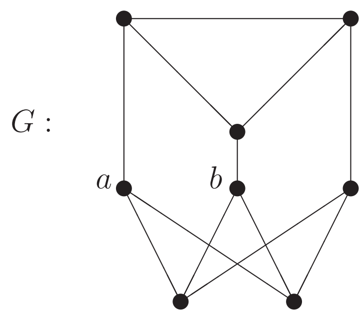

Figure 8: A 3-edge-i-critical graph of even order with $\delta(G) = 3$ which is not bicritical.

Note that the bound on the minimum degree stated in the hypotheses of Theorem 4.3. is best possible since there is a 3-edge-i-critical graph with minimum degree 3 having even order but is not bicritical. The graph G shown in Figure 8 is one of such graphs. Observe that $G - \{a, b\}$ has no perfect matching.

Recall that a graph is claw-free if it contains no induced subgraph isomorphic to $K_{1,3}$. Our next result shows that we can decrease the demand on connectivity in Theorem 4.3 if 3-edge-i-critical graphs of even order are claw-free.

Theorem 4.4. *Let G be a 2-connected 3-edge-i-critical claw-free graph of even order. If $\delta(G) \geq 3$, then G is bicritical.*

Proof. Suppose that G is not bicritical. Then, by applying similar arguments as in the proof of Theorem 4.3, there exists $S \subseteq V(G)$ where $c_0(G - S) = |S|$ and $|S| \leq 3$. Since G is 2-connected, $2 \leq |S| \leq 3$. Let $C_1, C_2, ..., C_{|S|}$ be the odd components of $G - S$.

Suppose that $|S| = 3$. Then, by Theorem 3.3.5, at least one component of

$G-S$ is a singleton and $G-S$ has no even components. Without loss of generality, we may assume that $|V(C_1)| = 1$ and that $V(C_1) = \{x_0\}$. Since $\delta(G) \geq 3$, x_0 is adjacent to every vertex of S. Since G is 2-connected, there are at least two vertices of S which are adjacent to vertices of C_2. Similarly, there are at least two vertices of S which are adjacent to vertices of C_3. Because $|S| = 3$, there must be a vertex of S, u say, such that u is adjacent to some vertex of C_2 and to some vertex of C_3. Thus u is a claw center in G, a contradiction. Hence, $|S| = 2$.

Let $S = \{u, v\}$. If $G - S$ contains an even component, H say, then $c(G - S) = 3$. Since G is 2-connected, each vertex of $\{u, v\}$ is adjacent to some vertex of every component C_1, C_2 and H. Thus u is a claw center in G, again a contradiction. Therefore, $G - S$ has no even components. So $G - S$ contains exactly two odd components. Note that $|V(C_i)| \geq 3$ for $1 \leq i \leq 2$ since $\delta(G) \geq 3$. For simplicity, we denote $N_G(x) \cap V(C_i)$ by $N_{C_i}(x)$.

Claim 1: For each $x \in S$, $G[N_{C_i}(x)]$ is complete for $1 \leq i \leq 2$.

Suppose to the contrary that there are $a, b \in N_{C_i}(x)$ such that $ab \notin E(G)$ for some $1 \leq i \leq 2$. Since G is 2-connected, there is $c \in C_j$ for $1 \leq j \neq i \leq 2$ such that $xc \in E(G)$. Hence, $G[\{x, a, b, c\}]$ is a claw centered at x, a contradiction. This proves our claim.

Claim 2: Suppose there is a vertex $x \in V(C_i)$ where $1 \leq i \leq 2$ and x dominates $V(C_i) \cup \{u, v\}$. Then for each $y \in V(C_j)$ where $1 \leq i \neq j \leq 2$ there is a vertex $z \in V(C_j) - N_G(y)$ such that $[x, z] \underset{i}{\rightarrow} y$. Further, z dominates $V(C_j) - \{y\}$.

Consider $G + xy$. Since $i(G + xy) = 2$, there exists a vertex $z_0 \in V(G) - \{x, y\}$ such that $[y, z_0] \underset{i}{\rightarrow} x$ or $[x, z_0] \underset{i}{\rightarrow} y$. In either case, $\{x, y, z_0\}$ is independent. We first suppose that $[y, z_0] \underset{i}{\rightarrow} x$. Then $z_0 \notin V(C_i) \cup \{u, v\}$. Thus $z_0 \in V(C_j)$. But then no vertex of $\{y, z_0\}$ dominates $V(C_i) - \{x\}$, a contradiction since $|V(C_i)| \geq 3$. Hence, $[x, z_0] \underset{i}{\rightarrow} y$. Consequently, z_0 dominates $V(C_j) - \{y\}$ and $z_0 \in V(C_j) - N_G(y)$. This proves our claim.

Claim 3: There is no vertex $x \in V(C_i)$, $1 \leq i \leq 2$ such that x dominates $V(C_i) \cup \{u, v\}$.

Suppose to the contrary that there is a vertex $x \in V(C_i)$ such that x dominates $V(C_i) \cup \{u, v\}$ for some $1 \leq i \leq 2$. Let $y_0 \in V(C_j)$ where $1 \leq i \neq j \leq 2$. Consider $G + xy_0$. By Claim 2, there is a vertex $z_0 \in V(C_j) - N_G(y_0)$ such that

z_0 dominates $V(C_j) - \{y_0\}$. We next consider $G + xz_0$. Then, by Claim 2, there is a vertex $z_0' \in V(C_j) - N_G(z_0)$ such that z_0' dominates $V(C_j) - \{z_0\}$. Hence, $z_0' = y_0$ since $V(C_j) - \{y_0\} \subseteq N_G(z_0)$. Since $|V(C_j)| \geq 3$ is odd, there is a vertex $y_1 \in V(C_j) - \{y_0, z_0\}$. By Claim 2, there is a vertex $z_1 \in V(C_j) - N_G(y_1)$ such that z_1 dominates $V(C_j) - \{y_1\}$. Clearly, $z_1 \notin \{y_0, z_0\}$. Again by Claim 2, there is a vertex $z_1' \in V(C_j) - N_G(z_1)$ such that z_1' dominates $V(C_j) - \{z_1\}$. By similar arguments as above, $z_1' = y_1$. Continuing in this manner, we can get a sequence of distinct vertices $y_0, z_0, y_1, z_1, ...$ such that y_r dominates $V(C_j) - \{z_r\}$ and z_r dominates $V(C_j) - \{y_r\}$ where $r \geq 0$. But this contradicts the fact that $|V(C_j)|$ is odd. This proves our claim.

Let $A = V(G) - (\{u, v\} \cup N_G(u) \cup N_G(v))$. Our next claim follows immediately from Claims 1 and 3.

Claim 4: If $A \cap V(C_i) = \phi$ where $1 \leq i \leq 2$, then $N_{C_i}(u) \cap N_{C_i}(v) = \phi$.

Claim 5: If $A \cap V(C_i) = \phi$ where $1 \leq i \leq 2$, then no vertex of $\{u, v\}$ dominates $V(C_i)$.

This claim follows by Claims 1 and 4 and the fact that $N_{C_i}(u) \neq \phi$ and $N_{C_i}(v) \neq \phi$.

Note that, by Theorem 2.3.4, either $A \cap V(C_1) = \phi$ or $A \cap V(C_2) = \phi$. We may assume without loss of generality that $A \cap V(C_1) = \phi$. Then the following claim follows by Claims 4 and 5.

Claim 6: $N_{C_1}(u) \cap N_{C_1}(v) = \phi$ and no vertex of $\{u, v\}$ dominates $V(C_1)$.

We now distinguish two cases according to $A \cap V(C_2)$.

Case 1: $A \cap V(C_2) = \phi$.

Then, by Claims 4 and 5, $N_{C_2}(u) \cap N_{C_2}(v) = \phi$ and no vertex of $\{u, v\}$ dominates $V(C_2)$. Further, $uv \in E(G)$ otherwise $i(G) = 2$. Since $|V(C_1)| \geq 3$, we may assume that $|N_{C_1}(v)| \geq 2$. Choose $a \in N_{C_1}(v)$. Then $au \notin E(G)$. We now consider $G + au$. Since $i(G + au) = 2$, there is $b \in V(G)$ such that $[u, b] \xrightarrow[i]{} a$ or $[a, b] \xrightarrow[i]{} u$. Note that $\{a, b, u\}$ is independent. If $[u, b] \xrightarrow[i]{} a$, then $b \in V(C_2)$ because $N_{C_2}(v) \neq \phi$ and so u dominates $V(C_1) - \{a\}$, contradicting the fact that $|N_{C_1}(v)| \geq 2$. Thus $[a,b]\underset{i}{\rightarrow}u$ which implies that $b \in N_{C_2}(v)$ and $bu \notin E(G)$. But then $G[\{v,u,a,b\}]$ is a claw centered at v, a contradiction. Hence, Case 1 cannot occur.

Case 2: $A \cap V(C_2) \neq \phi$.

Let $w \in A \cap V(C_2)$. Consider $G + uw$. Since $i(G + uw) = 2$, there is $z \in V(G) - \{u,w\}$ such that $[w,z]\underset{i}{\rightarrow}u$ or $[u,z]\underset{i}{\rightarrow}w$. In either case, $\{u,w,z\}$ is independent.

Subcase 2.1: $[w,z]\underset{i}{\rightarrow}u$.

Then z dominates $V(C_1)$ since $w \in V(C_2)$. Thus $z \in V(C_1) \cup \{v\}$. By Claim 6, $z \neq v$ and thus $z \in V(C_1)$. Since $wv \notin E(G)$, $zv \in E(G)$. Choose $a \in V(C_2)$ such that $au \in E(G)$. We now consider $G + az$. Since $i(G + az) = 2$, there is $x \in V(G) - \{a,z\}$ such that $[a,x]\underset{i}{\rightarrow}z$ or $[z,x]\underset{i}{\rightarrow}a$. In either case, $\{a,z,x\}$ is independent. We first assume that $[a,x]\underset{i}{\rightarrow}z$. Then x dominates $V(C_1)$ and thus $x \in V(C_1) \cup \{u\}$. Since $au \in E(G)$, $x \neq u$. Hence, $x \in V(C_1)$. Since z dominates $V(C_1)$, $zx \in E(G)$, contradicting the fact that $\{a,z,x\}$ is independent. Therefore, $[z,x]\underset{i}{\rightarrow}a$. Thus x dominates $V(C_2) - \{a\}$ and $ax \notin E(G)$. Clearly, $x \neq v$ since $zv \in E(G)$. Hence, $x \in V(C_2)$. Since $zu \notin E(G)$, $xu \in E(G)$. Therefore, $a, x \in N_{C_2}(u)$ and $ax \notin E(G)$. This contradicts Claim 1. Hence, Subcase 2.1 cannot occur.

Subcase 2.2: $[u,z]\underset{i}{\rightarrow}w$.

It follows by Claim 6 that $z \notin V(C_2)$. Then $z \in V(C_1) \cup \{v\}$.

Subcase 2.2.1: $z \in V(C_1)$.

Then, $zu \notin E(G)$ and u dominates $V(C_2) - \{w\}$. Consequently, $A = \{w\}$. Choose $a \in V(C_2)$ such that $av \in E(G)$. Thus $a \in N_G(u) \cap N_G(v)$. We now consider $G + az$. Since $i(G + az) = 2$, there is $b \in V(G) - \{a,z\}$ such that $[z,b]\underset{i}{\rightarrow}a$ or $[a,b]\underset{i}{\rightarrow}z$. In either case, $\{a,z,b\}$ is independent. We first suppose that $[z,b]\underset{i}{\rightarrow}a$. Then b dominates $V(C_2) - \{a\}$ and thus $b \in V(C_2)$. Since $zu \notin E(G)$, $bu \in E(G)$. So $a, b \in N_{C_2}(u)$ and $ab \notin E(G)$, contradicting Claim 1. Hence, $[a,b]\underset{i}{\rightarrow}z$. Then b dominates $V(C_1) - \{z\}$. Thus $b \in V(C_1)$ since $a \in N_{C_2}(u) \cap N_{C_2}(v)$. Hence, a dominates $V(C_2) \cup \{u,v\}$. But this contradicts Claim 3. Hence, Subcase 2.2.1

cannot occur.

Subcase 2.2.2: $z \in \{v\}$.

Then $[u, v] \underset{i}{\rightarrow} w$. Thus $uv \notin E(G)$. Furthermore, $A = \{w\}$. By Claim 6, $N_{C_1}(u) \cap N_{C_1}(v) = \phi$. We now consider $G + uv$. Since $i(G + uv) = 2$, there is $y \in V(G) - \{u, v\}$ such that $[u, y] \underset{i}{\rightarrow} v$ or $[v, y] \underset{i}{\rightarrow} u$. Without loss of generality, we may assume that $[u, y] \underset{i}{\rightarrow} v$. Then $\{u, v, y\}$ is independent. Thus $y = w$ and so $[u, w] \underset{i}{\rightarrow} v$. Hence, u dominates $V(C_1)$. But this contradicts Claim 6. Hence, Subcase 2.2.2 cannot occur and thus Subcase 2.2 cannot occur. This completes the proof of our theorem.

□

Let G_0 be a complete bipartite graph $K_{3,3}$ with bipartitioning sets $\{a, b, c\}$ and $\{x, y, z\}$. Then let G be a graph obtained from G_0 by replacing the vertex z with a complete graph K_n where $n \geq 2$ is odd and joining each vertex of $\{a, b, c\}$ to every vertex of K_n. Figure 9 illustrates the graph G.

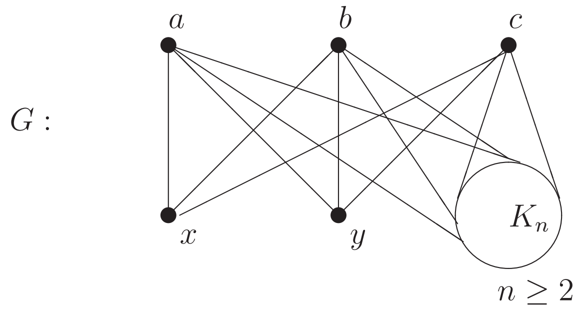

Figure 9: A 3-edge-i-critical $K_{1,4}$-free graph of even order which is not bicritical.

It is not difficult to see that G is 3-edge-i-critical. Note that G is not bicritical since $G - \{a, b\}$ has no perfect matching. Hence, the hypothesis that G is claw-free in Theorem 4.4 cannot be relaxed.

References

[1] N. Ananchuen, M.D. Plummer, Some results related to the toughness of 3-domination-critical graphs, Discrete Math. 272 (2003) 5-15.

[2] N. Ananchuen, M.D. Plummer, Matching properties in domination critical graphs, Discrete Math. 277 (2004) 1-13.

[3] N. Ananchuen, M.D. Plummer, 3-factor-criticality in domination critical graphs, Discrete Math. 307 (2007) 3006-3015.

[4] S. Ao, Independent domination critical graphs, M.sc Dissertation, University of Victoria, 1994.

[5] J.A. Bondy and U.S. Murty, Graph Theory with Applications, Elsevier Science Publishing Co., Inc, New York 1976.

[6] G. Chatrand, O.R. Oellermann, Applied and Algorithmic Graph Theory, Mcgraw-Hill, Inc., New York 1993.

[7] Y.J. Chen, F. Tian, A new proof of Wojcicka's conjecture, Discrete Appl. Math. 127 (2003) 545-554.

[8] O. Favaron, On k-factor-critical graphs, Discuss Math., Graph Theory 16 (1996) 41-51.

[9] O. Favaron, F. Tian, L. Zhang, Independence and Hamiltonicity in 3-domination-critical graphs, J. Graph Theory 25 (1997) 173-184.

[10] E. Flandrin, F. Tian, B. Wei, L. Zhang, Some properties of 3-domination critical graphs, Discrete Math. 205 (1999) 65-76.

[11] D. Hanson, Hamiltonian closures in domination critical graphs, J. Combin. Math. Combin. Comput. 13 (1993) 121-128.

[12] D.P. Sumner, Critical concepts in domination, Discrete Math. 86 (1990) 33-46

[13] D.P. Sumner, P. Blitch, Domination critical graphs, J. Combin. Theory Ser. B 34 (1983) 65-76.

[14] F. Tian, B. Wei, L. Zhang, Hamiltonicity in 3-domination-critical graphs with $\alpha = \delta + 2$, Discrete Appl. Math. 92 (1999) 57-70.

[15] X. Wen, D. Lou, Characterizing defect n-extendable graphs with different connectivities, Discrete Math. 307 (2007) 1898-1908.

[16] X. Wen, D. Lou, Characterizing defect n-extendable graphs and $(2n + 1)$-critical graphs, Discrete Math. 308 (2008) 5064-5069.

[17] E. Wojcicka, Hamiltonian properties of dominations-critical graphs, J. Graph Theory 14 (1990) 205-215.

[18] Y.F. Xue, Z.Q. Chen, Hamilton cycles in domination-critical graphs, J. Nanjing University (Natural Science edition, Special Issue on Graph Theory) 27 (1991) 58-62.

Biography

Name	Miss Watcharintorn Ruksasakchai
Address	144/5 Moo 3 Tambol Pakphreak, Amphur Muang, Kanchanaburi, 71000
Institution Attended	
2007	Bachelor of Science in Mathematics, Silpakorn University
2010	Master of Science in Mathematics, Silpakorn University