CHAPTER II

THEORETICAL BACKGROUND

This chapter devotes to the fundamental background in linear elastic fracture mechanics and the formulation of the boundary value problem associated with a cracked body. First, a stress field in the vicinity of the crack front and the definition of the stress intensity factors are briefly summarized. Next, a pair of weakly-singular weak-form boundary integral equations for the displacement and traction is presented. Finally, a symmetric weak formulation governing the three-dimensional crack problem is established.

2.1 Stress field near the crack front

Consider a crack embedded in a linearly elastic body as shown schematically in Figure 2.1. A reference local Cartesian coordinate system is chosen, for convenience, such that its origin is located at the crack front, the x_1-axis is normal to the crack front, the x_2-axis is normal to the crack surface and the x_3-axis is tangent to the crack front. Let (r,θ,x_3) denote a local cylindrical coordinate system defined based on the local (x_1,x_2,x_3) system as shown in Figure 2.1. By following the previous work by Westergaard (1939), Sneddon (1946), Irwin (1957) and Williams (1957), the stress field in the neighborhood of the crack front takes the following form

$$\sigma_{ij} = \left(\frac{k}{\sqrt{r}}\right)\tilde{\sigma}_{ij}(\theta) + \sum_{m=0}^{\infty} C_m r^{\frac{m}{2}} \hat{\sigma}_{ij}^{(m)}(\theta) \tag{2.1}$$

where σ_{ij} denote the Cartesian stress components; k is a constant depending on applied load, geometry of the body and crack, material properties, and location along the crack front; r is the distance from the crack front to a point of interest; $\tilde{\sigma}_{ij}$ is an angular dependent function; m is a non-negative integer; C_m are constants; and $\hat{\sigma}_{ij}^{(m)}$ are angular dependent functions.

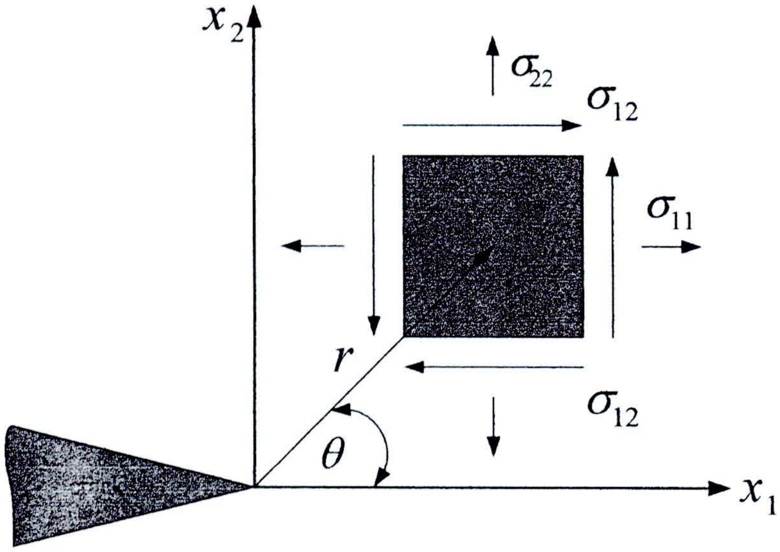

Figure 2.1 Schematic indicating stress field in the x_1-x_2 plane near the crack front

As apparent from above equation, the stress field near the crack front can be decomposed into two different parts where the first part (i.e. the second term on the right-hand side of (2.1)) remains finite and possesses a zero limit as r approaches zero and the second part (i.e. the first term on the right-hand side of (2.1)) is proportional to the inverse square-root function $1/\sqrt{r}$ which becomes infinite at the crack front. As approaching the crack front, the stress field is clearly dominated by the first term and, within the context of linear elastic fracture mechanics, the second term is typically discarded. The asymptotic stress field (generally known as the K-field) exhibits the singularity of order $1/\sqrt{r}$ and its characteristic can be completely determined by a single parameter k. Influence of loading conditions (including the magnitude and loading direction), properties of constituting materials, and geometry of the body and cracks on such local stress field is reflected through the constant k.

2.2 Stress intensity factors

The stress intensity factor, denoted by K, is an essential parameter in linear elastic fracture mechanics that is known to completely characterize the dominant stress field in the neighborhood of the crack front. This fracture data is directly related to the constant k in the expansion (2.1) via a simple relation:

$$K = k\sqrt{2\pi} \tag{2.2}$$

By inserting (2.2) into the first term on the right-hand side of (2.1), the dominant stress field now becomes

$$\sigma_{ij} = \left(\frac{K}{\sqrt{2\pi r}}\right)\tilde{\sigma}_{ij}(\theta) \tag{2.3}$$

From the eigen analysis, the above asymptotic stress field can be decomposed into three independent modes, i.e. the opening mode or mode I, the sliding mode or mode II and the tearing mode or mode III, corresponding to the different behavior of the relative crack-face displacement. More specifically, the component of the general relative crack-face displacement perpendicular to the crack surface, parallel to the crack surface and normal to the crack front, parallel to the crack surface and tangent to the crack front corresponds to the mode I, mode II and mode III, respectively, as shown in Figure 2.2. For general geometries and loading conditions, all three modes exist and the local stress field (2.3) can further be written in a form

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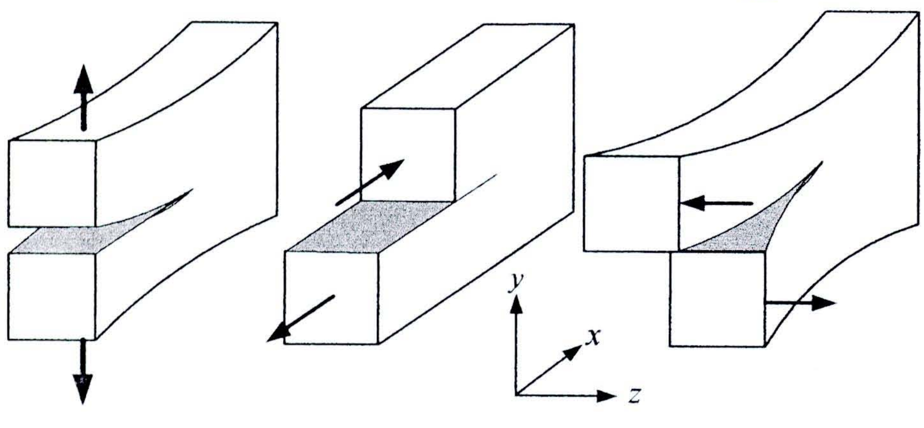
$$\sigma_{ij} = \left(\frac{K_I}{\sqrt{2\pi r}}\right)\tilde{\sigma}^{I}_{ij}(\theta) + \left(\frac{K_{II}}{\sqrt{2\pi r}}\right)\tilde{\sigma}^{II}_{ij}(\theta) + \left(\frac{K_{III}}{\sqrt{2\pi r}}\right)\tilde{\sigma}^{III}_{ij}(\theta) \tag{2.4}$$



(a) (b) (c)

Figure 2.2 Schematics indicating the relative crack-face displacement for (a) mode I or opening mode, (b) mode II or sliding mode, and (c) mode III or tearing mode

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where K_I, K_{II} and K_{III} are mode-I, mode-II and mode-III stress intensity factors, respectively, and $\tilde{\sigma}_{ij}^{I}$, $\tilde{\sigma}_{ij}^{II}$ and $\tilde{\sigma}_{ij}^{III}$ are corresponding functions describing the angular dependent behavior in the local region surrounding the crack front. The stress intensity factors for all three modes can be obtained from the local stress field resulting from solving a complete boundary value problem via following formulas

$$K_I = \lim_{r \to 0} \left(\sqrt{2\pi r} \sigma_{22}(r, \theta = 0) \right) \tag{2.5}$$

$$K_{II} = \lim_{r \to 0} \left(\sqrt{2\pi r} \sigma_{12}(r, \theta = 0) \right) \tag{2.6}$$

$$K_{III} = \lim_{r \to 0} \left(\sqrt{2\pi r} \sigma_{23}(r, \theta = 0) \right) \tag{2.7}$$

It is remarked that the definitions (2.5)-(2.7) apply to both isotropic and generally anisotropic materials. Barnett and Asaro (1972) and Xu (2000) proposed an alternative expression for determining the mixed-mode stress intensity factors for generally anisotropic media in terms of the relative crack-face displacement:

$$k_i = \frac{\sqrt{2\pi}}{4} B_{il} \lim_{x_1 \to 0^-} \left(\frac{\Delta u_l}{\sqrt{-x_1}} \right) \tag{2.8}$$

where k_i are related to the stress intensity factors by $k_1 = K_{II}$, $k_2 = K_I$, $k_3 = K_{III}$; Δu_l are components of relative crack-face displacement with respect to the local coordinate system (x_1, x_2, x_3) defined in Figure 2.1; and B_{il} are constants involving all elastic constants and the geometry of the crack front by

$$B_{il} = \frac{1}{2\pi} \int_0^{2\pi} \left[(\mathbf{a}, \mathbf{a})_{il} - (\mathbf{a}, \mathbf{b})_{im} (\mathbf{b}, \mathbf{b})_{mn}^{-1} (\mathbf{b}, \mathbf{a})_{nl} \right] d\phi \tag{2.9}$$

where **a** and **b** are orthonormal vectors contained in the plane defined by $x_3 = 0$; ϕ is the angle between the vector **a** and the x_1-axis; the operator $(\cdot,\cdot)$ is defined in terms of the elastic constants E_{ijkl} by $(\mathbf{a}, \mathbf{b})_{ij} = a_m E_{mijn} b_n$; and $(\mathbf{b}, \mathbf{b})^{-1}$ denotes the inverse of $(\mathbf{b}, \mathbf{b})$. The formula (2.8) can be used, when supplemented by positive features of the selected numerical scheme, to accurately and efficiently compute the stress intensity factor along the crack front for both isotropic and anisotropic cases. The explicit form of the angular dependent functions $\tilde{\sigma}_{ij}^{I}$, $\tilde{\sigma}_{ij}^{II}$ and $\tilde{\sigma}_{ij}^{III}$ for certain special cases (e.g. twodimensional problem with both plane stress and plane strain assumptions, anti-plane shear problem, etc.) can be found in textbooks for fundamental fracture mechanics (e.g. Anderson, 2005; Gdoutos, 2005; Barsom and Rolfe, 1999; Kanninen and Popelar, 1985; Hellan, 1984).

2.3 Boundary integral equations for cracked body

It is evident from sections 2.1 and 2.2 that the asymptotic and eigen analysis provide only information about the form of dominant elastic fields in the neighborhood of the crack front but leaving several important information such as loading conditions, geometries and material properties in terms of unknown constants termed the stress intensity factors. This essential fracture data can be determined once the complete boundary value problem associated with the entire body is solved. The local stress field in the region surrounding the crack front and the relative crack-face displacement resulting from such comprehensive stress analysis can be used to extract the stress intensity factors via the relations (2.5)-(2.7) or (2.8). In the present study, a numerical technique based on the weakly-singular, symmetric Galerkin boundary element method (SGBEM) is selected to perform such stress analysis. In this section, we briefly summarize a set of boundary integral equations essential for the development of the SGBEM.

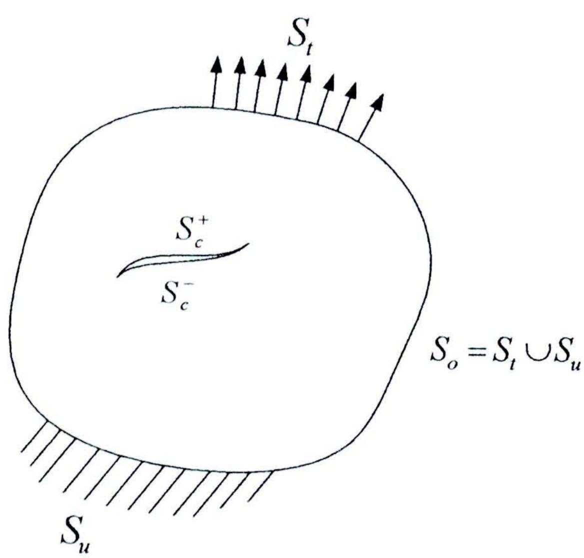

Figure 2.3 Schematic of three-dimensional cracked body

Consider a three-dimensional, finite, cracked body that is made of a generally anisotropic, linearly elastic material as shown in Figure 2.3. The boundary of the body can be decomposed into three surfaces $S_C^+ \cup S_C^-$, S_u, and S_t where $S_C^+ \cup S_C^-$ represents two geometrically coincident crack surfaces, S_u is a portion of the regular boundary on which the displacement is fully prescribed, and S_t is the remaining regular boundary on which the traction is prescribed. It is remarked that on the crack surface the traction is assumed to be known.

A pair of completely regularized boundary integral equations for the displacement and the traction applicable to a cracked body made of a generally anisotropy material was proposed by Rungamornrat and Mear (2008a). The final form of those two equations are given by

$$\begin{aligned}
\frac{1}{2}\int_{S_o} \tilde{t}_p(\mathbf{y})\, u_p(\mathbf{y})\, dS(\mathbf{y}) \quad &= \int_{S_o} \tilde{t}_p(\mathbf{y}) \int_{S_o} U_j^p(\boldsymbol{\xi}-\mathbf{y})\, t_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y}) \\
&+ \int_{S_o} \tilde{t}_p(\mathbf{y}) \int_{S} G_{mj}^p(\boldsymbol{\xi}-\mathbf{y})\, D_m v_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y}) \\
&- \int_{S_o} \tilde{t}_p(\mathbf{y}) \int_{S} n_i(\boldsymbol{\xi})\, H_{ij}^p(\boldsymbol{\xi}-\mathbf{y})\, v_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y})
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
-\int_{S} c(\mathbf{y})\, \tilde{v}_k(\mathbf{y})\, t_k(\mathbf{y})\, dS(\mathbf{y}) \quad &= \int_{S} D_t \tilde{v}_k(\mathbf{y}) \int_{S} C_{mj}^{tk}(\boldsymbol{\xi}-\mathbf{y})\, D_m v_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y}) \\
&+ \int_{S} D_t \tilde{v}_k(\mathbf{y}) \int_{S_o} G_{tk}^j(\boldsymbol{\xi}-\mathbf{y})\, t_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y}) \\
&+ \int_{S} \tilde{v}_k(\mathbf{y}) \int_{S_o} n_l(\mathbf{y})\, H_{lk}^j(\boldsymbol{\xi}-\mathbf{y})\, t_j(\boldsymbol{\xi})\, dS(\boldsymbol{\xi})\, dS(\mathbf{y})
\end{aligned} \tag{2.11}$$

where $S_o = S_u \cup S_t$, $S = S_u \cup S_t \cup S_c^+$, $\tilde{t}_p$ and $\tilde{v}_k$ are admissible test functions, $D_m(\cdot) = n_i \varepsilon_{ijm} \partial(\cdot)/\partial\xi_j$ denotes a surface differential operator, ε_{ijm} is a permutation symbol, $c(\mathbf{y})$ is a geometric dependent function defined by $c(\mathbf{y}) = 1/2$ for $\mathbf{y} \in S_o$ and $c(\mathbf{y}) = 1$ for $\mathbf{y} \in S_c^+$, $t_i(\boldsymbol{\xi})$ denotes the traction at point $\boldsymbol{\xi}$ on the boundary, $v_i(\boldsymbol{\xi})$ is the boundary data defined by

$$v_i(\boldsymbol{\xi}) = \begin{cases} u_i(\boldsymbol{\xi}) \ , & \boldsymbol{\xi} \in S_o \\ \Delta u_i(\boldsymbol{\xi}) \ , & \boldsymbol{\xi} \in S_c^+ \end{cases} \tag{2.12}$$

with $u_i(\xi)$ and $\Delta u_i(\xi) = u_i^+(\xi) - u_i^-(\xi)$ denoting the displacement on the regular boundary and the jump in the displacement across the crack surface. All four kernels appearing in above two integral equations are given explicitly by

$$H_{ij}^p(\xi - \mathbf{y}) = -\frac{1}{4\pi}\frac{(\xi_i - y_i)\delta_{pj}}{r^3} \tag{2.13}$$

$$U_i^p(\xi - \mathbf{y}) = K_{mp}^{mi}(\xi - \mathbf{y}) \tag{2.14}$$

$$G_{mj}^p(\xi - \mathbf{y}) = \varepsilon_{abm} E_{ajdc} K_{cp}^{bd}(\xi - \mathbf{y}) \tag{2.15}$$

$$C_{mj}^{tk}(\xi - \mathbf{y}) = A_{mjdn}^{tkoe} K_{dn}^{oe}(\xi - \mathbf{y}) \tag{2.16}$$

where δ_{ij} is a Kronecker delta and the constant A_{mjdn}^{tkoe} and the function $K_{jl}^{ik}(\xi - \mathbf{y})$ are defined by

$$A_{mjdn}^{tkoe} = \varepsilon_{pam}\varepsilon_{pbt}\left(E_{bknd}E_{ajeo} - \frac{1}{3}E_{ajkb}E_{dneo}\right) \tag{2.17}$$

$$K_{jl}^{ik}(\xi - \mathbf{y}) = \frac{1}{8\pi r^2}\oint_{\mathbf{z}\cdot\mathbf{r}=0}(\mathbf{z},\mathbf{z})_{kl}^{-1} z_i z_j ds(\mathbf{z}) \tag{2.18}$$

where $\mathbf{r} = \xi - \mathbf{y}$ denotes the position vector, $r = \sqrt{\mathbf{r}\cdot\mathbf{r}}$, $\mathbf{z}$ is a unit vector, $(\mathbf{z},\mathbf{z})_{kl} = z_m E_{mklp} z_p$, and the closed contour integral is defined on a unit circle on a plane normal to the position vector $\mathbf{r}$ (i.e. $\mathbf{z}\cdot\mathbf{r} = 0$). It is worth noting that the kernels $U_i^p(\xi - \mathbf{y})$, $G_{mj}^p(\xi - \mathbf{y})$ and $C_{mj}^{tk}(\xi - \mathbf{y})$ are material dependent and possess the same structure in terms of the line integral whereas the kernel $H_{ij}^p(\xi - \mathbf{y})$ is independent of elastic constants. In addition, all four kernels are singular only at $\xi = \mathbf{y}$ and are of order $\mathcal{O}(1/r)$ (see details of derivations and extensive discussion of these kernels in the work of Rungamornrat and Mear, 2008a). For the special case of isotropy, i.e. $E_{ijkl} = \mu\left[\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + 2\nu\delta_{ij}\delta_{kl}/(1-2\nu)\right]$ where μ is the elastic shear modulus and ν is Poisson's ratio, the closed contour integral (2.18) can be integrated explicitly and the kernels $U_i^p(\xi - \mathbf{y})$, $G_{mj}^p(\xi - \mathbf{y})$ and $C_{mj}^{tk}(\xi - \mathbf{y})$ now simply reduce to a closed form identical to those obtain by Li and Mear (1998) and Li *et al.* (1998):

$$U_i^p(\xi - \mathbf{y}) = \frac{1}{16\pi(1-\nu)\mu}\left(\frac{3-4\nu}{r}\delta_{ip} + \frac{(\xi_i - y_i)(\xi_p - y_p)}{r^3}\right) \quad (2.19)$$

$$G_{mj}^p(\xi - \mathbf{y}) = \frac{1}{8\pi(1-\nu)r}\left((1-2\nu)\varepsilon_{mpj} + \frac{(\xi_t - y_t)(\xi_p - y_p)}{r^2}\varepsilon_{tjm}\right) \quad (2.20)$$

$$C_{mj}^{tk}(\xi - \mathbf{y}) = \frac{\mu}{4\pi(1-\nu)r}\left((1-\nu)\delta_{tk}\delta_{mj} + 2\nu\delta_{km}\delta_{jt} - \delta_{kj}\delta_{tm} - \frac{(\xi_j - y_j)(\xi_k - y_k)}{r^2}\delta_{tm}\right) \quad (2.21)$$

The boundary integral equations (2.10) and (2.11) possess several positive features useful for the present study including that (i) they are cast in a weak form well-suited for establishing the symmetric weak formulation for the SGBEM discussed in the following section, (ii) all kernels are only weakly singular of $\mathcal{O}(1/r)$ allowing all involved integrals be interpreted in an ordinary sense and only requiring displacement data on the regular boundary and crack of the type C^0 for those integrals to be valid, (iii) they are applicable to cracks of general geometries and under arbitrary loading conditions, and (iv) they apply to both isotropic and generally anisotropic linearly elastic media.

2.4 Symmetric weak formulation for crack problem

To construct a symmetric weak formulation associated with the boundary value problem for a cracked body, a standard procedure similar to that employed by Li et al. (1998) and Rungamornrat and Mear (2008b) is employed. In such procedure, a pair of weakly singular, weak-form integral equations for the displacement and traction (2.10) and (2.11) is employed as follows. On the surface S_u, the displacement integral equation (2.10) is applied by choosing a test function such that $\tilde{t}_p = 0$ on S_t and on the surface S_t, the traction integral equation (2.11) is applied with a special choice of test function satisfying $\tilde{v}_p = 0$ on $S_u \cup S_c^+$. Finally, on a single crack surface S_c^+, the traction integral equation (2.11) is again applied by choosing a test function $\tilde{v}_p = \Delta\tilde{u}_p$ on S_c^+ and $\tilde{v}_p = 0$ on $S_o = S_u \cup S_t$. A set of weak-form equations resulting from appropriate applications to each surface is given in a concise form by (also see Li *et al.*, 1998; Rungamornrat and Mear, 2008b)

$$
\begin{aligned}
&\mathscr{A}_{uu}\left(\tilde{t},t\right) + \mathscr{B}_{ut}\left(\tilde{t},u\right) + \mathscr{B}_{uc}\left(\tilde{t},\Delta u\right) &&= \mathscr{R}_1(\tilde{t})\\
&\mathscr{B}_{ut}\left(t,\tilde{u}\right) + \mathscr{C}_{tt}\left(\tilde{u},u\right) + \mathscr{C}_{tc}\left(\tilde{u},\Delta u\right) &&= \mathscr{R}_2(\tilde{u}) \qquad (2.22)\\
&\mathscr{B}_{uc}\left(t,\Delta\tilde{u}\right) + \mathscr{C}_{ct}\left(\Delta\tilde{u},u\right) + \mathscr{C}_{cc}\left(\Delta\tilde{u},u\right) &&= \mathscr{R}_3(\Delta\tilde{u})
\end{aligned}
$$

where the linear operators $\mathscr{A}_{PQ}$, $\mathscr{B}_{PQ}$ and $\mathscr{C}_{PQ}$ (with $P,Q\in\{u,t,c\}$) are given in terms of weakly singular, double surface integrals by

$$\mathscr{A}_{PQ}(\boldsymbol{X},\boldsymbol{Y}) = \int_{S_P} X_k(\mathbf{y})\int_{S_Q} U_i^k(\boldsymbol{\xi}-\mathbf{y})Y_i(\boldsymbol{\xi})dS(\boldsymbol{\xi})dS(\mathbf{y}) \qquad (2.23)$$

$$
\begin{aligned}
\mathscr{B}_{PQ}(\boldsymbol{X},\boldsymbol{Y}) = &\int_{S_P} X_k(\mathbf{y})\int_{S_Q} G_{mj}^k(\boldsymbol{\xi}-\mathbf{y})D_m Y_j(\boldsymbol{\xi})dS(\boldsymbol{\xi})dS(\mathbf{y})\\
&-\int_{S_P} X_k(\mathbf{y})\int_{S_Q} n_m(\boldsymbol{\xi})H_{mj}^k Y_j(\boldsymbol{\xi})dS(\boldsymbol{\xi})dS(\mathbf{y})
\end{aligned} \qquad (2.24)
$$

$$\mathscr{C}_{PQ}(\boldsymbol{X},\boldsymbol{Y}) = \int_{S_P} D_t X_k(\mathbf{y})\int_{S_Q} C_{mj}^{tk}(\boldsymbol{\xi}-\mathbf{y})D_m Y_j(\boldsymbol{\xi})dS(\boldsymbol{\xi})dS(\mathbf{y}) \qquad (2.25)$$

and the operators $\mathscr{R}_1$, $\mathscr{R}_2$ and $\mathscr{R}_3$ involving the prescribed data on the boundary are given by

$$\mathscr{R}_1(\tilde{t}) = \mathscr{F}_u(\tilde{t},u_o) - \mathscr{A}_{ut}(\tilde{t},t_o) - \mathscr{B}_{uu}(\tilde{t},u_o) \qquad (2.26)$$

$$\mathscr{R}_2(\tilde{u}) = -\mathscr{F}_t(\tilde{u},t_o) - \mathscr{B}_{tt}(t_o,\tilde{u}) - \mathscr{C}_{tu}(\tilde{u},u_o) \qquad (2.27)$$

$$\mathscr{R}_3(\Delta\tilde{u}) = -2\mathscr{F}_c(\Delta\tilde{u},t_c) - \mathscr{B}_{tc}(t_o,\Delta\tilde{u}) - \mathscr{C}_{cu}(\Delta\tilde{u},u_o) \qquad (2.28)$$

in which the integral operator $\mathbf{F}_P$ (with $P\in\{u,c,t\}$) are given in terms of a regular single surface integral by

$$\mathscr{F}_P(\boldsymbol{X},\boldsymbol{Y}) = \frac{1}{2}\int_{S_P} X_i(\mathbf{y})Y_i(\mathbf{y})dS(\mathbf{y}). \qquad (2.29)$$

The symmetry of the weak formulation (2.22) should be obvious from the form of the integral operators $\mathscr{A}_{PQ}$ and $\mathscr{C}_{PQ}$ and the symmetry of the kernels $U_i^P(\boldsymbol{\xi}-\mathbf{y})$ and

$C_{mj}^{tk}(\boldsymbol{\xi}-\mathbf{y})$. This set of integral equations forms a complete boundary value problem for a body containing cracks in terms of unknown boundary data such as the traction on the surface S_u, the displacement on the surface S_t and the relative crack-face displacement on the crack surface S_c^+. For a special case of pure traction boundary value problems (i.e. $S_u = \phi$), the weak formulation (2.22) simply reduces to

$$\begin{aligned} \mathrm{C}_{tt}(\tilde{\mathbf{u}},\mathbf{u}) + \mathrm{C}_{tc}(\tilde{\mathbf{u}},\Delta\mathbf{u}) &= \mathrm{R}_2(\tilde{\mathbf{u}}) \\ \mathrm{C}_{ct}(\Delta\tilde{\mathbf{u}},\mathbf{u}) + \mathrm{C}_{cc}(\Delta\tilde{\mathbf{u}},\Delta\mathbf{u}) &= \mathrm{R}_3(\Delta\tilde{\mathbf{u}}) \end{aligned} \tag{2.30}$$

where $\mathcal{R}_2$ and $\mathcal{R}_3$ are now given by

$$\mathrm{R}_2(\tilde{\mathbf{u}}) = -\mathrm{F}_t(\tilde{\mathbf{u}},\mathbf{t}_o) - \mathrm{B}_{tt}(\mathbf{t}_o,\tilde{\mathbf{u}}) \tag{2.31}$$

$$\mathrm{R}_3(\Delta\tilde{\mathbf{u}}) = -2\mathrm{F}_c(\Delta\tilde{\mathbf{u}},\mathbf{t}_c) - \mathrm{B}_{tc}(\mathbf{t}_o,\Delta\tilde{\mathbf{u}}) \tag{2.32}$$

The formulation (2.22) is employed later as a basis for the development of the weakly singular SGBEM.