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Original Article

Existence of Moore-Penrose inverses in rings with involution

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Abstract

We give necessary and sufficient conditions for the existence of the Moore-Penrose inverse of an element in a ring with involution. If R is a ring with involution, we also investigate the existence of the Moore-Penrose inverse of the product $x_1x_2\cdots x_n$ where $x_1, x_2, \ldots, x_n$ are Moore-Penrose invertible elements of R.

Keywords: Moore-Penrose inverse, ring with involution

1. Introduction

The concept of Moore-Penrose inverses originally began on the work of E. H. Moore in the decade of 1910-1920. Moore studied the "general reciprocal" of any matrix and applied it to solve systems of linear equations (Moore, 1920). It was later rediscovered by R. Penrose in 1955 (Penrose, 1955), and is nowadays called the "Moore-Penrose inverse".

At the present time, Moore-Penrose inverses have been studied extensively in complex matrices (Baksalary & Trenkler, 2008; Ben-Israel & Greville, 2003; Cheng & Tian, 2003), linear operators on Banach or Hilbert spaces (Djord jević, 2007; Djordjević & Koliha, 2007), C^*-algebras (Harte & Mbekhta, 1992; Koliha, 1999) and have also been extended to any rings with involution (Koliha, Djordjević, & Cvetković, 2007; Mosić, & Djordjević, 2009; Mosić, Djordjević, & Koliha, 2009).

The purpose of this paper is two-fold. Firstly, we give necessary and sufficient conditions for an element in a ring with involution to be Moore-Penrose invertible. This generalizes the result of Kholiha *et al.* (Koliha, Djordjević, & Cvetković, 2007). Secondly, we investigate the existence of the Moore-Penrose inverse of the product $x_1x_2\cdots x_n$ where $x_1, x_2, \ldots, x_n$ are Moore-Penrose invertible elements in any ring with involution. For a ring R with involution $*$ and $a \in R$, let $a^\dagger$ denote the Moore-Penrose inverse of a (if it exists). We prove that if a is Moore-Penrose invertible and a normal element, i.e. $aa^* = a^*a$, then the product $x_1x_2\cdots x_n$ is always Moore-Penrose invertible for $x_1, x_2, \ldots, x_n \in \{a, a^*, a^\dagger, (a^\dagger)^*\}$. We also prove that if a is an EP element, i.e. a is Moore-Penrose invertible and $aa^\dagger = a^\dagger a$, then x^n is Moore-Penrose invertible for any $x \in \{a, a^*, a^\dagger, (a^\dagger)^*\}$ and for all $n \in N$. Finally, we show that if a is Moore-Penrose invertible, then $(aa^*)^n$, $(a^*a)^n$, $(a^*a^\dagger aa)^n$, $a(a^*a)^n$, and $a^*(aa^*)^n$ are also Moore-Penrose invertible for all $n \in N$.

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2. Preliminaries

Throughout this paper, R is an associative ring. An element $a \in R$, is *group invertible* if there is an element $a^\# \in R$ such that

$$aa^\# a = a, \quad a^\# aa^\# = a^\# \quad \text{and} \quad aa^\# = a^\# a.$$

$a^{\#}$ is the group inverse of a and it is uniquely determined by the above equations.We denote $R^{\#}$ the set of all group invertible elements of R . An *involution* $a \mapsto a^*$ in a ring R is an anti-isomorphism of degree 2, that is,

$(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^* a^*$.

Let R be a ring with involution $*$. An element $a \in R$ is *Moore-Penrose invertible* (or MP-invertible) if there is an element $a^\dagger \in R$ such that

$aa^\dagger a = a$, $a^\dagger a a^\dagger = a^\dagger$, $(aa^\dagger)^* = aa^\dagger$ and $(a^\dagger a)^* = a^\dagger a.$

$a^\dagger$ is the Moore-Penrose inverse of a and it is unique. We denote $R^\dagger$ the set of all Moore-Penrose invertible elements of R . An element $a \in R$ is *left *-cancellable* if $a^* ax = a^* ay$ implies $ax = ay$. It is *right *-cancellable* if $xaa^* = yaa^*$ implies $xa = ya$ and it is **-cancellable* if it is both left and right *-cancellable.

An element $a \in R$ satisfying $aa^* = a^* a$ is called *normal*. It is called *Hermitian* if $a = a^*$ and it is *EP* if $a \in R^{\#} \cap R^\dagger$ and $a^{\#} = a^\dagger$ .An element $p \in R$ is a *projection* if p is idempotent and Hermitian, that is, $p^2 = p = p^*$.

For a ring R , we denote $\mathbf{M}_n(R)$ the ring of $n \times n$ matrices over R with the usual matrix addition and multiplication. The ring of integers modulo a positive integer n is denoted by $\mathbf{Z}_n$.

We will use the following lemmas as characterizations of normal and EP elements.

Lemma 2.1 (Mosić, Djordjević, & Koliha, 2009) Let $a \in R^\dagger$. Then a is EP if and only if $aa^\dagger = a^\dagger a$.

Lemma 2.2 (Mosić & Djordjević, 2009) Let $a \in R^\dagger$. Then a is normal if and only if $aa^\dagger = a^\dagger a$ and $a^* a^\dagger = a^\dagger a^*$.

It should be noted that every Hermitian element is a normal element and every normal element in $R^\dagger$ is an EP element. The following theorem recalls some basic properties of the Moore-Penrose inverse.

Theorem 2.3 (Mosić & Djordjević, 2009) For any $a \in R^\dagger$, the following statements hold.

(1) $(a^\dagger)^\dagger = a$.

(2) $(a^*)^\dagger = (a^\dagger)^*$.

(3) $(a^* a)^\dagger = a^\dagger (a^\dagger)^*$.

(4) $(aa^*)^\dagger = (a^\dagger)^* a^\dagger$.

(5) $a^* = a^\dagger a a^* = a^* a a^\dagger$.

(6) $a^\dagger = (a^* a)^\dagger a^* = a^* (aa^*)^\dagger$.

(7) $(a^*)^\dagger = a(a^* a)^\dagger = (aa^*)^\dagger a$.

3. Main Results

Let R be a ring with involution $*$. We will give necessary and sufficient conditions for an element of R to be MP-invertible. We start with the following result.

Proposition 3.1 If $a \in R^{\#}$ then $a^* \in R^{\#}$ and $(a^*)^{\#} = (a^{\#})^*$.

Proof. Let $a \in R^{\#}$. Then we have $a^*(a^{\#})^* a^* = (aa^{\#} a)^* = a^*$, $(a^{\#})^* a^* (a^{\#})^* = (a^{\#} aa^{\#})^* = (a^{\#})^*$ and $a^*(a^{\#})^* = (a^{\#} a)^* = (aa^{\#})^* = (a^{\#})^* a^*$. Hence $a^* \in R^{\#}$ and $(a^*)^{\#} = (a^{\#})^*$.

Definition 3.2 An element $a \in R$ is *left supported by a projection* if $a = pa$ for some projection $p \in R$, it is *right supported by a projection* if $a = aq$ for some projection $q \in R$ and it is *supported by a projection* if it is both left and right supported by a projection.

Proposition 3.3 Let $a \in R^\dagger$. Then $aa^\dagger$ and $a^\dagger a$ are projections.

Proof. It is clear that $(aa^\dagger)^2 = (aa^\dagger a)a^\dagger = aa^\dagger$ and $(aa^\dagger)^* = aa^\dagger$. Thus $aa^\dagger$ is a projection. Similarly, $a^\dagger a$ is also a projection.

Theorem 3.4 For any $a \in R$, the following statements are equivalent:

(1) a is MP-invertible.

(2) a is left $*$-cancellable, right supported by a projection and a^*a is group invertible.

(3) a is right $*$-cancellable, left supported by a projection and aa^* is group invertible.

(4) a is $*$-cancellable, supported by a projection and both a^*a and aa^* are group invertible.

Proof.

(1) $\Rightarrow$ (2) , (3) , (4) Suppose that a is MP-invertible. Let $x, y \in R$ be such that $a^*ax = a^*ay$. Then $ax = aa^\dagger ax = (aa^\dagger)^* ax = (a^\dagger)^* a^*ax = (a^\dagger)^* a^*ay = (a^\dagger)^* a^*ay = aa^\dagger ay = ay$. Thus a is left $*$-cancellable. Let $p = aa^\dagger$ and $q = a^\dagger a$. Then p, q are projections and $pa = aq = a$. Thus a is left and right supported by a projection. Since a^*a and aa^* are Hermitian, a^*a and aa^* are EP elements. Thus a^*a and aa^* are group invertible. This proves (2) and (3). It is obvious that (2) and (3) implies (4). Hence (4) holds.

(2) $\Rightarrow$ (1) Suppose that a is left $*$-cancellable, right supported by a projection and a^*a is group invertible. Then $a = aq$ for some projection q. Let $b = (a^*a)^\# a^*$. Then $a^*aqba = a^*a(a^*a)^\# a^*a = a^*a = a^*aq$. Since a is left $*$-cancellable, $aba = aqba = aq = a$. It is clear that $bab = (a^*a)^\# a^*a(a^*a)^\# a^* = (a^*a)^\# a^* = b$, $(ab)^* = [a(a^*a)^\# a^*]^* = a(a^*a)^\# a^* = ab$ and $(ba)^* = [(a^*a)^\# a^*a]^* = a^*a(a^*a)^\# = (a^*a)^\# a^*a = ba$. Thus a is MP-invertible and $a^\dagger = b$.

(3) $\Rightarrow$ (1) Suppose that a is right $*$-cancellable, left supported by a projection and aa^* is group invertible. Then $a = pa$ for some projection p. Let $b = a^*(aa^*)^\#$. Then $abpaa^* = aa^*(aa^*)^\# aa^* = aa^* = paa^*$. Since a is right $*$-cancellable, $aba = abpa = pa = a$. It is clear that $bab = a^*(aa^*)^\# aa^*(aa^*)^\# = a^*(aa^*)^\# = b$, $(ab)^* = [aa^*(aa^*)^\#]^* = (aa^*)^\# aa^* = aa^*(aa^*)^\# = ab$ and $(ba)^* = [a^*(aa^*)^\# a]^* = a^*(aa^*)^\# a = ba$. Thus a is MP-invertible and $a^\dagger = b$.

(4) $\Rightarrow$ (1) It is trivial that (4) implies (3) and (3) implies (1). Thus (4) implies (1). This completes the proof.

Remark 3.5 If R is a ring with identity, then every element in R is clearly supported by the identity element. Thus Theorem 3.4 is a generalization of Proposition 1. in (Koliha, Djordjević, & Cvetković, 2007).

The following example shows that we cannot omit the condition that a is left (right) supported by a projection for an element a to be MP-invertible.

Example 3.6 Let $R = \{A \in \mathbf{M}_3(R) \mid a_{ij} = 0 \text{ for all } i \geq j\}$ with the usual matrix addition and multiplication.

For any $a = \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} \in R$, we define $a^* = \begin{pmatrix} 0 & z & y \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix}$. Then $*$ is an involution. A computation shows that $abc = 0$ for all $a, b, c \in R$. This implies that $a \in R^\#$ if and only if $a = 0$ and $a \in R^\dagger$ if and only if $a = 0$. Let $a = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Then $ax = 0$ for all $x \in R$. Thus $a^*ax = 0$ implies $ax = 0$ for all $x \in R$. Likewise, $xaa^* = 0$ implies $xa = 0$. Hence a is $*$-cancellable. It is clear that $aa^* = a^*a = 0$ which are group invertible. However, a is not MP-invertible.

Next, we investigate the existence of the Moore-Penrose inverse of the product $x_1x_2 \cdots x_n$ given that $x_1, x_2, ..., x_n \in R^\dagger$. It is worth noting that $R^\dagger$ is not closed under multiplication.

Example 3.7 Let $R = \mathbf{M}_2(Z_2)$ with the matrix transposition as an involution. Let $a = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then $a, b \in R^\dagger$ but $ab = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \notin R^\dagger$.

Definition 3.8 A subset Γ of $R^\dagger$ is called *star-dagger closed* if $x^* \in \Gamma$ and $x^\dagger \in \Gamma$ for all $x \in \Gamma$.

It is obvious that $R^\dagger$ is star-dagger closed. The following theorem shows that nontrivial star-dagger closed sets exist. For any $a \in R^\dagger$, we define $\Gamma(a) = \{a, a^\dagger, a^*, (a^\dagger)^*\}$.

Theorem 3.9 If $a \in R^\dagger$, then $\Gamma(a)$ is star-dagger closed.

Proof. Clearly, $a^*, (a^\dagger)^* \in \Gamma(a)$. Since $(a^*)^* = a \in \Gamma(a)$ and $[(a^\dagger)^*]^* = a^\dagger \in \Gamma(a)$, we have $x^* \in \Gamma(a)$ for all $x \in \Gamma(a)$. Obviously, $a^\dagger \in \Gamma(a)$. Since $(a^\dagger)^\dagger = a \in \Gamma(a)$, $(a^*)^\dagger = (a^\dagger)^* \in \Gamma(a)$ and $[(a^\dagger)^*]^\dagger = [(a^*)^\dagger]^\dagger = a^* \in \Gamma(a)$, we conclude that $x^\dagger \in \Gamma(a)$ for all $x \in \Gamma(a)$. Therefore, $\Gamma(a)$ is star-dagger closed.

Definition 3.10 A subset Γ of R is called a *commuting set* if $xy = yx$ for all $x, y \in \Gamma$.

Theorem 3.11 If $a \in R^\dagger$ is normal, then $\Gamma(a)$ is a commuting set.

Proof. Suppose that $a \in R^\dagger$ is normal. Then $aa^* = a^*a$, $aa^\dagger = a^\dagger a$ and $a^*a^\dagger = a^\dagger a^*$. Thus $a(a^\dagger)^* = (a^\dagger a^*)^* = (a^* a^\dagger)^* = (a^\dagger)^* a$. Similarly, $a^*(a^\dagger)^* = (a^\dagger a)^* = (aa^\dagger)^* = (a^\dagger)^* a^*$ and $a^\dagger (a^\dagger)^* = (a^* a)^\dagger = (aa^*)^\dagger = (a^\dagger)^* a^\dagger$. Hence $\Gamma(a)$ is a commuting set.

Theorem 3. 12 If $\Gamma \subseteq R^\dagger$ is a commuting and star-dagger closed set, then $xy \in R^\dagger$ for all $x, y \in \Gamma$. Moreover, $(xy)^\dagger = y^\dagger x^\dagger = x^\dagger y^\dagger$ for all $x, y \in \Gamma$.

Proof. Let $x, y \in \Gamma$. Then $x^*, y^*, x^\dagger, y^\dagger \in \Gamma$. Thus $(xy)(y^\dagger x^\dagger)(xy) = (xx^\dagger x)(yy^\dagger y) = xy$, $(y^\dagger x^\dagger)(xy)(y^\dagger x^\dagger) = (x^\dagger x x^\dagger)(y^\dagger y y^\dagger) = y^\dagger x^\dagger$, $(xyy^\dagger x^\dagger)^* = (xx^\dagger)^*(yy^\dagger)^* = xx^\dagger yy^\dagger = xyy^\dagger x^\dagger$ and $(y^\dagger x^\dagger xy)^* = (y^\dagger y)^*(x^\dagger x)^* = y^\dagger y x^\dagger x = y^\dagger x^\dagger xy$. This shows that $xy \in R^\dagger$ and $(xy)^\dagger = y^\dagger x^\dagger = x^\dagger y^\dagger$.

Theorem 3.13 If $\Gamma \subseteq R^\dagger$ is a commuting and star-dagger closed set, then $x_1 x_2 \cdots x_n \in R^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma$. Moreover, $\left(x_1 x_2 \cdots x_n\right)^\dagger = x_n^\dagger x_{n-1}^\dagger \cdots x_1^\dagger = x_1^\dagger x_2^\dagger \cdots x_n^\dagger$ for all $x_1, x_2, \ldots, x_n \in \Gamma$.

Proof. The proof is straightforward by Theorem 3.12 and induction.

Corollary 3.14 If $a \in R^\dagger$ is normal, then $x_1 x_2 \cdots x_n \in R^\dagger$ for all $x_1, x_2, ..., x_n \in \Gamma(a)$.

Proof. Since $\Gamma(a)$ is a commuting and star-dagger closed set, the result follows Theorem 3.13.

Corollary 3.15 If R is commutative, then $(R^\dagger, \cdot)$ is a subsemigroup of $(R, \cdot)$.

Proof. Since R is commutative, $R^\dagger$ is a commuting and star-dagger closed set. The result then follows Theorem 3.12.

Theorem 3.16 If a is an EP element, then $a^\dagger$, a^* and $(a^\dagger)^*$ are also EP elements.

Proof. Suppose that a is an EP element. Then $aa^\dagger = a^\dagger a$. Thus $a^\dagger (a^\dagger)^\dagger = a^\dagger a = aa^\dagger = (a^\dagger)^\dagger a^\dagger$. This means $a^\dagger$ is an EP element. We also have that $a^*(a^*)^\dagger = a^*(a^\dagger)^* = (a^\dagger a)^* = (aa^\dagger)^* = (a^*)^\dagger a^*$. Thus a^* is an EP element. This also implies $(a^\dagger)^*$ is an EP element.

Theorem 3.17 If a is an EP element, then $x^n \in R^\dagger$ for all $x \in \Gamma(a)$ and all $n \in N$.

Proof. Suppose that a is an EP element. By using Theorem 3.16, we know that $a, a^\dagger, a^*$ and $(a^\dagger)^*$ are EP elements. Thus it suffices to prove that $a^n \in R^\dagger$ for all $n \in N$. Since a is an EP element, $aa^\dagger = a^\dagger a$. Then $a^n (a^\dagger)^n a^n = (aa^\dagger a)^n = a^n$, $(a^\dagger)^n a^n (a^\dagger)^n = (a^\dagger a a^\dagger)^n = (a^\dagger)^n$, $[a^n (a^\dagger)^n]^* = [(aa^\dagger)^n]^* = [(aa^\dagger)^*]^n = (aa^\dagger)^n = a^n (a^\dagger)^n$ and $[(a^\dagger)^n a^n]^* = [(a^\dagger a)^n]^* = [(a^\dagger a)^*]^n = (a^\dagger a)^n = (a^\dagger)^n a^n$. This proves that $a^n \in R^\dagger$ and $(a^n)^\dagger = (a^\dagger)^n$ for all $n \in N$.

Theorem 3.18 Let $a \in R^\dagger$. Then the following elements are MP-invertible for all $n \in N$.

(1) $(aa^*)^n$,
(2) $(a^*a)^n$,
(3) $(a^* a^\dagger a a)^n$,
(4) $a(a^*a)^n$ and
(5) $a^*(aa^*)^n$.

Proof.

(1) We know that $aa^* \in R^\dagger$ is Hermitian. Thus aa^* is an EP element. By Theorem 3.17, $(aa^*)^n \in R^\dagger$ for all $n \in N$.

(2) We know that $a^*a \in R^\dagger$ is Hermitian. Thus a^*a is an EP element. By Theorem 3.17, $(a^*a)^n \in R^\dagger$ for all $n \in N$.

(3) Let $x = a^*a^\dagger aa$. Then $x^* = a^*a^*(a^\dagger)^*a$ and $xx^* = a^*a^\dagger aaa^*a^*(a^\dagger)^*a = a^*(a^\dagger a)^*aa^*(a^\dagger a)^*a = a^*a^*(a^\dagger)^*aa^*a^\dagger aa = x^*x$. Thus x is a normal element and hence an EP element. By Theorem 3.17, $(a^*a^\dagger aa)^n \in R^\dagger$ for all $n \in N$.

(4) Let $x = a(a^*a)^n = (aa^*)^n a$ and $y = [(a^*a)^\dagger]^n a^\dagger = a^\dagger[(aa^*)^\dagger]^n$. Since aa^* and a^*a are EP elements, we have $[(aa^*)^n]^\dagger = [(aa^*)^\dagger]^n$ and $[(a^*a)^n]^\dagger = [(a^*a)^\dagger]^n$. Then $xy = (aa^*)^n aa^\dagger[(aa^*)^n]^\dagger = (aa^*)^{n-1}(aa^*aa^\dagger)[(aa^*)^n]^\dagger = (aa^*)^n[(aa^*)^n]^\dagger$ and $yx = a^\dagger[(aa^*)^\dagger]^n a(a^*a)^n = a^\dagger[(aa^*)^\dagger]^{n-1}[(aa^*)^\dagger a](a^*a)^n = a^\dagger[(aa^*)^\dagger]^{n-1}(a^*)^\dagger(a^*a)^n = [a^\dagger(a^*)^\dagger]^n(a^*a)^n = [(a^*a)^n]^\dagger(a^*a)^n$. Thus, $xyx = a(a^*a)^n[(a^*a)^n]^\dagger(a^*a)^n = a(a^*a)^n = x$ and $yxy = a^\dagger[(aa^*)^n]^\dagger(aa^*)^n[(aa^*)^n]^\dagger = a^\dagger[(aa^*)^n]^\dagger = y$. Since xy and yx are projections, we also have $(xy)^* = xy$ and $(yx)^* = yx$. Therefore, $a(a^*a)^n \in R^\dagger$ and $[a(a^*a)^n]^\dagger = [(a^*a)^\dagger]^n a^\dagger$.

(5) Since $[a(a^*a)^n]^* = a^*(aa^*)^n$, we conclude that $a^*(aa^*)^n \in R^\dagger$.

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