MAXIMAL CLONES OF A MAJORITY ORDER

By

Udom Chotwattakawanit

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree

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โคลนใหญ่สุดเฉพาะกลุ่มของเซตอันดับเสียงข้างมาก

โดย
อุดม โชติวรรธกวณิช

วิทยานิพนธ์นี้เป็นส่วนหนึ่งของการศึกษาตามหลักสูตรปริญญาวิทยาศาสตรมหาบัณฑิต
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The Graduate School, Silpakorn University has approved and accredited the Thesis title of "Maximal clones of a majority order" submitted by Mr. Udom Chotwattakawanit as a partial fulfillment of the requirements for the degree of Master of Science in Mathematics

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Let $\boldsymbol{P} = (P; \leq)$ be an ordered set. For a positive number n, an operation $f: P^n \rightarrow P$ is called an order-preserving if $x_i \leq y_i$ in $\boldsymbol{P}$ for all $1 \leq i \leq n$ implies $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$. A ternary operation $m: P^3 \rightarrow P$ is a majority operation if for all $x, y \in P$, $m(y, x, x) = m(x, y, x) = m(x, x, y) = x$. An ordered set is called a majority ordered set if there is a majority which is an order-preserving. An ordered set $\boldsymbol{F} = (\{a_0, \dots, a_n\}; \leq)$ is said to be a fence if $a_{2i} < a_{2i+1} > a_{2i+2}$ or $a_{2i} > a_{2i+1} < a_{2i+2}$for all $0 \leq 2i \leq n-2$ and no other comparabilities. An ordered set $\boldsymbol{P}$ is a tree-like if there is a tree $\boldsymbol{T_P}$ such that $\boldsymbol{P}$ is obtained by replacing each interval $[q, q']$ which q' covers q in $\boldsymbol{T_P}$ by a lattice $L_{qq'}$ with a least element q and a greatest element q' so that any two such lattices intersect in at most one bound.

In this thesis, we characterize all maximal clones of a fence $\boldsymbol{F}$ with $|F| > 2$. We show all possibilities of maximal clones of an unbounded connected majority ordered set and then we characterize all maximal clones of an unbounded tree-like.

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Student's signature ....

Thesis Advisor's signature ....

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ให้ $\boldsymbol{P} = (P; \leq)$ เป็นเซตอันดับ สำหรับแต่ละจำนวนนับ n เราเรียกฟังก์ชัน $f: P^n \rightarrow P$ ว่าฟังก์ชันยืนยงอันดับถ้า $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ เมื่อใดก็ตามที่ $x_i \leq y_i$ ใน $\boldsymbol{P}$ สำหรับทุกๆ $1 \leq i \leq n$ และเรียกฟังก์ชันไตรวิภาค $m: P^3 \rightarrow P$ ว่าฟังก์ชันเสียงข้างมากถ้า $m(y, x, x) = m(x, y, x) = m(x, x, y) = x$ สำหรับทุกๆ $x, y \in P$ เซตอันดับเสียงข้างมากคือเซตอันดับซึ่งมีฟังก์ชันเสียงข้างมากเป็นฟังก์ชันยืนยงอันดับ $\boldsymbol{F} = (\{a_0, \dots, a_n\}; \leq)$ เป็นเฟนซ์ถ้า $a_{2i} < a_{2i+1} > a_{2i+2}$ หรือ $a_{2i} > a_{2i+1} < a_{2i+2}$ สำหรับทุกๆ $0 \leq 2i \leq n - 2$ และไม่มีความสัมพันธ์อื่นนอกจากนี้ เซตอันดับ $\boldsymbol{P}$ เป็นทรีไลค์ถ้ามีเซตอันดับทรี $\boldsymbol{T_P}$ ซึ่ง $\boldsymbol{P}$ เกิดจากการแทนที่แต่ละ $[q, q']$ ซึ่ง q' ปกคลุม q ใน $\boldsymbol{T_P}$ โดยแลตทิซ $L_{qq'}$ โดยที่มี q เป็นสมาชิกตัวน้อยสุดและ q' เป็นสมาชิกตัวมากสุด และสองแลตทิซดังกล่าวมีสมาชิกเหมือนกันอย่างมากตัวเดียวและต้องเป็นสมาชิกที่เป็นตัวมากสุดหรือตัวน้อยสุดในแลตทิซเท่านั้น

ในวิทยานิพนธ์นี้ เราเริ่มจากการจำแนกโคลนใหญ่สุดเฉพาะกลุ่มของเฟนซ์ $\boldsymbol{F}$ ซึ่ง $|F| > 2$ และแสดงโคลนใหญ่สุดเฉพาะกลุ่มที่เป็นไปได้ทั้งหมดของเซตอันดับเสียงข้างมากที่ต่อเนื่องและไม่มีขอบเขต และจบด้วยการจำแนกโคลนใหญ่สุดเฉพาะกลุ่มของเซตอันดับทรีไลค์

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ลายมือชื่อนักศึกษา..

ลายมือชื่ออาจารย์ที่ปรึกษาวิทยานิพนธ์ ....

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Chapter 1

Introduction

Let $O(A)$ denote the set of all finitary nonnullary operations on a finite set A. A subset C of $O(A)$ is called a *clone* if C contains all projection maps and is closed under superposition; that is, if $f_1, \dots, f_n$ are k-ary maps in C and g is an n-ary map in C for some positive integers k and n then $g(f_1, \dots, f_n) \in C$. It is a well-known fact that the set of all clones over a finite set is an ordered set with respect to inclusion; in fact, it is a complete lattice with the co-atom being *maximal clones*. Up to now, all possibilities to describe all clones on a finite set is only the set of cardinality 2 which is known as the lattice of all Boolean clones; but the lattice of all clones over a finite set whose cardinality more than 2 is an uncountably infinite; much of the lattice is unknown. Describing of some part of these lattices is still well known open problems. There are only finitely many maximal clones over a finite set and every proper subclone of the full clone is contained in a maximal one. I.G. Rosenberg [7] has classified all maximal clones over a finite set by finding six classes of relations such that maximal clones are just the clones $Pol(\rho)$ of operations preserving a relation ρ from one of these classes. Rosenberg's six classes of relations on a finite set A are the followings.

Class(1): The set of all bounded orders. These are reflexive, transitive and anti-symmetric binary relations $\rho \subseteq A \times A$ with $(0, x) \in \rho$ and $(x, 1) \in \rho$ for all $x \in A$ and for some $0, 1 \in A$.

Class(2): The set of all prime permutations. These are permutations on A which all of whose cycles have the same prime length.

Class(3): The class of all prime affine relations. A 4-ary relation $\rho \subseteq A^4$ is *affine* if we can define an abelian group operation, $+$, on A so that $(a, b, c, d) \in \rho$ if and only if $a + b = c + d$. An affine relation ρ is *prime* if $\langle A; + \rangle$ is an abelian group of prime power order. This class is empty unless $|A|$ is a prime power.

Class(4): The class of all non-trivial equivalence relations. These are reflexive, symmetric and transitive binary relations $\rho \subseteq A \times A$ which are neither the diagonal relation $\triangle_A := \{(a, a) \,|\, a \in A\}$ nor the universal relation $\nabla_A := A \times A$.

Class(5): The class of all relations which are k-regularly generated for some $3 \leq k \leq |A|$. For $3 \leq k \leq |A|$, a set $T = \{\Theta_1, \Theta_2, \dots, \Theta_m\}$ $(m \geq 1)$ of equivalence relations on A is *k-regular* if each Θ_i, $(1 \leq i \leq m)$ has exactly k equivalence classes and the intersection $\cap_{i=1}^{m} \varepsilon_i$ of arbitrary equivalence classes ε_i of Θ_i is nonempty. A k-ary relation $\rho = \{(a_1, \dots, a_k) | a_i \in A \text{ for all } i = 1, \dots, k\}$ is *k-regularly generated* by T if for each $i \in \{1, \dots, m\}$, at least two of the elements $a_1, \dots, a_k$ are

equivalent modulo Θ_i.
Class(6): The class of all central relations. A k-ary relation $\rho \subseteq A^k$ for some $k \geq 1$ is *totally reflexive* if $\{(a_1, \ldots, a_k) | a_i = a_j \text{ for some } i \neq j\} \subseteq \rho$; and is *totally symmetric* if for any permutation α on $\{1, \ldots, k\}$ we have $(a_1, \ldots, a_k) \in \rho$ if and only if $(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(k)}) \in \rho$. The center C_ρ of ρ is the set of all $a \in A$ such that $(a, a_2, \ldots, a_k) \in \rho$ for all $a_2, \ldots, a_k \in A$. We say that ρ is *central* if it is totally reflexive, totally symmetric and $\emptyset \neq C_\rho \subsetneq A$.

Let $\mathbf{P} = (P; \leq)$ be an ordered set. For a positive number n, an operation $f : P^n \longrightarrow P$ is called an *order-preserving* if $x_i \leq y_i$ in $\mathbf{P}$ for all $1 \leq i \leq n$ implies $f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_n)$. A ternary operation $m : P^3 \longrightarrow P$ is called a *majority operation* if for all $x, y \in P$,

$$m(y, x, x) = m(x, y, x) = m(x, x, y) = x.$$

An ordered set is called a *majority ordered set* if there is a majority order-preserving. An ordered set $\mathbf{F} = (\{a_0, a_1, \ldots, a_n\}; \leq)$ is said to be a *fence* (from a_0 to a_n) if $a_{2i} < a_{2i+1} > a_{2i+2}$ or $a_{2i} > a_{2i+1} < a_{2i+2}$ for all $0 \leq 2i \leq n-2$ which has no other comparabilities. An ordered set $\mathbf{P}$ is said to be *connected* if there is a fence from a to b as a subordered set for all $a, b \in P$. For each finite connected ordered set $\mathbf{P}$, the number $r(\mathbf{P}) := \max\left\{|F_a^b| - 1 \,|\, a, b \in P\right\}$ is said to be the *reach* of $\mathbf{P}$ where $\mathbf{F}_a^b$ denote a minimum size fence from a to b. The clone $Pol(\leq)$ on P is the set of all finitary order-preserving with respect to $\leq$; and we call $Pol(\leq)$ the *monotone clone* of $\mathbf{P}$.

The monotone clone of a finite ordered set is maximal if and only if the order is bounded. Davey et al. proved in [2] that if a finite ordered set $\mathbf{P}$ is disconnected then the nontrivial equivalence relation Θ whose blocks are connected components of $\mathbf{P}$ will give a maximal clone $Pol(\Theta)$ containing the monotone clone of the ordered set $\mathbf{P}$. C. Ratanaprasert [5] has shown that the monotone clone of a finite unbounded connected ordered set is a subclone of a maximal clone preserving only either k-regularly generated relations or central relations with arity more than 1 and also proved that if the monotone clone of the ordered set contains in a maximal clone preserving some k-regularly generated relations then the monotone clone contains no majority functions. J. Demetrovics and L. Rónyai [3] constructed an order preserving operation which is majority on a fence. Hence, the monotone clone of a fence is a subclone of a maximal clone preserving only central relations with arity more than 1. C. Ratanaprasert [6] constructed some maximal clones preserving binary central relations containing the monotone clone of a fence in the term of the distance function $d : P \times P \rightarrow \mathbb{N} \cup \{0\}$ which is defined by $d(a, b) = r(\mathbf{F}_a^b)$ for all $a, b \in P$. It is interesting whether there is a maximal clone preserving a central relation with arity more than 2 whose clone contains the monotone clone of a fence.

In chapter 2, we collect some important basic concepts which will be used in the sequel.

In chapter 3, we show that all maximal clones of an unbounded fence are the clones preserving central relations with arity only 2 and we also characterize these binary relations in terms of the distance function. Since a fence is a majority ordered set, one can ask whether the monotone clone of a finite unbounded connected majority ordered set is a subclone of a maximal clone preserving only a binary central relation.

A finite ordered set $\mathbf{P}$ is called a *tree* if the diagram of $\mathbf{P}$ is a tree graph. An ordered set $\mathbf{P}$ is a *tree-like* if there is a tree $\mathbf{T_P}$ such that $\mathbf{P}$ is obtained by replacing each interval $[q, q']$ which q' covers q in $\mathbf{T_P}$ by a lattice $L_{qq'}$ with a least element q and a greatest element q' so that any two such lattices intersect in at most one bound. Füredi Z. and Rosenberg I.G. [4] showed that the class of all tree-likes is a large class of connected majority ordered sets that need not be lattices.

In chapter 4, we show that all maximal clones of a finite unbounded connected majority ordered set are the clones preserving central relations with arity only 2 and we also show all possibilities of these binary central relations in terms of the distance function. Since tree-like is a majority ordered set, all maximal clones containing the monotone clone of a tree-like preserve only binary central relation. We also characterize these binary relations in terms of the distance function.

Chapter 2

Basic Concepts

In this chapter, we study related topics which will be referred in sequel. All theorems are stated without proofs.

2.1 Ordered sets and Lattices

In this section, we introduce and present some basic properties of an ordered set and a lattice.

Definition 2.1.1. Let P be a nonempty set. An *order* (or *partial order*) on P is a binary relation $\leq$ on P satisfying the following three conditions for all $x, y, z \in P$,

1. $x \leq x$, (reflexivity)
2. $x \leq y$ and $y \leq x$ imply $x = y$, (anti-symmetry)
3. $x \leq y$ and $y \leq z$ imply $x \leq z$. (transitivity)

A set P equipped with an order relation $\leq$ is said to be an *ordered set* (or *partially ordered set*) and denoted by $(P; \leq)$. Some authors use the shorthand *poset*. Usually we shall be a little slovenly and say simply '**P** is an ordered set '. An ordered set $(Q; \leq')$ is called a *subordered set* of $(P; \leq)$ if $Q \subseteq P$ and $\leq' = \leq|_{Q\times Q}$.

An order relation $\leq$ on P gives rise to a relation $<$ of *strictly inequality*: $x < y$ in P if and only if $x \leq y$ and $x \neq y$. For each $x, y \in P$, we say that x is comparable with y, and write $x \parallel y$, if $x \leq y$ or $y \leq x$.

Definition 2.1.2. Let $F = \{a_0, a_1, \ldots, a_n\}$. An ordered set $\mathbf{F} = (F; \leq)$ is said to be a *fence* (from a_0 to a_n) if $a_{2i} < a_{2i+1} > a_{2i+2}$ or $a_{2i} > a_{2i+1} < a_{2i+2}$ for all $0 \leq 2i \leq n-2$ which has no other comparabilities and the number n is called the *reach* of $\mathbf{F}$ and denote by $r(\mathbf{F})$. $\mathbf{F}$ is called an *up fence* if $a_{2i} < a_{2i+1} > a_{2i+2}$ for all $0 \leq 2i \leq n-2$ (from a_0 to a_n) and written by $\{a_0 < a_1 > a_2 < \ldots a_n\}$. $\mathbf{F}$ is called a *down fence* if $a_{2i} > a_{2i+1} < a_{2i+2}$ for all $0 \leq 2i \leq n-2$ (from a_0 to a_n) and written by $\{a_0 > a_1 < a_2 > \ldots a_n\}$.

Definition 2.1.3. Let $C_n = \{c_1, \ldots, c_{2n}\}$. An ordered set $\mathbf{C}_n = (C_n; \leq)$ is said to be a *crown* if $c_1 < c_2 > c_3 < \ldots > c_{2n-1} < c_{2n} > c_1$ which has no other comparabilities.

Definition 2.1.4. An ordered set **P** is called *connected* if there is a fence from a to b as a subordered set for all $a, b \in P$.

Definition 2.1.5. Let **P** be an ordered set and let $x, y \in P$. We say that x is *covered by* y (or y *cover* x), and write $x \prec y$ or $y \succ x$, if $x < y$ and $z = x$ for all $z \in P$ with $x \leq z < y$.

Observe that, if **P** is finite, $x < y$ if and only if there exists a finite sequence of covering relations $x = x_0 \prec x_1 \prec \ldots \prec x_n = y$. Thus, in the finite case, the order relation determines, and is determined by, the covering relation.

Definition 2.1.6. Let **P** be a finite ordered set. We can represent **P** by a configuration of circles (representing the elements of P) and interconnecting lines (indicating the covering relation). The construction goes as follows.

1. To each point $x \in P$, associate a point $P(x)$ of the euclidean plane $\mathbb{R}^2$, depicted by a small circle with center at $P(x)$.
2. For each covering pair $x \prec y$ in P, take a line segment $\ell(x, y)$ joining the circle at $P(x)$ to the circle at $P(y)$.
3. Carry out (*i*) and (*ii*) in such a way that
   (a) if $x \prec y$ then $P(x)$ is 'lower'than $P(y)$ (that is, in standard cartesian coordinates, has a strictly smaller second coordinate);
   (b) the circle at $P(z)$ does not intersect the line segment $\ell(x, y)$ if $z \neq x$ and $z \neq y$.

A configuration satisfying (1)-(3) is called a *diagram* (or *Hasse diagram*) of **P**.

Example 2.1.7. *Let* $\mathbf{F} = \{a_0 < a_1 > a_2 < a_3\}$ *and* $\mathbf{G} = \{b_0 > b_1 < b_2 < b_3\}$ *be up fence and down fence, respectively. Then the following picture shows diagrams of* **F** *and* **G**, *respectively.*

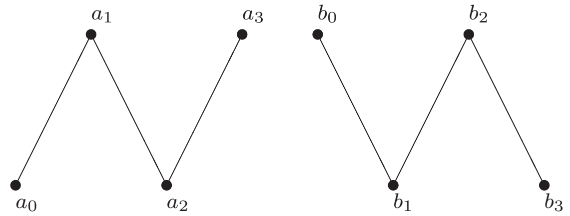

Figure 1. Examples of fences

Definition 2.1.8. Let **P** be an ordered set, n be a positive integer and $x_1, \ldots, x_{n+1}$ be distinct elements in P. A sequence $(x_1, \ldots, x_{n+1})$ is called a *path* (from x_1 to x_{n+1}) if $x_i \prec x_{i+1}$ or $x_i \succ x_{i+1}$ for all $i \in \{1, \ldots, n\}$.

Definition 2.1.9. Let **P** be a finite connected ordered set. We say that the diagram of **P** is a *tree graph* if there exists unique a path from a to b for all $a \neq b$ in P. If the diagram of **P** is a tree graph, P is called a *tree*.

Definition 2.1.10. Let $\mathbf{P}$ and $\mathbf{Q}$ be ordered sets and $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ be a function.

1. φ is called an *order-preserving* (or *monotone*) if $x \leq y$ in $\mathbf{P}$ implies $\varphi(x) \leq \varphi(y)$ in $\mathbf{Q}$.

2. φ is called an *order-embedding* if $x \leq y$ in $\mathbf{P}$ if and only if $\varphi(x) \leq \varphi(y)$ in $\mathbf{Q}$.

3. φ is called an *order-isomorphism* if it is an order-embedding mapping $\mathbf{P}$ onto $\mathbf{Q}$.

When $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ is an order-embedding we write $\varphi : \mathbf{P} \hookrightarrow \mathbf{Q}$. When there exists an order-isomorphism from $\mathbf{P}$ to $\mathbf{Q}$, we say that $\mathbf{P}$ is *isomorphic* to $\mathbf{Q}$ and write $\mathbf{P} \cong \mathbf{Q}$

Remark 2.1.11. [1] *Let $\mathbf{P}$, $\mathbf{Q}$ and $\mathbf{R}$ be ordered sets.*

1. *Let $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ and $\psi : \mathbf{Q} \longrightarrow \mathbf{R}$ be order-preservings. Then the composite map $\psi \circ \varphi$ is an order-preserving. More general the composition of a finite number of order-preservings is an order-preserving, if it is defined.*

2. *Let $\varphi : \mathbf{P} \hookrightarrow \mathbf{Q}$ and let $\varphi(P) = \{\varphi(x) \,|\, x \in P\}$. Then $\varphi(\mathbf{P}) \cong \mathbf{P}$.*

3. *An order-embedding is automatically one-to-one. An order-isomorphism is bijective.*

4. *$\mathbf{P}$ is isomorphic to $\mathbf{Q}$ if and only if there exist order-preservings $\varphi : \mathbf{P} \longrightarrow \mathbf{Q}$ and $\psi : \mathbf{Q} \longrightarrow \mathbf{P}$ such that $\varphi \circ \psi = id_Q$ and $\psi \circ \varphi = id_P$ (where $id_S : S \longrightarrow S$ denotes the* **identity map** *on S given by $id_S(x) = x$ for all $x \in S$).*

Proposition 2.1.12. [1] *Let $\mathbf{P}$ and $\mathbf{Q}$ be finite ordered sets. $\mathbf{P}$ is isomorphic to $\mathbf{Q}$ if and only if they can be drawn with identical diagram.*

Definition 2.1.13. Let $\mathbf{P_1}, \mathbf{P_2}, \ldots, \mathbf{P_n}$ be ordered sets. The *cartesian product* $P_1 \times P_2 \times \ldots \times P_n$ can be made into an ordered set by imposing the coordinatewise order defined by

$$(x_1, x_2, \ldots, x_n) \leq (y_1, y_2, \ldots, y_n) \Longleftrightarrow x_i \leq y_i \text{ in } P_i \text{ for all } i \in \{1, 2, \ldots, n\}.$$

Given an ordered set P, the notation P_n is used as shorthand for the n-fold product P^n.

Definition 2.1.14. Let A be a set. An operation $m : A^3 \longrightarrow A$ is called *majority* if

$$m(x, x, y) = m(x, y, x) = m(y, x, x) = x$$

for all $x, y \in A$.

Definition 2.1.15. Let $\mathbf{P} = (P; \leq)$ be an ordered set. We say that $\leq$ is a *majority order* if there exists a majority operation which is an order-preserving. If $\leq$ is a majority order, $\mathbf{P}$ is called a *majority ordered set.*

It is a fundamental property of the set of all real numbers, $\mathbb{R}$, that if I is a closed and bounded interval in $\mathbb{R}$, then every subset of I has both a least upper bound(or supremum) and a greatest lower bound(or infimum) in I. These concepts pertain to any ordered set.

Definition 2.1.16. Let $\mathbf{P}$ be an ordered set and $Q \subseteq P$.

1. $a \in Q$ is called a *maximal* element of Q if $a \leq x \in Q$ implies $a = x$.
2. $a \in Q$ is called a *minimal* element of Q if $a \geq x \in Q$ implies $a = x$.
3. $a \in Q$ is called the *maximum* element of Q if $a \geq x$ for every $x \in Q$.
4. $a \in Q$ is called the *minimum* element of Q if $a \leq x$ for every $x \in Q$.
5. $\mathbf{P}$ is said to be *bounded* if P has maximum and minimum elements and $\mathbf{P}$ is said to *unbounded* if $\mathbf{P}$ is not bounded.

Example 2.1.17. *Let X be an any set. The powerset $\mathscr{P}(X)$, consisting of all subsets of X, is ordered by the set inclusion: for $A, B \in \mathscr{P}(X)$, we define $A \leq B$ if and only if $A \subseteq B$. Moreover, X is a maximum element of $\mathscr{P}(X)$ and $\emptyset$ is a minimum element of $\mathscr{P}(X)$.*

Definition 2.1.18. Let $\mathbf{P}$ be an ordered set and let $S \subseteq P$. An element $x \in P$ is an *upper bound* of S if $s \leq x$ for all $s \in S$. A *lower bound* is defined dually. The set of all upper bounds of S is denoted by S^u(read as 'S upper') and the set of all lower bounds of S is denoted by S^l(read as 'S lower'):

$$S^u = \{x \in P | (\forall s \in S)\ s \leq x\} \ \ and \ \ S^l = \{x \in P | (\forall s \in S)\ s \geq x\}.$$

If S^u has the least element x then x is called the *least upper bound* of S and is denoted by $supS$. Equivalently, x is the least upper bound of S if

1. x is an upper bound of S and
2. $x \leq y$ for all upper bound y of S.

Dually, if S^l has the largest element, x, then x is called the *greatest lower bound* of S or the *infimum* of S and is denoted by $infS$.
Notation: We write $\vee S$ instead of $supS$ whenever $supS$ exists and $x \vee y$ (read as 'x joint y') in place of $sup\{x, y\}$ when it exists. Similarly we write $\wedge S$ instead of $infS$ whenever $infS$ exists and $x \wedge y$ (read as 'x meet y') in place of $inf\{x, y\}$ when it exists.

Definition 2.1.19. Let $\mathbf{P}$ be a non-empty ordered set.

1. If $x \vee y$ and $x \wedge y$ exist for all $x, y \in P$, then P is called a *lattice*.
2. If $\vee S$ and $\wedge S$ exist for all $S \subseteq P$, then P is called a *complete lattice*.

Definition 2.1.20. Let $\mathbf{L}$ be a lattice with the greatest element 1 and let $c \in L$. We say that c is a *co-atom* of L if no elements $x \in L$ such that $c < x < 1$.

2.2 Maximal clones

In this section, we will give some important concepts in clone theory which will be referred in the sequel.

Definition 2.2.1. Let A be a set and n be a positive integer. An *n-ary operation* on A is a mapping f from A^n into A and the integer n is called the *arity* of f. If $n = 1, 2$ or 3 then f is called a *unary, binary* or *ternary operation* on A, respectively. Moreover, f is called a *permutation* on A if f is a bijective unary operation.

For each positive integer n, let $O^n(A)$ denote the set of all n-ary operations on a set A and $O(A) := \cup_{n=1}^{\infty} O^n(A)$ denote the set of all operations on A. Since a function $e_j^n : A^n \longrightarrow A$ which is defined by

$$e_j^n(x_1, \ldots, x_n) = x_j$$

belongs to $O^n(A)$, we have that $O^n(A) \neq \emptyset$ for all positive integer n. The function e_j^n is called the *projection map* on A and let J_A denote the set of all projection maps on A.

Definition 2.2.2. Let A be a set. A subset C of $O(A)$ is called a *clone* over A if C contains all projection maps and is closed under superposition; that is, if $f_1, f_2, , , f_n$ are k-ary maps in C and g is an n-ary map in C for some positive integers k and n then the operation $g(f_1, \ldots, f_n)$ which is defined by

$$g(f_1, \ldots, f_n)(x_1, \ldots, x_k) = g(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k))$$

belongs to C.

Note that $O(A)$ is the greatest clone and is called the *full clone* over A.

Definition 2.2.3. Let A be a set. A clone C is called a *subclone* of a clone D if $C \subseteq D$. If there is no clones E with $C \subset E \subset D$, C is called a *maximal subclone* of D. A maximal subclone of $O(A)$ is called a *maximal clone.*

Definition 2.2.4. Let A be a set and h be a positive integer. A subset of A^h is called an *h-ary relation* on A.

Definition 2.2.5. Let A be a set, f be an n-ary operation and ρ be an h-ary relation on A. We say that f *preserves* ρ or ρ *is invariant* under f if

$$(f(x_{11}, \ldots, x_{n1}), \ldots, f(x_{1h}, \ldots, x_{nh})) \in \rho$$

whenever $(x_{11}, \ldots, x_{1h}), \ldots, (x_{n1}, \ldots, x_{nh}) \in \rho$.

Note that the set $Pol(\rho)$ of all operations preserving a relation ρ is a clone and $Pol(\rho)$ is said to *preserve* ρ. If $\mathbf{P} = (P; \leq)$ is an ordered set, $Pol(\leq)$ is called the *monotone clone* of $\mathbf{P}$.

There are only finitely many maximal clones over a finite set and every proper subclone of the full clone is contained in a maximal one. I.G. Rosenberg [7] has classified all maximal clones over a finite set A by finding six classes of relations such that maximal clones are just the clones $Pol(\rho)$ such that ρ is a relation in one of the following classes.

Class(1): The set of all bounded orders. These are reflexive, transitive and anti-symmetric binary relations $\rho \subseteq A \times A$ with $(0, x) \in \rho$ and $(x, 1) \in \rho$ for all $x \in A$ and for some $0, 1 \in A$.

Class(2): The set of all prime permutations. These are permutations on A which all of whose cycles have the same prime length.

Class(3): The class of all prime affine relations. A 4-ary relation $\rho \subseteq A^4$ is *affine* if we can define an abelian group operation, $+$, on A so that $(a, b, c, d) \in \rho$ if and only if $a + b = c + d$. An affine relation ρ is *prime* if $\langle A; + \rangle$ is an abelian group of prime power order. This class is empty unless $|A|$ is a prime power.

Class(4): The class of all non-trivial equivalence relations. These are reflexive, symmetric and transitive binary relations $\rho \subseteq A \times A$ which are neither the diagonal relation $\triangle_A := \{(a, a) \mid a \in A\}$ nor the universal relation $\nabla_A := A \times A$.

Class(5): The class of all relations which are k-regularly generated for some $3 \leq k \leq |A|$. For $3 \leq k \leq |A|$, a set $T = \{\Theta_1, \Theta_2, \ldots, \Theta_m\}$ $(m \geq 1)$ of equivalence relations on A is *k-regular* if each Θ_i, $(1 \leq i \leq m)$ has exactly k equivalence classes and the intersection $\cap_{i=1}^{m} \varepsilon_i$ of arbitrary equivalence classes ε_i of Θ_i is nonempty. A k-ary relation $\rho = \{(a_1, \ldots, a_k) | a_i \in A$ for all $i = 1, \ldots, k\}$ is *k-regularly generated* by T if for each $i \in \{1, \ldots, m\}$, at least two of the elements $a_1, \ldots, a_k$ are equivalent modulo Θ_i.

Class(6): The class of all central relations. A k-ary relation $\rho \subseteq A^k$ for some $k \geq 1$ is *totally reflexive* if $\{(a_1, \ldots, a_k) | a_i = a_j$ for some $i \neq j\} \subseteq \rho$; and is totally symmetric if for any permutation α on $\{1, \ldots, k\}$ we have $(a_1, \ldots, a_k) \in \rho$ if and only if $(a_{\alpha(1)}, a_{\alpha(2)}, \ldots, a_{\alpha(k)}) \in \rho$. The center C_ρ of ρ is the set of all $a \in A$ such that $(a, a_2, \ldots, a_k) \in \rho$ for all $a_2, \ldots, a_k \in A$. We say that ρ is *central* if it is totally reflexive, totally symmetric and $\emptyset \neq C_\rho \subsetneq A$.

Chapter 3

All Maximal Clones of a Fence

Let $n \geq 2$ be an integer. An ordered set $\mathbf{F} = (\{a_0, a_1, \ldots, a_n\}\,;\leq)$ is said to be a *fence* if $a_{2i} < a_{2i+1} > a_{2i+2}$ or $a_{2i} > a_{2i+1} < a_{2i+2}$ for all $0 \leq 2i \leq n-2$ which has no other comparabilities. J. Demetrovics and L. Rónyai [3] constructed a majority order-preserving operation on $\mathbf{F}$ which implies that $\mathbf{F}$ is a majority ordered set. C. Ratanaprasert [5] showed that the monotone clone of a finite unbounded majority ordered set is a subclone of a maximal clone preserving only a central relation with arity more than 1. Hence, so is the monotone clone of a fence $\mathbf{F}$.

In this chapter, we characterize all central relations on F which is equivalent to characterize all maximal clones containing the monotone of $\mathbf{F}$. To avoid confusion, let $\leq^*$ denote the usual order of integers.

3.1 All Ordered Sets satisfy the Preservation Distance Property

C. Ratanaprasert [6] constructed some binary central relations ρ whose $Pol(\rho)$ contains the monotone clone of a fence $\mathbf{F}$ in the term of the distance function $\tilde{d} : F \times F \longrightarrow \mathbb{N} \cup \{0\}$ which is defined by

$$\tilde{d}(a_i, a_j) = |i - j|$$

for all $i, j \in \{0, \ldots, n\}$. By the triangle inequality property of the absolute value on the set of all real numbers, the distance function $\tilde{d}$ satisfies this property; that is,

$$\tilde{d}(x, z) \leq^* \tilde{d}(x, y) + \tilde{d}(y, z)$$

for all $x, y, z \in F$.

An ordered set $\mathbf{P}$ is said to be *connected* if there is a fence from a to b as a subordered set for all $a, b \in P$. For each $a, b \in P$, let $\mathbf{F}_a^b$ denote a minimum size fence from a to b. It is known that the number $r(\mathbf{P}) =: \max\left\{|F_a^b| - 1 \,|\, a, b \in P\right\}$ is the *reach* of $\mathbf{P}$. The distance function $d : P \times P \longrightarrow \mathbb{N} \cup \{0\}$ is defined by $d(a, b) = r(\mathbf{F}_a^b)$ for all $a, b \in P$. If an ordered set $\mathbf{P}$ is the fence $\mathbf{F}$ then it is clearly that $d = \tilde{d}$. As $\tilde{d}$ satisfies the triangle inequality, we will also prove that d satisfies the triangle inequality.

Theorem 3.1.1. *Let* $\mathbf{P}$ *be a connected ordered set. Then*

$$d(a,b) \leq^* d(a,c) + d(c,b)$$

for all $a,b,c \in P$.

Proof. We will prove by induction on the distance between a and c. If $a = c \in P$ then $d(a,c) = 0$ and $d(a,b) = d(a,c) + d(c,b)$ for all $b \in P$. Now, let $k \in \mathbb{N} \cup \{0\}$ and we suppose that the inequality holds for all $a,b,c \in P$ with $d(a,c) = k$. Let $a,b,c \in P$ with $d(a,c) = k+1$. and $\mathbf{F}_a^c = (\{a_0,\dots,a_{k+1}\};\leq)$. Then $d(a,a_k) = k$ and $d(a,c) = d(a,a_k)+1$. By the induction hypothesis, we have that $d(a,b) \leq^* d(a,a_k) + d(a_k,b)$. Since a_k is comparable with c, $d(a_k,b) \leq^* d(c,b)+1$. Hence, $d(a,b) \leq^* d(a,a_k) + d(a_k,b) \leq^* d(a,c) - 1 + d(c,b) + 1 = d(a,c) + d(c,b)$. □

Corollary 3.1.2. *Let* $\mathbf{P}$ *be a connected ordered set and* x *be comparable with* y *in* $\mathbf{P}$ *and* $a \in P$. *Then*

1. $|\, d(a,x) - d(a,y)| \leq^* 1$.
2. *if* $\mathbf{F}$ *is a fence from* a *to* x, *there is a subfence set* $\hat{\mathbf{F}}$ *from* a *to* y *of* $\mathbf{F} \cup \{y\}$ *such that* $r(\hat{\mathbf{F}}) \leq^* r(\mathbf{F}) + 1$.

Definition 3.1.3. A connected ordered set $\mathbf{P}$ is said to satisfy *the preservation distance property* (PDP) if

$$d(f(a_1,...,a_n), f(b_1,...,b_n)) \leq^* \max\{d(a_1,b_1),...,d(a_n,b_n)\}$$

for all $a_1,...,a_n,b_1,...,b_n \in P$ and order-preserving $f : P^n \to P$.

In [6], C. Ratanaprasert has shown that all fences satisfy PDP. We will now characterize all finite connected ordered sets satisfying PDP by a graph of a specific subordered set. Moreover, the characterization can be shown in term of notations which are defined as follows.

Let $\mathbf{P}$ be a finite connected ordered set and $a,b \in P$. We call $\mathbf{F}_a^b$ a *minimum size up fence* if $\mathbf{F}_a^b$ is an up fence and denote by ${}^0\mathbf{F}_a^b$; and dually, we call $\mathbf{F}_a^b$ a *minimum size down fence* if $\mathbf{F}_a^b$ is a down fence and denote by ${}^1\mathbf{F}_a^b$. For each $k \in \{0,\dots,r(\mathbf{P})\}$ and $i \in \{0,1\}$, we define

$$\begin{aligned} \Gamma_k &= \{(a,b) \in P \times P \,|\, d(a,b) = k\}, \\ {}^0\Gamma_k &= \left\{(a,b) \in \Gamma_k \,|\, {}^0\mathbf{F}_a^b \text{ exists}\right\}, \\ {}^1\Gamma_k &= \left\{(a,b) \in \Gamma_k \,|\, {}^1\mathbf{F}_a^b \text{ exists}\right\}, \\ \Theta_k &= \{(a,b) \in P \times P \,|\, d(a,b) \leq^* k\}. \end{aligned}$$

Note that ${}^0\Gamma_k \cup {}^1\Gamma_k = \Gamma_k$ and $\Theta_{k-1} \cup \Gamma_k = \Theta_k$ for all $k \in \{1,\dots,r(\mathbf{P})\}$.

Example 3.1.4. *Let* $\mathbf{F} = \{a_0 < a_1 > a_2 < a_3\}$ *be a fence. Then*

$$\Theta_1 = \{(a_0,a_0),(a_0,a_1),(a_1,a_0),(a_1,a_1),(a_1,a_2),(a_2,a_2),(a_2,a_3),(a_3,a_2),(a_3,a_3)\},$$

$$\Gamma_2 = \{(a_0,a_2),(a_2,a_0),(a_1,a_3),(a_3,a_1)\},$$

$${}^0\Gamma_2 = \{(a_0,a_2),(a_2,a_0)\}$$

and

$${}^1\Gamma_2 = \{(a_1,a_3),(a_3,a_1)\}.$$ □

For convenient, if $\mathbf{P}$ is an ordered set and ρ is a relation on P then we say that ρ is *compatible with* $\mathbf{P}$ or ρ is a *compatible relation with* $\mathbf{P}$ whenever $Pol(\rho)$ contains the monotone clone of $\mathbf{P}$.

Definition 3.1.5. A *diamond lattice* $\mathbf{D}$ is a set $D = \{0, 1, a, b\}$ together with the order $0 < a, b < 1$ which has no other comparabilities.

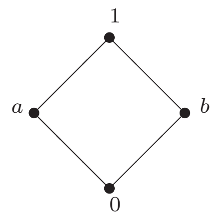

Figure 2. Diamond lattice

Theorem 3.1.6. *Let* $\mathbf{P}$ *be a connected ordered set. Then the followings are equivalent:*

1. $\mathbf{P}$ *satisfies PDP.*
2. Θ_k *is compatible with* $\mathbf{P}$ *for all* $k \in \{0, \ldots, r(\mathbf{P})\}$.
3. $\mathbf{P}$ *has no subordered sets which are isomorphic to a diamond lattice.*
4. ${}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$.

Proof. (1) $\Rightarrow$ (2). Let $k \in \{0, \ldots, r(\mathbf{P})\}$, f be an n-ary order-preserving on P and $(x_i, y_i) \in \Theta_k$ for all $i \in \{0, \ldots, n\}$. Since $\mathbf{P}$ satisfies PDP,

$$d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \leq^* \max\{d(x_1, y_1), \ldots, d(x_n, y_n))\} \leq^* k.$$

Hence, $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \Theta_k$. Therefore, Θ_k is compatible with $\mathbf{P}$.

(2) $\Rightarrow$ (3). Suppose that $\mathbf{P}$ has a diamond lattice $\mathbf{D} = (\{0, a, b, 1\}; \leq)$ as a subordered set. We will show that $Pol(\Theta_1)$ does not contain the monotone clone of $\mathbf{P}$. We define a function $f : P^2 \longrightarrow P$ by

$$f(x, y) = \begin{cases} 0 & \text{if } y \ngeq 1 \text{ and } x \leq 0, \\ a & \text{if } y \geq 1 \text{ and } x \leq 0, \\ b & \text{if either } y \ngeq 1,\ x \nleq 0 \text{ and } x \leq b \text{ or } x \nleq b \text{ and } y \ngeq 1, \\ 1 & \text{if either } y \geq 1,\ x \nleq 0 \text{ and } x \leq b \text{ or } x \nleq b \text{ and } y \geq 1. \end{cases}$$

To prove that f is an order-preserving, let x, y, s, t with $x \leq s$ and $y \leq t$.

Case 1 : $x \leq 0$.

Case 1.1 : $y \ngeq 1$. Then $f(x, y) = 0 \leq f(s, t) \in \{0, a, b, 1\}$.

Case 1.2 : $y \geq 1$. Then $t \geq 1$. Hence, $f(x, y) = a \leq 1 = f(s, t)$.

Case 2 : $x \nleq 0$ and $x \leq b$. Then $s \nleq 0$.

Case 2.1 : $t \ngeq 1$. Then $y \ngeq 1$. Hence, $f(x, y) = b = f(s, t)$,

Case 2.2 : $t \geq 1$. Then $f(s, t) = 1 \geq f(x, y) \in \{0, a, b, 1\}$.

Case 3 : $x \nleq 0$ and $x \nleq b$. Then $s \nleq b$

Case 3.1 : $t \ngeq 1$. Then $y \ngeq 1$. Hence, $f(x, y) = b = f(s, t)$,

Case 3.2 : $t \geq 1$. Then $f(s, t) = 1 \geq f(x, y) \in \{0, a, b, 1\}$.

In any cases, f is an order-preserving. One can see that $d(f(b, b), f(0, 1)) = d(b, a) = 2$; but $d(b, 0) = d(b, 1) = 1$. Hence $f \notin Pol(\Theta_1)$.

(3) $\Rightarrow$ (4). It is trivial.

(4) $\Rightarrow$ (1). Assume that ${}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$. Let f be an n-ary order-preserving on F and $x_1, \ldots, x_n, y_1, \ldots, y_n \in P$ such that $d(x_i, y_i) = m_i$ for all $0 \leq^* i \leq^* n$ and let $m = \max\{m_1, \ldots, m_n\}$.

Case 1 : $m = 1$. We may assume that there is a positive integer r such that $x_i \leq y_i$ for all $1 \leq^* i \leq^* r$ and $x_i \geq y_i$ for $r + 1 \leq^* i \leq^* n$. Then

$$f(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n) \leq f(x_1, \ldots, x_n) \leq f(y_1, \ldots, y_r, x_{r+1}, \ldots, x_n)$$

and

$$f(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n) \leq f(y_1, \ldots, y_n) \leq f(y_1, \ldots, y_r, x_{r+1}, \ldots, x_n).$$

By the assumption that ${}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$, the four elements $f(x_1, \ldots, x_r, y_{r+1}, \ldots, y_n)$, $f(x_1, \ldots, x_n)$, $f(y_1, \ldots, y_r, x_{r+1}, \ldots, x_n)$ and $f(y_1, \ldots, y_n)$ are comparable. Therefore, $d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \leq^* 1$.

Case 2 : $m >^* 1$. Let $\mathbf{F}_i = (\{a_i^0, a_i^1, \ldots, a_i^{m_i}\}; \leq)$ is a fence from x_i to y_i for all $1 \leq^* i \leq^* n$. By Case 1 and $d(a_i^j, a_i^{j+1}) \leq^* 1$ for all $1 \leq^* i \leq^* n$ and $0 \leq^* j \leq^* m_i - 1$, we have $d(f(a_1^j, \ldots, a_n^j), f(a_1^{j+1}, \ldots, a_n^{j+1})) \leq^* 1$ for all $0 \leq^* j \leq^* m - 1$ where $a_i^k = a_i^{m_i}$ for all $k \geq^* m_i$. By triangle inequality, $d(f(a_1^1, \ldots, a_n^1), f(a_1^m, \ldots, a_n^m)) \leq^* m$. $\square$

Corollary 3.1.7. *Let* $\mathbf{P}$ *be a finite connected ordered set. The relation* Θ_1 *is compatible with* $\mathbf{P}$ *if and only if* $\mathbf{P}$ *has no subordered sets which are isomorphic to a diamond lattice.*

3.2 All Maximal Clones of a Fence

By the preservation distance property of a fence $\mathbf{F}$, the relations Θ_k is compatible with $\mathbf{F}$ for all $k \in \{0, \ldots, r(\mathbf{F})\}$. For a suitable positive number k, one can see some central relations from these binary relations; for instance, $\Theta_{r(\mathbf{F})-1}$ is a central relation whose the maximal clone $Pol(\Theta_{r(\mathbf{F})-1})$ contains the monotone clone of $\mathbf{F}$. Arising now is whether there is a maximal clone preserving a central relation with arity greater than 2 whose clone contains the monotone clone of a fence. We are going to show that the answer is not affirmative; that is, if ρ is a central relation whose $Pol(\rho)$ contains the monotone clone of a fence then it is binary. For convenient, if $\mathbf{F} = (F = \{a_0, \ldots, a_n\}; \leq)$ is a fence, we denote

$$a_i] = \{a_j \,|\, j \in \{0, \ldots, i\}\}$$

and

$$[a_i = \{a_j \,|\, j \in \{i, \ldots, n\}\}$$

for all $i \in \{0, \ldots, n\}$.

Lemma 3.2.1. *Let* $\mathbf{F}$ *be a fence. Then the following two binary operations* $\varphi^+ : F \times F \to F$ *and* $\varphi^- : F \times F \to F$ *which are defined respectively by*

$$\varphi^-(x, y) = \begin{cases} x & \text{if } x \in y], \\ y & \text{if } x \in [y \end{cases}$$

and

$$\varphi^+(x, y) = \begin{cases} x & \text{if } x \in [y, \\ y & \text{if } x \in y], \end{cases}$$

are order-preservings.

Proof. By dually, we will show only that φ^+ is an order-preserving. Let $a, b, c, d \in F = \{a_0, \ldots, a_n\}$ with $a < b$ and $c < d$. Then there exist $i, j \in \{0, \ldots, n-1\}$ such that $\{a, b\} = \{a_i, a_{i+1}\}$ and $\{c, d\} = \{a_j, a_{j+1}\}$. With out loss of generality, we may assume that $i \leq^* j$. If $i <^* j$ then $i + 1 \leq^* j$; and so

$$\varphi^+(a, c) = c < d = \varphi^+(b, d).$$

Note that a and c are minimal and b and d are maximal. Hence, if $i = j$ then $\{a, b\} = \{c, d\}$; and so, $a = c$ and $b = d$ imply that

$$\varphi^+(a, c) = c < d = \varphi^+(b, d).$$

Hence, φ^+ is an order-preserving. □

Theorem 3.2.2. *Let* ρ *be a central compatible relation with a fence* $\mathbf{F}$. *Then*

1. ρ *is binary, and*

2. *the center* C_ρ *is convex; that is, for each* $x, y \in C_\rho$ *with* $x \in y]$ *and for all* $z \in F$, *if* $z \in [x \cap y]$ *then* $z \in C_\rho$.

Proof. (1) Suppose that ρ is an n-ary relation for some positive number n with $n \geq 3$. Let $a \in C_\rho$ and let $(x_1, \ldots, x_n) \in F^n$. Since $n \geq 3$, at least two elements in the set $\{x_1, \ldots, x_n\}$ belong to either $[a$ or $a]$. We assume that $x_1, x_2 \in [a$ or $x_1, x_2 \in a]$. Since $a \in C_\rho$, we obtain $(x_1, a, x_3, \ldots, x_n), (a, x_2, \ldots, x_n) \in \rho$. If $x_1, x_2 \in [a$ then Lemma 3.2.1 implies that $\varphi^+ \in Pol(\rho)$ and

$$(x_1, \ldots, x_n) = (\varphi^+(x_1, a), \varphi^+(a, x_2), \varphi^+(x_3.x_3) \ldots, \varphi^+(x_n, x_n)) \in \rho.$$

Similarly, if $x_1, x_2 \in a]$ then Lemma 3.2.1 implies that $\varphi^- \in Pol(\rho)$ and

$$(x_1, \ldots, x_n) = (\varphi^-(x_1, a), \varphi^-(a, x_2), \varphi^-(x_3.x_3) \ldots, \varphi^-(x_n, x_n)) \in \rho.$$

In either cases, $F^n = \rho$, a contradiction.

(2) Let $x, y \in C_\rho$ with $x \in y]$ and $z \in [x \cap y]$. Since ρ is binary and symmetric, it is enough to show that $(a, z) \in \rho$ for all $a \in F$. Let $a \in F$. Then $a \in [z$ or $a \in z]$. Since $x, y \in C_\rho$, we have $(a, x), (x, z), (a, y), (y, z) \in \rho$. If $a \in [z \subseteq [x$ then Lemma 3.2.1 implies that $\varphi^+ \in Pol(\rho)$ and

$$(a, z) = (\varphi^+(x, a), \varphi^+(z, x)) \in \rho.$$

Similarly, if $a \in z] \subseteq y]$ then Lemma 3.2.1 implies that $\varphi^- \in Pol(\rho)$ and

$$(a, z) = (\varphi^-(y, a), \varphi^-(z, y)) \in \rho.$$

In any cases, $(a, z) \in \rho$. Thus $z \in C_\rho$. □

Theorem 3.2.2 shows that all maximal clones containing the monotone clone of a fence are the clones preserving only binary central compatible relations with the fence. Finding all binary compatible relations with a fence is a very first process of finding all maximal clones containing the monotone clone of the fence. We will start finding all binary compatible relations with a fence via the following lemmata.

Lemma 3.2.3. *Let* $\mathbf{F} = (F = \{a_0, \ldots, a_n\}; \leq)$ *be a fence and* $0 \leq k \leq n$. *Then the following two unary operations* $\eta_k^+ : F \longrightarrow F$ *and* $\eta_k^- : F \longrightarrow F$ *whcih are defined respectively by*

$$\eta_k^+(a_r) = \begin{cases} a_k & \text{if } r \geq^* k, \\ a_r & \text{if } r <^* k \end{cases}$$

and

$$\eta_k^-(a_r) = \begin{cases} a_k & \text{if } r \leq^* k, \\ a_r & \text{if } r >^* k \end{cases}$$

are order-preservings.

Proof. By dually, we will show that η_k^+ is an order-preserving. Let $a < b$. Then there exists $i \in \{0, \ldots, n-1\}$ with $\{a, b\} = \{a_i, a_{i+1}\}$. If $i <^* k$ then $i+1 \leq^* k$; and so $\eta_k^+(a) = a < b = \eta_k^+(b)$. If $i \geq^* k$ then $i + 1 \geq^* k$; and so $\eta_k^+(a) = a_k = \eta_k^+(b)$. Hence η_k^+ is an order-preserving. □

Demetrovics J. and Rónyai L. [3] showed that if $\mathbf{F}$ is a subfence of a fence $\mathbf{G}$ then $\mathbf{F}$ is a retract of $\mathbf{G}$; that is, there exists an order-preserving g from $\mathbf{G}$ onto $\mathbf{F}$ such that $g(g(x)) = g(x)$ for all $x \in G$. Now, we will state and prove the results.

Corollary 3.2.4. [3] *Let* $\mathbf{F} = (\{a_0, a_1, \ldots, a_m\}; \leq)$ *and* $\mathbf{G} = (\{b_0, b_1, \ldots, b_n\}; \leq)$ *be fences.*

1. *If* $m \leq^* n$ *and both* $\mathbf{F}$ *and* $\mathbf{G}$ *are up fences or down fences then there exists an order-preserving* g *from* $\mathbf{G}$ *onto* $\mathbf{F}$ *with* $g(b_0) = a_0$ *and* $g(b_n) = a_m$.
2. *If* $m <^* n$ *then there exists an order-preserving* g *from* $\mathbf{G}$ *onto* $\mathbf{F}$ *with* $g(b_0) = a_0$ *and* $g(b_n) = a_m$.

3. *If* $\mathbf{F}$ *is a subfence of* $\mathbf{G}$ *then there exists an order-preserving* g *from* $\mathbf{G}$ *onto* $\mathbf{F}$ *with* $g\downharpoonright_F = id_F$.

Proof. (1) Suppose that $m \leq^* n$ and both $\mathbf{F}$ and $\mathbf{G}$ are up fences or down fences. By Lemma 3.2.3, η_m^+ is an order-preserving from $\mathbf{G}$ onto subordered set $\mathbf{G}_1 = (\{b_0, b_1, \ldots, b_m\}, \leq)$ of $\mathbf{G}$ such that $\eta_m^+(b_0) = b_0$ and $\eta_m^+(b_n) = b_m$. Moreover, both $\mathbf{F}$ and $\mathbf{G}_1$ are up fences or down fences; so the operation $f : \mathbf{G}_1 \longrightarrow \mathbf{F}$ which is defined by $f(b_i) = a_i$ for all $0 \leq i \leq m$ is an order-preserving. Hence, $f \circ \eta_m^+$ is an order-preserving from $\mathbf{G}$ onto $\mathbf{F}$ with $(f \circ \eta_m^+)(b_0) = a_0$ and $(f \circ \eta_m^+)(b_n) = a_m$.

(2) Suppose that $m <^* n$ and either $\mathbf{F}$ or $\mathbf{G}$ is an up fence, but not both. Then Lemma 3.2.3 implies that η_1^- is an order-preserving from $\mathbf{G}$ onto subordered set $\mathbf{G}_2 = (\{b_1, b_2, \ldots, b_n\}, \leq)$ of $\mathbf{G}$ such that $\eta_1^-(b_0) = b_1$ and $\eta_1^-(b_n) = b_n$. One can see that both $\mathbf{F}$ and $\mathbf{G}_2$ are up fences or down fences. By (1), there exists an order-preserving g from $\mathbf{G}_2$ onto $\mathbf{F}$ with $g(b_1) = a_0$ and $g(b_n) = a_m$. Hence $g \circ \eta_1^-$ is an order-preserving from $\mathbf{G}$ onto $\mathbf{F}$ with $\left(g \circ \eta_1^-\right)(b_0) = a_0$ and $\left(g \circ \eta_1^-\right)(b_n) = a_m$.

(3) We may assume that $\mathbf{F} = (\{b_i, b_{i+1}, \ldots, b_{i+j}\}; \leq)$ for some $i, j \in \{0, \ldots, n\}$ with $i + j \leq n$. Then $\eta_i^- \circ \eta_{i+m}^+$ is an order-preserving from $\mathbf{G}$ onto $\mathbf{F}$ with $\eta_i^- \circ \eta_{i+m}^+ \downharpoonright_F = id_F$ □

Lemma 3.2.5. *Let* ρ *be a binary compatible relation with a fence* $\mathbf{F}$.

1. ${}^i\Gamma_k \cap \rho \neq \emptyset$ *if and only if* $\emptyset \neq {}^i\Gamma_k \subseteq \rho$ *for all* $i \in \{0, 1\}$ *and* $k \in \{0, \ldots, r(\mathbf{F})\}$.
2. *For each* $k \in \{1, \ldots, r(\mathbf{F})\}$, *if* $\Gamma_k \cap \rho \neq \emptyset$ *then* $\Theta_{k-1} \subseteq \rho$.

Proof. (1) Suppose that $(x, y) \in {}^i\Gamma_k \cap \rho$ and $(a, b) \in {}^i\Gamma_k$. Let $\mathbf{F}_x^y$ and $\mathbf{F}_a^b$ be subfences from x to y and a to b of $\mathbf{F}$, respectively. By Corollary 3.2.4 (3), there is an order-preserving $g : \mathbf{F} \longrightarrow \mathbf{F}_x^y$ with $g \downharpoonright_{F_x^y} = id_{F_x^y}$. Since both $\mathbf{F}_x^y$ and $\mathbf{F}_a^b$ are up fences or down fences with $r(\mathbf{F}_x^y) = r(\mathbf{F}_a^b)$ and by Corollary 3.2.4 (1), there is an order-preserving $h : \mathbf{F}_x^y \longrightarrow \mathbf{F}_a^b$ such that $h(x) = a$ and $h(y) = b$. Hence, $h \circ g$ is an order-preserving with $(h \circ g)(x) = a$ and $(h \circ g)(y) = b$. Since $h \circ g \in Pol(\rho)$ and $(x, y) \in \rho$, we have $(a, b) \in \rho$. The converse is trivial.

(2) Let $(x, y) \in \Gamma_k \cap \rho$ and $(a, b) \in \Theta_{k-1}$. Let $\mathbf{F}_x^y$ and $\mathbf{F}_a^b$ be subfences from x to y and a to b of $\mathbf{F}$, respectively. By Corollary 3.2.4 (3), there is an order-preserving $g : \mathbf{F} \longrightarrow \mathbf{F}_x^y$ with $g \downharpoonright_{F_x^y} = id_{F_x^y}$. Since $r(\mathbf{F}_a^b) <^* r(\mathbf{F}_x^y)$ and by Corollary 3.2.4 (2), there is an order-preserving $h : \mathbf{F}_x^y \longrightarrow \mathbf{F}_a^b$ such that $h(x) = a$ and $h(y) = b$. Hence, $h \circ g$ is an order-preserving with $(h \circ g)(x) = a$ and $(h \circ g)(y) = b$. Since $h \circ g \in Pol(\rho)$ and $(x, y) \in \rho$, $(a, b) \in \rho$. □

The following theorem gives a characterization of all binary compatible relations with a fence $\mathbf{F}$ in terms of Θ_k and ${}^i\Gamma_k$ for some $0 \leq k \leq n$ and $i \in \{0, 1\}$.

Theorem 3.2.6. *A non empty binary relation* ρ *is compatible with a fence* $\mathbf{F}$ *if and only if* ρ *is one of the following relations:*

$$\Theta_j \ , \ \Theta_{k-1} \cup {}^0\Gamma_k \text{ or } \Theta_{k-1} \cup {}^1\Gamma_k$$

for some $j \in \{0, \ldots, r(\mathbf{F})\}$ *and* $k \in \{1, \ldots, r(\mathbf{F})\}$.

Proof. ($\Longrightarrow$) Let ρ be a non empty binary compatible relation with $\mathbf{F}$. It is clear that all constant functions are order-preservings. Thus, ρ is refexive; that is, $\Theta_0 \subseteq \rho$. Let $(x, y) \in \rho$ such that $d(x, y) \geq^* d(a, b)$ for all $(a, b) \in \rho$. We let $k = d(x, y)$. Then $\rho \subseteq \Theta_k$ and $({}^0\Gamma_k \cup {}^1\Gamma_k) \cap \rho = \Gamma_k \cap \rho \neq \emptyset$. Therefore, ${}^i\Gamma_k \cap \rho \neq \emptyset$ for some $i \in \{0, 1\}$. If $k = 0$ then $\rho = \Theta_0$. We consider the case that k is positive. Hence, Lemma 3.2.5 implies that ${}^i\Gamma_k \subseteq \rho$ and $\Theta_{k-1} \subseteq \rho$. Let $j \in \{0, 1\}$ with $j \neq i$. If ${}^j\Gamma_k \cap \rho = \emptyset$ then $\rho \subseteq \Theta_{k-1} \cup {}^i\Gamma_k \subseteq \rho$. Hence, $\rho = \Theta_{k-1} \cup {}^i\Gamma_k$. If ${}^j\Gamma_k \cap \rho \neq \emptyset$, Lemma 3.2.5 implies that ${}^j\Gamma_k \subseteq \rho$; and so, $\Theta_k = \Theta_{k-1} \cup {}^i\Gamma_k \cup {}^j\Gamma_k \subseteq \rho \subseteq \Theta_k$. Therefore, $\rho = \Theta_k$.

($\Longleftarrow$) By Theorem 3.1.6 and dually, it suffices to show that $Pol(\Theta_{k-1} \cup {}^0\Gamma_k)$ contains the monotone clone of the fence. Let f be an n-ary order-preserving defined on F and $(x_i, y_i) \in \Theta_{k-1} \cup {}^0\Gamma_k$ for all $i \in \{1, \ldots, n\}$. Let $d(x_i, y_i) = m_i$ and $\mathbf{F}_i = (\{a_i^0, a_i^1, \ldots, a_i^{m_i}\}; \leq)$ be a fence from x_i to y_i for all $i \in \{1, \ldots, n\}$. Then $m_i \leq^* k$ for all $i \in \{1, \ldots, n\}$. We may assume that there is an $r \in \{1, \ldots, n\}$ such that $a_i^0 \leq a_i^1$ for all $i \in \{1, \ldots, r\}$ and $a_i^0 \geq a_i^1$ for all $i \in \{r+1, \ldots, n\}$. Since f is an order-preserving,

$$
\begin{aligned}
f(x_1, \ldots, x_n) &= f(a_1^0, \ldots, a_n^0) \\
&\leq f(a_1^1, \ldots, a_r^1, a_{r+1}^0, \ldots, a_n^0) \\
&\geq f(a_1^2, \ldots, a_r^2, a_{r+1}^1, \ldots, a_n^1) \\
&\leq f(a_1^3, \ldots, a_r^3, a_{r+1}^2, \ldots, a_n^2) \\
&\vdots \\
&\quad f(a_1^k, \ldots, a_r^k, a_{r+1}^{k-1}, \ldots, a_n^{k-1}) \\
&= f(y_1, \ldots, y_n).
\end{aligned}
$$

where $a_i^l = y_i$ for all $l \geq^* m_i$. Hence, either there is an up fence from $f(x_1, \ldots, x_n)$ to $f(y_1, \ldots, y_n)$ with the reach k or $d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) <^* k$. So, $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \Theta_{k-1} \cup {}^0\Gamma_k$. Therefore, $Pol(\Theta_{k-1} \cup {}^0\Gamma_k)$ contains the monotone clone of $\mathbf{F}$. □

The lattice of all binary compatible relations with a fence is illustrated in Figure 3.

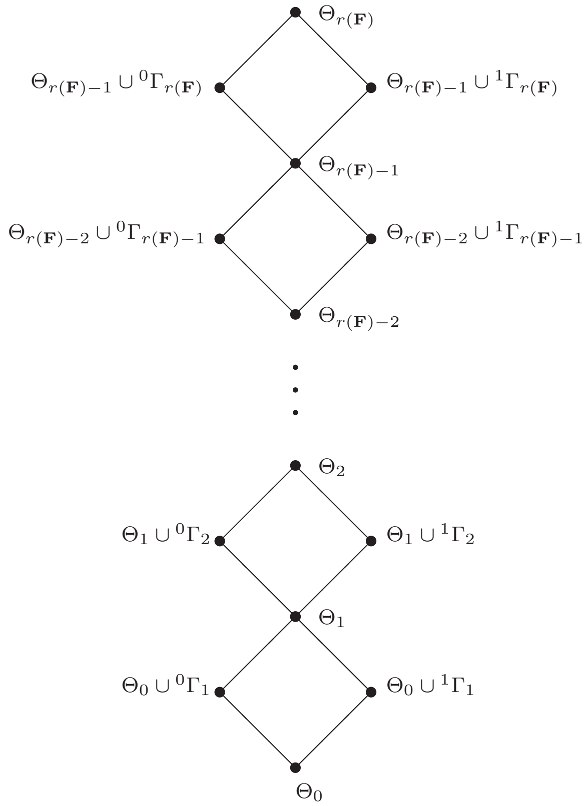

Figure 3. The lattice of all binary compatible relations with a fence $\mathbf{F}$

Recall that a binary relation ρ is central on a set A if ρ is reflexive, symmetric and the center of ρ is neither empty nor A. Even though all relations in Theorem 3.2.6 are binary compatible with a fence, but some of these relations are not central; for instances, Θ_0 and $\Theta_0 \cup {}^0\Gamma_1$ are not central since $C_{\Theta_0} = \emptyset$ and $\Theta_0 \cup {}^0\Gamma_1$ is not symmetric. Since all relations in Theorem 3.2.6 contain Θ_0, those are reflexive. So, it is interesting to find all relations in Theorem 3.2.6 which are symmetric and the center is neither empty nor F. Next, we will give a characterization of all binary central compatible relations with a fence via the following lemmata.

Lemma 3.2.7. *Let* $\mathbf{F}$ *be a fence.*

1. *The relation* Θ_j *is symmetric for all* $j \in \{0, \dots, r(\mathbf{F})\}$.
2. *For* $i \in \{0, 1\}$ *and* $k \in \{1, \dots, r(\mathbf{F})\}$, *the relation* $\Theta_{k-1} \cup {}^i\Gamma_k$ *is symmetric if and only if* k *is an even number.*

Proof. By dually, we may assume that $\mathbf{F} = \left\{a_0 < a_1 > a_2 \dots < (>) a_{r(\mathbf{F})}\right\}$ is an up fence.

(1) It is clear that Θ_j is symmetric for all $0 \leq^* j \leq^* r(\mathbf{F})$ since $d(a, b) = d(b, a)$ for all $a, b \in F$.

(2) If $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric for some $0 <^* k \leq^* r(\mathbf{F})$ then $(a_{2j+i}, a_{2j+i\pm k})$ and $(a_{2j+i\pm k}, a_{2j+i})$ are in ${}^i\Gamma_k$; and so $2j + i \pm k = 2m + i$, that is, $k = \pm 2(j - m)$. Conversely, suppose that $k = 2l$ for some $0 <^* 2l \leq^* r(\mathbf{F})$. Since Θ_{2l-1} is

symmetric, it suffices to show that ${}^{i}\Gamma_{2l}$ is symmetric. Since $(a_{2j+i}, a_{2j+i\pm 2l})$ and $(a_{2(j\pm l)+i}, a_{2j+i})$ are in ${}^{i}\Gamma_{2l}$, we have that ${}^{i}\Gamma_{2l}$ is symmetric. □

Recall that a is in the center C_ρ of a binary relation ρ on a set A if $(a, b), (b, a) \in \rho$ for all $b \in A$ and the ceiling function $\lceil\,\rceil$ maps a real number x to the smallest integer which is greater than x. For a fence $\{a_0 < a_1 > a_2 < a_3 > a_4\}$, one can see that the center of the relation $\Theta_1 \cup {}^{0}\Gamma_2$ is not empty. But, for a fence $\{b_0 > b_1 < b_2 > b_3 < b_4\}$, the center of the relation $\Theta_1 \cup {}^{0}\Gamma_2$ is empty. Hence, for a fence $\mathbf{F}$ and $i \in \{0, 1\}$, the center of the relation $\Theta_{\lceil \frac{r(\mathbf{F})}{2} \rceil - 1} \cup {}^{i}\Gamma_{\lceil \frac{r(\mathbf{F})}{2} \rceil}$ is not necessary empty. The following lemma gives necessary conditions for a positive number k such that the center of the relation $\Theta_{k-1} \cup {}^{i}\Gamma_k$ is empty or not for all $i \in \{0, 1\}$.

Lemma 3.2.8. *Let* $\mathbf{F}$ *be a fence.*

1. *The center of the relation* Θ_k *is not empty if and only if* $k \geq^* \left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$.

2. *For* $i \in \{0, 1\}$, *the center of the relation* $\Theta_{k-1} \cup {}^{i}\Gamma_k$ *is not empty if* $k >^* \left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$, *and the center of the relation* $\Theta_{k-1} \cup {}^{i}\Gamma_k$ *is empty if* $k <^* \left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$.

Proof. Let $\mathbf{F} = \big(\{a_0, a_1, \ldots, a_{r(\mathbf{F})}\}, \leq\big)$ be a fence.

(1) If $k <^* r(\mathbf{F})/2$ and $a_i \in C_{\Theta_k}$ then $d(a_0, a_i), d(a_i, a_{r(\mathbf{F})}) \leq^* k <^* r(\mathbf{F})/2$; and so

$$r(\mathbf{F}) = d(a_0, a_{r(\mathbf{F})}) \leq^* d(a_0, a_i) + d(a_i, a_{r(\mathbf{F})}) <^* r(\mathbf{F}),$$

a contradiction. Conversely, suppose that $k \geq^* r(\mathbf{F})/2$. For each $0 \leq m \leq r(\mathbf{F})$, we see that

$$-k \leq^* m - k \leq^* r(\mathbf{F}) - k \leq^* \frac{r(\mathbf{F})}{2} \leq^* k.$$

So, $d(a_m, a_k) = d(a_k, a_m) = |k - m| \leq^* k$ for all $0 \leq^* m \leq^* r(\mathbf{F})$.

(2) It is easy to see that if ρ_1 and ρ_2 are relations on F with $\rho_1 \subseteq \rho_2$ then $C_{\rho_1} \subseteq C_{\rho_2}$. If $k - 1 \geq^* r(\mathbf{F})/2$ then (1) and $\Theta_{k-1} \subseteq \Theta_{k-1} \cup {}^{i}\Gamma_k$ imply that the center of $\Theta_{k-1} \cup {}^{i}\Gamma_k$ is not empty. If $k <^* r(\mathbf{F})/2$ then (1) and $\Theta_{k-1} \cup {}^{i}\Gamma_k \subseteq \Theta_k$ imply that the center of $\Theta_{k-1} \cup {}^{i}\Gamma_k$ is empty. □

Let $\mathbf{F}$ be a fence. It is clear that the center of the relation $\Theta_{r(\mathbf{F})}$ is F; so, $\Theta_{r(\mathbf{F})}$ is not central. If $r(\mathbf{F})$ is an odd number then Lemma 3.2.7 (2) implies that $\Theta_{r(\mathbf{F})-1} \cup {}^{i}\Gamma_{r(\mathbf{F})}$ is not symmetric for all $i \in \{0, 1\}$. Suppose that that $r(\mathbf{F}) = 2n$ for some positive number n and $\mathbf{F}$ is an up fence. Then ${}^{1}\Gamma_{r(\mathbf{F})} = \emptyset$ and $\Theta_{r(\mathbf{F})-1} \cup {}^{0}\Gamma_{r(\mathbf{F})} = \nabla_F$. Hence, $\Theta_{r(\mathbf{F})-1} \cup {}^{1}\Gamma_{r(\mathbf{F})}$ and $\Theta_{r(\mathbf{F})-1} \cup {}^{0}\Gamma_{r(\mathbf{F})}$ are not central.

Remark 3.2.9. *Let* $\mathbf{F}$ *be a fence and* $i \in \{0, 1\}$. *Then the relations* $\Theta_{r(\mathbf{F})}$ *and* $\Theta_{r(\mathbf{F})-1} \cup {}^{i}\Gamma_{r(\mathbf{F})}$ *are not central.*

The following theorem shows a characterization of all binary central compatible relations with a fence.

Theorem 3.2.10. *Let* $\mathbf{F}$ *be a fence.*

1. *Θ_j is central if and only if* $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil \leq^* j <^* r(\mathbf{F})$.

2. *For $i \in \{0,1\}$ and $k \in \{1, \ldots, r(\mathbf{F})\}$, $\Theta_{k-1} \cup {}^i\Gamma_k$ is central if and only if k is an even number satisfying one of the followings:*

   (a) $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil <^* k <^* r(\mathbf{F})$,

   (b) $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil = k$ *with* $r(\mathbf{F}) = 2k - 1$

   (c) $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil = k$ *with $r(\mathbf{F}) = 2k$ and either $\mathbf{F}$ is an up fence if $i = 0$ or $\mathbf{F}$ is a down fence if $i = 1$.*

Proof. Let $\mathbf{F} = \big(\{a_0, a_1, \ldots, a_{r(\mathbf{F})}\}, \leq\big)$ be a fence.

(1) Lemma 3.2.7 (1), Lemma 3.2.8 (1) and Remark 3.2.9 imply that Θ_j is central if and only if j is a positive integer with $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil \leq^* j \leq^* r(\mathbf{F}) - 1$.

(2) By dually, let $i = 0$.

($\Rightarrow$) Suppose that $\Theta_{k-1} \cup {}^0\Gamma_k$ is central. Lemma 3.2.7 (2) implies that k is even. By Lemma 3.2.8 and Remark 3.2.9, it suffices to consider the case that $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil = k$ which implies that $r(\mathbf{F}) = 2k$ or $2k - 1$. One can see that if $a \in C_{\Theta_{k-1} \cup {}^0\Gamma_k}$ then $d(a, b) \leq^* k$ for all $b \in F$. Suppose that $r(\mathbf{F}) = 2k$; but $\mathbf{F}$ is down fence. Then there is no up fences from a_k to a_0; that is $(a_k, a_0) \notin {}^0\Gamma_k$. Therefore, $a_k \notin C_{\Theta_{k-1} \cup {}^0\Gamma_k}$. Since there exists unique a_k such that $d(a_k, b) \leq^* k$ for all $b \in F$, the set $C_{\Theta_{k-1} \cup {}^0\Gamma_k}$ is empty, a contradiction.

($\Leftarrow$) Suppose that k is even number. If $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil <^* k <^* r(\mathbf{F})$ then Lemma 3.2.8 (2) implies that $\Theta_{k-1} \cup {}^0\Gamma_k$ is central. If $r(\mathbf{F}) = 2k - 1$, it is clearly that $(a_k, a_0) \in {}^0\Gamma_k$ and $(a_k, b) \in \Theta_{k-1}$ for all $b \in F \setminus \{a_0\}$; and so, $a_k \in C_{\Theta_{k-1} \cup {}^0\Gamma_k}$. Similarly, if $r(\mathbf{F}) = 2k$ and $\mathbf{F}$ is up fence, $(a_k, a_0), (a_k, a_{2k}) \in {}^0\Gamma_k$ and $(a_k, b) \in \Theta_{k-1}$ for all $b \in F \setminus \{a_0, a_{2k}\}$, and so, $a_k \in C_{\Theta_{k-1} \cup {}^0\Gamma_k}$. □

Recall that a is *congruence* to b *modulo* m, and we write $a \equiv b \pmod{m}$ if there is an integer k such that $a = b + km$ for intergers a, b, m with $m >^* 0$. Figure 4 - 7 illustrate the lattice of all central compatible relations with a fence $\mathbf{F}$.

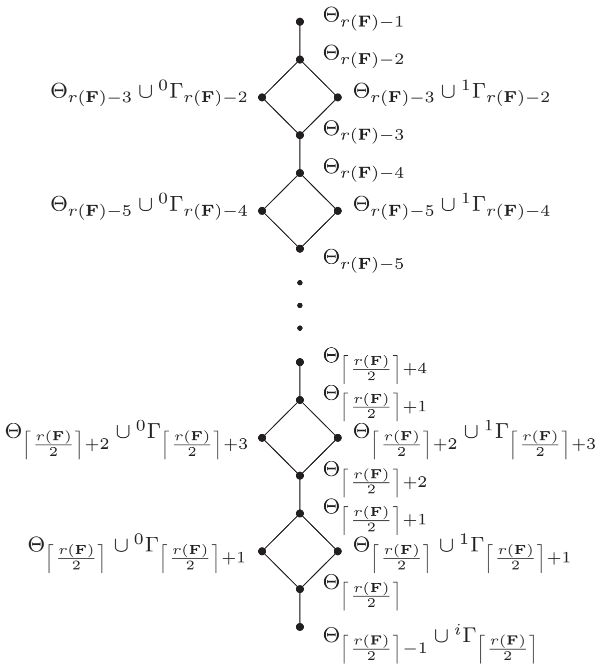

Figure 4. The lattice of all central compatible relations with a fence $\mathbf{F}$ such that $r(\mathbf{F}) \equiv 0 \pmod 4$

In Figure 4, the numbers $r(\mathbf{F})$ and $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$ are even; that is, $r(\mathbf{F}) \equiv 0 \pmod 4$ and if $\mathbf{F}$ is up fence, we let $i = 0$ and if $\mathbf{F}$ is down fence, we let $i = 1$.

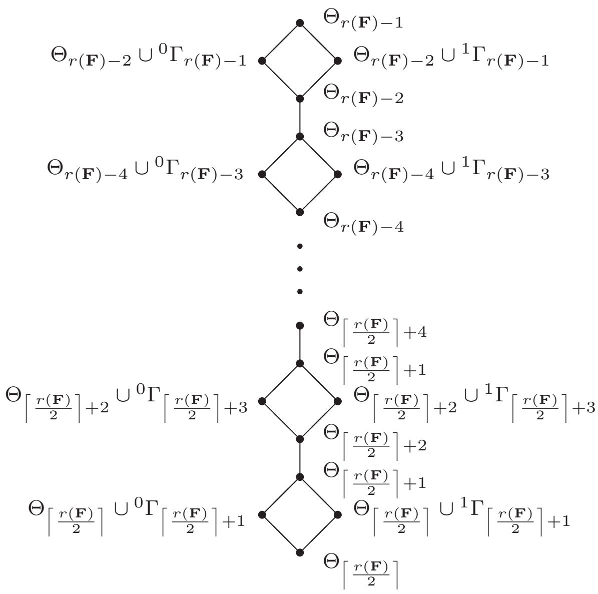

Figure 5. The lattice of all central compatible relations with a fence $\mathbf{F}$ such that $r(\mathbf{F}) \equiv 1 \pmod 4$

In Figure 5, the numbers $r(\mathbf{F})$ and $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$ are odd; that is, $r(\mathbf{F}) \equiv 1 \pmod 4$.

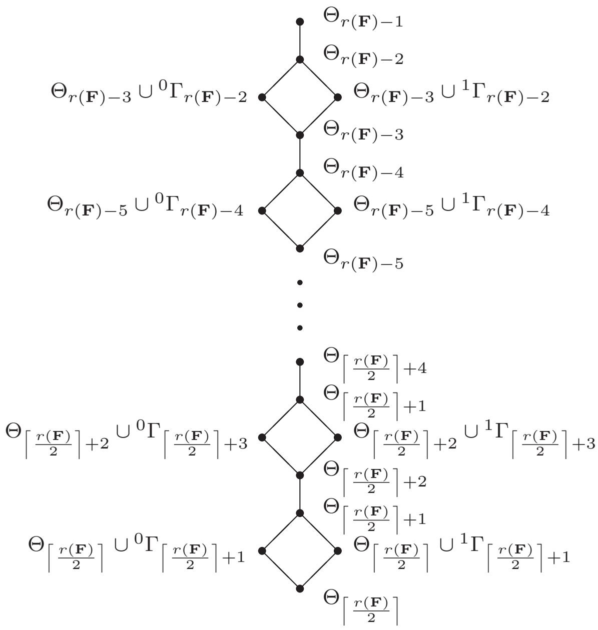

Figure 6. The lattice of all central compatible relations with a fence $\mathbf{F}$ such that $r(\mathbf{F}) \equiv 2 \pmod 4$

In Figure 6, the number $r(\mathbf{F})$ is even and the number $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$ is odd; that is, $r(\mathbf{F}) \equiv 2 \pmod 4$.

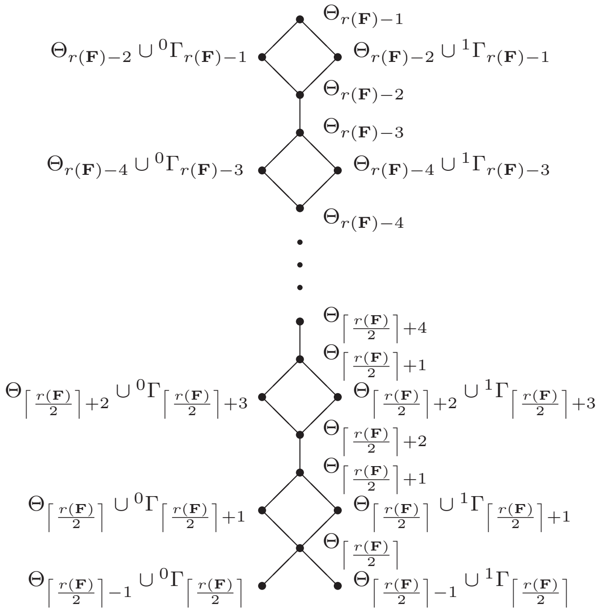

Figure 7. The lattice of all central compatible relations with a fence $\mathbf{F}$ such that $r(\mathbf{F}) \equiv 3 \pmod 4$

In Figure 7, the number $r(\mathbf{F})$ is odd and the number $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil$ is even; that is, $r(\mathbf{F}) \equiv 3 \pmod 4$.

Chapter 4

All Maximal Clones of a Majority Ordered Set

A ternary operation $m : P^3 \longrightarrow P$ is called *majority* if for each $x, y \in P$,

$$m(y, x, x) = m(x, y, x) = m(x, x, y) = x.$$

An ordered set is called a *majority ordered set* if the monotone clone contains a majority order-preserving. C. Ratanaprasert [5] has shown that the monotone clone of a finite unbounded connected majority ordered set is a subclone of a maximal clone preserving a central relation with arity more than 1. Füredi Z. and Rosenberg I.G. [4] introduced the class of all tree-likes which is a large class of connected majority ordered sets that need not be lattices. J. Demetrovics and L. Rónyai [3] proved that a fence is a majority ordered set. In Chapter 3, we characterized all maximal clones of a fence. In this chapter, we find all possibilities of central compatible relations with a finite unbounded connected majority ordered set in term of the distance function. And then we characterize all central compatible relations with an unbounded tree-like.

Throughout this chapter, we will consider a finite connected ordered set having the reach more than 1.

4.1 Maximal Clones of a Majority Ordered Set

In chapter 3, we proved that all maximal clones of an unbounded fence are clones preserving central relations with arity only 2. As we already knew that a fence is a majority order, it is interesting whether a central compatible relations with a majority ordered set is also only binary. We are going to prove the affirmative answer.

Theorem 4.1.1. *If ρ is a central compatible relation with a majority ordered set then ρ is binary.*

Proof. Suppose that ρ is an n-ary central compatible relation with a majority ordered set $\mathbf{P}$ for some $n \geq 3$. Let $m : P^3 \longrightarrow P$ be a majority operation on P such that $m \in Pol(\rho)$. Let $a \in C_\rho$ and $(x_1, \ldots, x_n) \in P^n$. Then

$$(a, x_2, x_3, \ldots, x_n), (x_1, a, x_3, \ldots, x_n), (x_1, x_2, a, x_4, \ldots, x_n) \in \rho.$$

Since m is a majority operation, we have that $x_1 = m(a, x_1, x_1)$, $x_2 = m(x_2, a, x_2)$, $x_3 = m(x_3, x_3, a)$ and $x_i = m(x_i, x_i, x_i)$ for all $i \in \{4, \ldots, n\}$; and so, $(x_1, \ldots, x_n)$ belongs to ρ. Thus, $P^n = \rho$, a contradiction. $\square$

The above theorem shows that all maximal clones of a majority ordered set are clones preserving central relations with arity only 2. In order to find all maximal clones of a majority ordered set, it is first to find all binary central compatible relations with the ordered set. Now, we will show all possibilities of binary compatible relations with an ordered set.

Let $\mathbf{P}$ be an ordered set. For each $a, b \in P$ and $i \in \{0, 1\}$, let ${}^i\mathbf{F}_a^b = (\{a = a_0, \ldots a_k = b\}, \leq)$ and define ${}^if_a^b : P \to {}^iF_a^b$ by

$$ {}^if_a^b(x) = \begin{cases} a_r & \text{if } (a, x) \in {}^i\Gamma_r \setminus {}^j\Gamma_r \text{ for some } 0 \leq^* r \leq^* k, \\ a_{r-1} & \text{if } (a, x) \in {}^j\Gamma_r \text{ for some } 1 \leq^* r \leq^* k, \\ b & \text{if } d(a, x) >^* k \end{cases} $$

where $j \in \{0, 1\}$ with $j \neq i$. The following lemma gives that ${}^if_a^b$ is an order-preserving.

Lemma 4.1.2. *Let* $\mathbf{P}$ *be an ordered set.*

1. *The operation* ${}^if_a^b$ *is an order-preserving.*
2. *If* $(a, b) \in {}^i\Gamma_k \setminus {}^j\Gamma_k$ *then* ${}^if_a^b \restriction_{{}^i\mathbf{F}_a^b} = id_{{}^i\mathbf{F}_a^b}$.

Proof. (1) By dually, assume that $i = 0$ and let $f = {}^0f_a^b$. Note that $a_{2i} < a_{2i+1} > a_{2i+2}$ for all $0 \leq^* 2i \leq^* k-2$. Let $x < y$. If $d(a, x) >^* k+1$ or $d(a, y) >^* k+1$ then Corollary 3.1.2 implies that $f(x) = f(y) = a_k$. Suppose that $(a, x), (a, y) \in \Theta_{k+1}$. We will consider the following cases.

Case 1 : $(a, x) \in {}^0\Gamma_{2i} \setminus {}^1\Gamma_{2i}$ for some i with $0 \leq^* 2i \leq^* k + 1$. Then $f(x) = a_{2i}$. By Corollary 3.1.2, $(a, y) \in \Gamma_{2i-1} \cup \Gamma_{2i} \cup \Gamma_{2i+1}$. We will show that $(a, y) \notin {}^1\Gamma_{2i-1}$. Suppose that ${}^1\mathbf{F}_a^y = \{a = y_0 > y_1 < y_2 \ldots y_{2i-2} > y_{2i-1} = y\}$ exists. Since $x < y < y_{2i-2}$ and by Corollary 3.1.2, $d(a, x) \leq^* d(a, y_{2i+2}) + 1 = 2i - 1$, a contradiction. Thus, $(a, y) \in ({}^0\Gamma_{2i-1} \setminus {}^1\Gamma_{2i-1}) \cup \Gamma_{2i} \cup \Gamma_{2i+1}$. Hence, $f(y) \in \{a_{2i-1}, a_{2i}, a_{2i+1}\}$. Since $a_0 < a_1 > a_2 \ldots a_k$, we have $f(x) \leq f(y)$.

Case 2 : $(a, x) \in {}^1\Gamma_{2i}$ for some i with $0 \leq^* 2i \leq^* k+1$. Then $f(x) = a_{2i-1}$. By Corollary 3.1.2, $(a, y) \in \Gamma_{2i-1} \cup \Gamma_{2i} \cup \Gamma_{2i+1}$. Assume that ${}^1\mathbf{F}_a^x = \{x_0 > x_1 < \ldots > x_{2i-1} < x_{2i} = x\}$. Since $x_{2i-1} < x < y$ and by Corollary 3.1.2, $d(a, y) \leq^* d(a, x_{2i-1}) + 1 = 2i$; and so, $(a, y) \in \Gamma_{2i-1} \cup \Gamma_{2i}$. Moreover, if $d(a, y) = 2i$ then $(a, y) \in {}^1\Gamma_{2i}$. We will show that $(a, y) \notin {}^1\Gamma_{2i-1}$. Suppose that ${}^1\mathbf{F}_a^y = \{y_0 > y_1 < y_2 > \ldots < y_{2i-2} > y_{2i-1} = y\}$ exists. Since $y_{2i-2} > y > x$ and by Corollary 3.1.2, $d(a, x) \leq^* d(a, y_{2i-2}) + 1 = 2i - 1$, a contradiction. Therefore, $(a, y) \in ({}^0\Gamma_{2i-1} \setminus {}^1\Gamma_{2i-1}) \cup {}^1\Gamma_{2i}$. Hence, $f(y) = a_{2i-1} = f(x)$

Case 3 : $(a, x) \in {}^0\Gamma_{2i+1} \setminus {}^1\Gamma_{2i+1}$ for some i with $0 \leq^* 2i \leq^* k$ Then $f(x) = a_{2i+1}$. Similarly with the proof of Case 2, we obtain $(a, y) \in {}^0\Gamma_{2i+1} \setminus {}^1\Gamma_{2i+1}$. Hence, $f(x) = f(y)$.

Case 4 : $(a, x) \in {}^1\Gamma_{2i+1}$ for some i with $0 \leq^* 2i \leq^* k$. Then $f(x) = a_{2i}$. Similarly with the proof of Case 1, we obtain $(a, y) \in \Gamma_{2i} \cup \Gamma_{2i+1} \cup {}^1\Gamma_{2i+2}$. Hence, $f(x) \leq f(y)$.
For any cases, we conclude that $f(x) \leq f(y)$.

(2) It is clear by the definition of the operations that if $(a, b) \in {}^i\Gamma_k \setminus {}^j\Gamma_k$ then ${}^if_a^b \restriction_{{}^j\mathbf{F}_a^b} = id_{{}^i\mathbf{F}_a^b}$. □

Lemma 4.1.3. *Let ρ be a binary compatible relation with an ordered set $\mathbf{P}$.*

1. *For each $k \in \{0, \dots, r(\mathbf{P})\}$ and $i, j \in \{0, 1\}$ with $i \neq j$, $({}^i\Gamma_k \setminus {}^j\Gamma_k) \cap \rho \neq \emptyset$ if and only if $({}^i\Gamma_k \setminus {}^j\Gamma_k) \neq \emptyset$ and ${}^i\Gamma_k \subseteq \rho$.*
2. *For each $k \in \{1, \dots, r(\mathbf{P})\}$, if $\Gamma_k \cap \rho \neq \emptyset$ then $\Theta_{k-1} \subseteq \rho$.*

Proof. (1) By dually, we may assume that $(x, y) \in ({}^0\Gamma_k \setminus {}^1\Gamma_k) \cap \rho$ and ${}^0\mathbf{F}_x^y = \{x_0 < x_1 > \dots x_k\}$ is a fence from x to y. Then $({}^i\Gamma_k \setminus {}^j\Gamma_k) \neq \emptyset$. By Lemma 4.1.2, the operation ${}^0f_x^y$ is an order-preserving with ${}^0f_x^y \restriction_{{}^0\mathbf{F}_x^y} = id_{{}^0\mathbf{F}_x^y}$. Let $(a, b) \in {}^0\Gamma_k$ and ${}^0\mathbf{F}_a^b = \{a_0 < a_1 > \dots a_k\}$. By Corollary 3.2.4, there is an order-preserving g from ${}^0\mathbf{F}_x^y$ onto ${}^0\mathbf{F}_a^b$ with $g(x) = a$ and $g(y) = b$. Hence, $g \circ {}^0f_x^y$ is an order-preserving with $g \circ {}^0f_x^y(x) = a$ and $g \circ {}^0f_x^y(y) = b$. Since $(x, y) \in \rho$ and $g \circ {}^0f_x^y \in Pol(\rho)$, we obtain $(a, b) \in \rho$.

(2) Let $(x, y) \in \Gamma_k \cap \rho$. By dually, we may assume that ${}^0\mathbf{F}_x^y = \{x_0 < x_1 > \dots x_k\}$ exists. It is clear that $(x_0, x_m) \in {}^0\Gamma_m \setminus {}^1\Gamma_m$ and $(x_1, x_{n+1}) \in {}^1\Gamma_n \setminus {}^0\Gamma_n$ for all $m, n \in \{1, \dots, k-1\}$. By Lemma 4.1.2, the operations ${}^0f_{x_0}^{x_m}$ and ${}^1f_{x_1}^{x_{n+1}}$ are order-preservings with ${}^0f_{x_0}^{x_m}(x) = x_0$, ${}^0f_{x_0}^{x_m}(y) = x_m$, ${}^1f_{x_1}^{x_n}(x) = x_1$ and ${}^1f_{x_1}^{x_{n+1}}(y) = x_{n+1}$ for all $m, n \in \{1, \dots, k-1\}$. Since ${}^0f_{x_0}^{x_m}, {}^1f_{x_1}^{x_{n+1}} \in Pol(\rho)$ and $(x, y) \in \rho$, we get $(x_0, x_m), (x_1, x_{n+1}) \in \rho$ which implies that $({}^0\Gamma_m \setminus {}^1\Gamma_m) \cap \rho \neq \emptyset$ and $({}^1\Gamma_n \setminus {}^0\Gamma_n) \cap \rho \neq \emptyset$ for all $m, n \in \{1, \dots, k-1\}$. By (1), $\Gamma_m = {}^0\Gamma_m \cup {}^1\Gamma_m \subseteq \rho$ for all $m \in \{1, \dots, k-1\}$. It is clearly that $\Gamma_0 \subseteq \rho$. Therefore, $\Theta_{k-1} \subseteq \rho$. □

The following theorem shows all possibilities of binary compatible relations with an ordered set in terms of the distance function.

Theorem 4.1.4. *Let ρ be a binary compatible relation with an ordered set $\mathbf{P}$. Then ρ is one of the following relations:*

$$\Theta_{r(\mathbf{P})}\ ,\ \Theta_{k-1} \cup {}^i\Gamma_k \text{ or } \Theta_{\mathrm{k-1}} \cup \mathrm{A_k}$$

for some $k \in \{1, \dots, r(\mathbf{P})\}$ and $i \in \{0, 1\}$ where $A_k \subseteq {}^0\Gamma_k \cap {}^1\Gamma_k$; especially, $A_2 = {}^0\Gamma_2 \cap {}^1\Gamma_2$.

Proof. Since all constant functions are order-preservings, ρ is refexive; so, $\Theta_0 \subseteq \rho$. Let $(x, y) \in \rho$ such that $d(x, y) \geq^* d(a, b)$ for all $(a, b) \in \rho$ and let $k = d(x, y)$. Then $\rho \subseteq \Theta_k$ and $\Gamma_k \cap \rho \neq \emptyset$. If $k = 0$ then $\rho = \Theta_0 = \Theta_0 \cup A_1$ where $A_1 = \emptyset \subseteq {}^0\Gamma_1 \cap {}^1\Gamma_1$. We consider the case that k is positive. Then, Lemma 4.1.3 (2) implies that $\Theta_{k-1} \subseteq \rho$.

If $({}^0\Gamma_k \setminus {}^1\Gamma_k) \cap \rho \neq \emptyset$ and $({}^1\Gamma_k \setminus {}^0\Gamma_k) \cap \rho \neq \emptyset$, Lemma 4.1.3 (1) implies that $\Theta_{k-1} \cup {}^0\Gamma_k \cup {}^1\Gamma_k \subseteq \rho$. Since $\Theta_{k-1} \cup {}^0\Gamma_k \cup {}^1\Gamma_k = \Theta_k$ and $\rho \subseteq \Theta_k$, we have that $\rho = \Theta_k = \Theta_k \cup A_{k+1}$ where $A_{k+1} = \emptyset \subseteq {}^0\Gamma_{k+1} \cap {}^1\Gamma_{k+1}$.

If $({}^0\Gamma_k \setminus {}^1\Gamma_k) \cap \rho \neq \emptyset$ and $({}^1\Gamma_k \setminus {}^0\Gamma_k) \cap \rho = \emptyset$, Lemma 4.1.3 (1) implies that $\Theta_{k-1} \cup {}^0\Gamma_k \subseteq \rho$ Since $\rho \subseteq \Theta_k = \Theta_{k-1} \cup {}^0\Gamma_k \cup ({}^1\Gamma_k \setminus {}^0\Gamma_k)$ and $({}^1\Gamma_k \setminus {}^0\Gamma_k) \cap \rho = \emptyset$, we have that $\rho \subseteq \Theta_{k-1} \cup {}^0\Gamma_k$; so, $\rho = \Theta_{k-1} \cup {}^0\Gamma_k$.

If $({}^0\Gamma_k \setminus {}^1\Gamma_k) \cap \rho = \emptyset$ and $({}^1\Gamma_k \setminus {}^0\Gamma_k) \cap \rho \neq \emptyset$ then similarly, $\rho = \Theta_{k-1} \cup {}^1\Gamma_k$.

If $({}^0\Gamma_k \setminus {}^1\Gamma_k) \cap \rho = \emptyset$ and $({}^1\Gamma_k \setminus {}^0\Gamma_k) \cap \rho = \emptyset$ then $\rho = \Theta_{k-1} \cup A_k$ where $A_k = \rho \cap ({}^0\Gamma_k \cap {}^1\Gamma_k)$.

Now, assume that $\rho = \Theta_1 \cup A_2$. If ${}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$ then $A_2 \subseteq {}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$; so, $A_2 = {}^0\Gamma_2 \cap {}^1\Gamma_2 = \emptyset$. Let ${}^0\Gamma_2 \cap {}^1\Gamma_2 \neq \emptyset$. By Corollary 3.1.7, Θ_1 is not compatible with $\mathbf{P}$. Hence, $A_2 \neq \emptyset$. Let $(a, b) \in A_2$. Since $A_2 \subseteq {}^0\Gamma_2 \cap {}^1\Gamma_2$, there exist $0, 1 \in P$ such that $a < 1 > b$ and $a > 0 < b$. Note that that the binary operation $f : P^2 \longrightarrow \{a, b, 0, 1\}$ which is defined as in the proof of Theorem 3.1.6 is an order-preserving with $f(0, 1) = a$ and $f(b, b) = b$. To show that ${}^0\Gamma_2 \cap {}^1\Gamma_2 \subseteq A_2$, let $(\tilde{a}, \tilde{b}) \in {}^0\Gamma_2 \cap {}^1\Gamma_2$ and let $\tilde{0}, \tilde{1} \in P$ with $\tilde{a} < \tilde{1} > \tilde{b}$ and $\tilde{a} > \tilde{0} < \tilde{b}$. Define a unary operation $g : \{a, b, 0, 1\} \longrightarrow \{\tilde{a}, \tilde{b}, \tilde{0}, \tilde{1}\}$ by $g(x) = \tilde{x}$ for all $x \in \{a, b, 0, 1\}$. It is clearly that g is an order-preserving. Since $g \circ f \in Pol(\Theta_1 \cup A_2)$ and $(0, b), (1, b) \in \Theta_1$, we have

$$(\tilde{a}, \tilde{b}) = (g \circ f(0, 1), g \circ f(b, b)) \in \Theta_1 \cup A_2$$

which implies that $(\tilde{a}, \tilde{b}) \in A_2$. Therefore, $A_2 = {}^0\Gamma_2 \cap {}^1\Gamma_2$. □

Figure 8 illustrates the lattice of all possibilities of binary compatible relations with an ordered set.

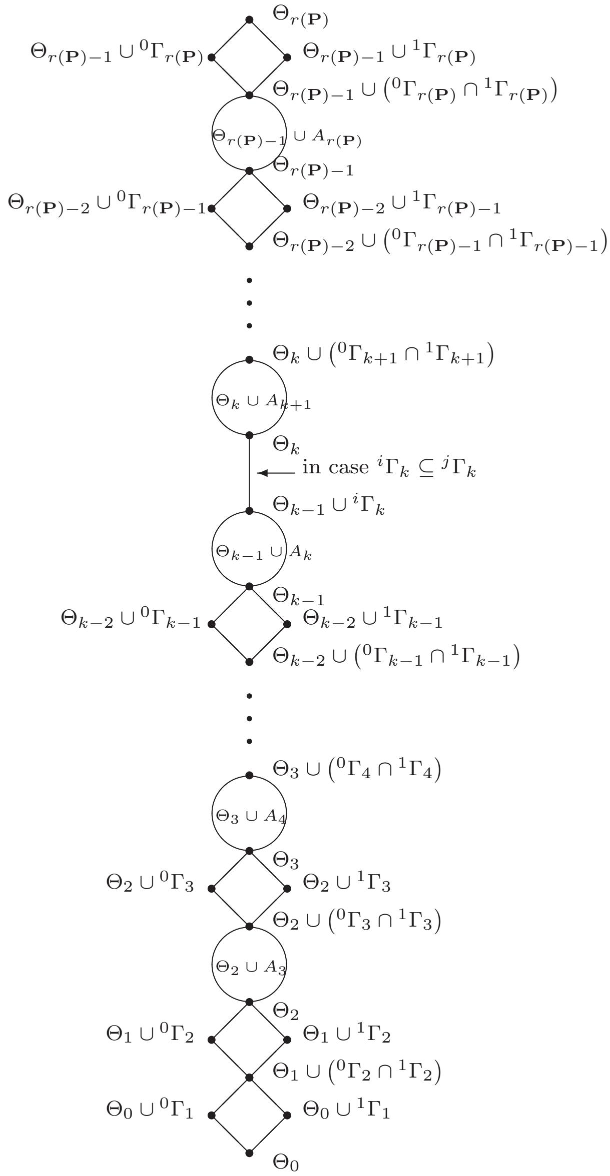

Figure 8. The lattice of all possibilities of binary compatible relations with an ordered set **P**

4.2 All Maximal Clones of a Tree-like

An ordered set **P** is a *tree* if the diagram of **P** is a tree graph. An ordered set **P** is a *tree-like* if there is a tree $\mathbf{T_P}$ such that **P** is obtained by replacing each interval $[q, q']$ which q' covers q in $\mathbf{T_P}$ by a lattice $L_{qq'}$ with a least element q and a greatest element q' so that any two such lattices intersect in at most one bound. Füredi Z., Rosenberg I.G. [4] showed that the class of all tree-likes is a large class of majority ordered sets that need not be lattices. It is clear that trees and lattices are tree-likes. J. Demetrovics and L. Rónyai [3] proved that a crown is not a majority ordered set. Hence, crowns are a class of orsered set which is not tree-like. We are going to show a class of ordered sets which are not a tree-like and a crown.

Note that every tree has no crowns as subordered sets and for each two elements a and b in a tree, all fences from a to b are minimal.

Proposition 4.2.1. *If* $\mathbf{P}$ *is a tree-like, then* ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ *for all* $k \in \{3, \ldots, r(\mathbf{P})\}$.

Proof. Suppose that there is $(a,b) \in {}^0\Gamma_k \cap {}^1\Gamma_k$ for some $k \in \{3, \ldots, r(\mathbf{P})\}$. Let ${}^0\mathbf{F}_a^b = \{a_0 < a_1 > a_2 < \ldots a_k\}$ and ${}^1\mathbf{F}_a^b = \{b_0 > b_1 < b_2 > \ldots b_k\}$. By the properties of $\mathbf{P}$, we may assume that $a_1, \ldots, a_{k-1}, b_1, \ldots, b_{k-1} \in \mathbf{T_P}$. Then $a_1 > a_0 = a = b_0 > b_1$ and a_{k-1} is comparable with b_{k-1}. The minimality of ${}^0\mathbf{F}_a^b$ and ${}^1\mathbf{F}_a^b$ implies that a_i is not comparable with b_j if $i \neq j$. Let $m = \min\{i \in \{2, 3, \ldots, k\} \mid a_i \text{ is comparable with } b_i\}$. Then $\mathbf{T_P}$ contains the crown $\mathbf{C}_m = (\{a_1, \ldots, a_m, b_1, \ldots, b_m\}; \leq\restriction_{C_m})$ as a subordered set, a contradiction. □

Now, we will give a characterization of all finite connected ordered sets $\mathbf{P}$ satisfying ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ for all $k \in \{3, \ldots, r(\mathbf{P})\}$ by a graph of its subordered sets which will define as follows.

Definition 4.2.2. Let $\mathbf{P}$ be an ordered set and $a, b \in P$. A subordered set ${}^0\mathbf{F}_a^b \cup {}^1\mathbf{F}_a^b$ of $\mathbf{P}$ is called a *skew-diamond (from a to b)* and denoted by $\mathbf{D}_a^b$ if $(a,b) \in {}^0\Gamma_k \cap {}^1\Gamma_k$ for some $k \in \{2, \ldots, r(\mathbf{P})\}$.

Note that a skew-diamond D_a^b with $d(a,b) = 2$ is a diamond lattice. If $d(a,b) > 2$, a skew-diamond D_a^b is called a *proper skew-diamond.*

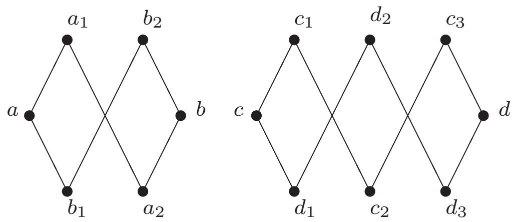

Figure 9. Examples of skew-diamonds

Next theorem is a characterization of a finite connected ordered set $\mathbf{P}$ satisfying ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ for all $k \in \{3, \ldots, r(\mathbf{P})\}$. It can be proved directly by the definition.

Theorem 4.2.3. *Let* $\mathbf{P}$ *be an ordered set. The followings are equivalent:*

1. $\mathbf{P}$ *satisfy* ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ *for all* $k \in \{3, \ldots, r(\mathbf{P})\}$.
2. $\mathbf{P}$ *has no proper skew-diamonds as subordered sets.* □

Proposition 4.2.1 and Theorem 4.2.3 imply that an ordered set is not tree-like if the ordered set has a proper skew-diamond as a subordered set.

Theorem 4.1.1 implies that all maximal clones of a majority ordered set are clones preserving central relations with arity only 2. Hence, by Theorem 4.1.4, we obtain all possibilities of binary compatible relations with an ordered set. Now, we are going to characterize all binary compatible relations with a tree-like.

Theorem 4.2.4. *Let* $\mathbf{P}$ *be a tree-like. A binary relation* ρ *is compatible with* $\mathbf{P}$ *if and only if* ρ *is one of the following relations:*

$$\Theta_j,\ \Theta_1 \cup \left({}^0\Gamma_2 \cap {}^1\Gamma_2\right) \text{ or } \Theta_{\mathrm{k}-1} \cup {}^{\mathrm{i}}\Gamma_{\mathrm{k}}$$

for some $j \in \{0, \ldots, r(\mathbf{P})\}$ *with* $j \neq 1$*,* $k \in \{1, \ldots, r(\mathbf{P})\}$ *and* $i \in \{0, 1\}$*.*

Proof. If ρ is a binary compatible relation with $\mathbf{P}$, ρ is one of those relations in Theorem 4.1.4. Since $\mathbf{P}$ is tree-like and by Proposition 4.2.1, ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ for all $k \in \{3, \ldots, r(\mathbf{P})\}$; and so, $A_k = \emptyset$ for all $k \in \{3, \ldots, r(\mathbf{P})\}$. Hence, ρ is a relation in this theorem. Conversely, we will show that $Pol(\Theta_{k-1} \cup {}^0\Gamma_k)$ contains the monotone clone of $\mathbf{P}$. Let f be an n-ary order-preserving defined on P and $(x_i, y_i) \in \Theta_{k-1} \cup {}^0\Gamma_k$ for all $i \in \{1, \ldots, n\}$. Let $d(x_i, y_i) = m_i$ and $\mathbf{F}_i = (\{a_i^0, a_i^1, \ldots, a_i^{m_i}\}; \leq)$ is a minimum size fence from x_i to y_i for all $i \in \{1, \ldots, n\}$. Then $m_i \leq^* k$ for all $i \in \{1, \ldots, n\}$. We may assume that there is an $r \in \{1, \ldots, n\}$ such that $a_i^0 \leq a_i^1$ for all $i \in \{1, \ldots, r\}$ and $a_i^0 \geq a_i^1$ for all $i \in \{r+1, \ldots, n\}$. Since f is an order-preserving,

$$\begin{aligned} f(x_1, \ldots, x_n) &= f(a_1^0, \ldots, a_n^0) \\ &\leq f(a_1^1, \ldots, a_r^1, a_{r+1}^0, \ldots, a_n^0) \\ &\geq f(a_1^2, \ldots, a_r^2, a_{r+1}^1, \ldots, a_n^1) \\ &\leq f(a_1^3, \ldots, a_r^3, a_{r+1}^2, \ldots, a_n^2) \\ &\vdots \\ &\ f(a_1^k, \ldots, a_r^k, a_{r+1}^{k-1}, \ldots, a_n^{k-1}) \\ &= f(y_1, \ldots, y_n) \end{aligned}$$

where $a_i^l = y_i$ for all $l \geq^* m_i$. Hence, either there is an up fence from $f(x_1, \ldots, x_n)$ to $f(y_1, \ldots, y_n)$ with the reach k or $d(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) <^* k$ Therefore, $(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \Theta_{k-1} \cup {}^0\Gamma_k$. Hence, $Pol(\Theta_{k-1} \cup {}^0\Gamma_k)$ contains the monotone clone of $\mathbf{P}$; that is, $\Theta_{k-1} \cup {}^0\Gamma_k$ is compatible with $\mathbf{P}$. By dually, $\Theta_{k-1} \cup {}^1\Gamma_k$ is compatible with $\mathbf{P}$. It follow that $\Theta_{k-1} \cup ({}^0\Gamma_k \cap {}^1\Gamma_k) = (\Theta_{k-1} \cup {}^0\Gamma_k) \cap (\Theta_{k-1} \cup {}^1\Gamma_k)$ is compatible with $\mathbf{P}$. Hence, $\Theta_1 \cup ({}^0\Gamma_2 \cap {}^1\Gamma_2)$ is compatible with $\mathbf{P}$. By Proposition 4.2.1, ${}^0\Gamma_k \cap {}^1\Gamma_k = \emptyset$ for all $k \in \{3, \ldots, r(\mathbf{P})\}$. Thus, Θ_j is compatible with $\mathbf{P}$ for all $j \in \{2, \ldots, r(\mathbf{P}) - 1\}$. Since $Pol(\Theta_0) = Pol(\Theta_{r(\mathbf{P})}) = O(P)$, Θ_j is compatible with $\mathbf{P}$ for all $j \in \{0, \ldots, r(\mathbf{P})\}$ with $j \neq 1$. □

The lattice of all binary compatible relations with a tree-like is illustrated in Figure 10.

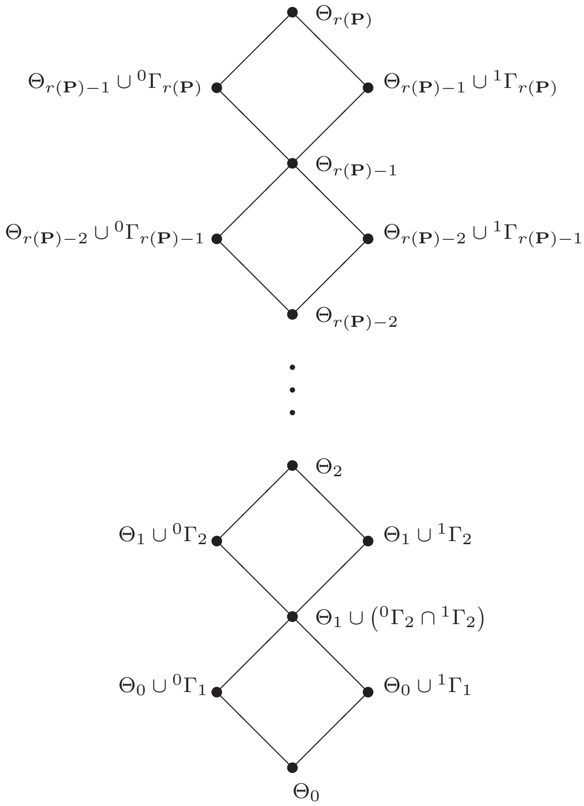

Figure 10. The lattice of all binary compatible relations with a tree-like $\mathbf{P}$

One can see that the lattice of all binary compatible relations with a tree-like $\mathbf{P}$ is isomorphic to the lattice of all binary compatible relations with a fence $\mathbf{T}$ if $r(\mathbf{P}) = r(\mathbf{F})$.

By the proof of the above theorem, we have the following corollary.

Corollary 4.2.5. *Let* $\mathbf{P}$ *be an ordered set. Then the relations*

$$\Theta_{k-1} \cup {}^i\Gamma_k \text{ and } \Theta_{k-1} \cup \left({}^0\Gamma_k \cap {}^1\Gamma_k\right)$$

are compatible with $\mathbf{P}$ *for all* $k \in \{1, \ldots, r(\mathbf{P})\}$ *and* $i \in \{0, 1\}$. □

Even though all relations in Theorem 4.2.4 are binary compatible with a tree-like, but some of these relations are not central; for instances, Θ_0 and $\Theta_0 \cup {}^0\Gamma_1$ are not central since $C_{\Theta_0} = \emptyset$ and $\Theta_0 \cup {}^0\Gamma_1$ is not symmetric. We close our story by giving a characterization of all central compatible relations with a tree-like via the following lemmata.

Lemma 4.2.6. *If* $\mathbf{T}$ *be a tree, the center of the relation* $\Theta_{\lceil \frac{r(\mathbf{T})}{2} \rceil}$ *is not empty.*

Proof. Suppose that $r(\mathbf{T}) = 2m$ for some positive integer m. Let $a, b \in T$ with $d(a, b) = r(\mathbf{T})$. We may assume that $(\{a = a_0, \ldots, a_{2m} = b\}; \leq)$ is a fence from a to b such that $a_{m-1} < a_m > a_{m+1}$ and a_m is a minimum element of $\{a_{m-1}, a_{m+1}\}^u$ in $\mathbf{T}$.

Claim 1: If $(a_m, x) \in {}^1\Gamma_r$ then $r \leq^* m$ and if $(a_m, y) \in {}^0\Gamma_s$ then $s \leq^* m{+}1$. To prove Claim 1, let $\{a_m = x_0 > x_1 < x_2 > \ldots x_r = x\}$ and $\{a_m = y_0 < y_1 >$

$y_2 < \ldots y_s = y\}$ be fences from a_m to x and a_m to y, respectively for some $x, y \in T$. By choosing a_m and property of $\mathbf{T}$, x_1 is not comparable with a_{m-1} or a_{m+1}. Similarly, y_2 is not comparable with a_{m-1} or a_{m+1}. We may assume that x_1 and y_2 are not comparable with a_{m-1}. By the property of $\mathbf{T}$ again, $(\{a_0, \ldots, a_{m-1}, a_m, x_1, x_2, \ldots x_r = x\}; \leq)$ and $(\{a_0, \ldots, a_{m-1}, y_1, y_2, \ldots y_s = y\}; \leq)$ are fences from a_0 to x and a_0 to y, respectively. Hence, $m + r = d(a_0, x) \leq^* 2m$ and $m + s - 1 = d(a_0, y) \leq^* 2m$; that is, $r \leq^* m$ and $s \leq^* m + 1$.

By Claim 1, if there is no $b \in T$ such that $(a_m, b) \in {}^0\Gamma_{m+1}$ then $a_m \in C_{\Theta_m}$. Suppose that there is $b \in T$ such that $(a_m, b) \in {}^0\Gamma_{m+1}$ and let $\{a_m = b_0 < b_1 > b_2 < \ldots b_{m+1} = b\}$.

Claim 2: If $(b_1, x) \in {}^1\Gamma_r$ then $r \leq^* m$ and if $(b_1, y) \in {}^0\Gamma_s$ then $s \leq^* m + 1$. By the same argument as the proof of Claim 1, we conclude that if $(b_1, y) \in {}^0\Gamma_s$ then $s \leq^* m + 1$. Let $\{b_1 = x_0 > x_1 < x_2 > \ldots x_r = x\}$ be a fence from b_1 to x for some $x \in T$. If x_1 is comparable with a_{m-1} and a_{m+1} then x_1 is not comparable with b_2 which implies by the property of $\mathbf{T}$ that $(\{x_r, \ldots, x_1, x_0, b_2, \ldots, b_{m+1}\}; \leq)$ is a fence; and so, $r + m = d(x_r, b_{m+1}) \leq^* 2m$; that is, $r \leq^* m$. Assume that x_1 is not comparable with a_{m-1}. By the property of $\mathbf{T}$, $(\{a_0, \ldots, a_{m-1}, x_0, x_1, x_2, \ldots x_r = x\}; \leq)$ is a fence; and so, $r + m = d(a_0, x_r) \leq^* 2m$; that is, $r \leq^* m$.

By Claim 2, if there is no $b \in T$ such that $(b_1, b) \in {}^0\Gamma_{m+1}$ then $a_m \in C_{\Theta_m}$. Suppose that there is $b \in T$ such that $(b_1, b) \in {}^0\Gamma_{m+1}$. We continue this process. Since T is finite, the process will be end. So, there is $c \in T$ such that if $(b, x) \in {}^1\Gamma_r$ then $r \leq^* m$ and if $(b, y) \in {}^0\Gamma_s$ then $s \leq^* m$. Hence, $c \in C_{\Theta_m}$. Therefore, we can conclude that if $\mathbf{T}$ is a tree with $r(\mathbf{F}) = 2m$ for some positive integer m then $C_{\Theta_m} \neq \emptyset$.

Next, we suppose that $r(\mathbf{T}) = 2m - 1$ for some positive integer m. Let $a, b \in T$ with $r(\mathbf{F}_a^b) = 2m - 1$. By dually, we may assume that $\mathbf{F}_a^b$ is up fence. Let $T' = T \cup \{x\}$ for some $x \notin T$ and let $\leq' = \leq \cup \{(y, x) \mid y \leq a \text{ or } y = x\}$. Then $\mathbf{T}' = (T'; \leq')$ is a tree with $r(\mathbf{T}') = 2m$. By the above fact, there is $c \in T'$ such that $d'(c, y) \leq^* m$ for all $y \in T'$ where d' is the distance function on $T' \times T'$. Note that $d = d' \lfloor_{T \times T}$ and $c \neq x$. Hence, $d(c, y) \leq^* m$ for all $y \in T$. Therefore, $c \in C_{\Theta_m}$. □

We see that all relations in Theorem 4.2.4 contain Θ_0. Hence, those relations are reflexive. It is interesting to see which all relations in Theorem 4.2.4 are symmetric and the center is neither empty nor P.

Lemma 4.2.7. *Let* $\mathbf{P}$ *be a tree-like.*

1. *The relation* Θ_j *is symmetric for all* $j \in \{0, \ldots, r(\mathbf{P})\}$.
2. *For* $i \in \{0, 1\}$ *and* $k \in \{1, \ldots, r(\mathbf{P})\}$, *the relation* $\Theta_{k-1} \cup {}^i\Gamma_k$ *is symmetric if and only if* k *is an even number.*

Proof. (1) We see that $d(a, b) = d(b, a)$ for all $a, b \in P$. Hence, Θ_j is symmetric for all $j \in \{0, \ldots, r(\mathbf{P})\}$.

(2) Let $i \in \{0,1\}$ and $k \in \{1,\ldots,r(\mathbf{P})\}$. If k is even, it is clear that ${}^i\Gamma_k$ is symmetric; and so, $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric. Suppose that $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric and k is odd. Let $(a,b) \in {}^i\Gamma_k$. Since $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric, $(b,a) \in {}^i\Gamma_k$. If $k=1$, the anti-symmetricity of the order implies that $a=b$, a contradiction. Assume that $k \geq^* 3$. Since $(b,a) \in {}^i\Gamma_k$ and k is odd, $(a,b) \in {}^j\Gamma_k$ for $\{i,j\} = \{0,1\}$. Thus, $(a,b) \in {}^i\Gamma_k \cap {}^j\Gamma_k$ which contradicts to Proposition 4.2.1. Therefore, k is even. □

Follows from Theorem 3.2.10, the center of the relation $\Theta_{\lceil \frac{r(\mathbf{P})}{2} \rceil -1} \cup {}^i\Gamma_{\lceil \frac{r(\mathbf{P})}{2} \rceil}$ is not necessary empty if $\mathbf{P}$ is a tree-like. The following lemma shows necessary conditions for a positive number k such that the center of the relation $\Theta_{k-1} \cup {}^i\Gamma_k$ is empty or not for all $i \in \{0,1\}$.

Lemma 4.2.8. *Let* $\mathbf{P}$ *be a tree-like and* $k \in \{0,\ldots,r(\mathbf{P})\}$.

1. *The center of the relation* Θ_k *is not empty if and only if* $k \geq^* \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$.
2. *For* $i \in \{0,1\}$, *the center of the relation* $\Theta_{k-1} \cup {}^i\Gamma_k$ *is not empty if* $k >^* \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$, *and the center of the relation* $\Theta_{k-1} \cup {}^i\Gamma_k$ *is empty if* $k <^* \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$.

Proof. (1) Suppose that $c \in C_{\Theta_k}$ with $k <^* \frac{r(\mathbf{P})}{2}$ and let $a,b \in P$ with $d(a,b) = r(\mathbf{P})$. Then

$$r(\mathbf{P}) = d(a,b) \leq^* d(a,c) + d(c,b) \leq^* 2k <^* r(\mathbf{P}),$$

a contradiction. Hence, $k \geq^* \frac{r(\mathbf{P})}{2}$; and so, $k \geq^* \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$. Conversely, suppose that $k \geq^* \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$. By Lemma 4.2.6, there is $c \in \mathbf{T_P}$ such that $d(c,b) \leq^* \left\lceil \frac{r(\mathbf{T_P})}{2} \right\rceil = \left\lceil \frac{r(\mathbf{P})}{2} \right\rceil \leq^* k$ for all $b \in \mathbf{T_P}$. Let $a \in \mathbf{P}$ and let $\{a = a_0 < a_1 > \ldots, a_m = c\}$ be a minimum size fence from a to c. Assume that $a_1,\ldots,a_m \in \mathbf{T_P}$. By the properties of $\mathbf{P}$, there is $a'_0 \in \mathbf{T_P}$ such that $a \in L_{a'_0 a_1}$. Therefore, $d(a,c) = d(a'_0,c) \leq^* k$. It follows that $c \in C_{\Theta_k}$.

(2) If $k-1 \geq^* r(\mathbf{P})/2$ then (1) and $\Theta_{k-1} \subseteq \Theta_{k-1} \cup {}^i\Gamma_k$ imply that the center of $\Theta_{k-1} \cup {}^i\Gamma_k$ is not empty. Also, if $k <^* r(\mathbf{P})/2$ then (1) and $\Theta_{k-1} \cup {}^i\Gamma_k \subseteq \Theta_k$ imply that the center of $\Theta_{k-1} \cup {}^i\Gamma_k$ is empty. □

The following theorem gives a characterization of all binary central compatible relations with a tree-like.

Theorem 4.2.9. *Let* $\mathbf{P}$ *be a tree-like.*

1. Θ_j *is central if and only if* $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil \leq^* j <^* r(\mathbf{P})$.
2. *For* $i \in \{0,1\}$, $\Theta_{k-1} \cup {}^i\Gamma_k$ *is central if and only if* k *is an even number satisfying one of the followings:*
   (a) $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil = k$ *and there exists* $a \in C_{\Theta_k}$ *such that* $(a,b) \in {}^i\Gamma_k$ *for all* $b \in P$ *with* $(a,b) \in \Gamma_k$,
   (b) $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil <^* k <^* r(\mathbf{P})$ *or*

(c) $r(\mathbf{P}) = k$ *and* $\emptyset \neq {}^i\Gamma_k \neq \Gamma_k$.

3. $\Theta_1 \cup ({}^0\Gamma_2 \cap {}^1\Gamma_2)$ *is central if and only if either*

(a) $r(\mathbf{P}) = 2$ *and* $({}^0\Gamma_2 \cap {}^1\Gamma_2) \neq \Gamma_2$, *or*

(b) $r(\mathbf{P}) \in \{3, 4\}$ *and there exists* $a \in C_{\Theta_2}$ *such that* $(a, b) \in {}^0\Gamma_2 \cap {}^1\Gamma_2$ *for all* $b \in P$ *with* $(a, b) \in \Gamma_2$.

Proof. (1) It is clearly that $\Theta_{r(\mathbf{F})}$ is not central. Lemma 4.2.7 (1) and Lemma 4.2.8 (1) imply that Θ_j is central if and only if j is a positive integer with $\left\lceil \frac{r(\mathbf{F})}{2} \right\rceil \leq^* j \leq^* r(\mathbf{F}) - 1$.

(2) ($\Rightarrow$) Suppose that $\Theta_{k-1} \cup {}^i\Gamma_k$ is central. Then $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric. Lemma 4.2.7 (2) implies that k is an even number. By Lemma 4.2.8 (2), $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil \leq^* k \leq^* r(\mathbf{P})$. If $k = r(\mathbf{P})$ but ${}^i\Gamma_{r(\mathbf{P})} = \Gamma_{r(\mathbf{P})}$ then $\Theta_{r(\mathbf{P})-1} \cup {}^i\Gamma_{r(\mathbf{P})} = \Theta_{r(\mathbf{P})}$ is not central, a contradiction. Hence, if $r(\mathbf{P}) = k$ then ${}^i\Gamma_k \neq \Gamma_k$. Suppose that $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil = k$. Let $a \in C_{\Theta_{k-1} \cup {}^i\Gamma_k}$. Then $(a, b) \in {}^i\Gamma_k$ for all $b \in P$ with $(a, b) \in \Gamma_k$. We see that $a \in C_{\Theta_{k-1} \cup {}^i\Gamma_k} \subseteq C_{\Theta_k}$.

($\Leftarrow$) Let k be an even number. Then Lemma 4.2.7 (2) implies that $\Theta_{k-1} \cup {}^i\Gamma_k$ is symmetric. By Lemma 4.2.8 (2), if $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil <^* k \leq^* r(\mathbf{P})$ the set $C_{\Theta_{k-1} \cup {}^i\Gamma_k}$ is not empty. We see that if $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil <^* k <^* r(\mathbf{P})$ then $C_{\Theta_{k-1} \cup {}^i\Gamma_k} \neq P$; and so $\Theta_{k-1} \cup {}^i\Gamma_k$ is central. If ${}^i\Gamma_{r(\mathbf{P})} \neq \Gamma_{r(\mathbf{P})}$ then there are $a, b \in P$ such that $d(a, b) = r(\mathbf{P})$ but $(a, b) \notin {}^i\Gamma_{r(\mathbf{P})}$; and so $a \notin C_{\Theta_{r(\mathbf{P})-1} \cup {}^i\Gamma_{r(\mathbf{P})}}$ and $\Theta_{r(\mathbf{P})-1} \cup {}^i\Gamma_{r(\mathbf{P})}$ is central. Suppose that $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil = k$ and there exists $a \in C_{\Theta_k}$ such that $(a, b) \in {}^i\Gamma_k$ for all $b \in P$ with $(a, b) \in \Gamma_k$. It suffices to show that $a \in C_{\Theta_{k-1} \cup {}^i\Gamma_k}$. Let $b \in P$. Since $a \in C_{\Theta_k}$, we get $d(a, b) \leq^* k$. If $d(a, b) = k$ then by the assumption, we have that $(a, b) \in {}^i\Gamma_k$. Therefore, $(a, b) \in \Theta_{k-1} \cup {}^i\Gamma_k$. Hence, $a \in C_{\Theta_{k-1} \cup {}^i\Gamma_k}$.

(3) It is clear that $\Theta_1 \cup ({}^0\Gamma_2 \cap {}^1\Gamma_2)$ is symmetric. If $r(\mathbf{P}) \geq^* 5$ then $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil = 3$; and so $C_{\Theta_1 \cup ({}^0\Gamma_2 \cap {}^1\Gamma_2)} \subseteq C_{\Theta_2} = \emptyset$. Similarly with (2). $\square$

Figure 11 -13 illustrate the lattice of all central compatible relations with a tree-like $\mathbf{P}$ with $2 \leq^* r(\mathbf{P}) <^* 5$.

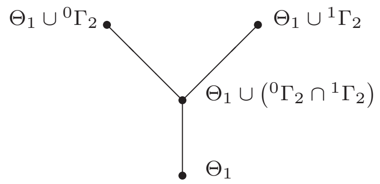

Figure 11. The lattice of all central compatible relations with a tree-like $\mathbf{P}$ such that $r(\mathbf{P}) = 2$

Figure 11 shows the lattice of all central compatible relations with a tree-like **P** such that $r(\mathbf{P}) = 2$ and these relations are distinct central.

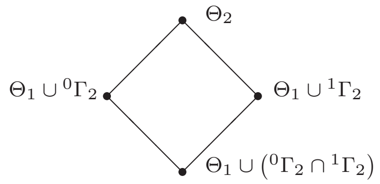

Figure 12. The lattice of all central compatible relations with a tree-like **P** such that $r(\mathbf{P}) = 3$

Figure 12 shows the lattice of all central compatible relations with a tree-like **P** such that $r(\mathbf{P}) = 3$ and these relations are distinct central.

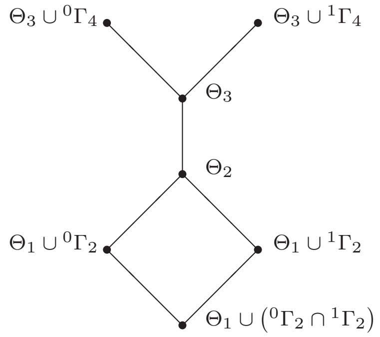

Figure 13. The lattice of all central compatible relations with a tree-like **P** such that $r(\mathbf{P}) = 4$

Figure 13 shows the lattice of all central compatible relations with a tree-like **P** such that $r(\mathbf{P}) = 4$ and these relations are distinct central.

Figure 14 - 17 illustrate the lattice of all central compatible relations with a tree-like **P** with $r(\mathbf{P}) \geq^* 5$.

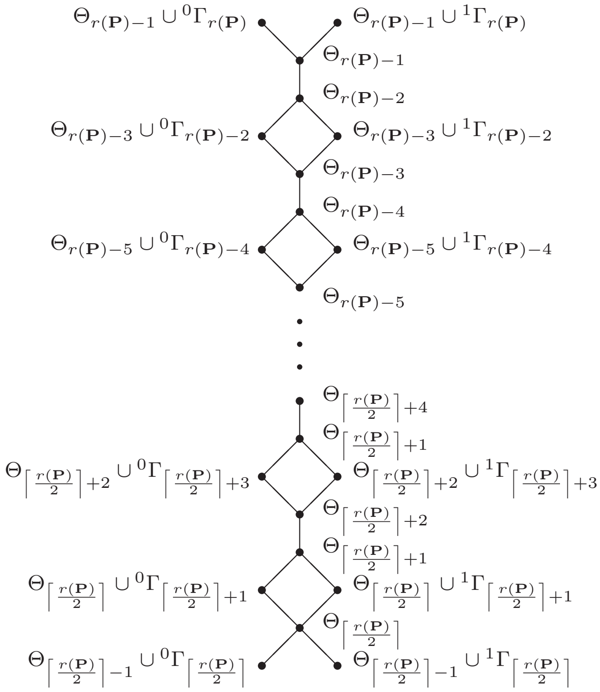

Figure 14. The lattice of all central compatible relations with a fence $\mathbf{P}$ such that $r(\mathbf{P}) \equiv 0 \pmod 4$

In Figure 14, the numbers $r(\mathbf{P})$ and $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$ are even; that is, $r(\mathbf{P}) \equiv 0 \pmod 4$ and both $\Theta_{\lceil \frac{r(\mathbf{P})}{2} \rceil-1} \cup {}^0\Gamma_{\lceil \frac{r(\mathbf{P})}{2} \rceil}$ and $\Theta_{\lceil \frac{r(\mathbf{P})}{2} \rceil-1} \cup {}^0\Gamma_{\lceil \frac{r(\mathbf{P})}{2} \rceil}$ are central and both ${}^0\Gamma_{r(\mathbf{P})}$ and ${}^1\Gamma_{r(\mathbf{P})}$ are not empty and $\Gamma_{r(\mathbf{P})}$. By the properties of a fence, we see that Figure 14 is improved to Figure 4.

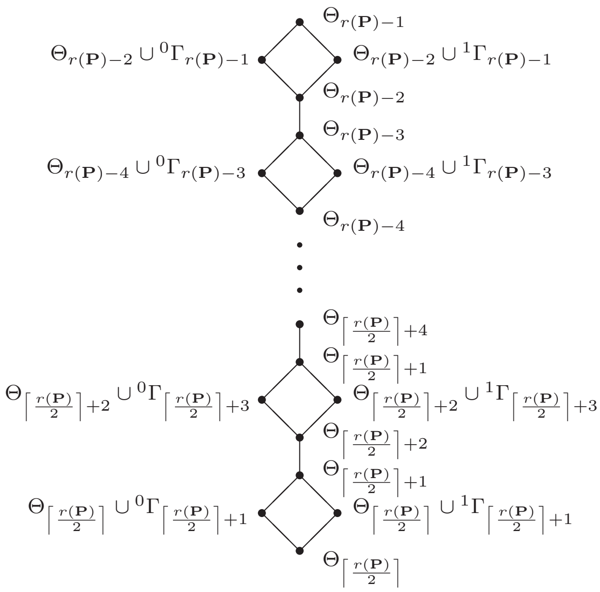

Figure 15. The lattice of all central compatible relations with a tree-like $\mathbf{P}$ such that $r(\mathbf{P}) \equiv 1 \pmod 4$

In Figure 15, the numbers $r(\mathbf{P})$ and $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$ are odd; that is, $r(\mathbf{P}) \equiv 1 \pmod 4$. Figure 15 is look like Figure 5 for the same reaches.

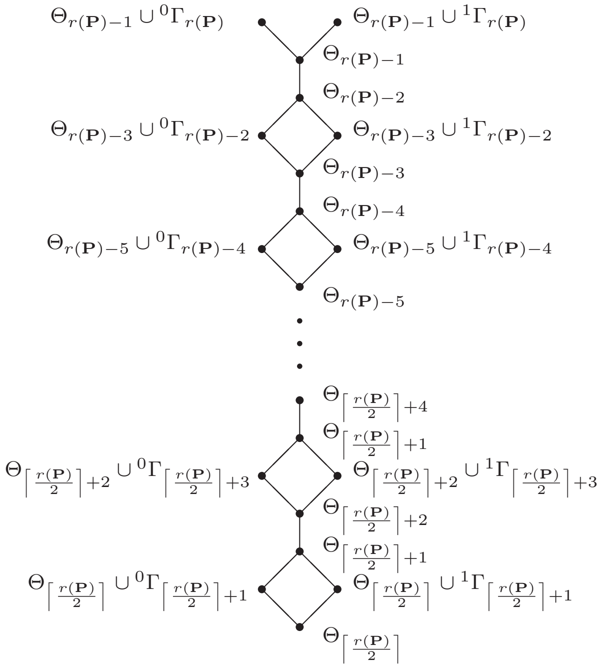

Figure 16. The lattice of all central compatible relations with a fence **F** such that $r(\mathbf{P}) \equiv 2 \pmod 4$

In Figure 16, the number $r(\mathbf{P})$ is even and the number $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$ is odd; that is, $r(\mathbf{P}) \equiv 2 \pmod 4$ and both ${}^0\Gamma_{r(\mathbf{P})}$ and ${}^1\Gamma_{r(\mathbf{P})}$ are not empty and $\Gamma_{r(\mathbf{P})}$. For a fence $\mathbf{F}$ with $r(\mathbf{F}) \equiv 2 \pmod 4$, we see that ${}^0\Gamma_{r(\mathbf{P})}$ and ${}^1\Gamma_{r(\mathbf{P})}$ are either empty or $\Gamma_{r(\mathbf{P})}$. So, Figure 16 is improved to Figure 6

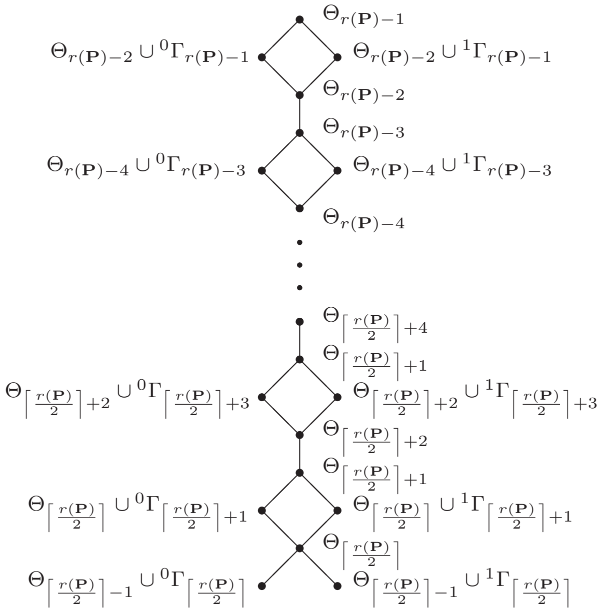

Figure 17. The lattice of all central compatible relations with a fence **P** such that $r(\mathbf{P}) \equiv 3 \pmod 4$

In Figure 17, the number $r(\mathbf{P})$ is odd and the number $\left\lceil \frac{r(\mathbf{P})}{2} \right\rceil$ is even; that is, $r(\mathbf{P}) \equiv 3 \pmod 4$ and both $\Theta_{\lceil \frac{r(\mathbf{P})}{2} \rceil - 1} \cup {}^{0}\Gamma_{\lceil \frac{r(\mathbf{P})}{2} \rceil}$ and $\Theta_{\lceil \frac{r(\mathbf{P})}{2} \rceil - 1} \cup {}^{1}\Gamma_{\lceil \frac{r(\mathbf{P})}{2} \rceil}$ are central. We see that Figure 17 is look like Figure 7.

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APPENDIX

List of Symbols

$Pol(\rho)$	the set of all operations preserving a relation ρ
$\mathbf{F}_a^b$	a minimum size fence from a to b
${}^0\mathbf{F}_a^b$	a minimum size up fence from a to b
${}^1\mathbf{F}_a^b$	a minimum size down fence from a to b
$r(\mathbf{P})$	the maximum of the set of all integers $\|\mathbf{F}_a^b\| - 1$ for some $a, b \in P$
Γ_k	the set of all $(a, b) \in P \times P$ such that $d(a, b) = k$
${}^0\Gamma_k$	the set of all $(a, b) \in \Gamma_k$ such that ${}^0\mathbf{F}_a^b$ sxists
${}^1\Gamma_k$	the set of all $(a, b) \in \Gamma_k$ such that ${}^1\mathbf{F}_a^b$ sxists
Θ_k	the set of all $(a, b) \in P \times P$ such that $d(a, b) \leq k$

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