

Original Article

Stability and Hopf bifurcation on an SEIR delayed model with logistic growth

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Received: 14 February 2017; Revised: 1 May 2017; Accepted: 22 May 2017

Abstract

This paper investigates the stability and Hopf bifurcation of *SEIR* delay model with logistic growth. Firstly, the existence and uniqueness of equilibrium point are analyzed. For the study of the stability of the equilibrium point time delay (τ) was chosen as the bifurcation parameter. By considering the roots of characteristic equations, it was found that disease-free equilibrium is locally asymptotically stable for all $\tau \geq 0$. The endemic equilibrium of the model is conditionally stable. Hopf bifurcation will occur when the bifurcation parameter passes through a critical value. Moreover, stability and direction of Hopf bifurcation are obtained by using the normal form theory and the center manifold reduction. Finally, the numerical solutions are simulated to verify the theoretical results.

Keywords: SEIR model, time delay, logistic growth, local stability, Hopf bifurcation

1. Introduction

Outbreak of a disease is a serious danger for human lives. Many diseases such as Spanish flu, cholera, severe acute respiratory syndrome (SARS), and measles have caused many deaths of humans (Patterson & Pyle, 1991; Johnson & Mueller, 2002; Smith, 2006; Taubenburger & Morenst, 2006; World Health Organization [WHO], 2010). While a doctor tries to treat patients to recover from diseases, the scientist attempts to find a method in prevention or control the outbreak of a disease. They construct a system of equations to describe the phenomenon of disease, which is called *epidemic model*. Usually, an epidemic model is used to study the effect of each parameter to the number of infected people. Controlling these parameters help to decrease the number of infected people or control the infected people when a disease outbreak occurs. The epidemic model has several types depending on individuals in the model. The model in this research used the assumption that when susceptible individuals (s) get the

disease, it incubate inside in individuals for a period of time before becoming infectious. This period is called latent period and individuals in this period are exposed individuals (E). After passing this period, individuals become infected (I) and/or recovered (R) individuals, respectively. Thus, this paper studies the dynamical behavior based on the SEIR epidemic model.

There are many researches about dynamic behaviors of SEIR model. Greenhalgh (1992) analyzed the SEIR epidemic model when the death rate depends on the number of individuals in the population. Li and Muldowney (1995) studied the global stability of SEIR model with nonlinear incidence rate. Zhang *et al.* (2006) studied the global stability and dynamics of an SEIR epidemic model with immigration in different individuals. Li *et al.* (2006) studied the global stability of an SEIR model with constant immigration. Li and Jin (2005) studied the global stability of epidemic model with infectious force in latent, infected and immune period. Massad *et al.* (2007) used the SEIR model with logistic growth and infectious forced in infected and latent period to predict the number of patient from influenza in Brazil.

Time delay is introduced in the epidemic model to study changes in the dynamic behavior. Usually, the time delay parameter is chosen to be a bifurcation parameter. The

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dynamic behaviors of the model may change when the bifurcation parameter increases or decreases. There are some studies about mathematical models with time delay. Khan and Greenhalgh (1999) analyzed the Hopf bifurcation of an epidemic model with time delay in vaccination. Wan and Wei (2010) analyzed Hopf bifurcation in a food-limited population model with time delay. Zhang *et al.* (2010) analyzed the stability of Hopf bifurcation of an epidemic model with stage structure. Chinviriyasit (2015) studied the effect time delay of an SEIR epidemic model with nonlinear incidence rate. Misra *et al.* (2012) constructed a delay mathematical model to analyze the controlling of cholera epidemic.

The aim of this paper is to modify the model of Massad *et al.* (2007) with some assumptions. First, the sensitivity of infection from exposed and infected individuals not equal. Thus, the value of contact rate from exposed and infected individuals should be different. Second, when susceptible individuals contact the infectious individuals, they do not become infectious immediately. Therefore, the time delay is introduced into the model. Furthermore, the dynamics of the model changes when the time delay is included which is interesting behavior. Formulation and existence of equilibrium point of the model are illustrated in Section 2. The stability of disease-free and endemic equilibrium are analyzed in Section 3 and 4. Further, the direction of Hopf bifurcation is illustrated in Section 5. Numerical results are shown in Section 6 and conclusion of this paper is presented in Section 7.

2. Model Formulation and Equilibrium Point

2.1 Model formulation

In this research, the model of Massad *et al.* (2007) is modified. The total population at time t , denoted by $N(t)$, is subdivided into four individuals: susceptible ($S(t)$), exposed ($E(t)$), infected ($I(t)$) and recovered ($R(t)$), thus

$$N(t) = S(t) + E(t) + I(t) + R(t) \tag{1}$$

The susceptible individuals is increased by the logistic growth term when r is birth rate and K is carrying capacity. Susceptible individuals decrease by acquiring infection from both exposed individuals and infected individuals. It is assumed that susceptible individuals are exposed at a time $t - \tau$ and became infective at a time τ later. Thus, the rate at which susceptible individuals in contact with the virus progress to the latent stage is given by

$$\frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)},$$

where τ is a latent time delay, $\beta_1 = \beta\beta_E$ and $\beta_2 = \beta\beta_I$. The β be the average number of sufficient contact s to transmit infection in unit time per infective individual in the population (of size N). The parameters β_E and β_I account for the ability to cause infection by exposed individuals ($0 \leq \beta_E \leq 1$) and by infected individuals ($0 \leq \beta_I \leq 1$), respectively. Further, the

population of susceptible individuals is decreased by natural death (at rate μ). Thus, the rate of change of the susceptible population is given by

$$\frac{dS(t)}{dt} = rN(t) \left(1 - \frac{N(t)}{K} \right) - \frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)} S(t - \tau) - \mu S(t). \tag{2}$$

The population of exposed individuals is generated by infection of susceptible individuals. This population decrease by development of disease symptoms (at rate σ), recovery from the disease (at rate κ) and natural death (at rate μ). This gives

$$\frac{dE(t)}{dt} = \frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)} S(t - \tau) - (\mu + \sigma + \kappa) E(t). \tag{3}$$

The infected individuals is increased at rate σ . This individuals decreased by natural death (at rate μ), disease-induced death (at rate α) and recovery from the disease (at rate γ). Thus, the rate of change in this individual is given by

$$\frac{dI(t)}{dt} = \sigma E(t) - (\mu + \alpha + \gamma) I(t). \tag{4}$$

Finally, the population of recover individuals is generated by the recovery from the disease in exposed and infected individuals at rate κ and γ , respectively. This population decreased by natural death at rate μ . Thus,

$$\frac{dR(t)}{dt} = \kappa E(t) + \gamma I(t) - \mu R(t). \tag{5}$$

Thus the model for the transmission dynamics of an infectious disease with time delay is given by the following nonlinear system of delay differential equations:

$$\frac{dS(t)}{dt} = rN(t) \left(1 - \frac{N(t)}{K} \right) - \frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)} S(t - \tau) - \mu S(t), \tag{6}$$

$$\frac{dE(t)}{dt} = \frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)} S(t - \tau) - (\mu + \sigma + \kappa) E(t),$$

$$\frac{dI(t)}{dt} = \sigma E(t) - (\mu + \alpha + \gamma) I(t),$$

$$\frac{dR(t)}{dt} = \kappa E(t) + \gamma I(t) - \mu R(t).$$

The initial condition of (6) is given as

$$S(\theta) = \phi_1(\theta), E(\theta) = \phi_2(\theta), I(\theta) = \phi_3(\theta), R(\theta) = \phi_4(\theta), \theta \in [-\tau, 0], \tag{7}$$

where $\phi = [\phi_1, \phi_2, \phi_3, \phi_4] \in \mathcal{C}$ such that $\phi_i(\theta) = \phi_i(0) \geq 0$ for $\theta \in [-\tau, 0], i = 1, 2, 3, 4$, and \mathcal{C} denotes the Banach space $\mathcal{C}([-\tau, 0], \mathbb{R}_+^4)$ of continuous functions mapping the interval $[-\tau, 0]$ into \mathbb{R}_+^4 .

The basic dynamical feature of the model (6) will be explored and the following lemmas are established.

Lemma 1: All solution $(S(t), E(t), I(t), R(t))$ of the model (6) with initial condition (7) are positive for all $t \geq 0$ when $N(t) < K$.

Proof: From the second equation of the model (6), we have

$$\frac{dE}{dt} \geq -(\mu + \sigma + \kappa)E(t).$$

Hence,

$$E(t) \geq E(0)e^{-(\mu + \sigma + \kappa)t} > 0, \text{ for all } t > 0. \tag{8}$$

Similarly, using the same approach as for $E(t)$, it can be shown that $I(t) > 0$ and $R(t) > 0$ for all $t > 0$. Next, we assume that $N(t) < K$ and there exist a constant \tilde{K} which

$$\frac{\beta_1 E(t - \tau) + \beta_2 I(t - \tau)}{N(t - \tau)} S(t - \tau) = \tilde{K} \frac{\beta_1 E(t) + \beta_2 I(t)}{N(t)} S(t), \tag{9}$$

the first equation in the model (6) is rewritten

$$\begin{aligned} \frac{dS(t)}{dt} &= rN(t) \left(1 - \frac{N(t)}{K} \right) - \tilde{K} \frac{\beta_1 E(t) + \beta_2 I(t)}{N(t)} S(t) - \mu S(t), \\ &\geq -\tilde{K} \frac{\beta_1 E(t) + \beta_2 I(t)}{N(t)} S(t) - \mu S(t). \end{aligned} \tag{10}$$

Therefore,

$$S(t) \geq S(0) \exp \left\{ -\int_0^t \left(\tilde{K} \frac{\beta_1 E(u) + \beta_2 I(u)}{N(u)} + \mu \right) du \right\} > 0 \text{ for all } t > 0. \tag{11}$$

Thus, $S(t), E(t), I(t)$ and $R(t)$ are positive for all $t \geq 0$.

Lemma 2: Let $r > \mu$, the closed set

$$\Omega = \left\{ (S, E, I, R) \in R_+^4 : 0 \leq S + E + I + R \leq \frac{K(r - \mu)}{r} \right\}, \tag{12}$$

is positively invariant.

Proof:

Adding all equations in (6) gives

$$\frac{dN(t)}{dt} = rN(t) \left(1 - \frac{N(t)}{K} \right) - \mu N(t) - \alpha I(t). \tag{13}$$

Since $\frac{dN}{dt} \leq \frac{rN}{K} \left[\frac{K(r - \mu)}{r} - N \right]$, it follows that $\frac{dN}{dt} \leq 0$ if $N(t) \geq \frac{K(r - \mu)}{r}$. By standard comparison theorem, it can be shown that

$$N(t) \leq \frac{K(r - \mu)}{r + \frac{r}{N(0)} \left[\frac{K(r - \mu)}{r} - N(0) \right] e^{-(r - \mu)t}}. \tag{14}$$

In particular, $N(t) \leq \frac{K(r-\mu)}{r}$ if $N(0) \leq \frac{K(r-\mu)}{r}$. Thus, Ω is positively invariant. Further, if $N(0) > \frac{K(r-\mu)}{r}$, either the solution of the model (6) enters Ω in finite time, or $N(t)$ approaches $\frac{K(r-\mu)}{r}$ and the variables E, I and R approach to zero.

Hence, the region Ω attracts all solutions in R_+^4 (i.e., all solutions in R_+^4 eventually enter Ω).

Thus, the model (6) is well-posed epidemiologically and mathematically in Ω (Hethcote, 2000). Hence, it is sufficient to study the dynamics of the model in Ω .

2.2 Equilibria of the model

The equilibria of the model (6) can find by setting the right hand side of the model (6) equal to zeros. Let (S^*, E^*, I^*, R^*) be any arbitrary equilibrium point of the model (6) and $N^* = S^* + E^* + I^* + R^*$. By solving equation at steady state, the equilibrium point of the model (6) is given by

$$S^* = \frac{rN^*}{\lambda^* + \mu} \left(1 - \frac{N^*}{K} \right), E^* = \frac{\lambda^* S^*}{k_1}, I^* = \frac{\sigma E^*}{k_2}, R^* = \frac{\kappa E^* + \gamma I^*}{\mu}, \tag{15}$$

where $k_1 = \mu + \sigma + \kappa$ and $k_2 = \mu + \alpha + \gamma$.

Note that, λ^* is the force of infection at steady state, can be expressed as

$$\lambda^* = \frac{\beta_1 E^* + \beta_2 I^*}{N^*}. \tag{16}$$

For convenience in computation, (15) is rewrite in terms of $\lambda^* S^*$ as shown in below:

$$E^* = \frac{\lambda^* S^*}{k_1}, I^* = \frac{\sigma \lambda^* S^*}{k_1 k_2}, R^* = \frac{(\kappa k_2 + \sigma \gamma) \lambda^* S^*}{\mu k_1 k_2}. \tag{17}$$

Substituting (17) in (16) gives

$$\frac{\lambda^* S^*}{\mu k_1 k_2} \left\{ \lambda^* (k_1 k_2 - \sigma \alpha) + \mu k_1 k_2 - \mu (\beta_1 k_2 + \beta_2 \sigma) \right\} = 0, \tag{18}$$

Observe that (18) has two solutions. First, $\lambda^* S^* = 0$ this yield $E^* = I^* = 0$. Substitute these results into (15), the *disease-free equilibrium (DFE)* of the model (6) is presented.

$$\mathcal{E}_0 = (S_0, E_0, I_0, R_0) = \left(\frac{K(r-\mu)}{r}, 0, 0, 0 \right), \tag{19}$$

The *endemic equilibrium (EE)* can find by solving the remaining terms in (18), this give

$$\lambda^* = \frac{\mu k_1 k_2 (R_0 - 1)}{k_1 k_2 - \sigma \alpha}. \tag{20}$$

where R_0 is called basic reproduction number, given by

$$R_0 = \frac{\beta_1 k_2 + \beta_2 \sigma}{k_1 k_2}. \tag{21}$$

From (20), it follow that $\lambda^* > 0$ if $R_0 > 1$. Thus, the model (6) has a unique endemic equilibrium if $R_0 > 1$. The each components of this equilibrium are obtained by substituting (20) into (15). In the case $R_0 < 1$ ($\lambda^* < 0$), the model (6) has no positive

equilibrium which is not biological feasible. Further, if $R_0 = 1$, then $\lambda^* = 0$ corresponds to disease-free equilibrium. These results are summarized below.

Theorem 3: If $R_0 > 1$, the model (6) has a unique endemic equilibrium, given by $\mathcal{E}^* = (S^*, E^*, I^*, R^*)$ when

$$S^* = \frac{K[rR_0(k_1k_2 - \sigma\alpha) - \mu(R_0k_1k_2 - \sigma\alpha)]}{(k_1k_2 - \sigma\alpha)rR_0^2}, E^* = \frac{\mu k_2(R_0 - 1)S^*}{k_1k_2 - \sigma\alpha}, \tag{22}$$

$$I^* = \frac{\mu\sigma(R_0 - 1)S^*}{k_1k_2 - \sigma\alpha}, R^* = \frac{(\kappa k_2 + \sigma\gamma)(R_0 - 1)S^*}{k_1k_2 - \sigma\alpha}.$$

Further, the model (6) has no endemic equilibrium when $R_0 \leq 1$.

3. Stability of Disease-Free Equilibrium

It is known that the stability of the equilibrium is determined by considering the roots of the Jacobian of the model (6) evaluated at the equilibrium point. The equilibrium point is absolutely stable if all roots have negative real parts. On the other hand, it unstable if there exist at least one positive real root. Thus, to analysis the stability of disease-free equilibrium, the Jacobian of the model (6) evaluated at \mathcal{E}_0 is constructed.

$$J(\mathcal{E}_0) = \begin{bmatrix} -r + \mu & -r + 2\mu - \beta_1 e^{-\lambda\tau} & -r + 2\mu - \beta_2 e^{-\lambda\tau} & -r + 2\mu \\ 0 & -k_1 + \beta_1 e^{-\lambda\tau} & \beta_2 e^{-\lambda\tau} & 0 \\ 0 & \sigma & -k_2 & 0 \\ 0 & \kappa & \gamma & -\mu \end{bmatrix} \tag{23}$$

Two eigenvalues of (23) are $\lambda_1 = -r + \mu$, $\lambda_2 = -\mu$ and the other eigenvalues λ_3 and λ_4 are roots of transcendental equation

$$\lambda^2 + (k_1 + k_2 - \beta_1 e^{-\lambda\tau})\lambda + k_1k_2 - \beta_1k_2e^{-\lambda\tau} - \beta_2\sigma e^{-\lambda\tau} = 0, \tag{24}$$

For $\tau = 0$, (24) is reduced to

$$\lambda^2 + \left(k_2 + k_1(1 - R_0) + \frac{\beta_2\sigma}{k_2} \right)\lambda + k_1k_2(1 - R_0) = 0, \tag{25}$$

It is obvious that, all roots of (25) are negative real part when $R_0 < 1$. Thus, if $R_0 < 1$, \mathcal{E}_0 is locally asymptotically stable when $\tau = 0$.

For $\tau > 0$, the existence of pure imaginary root of (24) is investigated. Let $\lambda = i\omega(\omega > 0)$ be root of (24), separating the real and imaginary parts, we have

$$\left. \begin{aligned} (\beta_1k_2 + \beta_2\sigma)\cos\omega\tau + \beta_1\omega\sin\omega\tau &= k_1k_2 - \omega^2, \\ \beta_1\omega\cos\omega\tau - (\beta_1k_2 + \beta_2\sigma)\sin\omega\tau &= (k_1 + k_2)\omega, \end{aligned} \right\} \tag{26}$$

Squaring and adding two equations of (26), we have

$$\omega^4 + \left(\left(\frac{\beta_2 \sigma}{k_2} + k_1(1 - R_0) \right) (k_1 + \beta_1) + k_2^2 \right) \omega^2 + k_1 k_2 (1 - R_0) (k_1 k_2 + \beta_1 k_2 + \beta_2 \sigma) = 0, \tag{27}$$

Obvious that (27) has no positive real roots if $R_0 < 1$. Thus, (24) has no purely imaginary roots. Furthermore, the existence of positive real roots of (24) is considered when $\tau > 0$. It is obvious that $\lambda^2 + (k_1 + k_2)\lambda + k_1 k_2 = 0$ has no positive real roots. By Lemma 3.1 (Tipsri & Chinviriyasit, 2015), all roots of (24) are not positive real roots for all $\tau > 0$. From these results, the following Theorem is established.

Theorem 4: *If $R_0 < 1$, the disease-free equilibrium, \mathcal{E}_0 is absolutely stable for all $\tau \geq 0$.*

4. Stability of Endemic Equilibrium and Bifurcation Analysis

In this section, the stability of endemic equilibrium is analyzed by observing the eigenvalues of model (6). Let

$a^* = r \left(\frac{2N^*}{K} - 1 \right)$ and $q^* = \frac{\beta_1 E^* + \beta_2 I^*}{N^*}$, the Jacobian matrix of the model (6) evaluated at \mathcal{E}^* is given by

$$J(\mathcal{E}^*) = \begin{bmatrix} J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ 0 & \sigma & -k_2 & 0 \\ 0 & \kappa & \gamma & -\mu \end{bmatrix}, \tag{28}$$

where

$$J_{11} = J_{14} - \mu - q^* e^{-\lambda \tau}, \quad J_{12} = J_{14} - \frac{\beta_1 e^{-\lambda \tau}}{R_0}, \quad J_{13} = J_{14} - \frac{\beta_2 e^{-\lambda \tau}}{R_0}, \quad J_{14} = -a^* + \frac{q^* e^{-\lambda \tau}}{R_0},$$

$$J_{21} = -J_{11} - a^* - \mu, \quad J_{22} = -J_{12} - a^* - k_1, \quad J_{23} = -J_{13} - a^*, \quad J_{24} = -J_{14} - a^*.$$

The eigenvalues of $J(\mathcal{E}^*)$ are the roots of the polynomial

$$\lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0 + [b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0] e^{-\lambda \tau} = 0, \tag{29}$$

where

$$a_0 = \mu^2 k_1 k_2 + \mu k_1 k_2 a^*, \quad a_1 = \mu^2 (k_1 + k_2) + 2\mu k_1 k_2 + (k_1 k_2 + \mu k_1 + \mu k_2) a^*,$$

$$a_2 = \mu^2 + 2\mu (k_1 + k_2) + k_1 k_2 + (k_1 + k_2 + \mu) a^*, \quad a_3 = k_1 + k_2 + 2\mu + a^*,$$

$$b_0 = (\mu k_1 k_2 R_0 - 2\mu k_1 k_2) a^* + \mu q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) - \mu^2 k_1 k_2,$$

$$b_1 = \left((k_1 + k_2) q^* - k_1 k_2 - \frac{\beta_1 \mu}{R_0} \right) a^* - 2\mu k_1 k_2 - \frac{\beta_1 \mu^2}{R_0} + \mu q^* (k_1 + k_2) + q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right),$$

$$b_2 = q^* (k_1 + k_2 + \mu) + \left(q^* - \frac{\beta_1}{R_0} \right) a^* - 2\mu \frac{\beta_1}{R_0} - k_1 k_2, \quad b_3 = q^* - \frac{\beta_1}{R_0}.$$

First, the stability of endemic equilibrium, \mathcal{E}^* at $\tau = 0$ is analyzed. Replace $\tau = 0$ in (29), this gives

$$\lambda^4 + (a_3 + b_3)\lambda^3 + (a_2 + b_2)\lambda^2 + (a_1 + b_1)\lambda + (a_0 + b_0) = 0, \tag{30}$$

where

$$\begin{aligned} a_0 + b_0 &= \mu k_1 k_2 (R_0 - 1) a^* + \mu q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right), \quad a_3 + b_3 = k_2 + 2\mu + a^* + q^* + \frac{\beta_2 \sigma}{R_0 k_2}, \\ a_1 + b_1 &= \mu^2 k_2 + \mu k_2 a^* + (k_1 + k_2) q^* a^* + \mu q^* (k_1 + k_2) + q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) + \mu (\mu + a^*) \frac{\beta_2 \sigma}{R_0 k_2}, \\ a_2 + b_2 &= \mu^2 + 2\mu k_2 + (k_2 + \mu) a^* + (k_1 + k_2 + \mu) q^* + a^* q^* + (2\mu + a^*) \frac{\beta_2 \sigma}{R_0 k_2}. \end{aligned}$$

Let $h_i = a_i + b_i$ when $i = 0, 1, 2, 3$. Obvious that the coefficients $h_i (i = 0, 1, 2, 3)$ are all positive if $R_0 > 1$ and $2N^* > K$ (note that $q^* > 0$ if $R_0 > 1$). Furthermore it can be shown that

$$\begin{aligned} h_2 h_3 - h_1 &= (2\mu + a^* + q^*)(\mu + a^*)\mu + (k_1 + k_2)q^* \left(q^* + \frac{\beta_2 \sigma}{R_0 k_2} \right) + k_2^2 q^* + \frac{q^* \sigma \alpha}{R_0} \\ &\quad + \left(2\mu + k_2 + a^* + q^* + \frac{\beta_2 \sigma}{R_0 k_2} \right) \left(2\mu k_2 + k_2 a^* + (\mu + a^*)q^* + (a^* + 2\mu) \frac{\beta_2 \sigma}{R_0 k_2} \right), \\ &> 0. \end{aligned}$$

$$\begin{aligned} h_1 h_2 h_3 - h_1^2 - h_0 h_3^2 &= h_1 (h_2 h_3 - h_1) - h_0 h_3^2, \\ &= \left[\mu k_2 (\mu + a^*) + q^* (k_1 + k_2) (\mu + a^*) + q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) + (\mu + a^*) \frac{\mu \beta_2 \sigma}{R_0 k_2} \right] \\ &\quad \left[\mu (\mu + a^*) (2\mu + a^* + q^*) + (k_1 + k_2) q^* \left(q^* + \frac{\beta_2 \sigma}{R_0 k_2} \right) + q^* k_2^2 + \frac{q^* \sigma \alpha}{R_0} \right] \\ &\quad + \left(k_2 + 2\mu + a^* + q^* + \frac{\beta_2 \sigma}{R_0 k_2} \right) \left\{ \mu q^* \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) \left(\alpha + \gamma + \frac{\beta_2 \sigma}{R_0 k_2} \right) \right. \\ &\quad + \mu q^* (a^* + q^*) \frac{\sigma \alpha}{R_0} + q^* a^* \sigma \alpha \left(k_2 + q^* + \frac{\beta_2 \sigma}{R_0 k_2} \right) \left(1 - \frac{1}{R_0} \right) + q^* a^* \sigma \alpha (2\mu + a^*) \\ &\quad + q^* \mu^2 k_1 k_2 + q^* k_1 (\mu + a^*) \left((2\mu + a^*) \frac{\beta_2 \sigma}{R_0 k_2} + q^* (\mu + a^*) \right) + \left[\mu k_2 (\mu + q^*) \right. \\ &\quad \left. \left. + a^* k_2 (\mu + q) + (\mu + a^*) \frac{\mu \beta_2 \sigma}{R_0 k_2} \right] \left[(2\mu + a^*) \left(k_2 + \frac{\beta_2 \sigma}{R_0 k_2} \right) + q^* (\mu + a^*) \right] \right\}, \\ &> 0. \end{aligned}$$

It is seen that $a_0 + b_0, a_3 + b_3$ and above results are positive when $R_0 < 1$. Hence, by the Routh-Hurwitz criterion, all roots of (30) have negative real parts when $R_0 > 1$ and the following Lemma is established.

Lemma 5: *If $R_0 > 1$ and $2N^* > K$, then the endemic equilibrium, \mathcal{E}^* of the system (6) is asymptotically stable at $\tau = 0$.*

Next, the stability of endemic equilibrium for the case $\tau > 0$ is analyzed. First, we show that (29) has no positive real roots by considering the roots of polynomial

$$f(\lambda) = \lambda^4 + a_3 \lambda^3 + a_2 \lambda^2 + a_1 \lambda + a_0. \tag{31}$$

The coefficients $a_i (i = 0, 1, 2, 3)$ are all positive when $2N^* > K$. Further it can be shown that

$$\begin{aligned} a_2 a_3 - a_1 &= (k_1 + k_2)(k_1 k_2 + (2\mu + a^*)(k_1 + k_2)) + (2\mu + a^*)(\mu(\mu + a^*) + (2\mu + a^*)(k_1 + k_2)), \\ &> 0, \\ a_1(a_2 a_3 - a_1) - a_3^2 a_0 &= (2\mu + a^*)(k_1 + k_2)k_1 k_2(k_1 k_2 + (2\mu + a^*)^2 + (k_1 + k_2)(2\mu + a^*)) + (2\mu + a^*)^2 k_1 k_2 \\ &\quad + \mu(\mu + a^*)(k_1 + k_2)(2\mu + a^*)(k_1^2 + k_2^2 + (2\mu + a^*)(k_1 + k_2) + 1 + \mu(\mu + a^*)), \\ &> 0. \end{aligned}$$

By Routh-Hurwitz criterion, $f(\lambda)$ has no positive real roots. Applying Lemma 3.1 in Tipsri and Chinviriyasit (2015), (29) has no positive real roots for $\tau \geq 0$.

Next, the distribution of roots of (29) is investigated by assuming that it has purely imaginary roots. Replacing $\lambda = i\omega (\omega > 0)$ in (29), this gives

$$\omega^4 - a_3 \omega^3 i - a_2 \omega^2 + a_1 \omega i + a_0 + (-b_3 \omega^3 i - b_2 \omega^2 + b_1 \omega i + b_0) \times (\cos \omega \tau - i \sin \omega \tau) = 0. \tag{32}$$

Separating the real and imaginary parts, this gives

$$\left. \begin{aligned} (-b_2 \omega^2 + b_0) \cos \omega \tau + (b_1 \omega - b_3 \omega^3) \sin \omega \tau &= -\omega^4 + a_2 \omega^2 - a_0, \\ (-b_3 \omega^3 + b_1 \omega) \cos \omega \tau + (b_2 \omega^2 - b_0) \sin \omega \tau &= a_3 \omega^3 - a_1 \omega. \end{aligned} \right\} \tag{33}$$

Squaring and adding both equation of (33), gives

$$\omega^8 + e_3 \omega^6 + e_2 \omega^4 + e_1 \omega^2 + e_0 = 0, \tag{34}$$

where

$$\begin{aligned} e_0 &= a_0^2 - b_0^2, \quad e_1 = a_1^2 - 2a_0 a_2 - b_1^2 + 2b_0 b_2, \\ e_2 &= a_2^2 + 2a_0 a_3 - b_2^2 + 2b_1 b_3 - 2a_1 a_3, \quad e_3 = a_3^2 - b_3^2 - 2a_2. \end{aligned}$$

Let $z = \omega^2$, (34) can be rewritten as

$$h(z) = z^4 + e_3 z^3 + e_2 z^2 + e_1 z + e_0 = 0, \tag{35}$$

where

$$\begin{aligned} e_0 &= (a_0 + b_0)(a_0 - b_0), \\ &= \left\{ a^* \mu k_1 k_2 (R_0 - 1) + \mu q \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) \right\} \left\{ \mu k_1 k_2 (a^* (3 - R_0) + 2\mu) - \mu q \left(k_1 k_2 - \frac{\sigma \alpha}{R_0} \right) \right\}, \\ e_1 &= (a_1^2 - 2a_0 a_3) + (2b_0 b_2 - b_1^2), \\ &= \mu^2 (\mu + a^*)^2 \left\{ \left(k_1 - \frac{\beta_1}{R_0} \right) \left(k_1 + \frac{\beta_1}{R_0} \right) \right\} + (\mu + a^*)^2 \left\{ q^* k_1 k_2 \left(k_1 + k_2 - 2\mu - 2 \frac{\beta_1}{R_0} \right) \right. \\ &\quad \left. + 2\sigma \alpha q^* \left(\frac{\beta_1}{R_0} - q^* \right) + q^* k_1^2 (k_2 - q^*) + q^* k_2^2 (k_1 - q^*) + 2\mu q^* (k_1 + k_2) \frac{\beta_1}{R_0} + \mu^2 k_2^2 \right. \\ &\quad \left. + (R_0 - 1) \left\{ \left(\frac{\sigma \alpha q^*}{R_0} \right)^2 (R_0 + 1) + 2\mu^2 k_1^2 k_2^2 + \frac{2\mu q^* \sigma \alpha k_1 k_2}{R_0} + \frac{2a^* q^* \sigma \alpha k_2 (k_1 - q^*)}{R_0} \right\} \right. \\ &\quad \left. + \frac{2\mu \sigma \alpha q^{*2} (\mu + a^*)}{R_0} \right\} - \mu^2 (R_0 - 1)^2 \left\{ k_1^2 k_2^2 + \frac{2\sigma \alpha k_1 k_2}{R_0 (k_1 k_2 - \sigma \alpha)} \left(\frac{k_1^2 k_2^2 (R_0 - 1)}{k_1 k_2 - \sigma \alpha} \right) \right. \\ &\quad \left. + \frac{a^* k_1 k_2 (k_1 + k_2) (R_0 - 1)}{k_1 k_2 - \sigma \alpha} + \frac{\mu \beta_1}{R_0} \right\}, \end{aligned} \tag{37}$$

$$\begin{aligned}
 e_2 &= (a_2^2 + 2a_0 - 2a_1a_3) + (2b_1b_3 - b_2^2), \\
 &= \mu^2(k_1^2 + k_2^2) + (\mu + a^*)^2 \left\{ \left(k_1 - \frac{\beta_1}{R_0} \right) \left(k_1 + \frac{\beta_1}{R_0} \right) \right\} + q^* k_1^2 (2k_2 - q^*) \\
 &\quad + k_2^2 \left\{ (\mu + a^* + q^*)(\mu + a^* - q^*) \right\} + \mu^2 \left\{ \left(\mu + a^* + \frac{\beta_1}{R_0} \right) \left(\mu + a^* - \frac{\beta_1}{R_0} \right) \right\} \\
 &\quad + \frac{2\sigma\alpha q^*}{R_0} \left(\frac{\beta_1}{R_0} - q^* \right) + \frac{2\mu q^* \beta_1 (k_1 + k_2)}{R_0} + 2q^* k_1 k_2 \left(k_2 - \mu - \frac{\beta_1}{R_0} \right) \\
 &\quad + (\mu + a^*)^2 q^* \left(\frac{\beta_1}{R_0} - q^* \right) + q^* \frac{\beta_1}{R_0} (\mu + a^*)^2,
 \end{aligned} \tag{38}$$

$$\begin{aligned}
 e_3 &= a_3^2 - b_3^2 - 2a_2, \\
 &= (k_1 + k_2 + 2\mu + a^*)^2 - \left(q^* - \frac{\beta_1}{R_0} \right)^2 - 2(\mu^2 + 2\mu(k_1 + k_2) + k_1 k_2 + (k_1 + k_2 + \mu)a^*), \\
 &= k_1^2 + k_2^2 + 2\mu^2 + 2a^* \mu + a^{*2} - q^{*2} + \frac{2q^* \beta_1}{R_0} - \left(\frac{\beta_1}{R_0} \right)^2, \\
 &= \frac{\beta_2 \sigma}{R_0 k_2} \left(k_1 + \frac{\beta_1}{R_0} \right) + (k_2 + q^*)(k_2 - q^*) + 2\mu^2 + 2\mu a^* + a^{*2} + \frac{2q^* \beta_1}{R_0}.
 \end{aligned} \tag{39}$$

Applying with Lemma of Zhang, Cao, and Xu (2015), the roots of (34) is distributed by analyzing $h(z)$ in (35). For $e_0 < 0$, it follows that $h(0) = e_0 < 0$, then as $\lim_{z \rightarrow +\infty} h(z) = +\infty$, thus $h(z) = 0$ has at least one positive root. Further, differentiating (35) with respect to z give

$$h'(z) = 4z^3 + 3e_3 z^2 + 2e_2 z + e_1. \tag{40}$$

It is seen that if all coefficients $e_i (i=1,2,3) > 0$ then $h'(z) > 0$ implies that $h(z)$ is monotonically increasing in $(0, +\infty)$. Thus, there exists a unique positive number z_0 such that $h(z_0) = 0$. By these results the following Lemma is established.

Lemma 6: For the characteristic Equation (29), and the conditions

$$\mu + a^* > \frac{\beta_1}{R_0} > q^*, \tag{41}$$

$$k_1 > \mu + \frac{\beta_1}{R_0}, \tag{42}$$

$$k_2 > \mu + \frac{\beta_1}{R_0}. \tag{43}$$

are satisfied, the following results hold:

- (i) If $e_0 > 0$, then (29) has no pure imaginary root for $\tau > 0$.
- (ii) If $e_0 < 0$, then (29) has a pair of purely imaginary roots $\pm i\omega_0$ when $\tau = \tau_c^j, j = 0, 1, \dots$, where

$$\begin{aligned}
 \tau_c^j &= \frac{1}{\omega_0} \left[\arccos \left\{ \frac{(b_2 - a_3 b_3) \omega_0^6 + (a_1 b_3 + a_3 b_1 - a_2 b_2 - b_0) \omega_0^4}{(b_3 \omega_0^2 - b_1)^2 \omega_0^2 + (b_0 \omega_0^2 - b_0) \omega_0^2} \right. \right. \\
 &\quad \left. \left. + \frac{(a_0 b_2 + a_2 b_0 - a_1 b_1) \omega_0^2 - a_0 b_0}{(b_3 \omega_0^2 - b_1)^2 \omega_0^2 + (b_0 \omega_0^2 - b_0) \omega_0^2} \right\} + 2j\pi \right], j = 0, 1, \dots, n.
 \end{aligned} \tag{44}$$

and $\omega_0 = \sqrt{z_0}$ with z_0 being the unique positive zero of (35).

From, Lemma 3.5 (Tipsri&Chinviriyasit, 2015) and Lemma 6 the following theorem is established.

Theorem 7: If $R_0 > 1, 2N^* > K$ and conditions (41)-(43) are satisfied, then the following results of endemic equilibrium \mathcal{E}^* hold:

(i) \mathcal{E}^* is absolutely stable for $\tau \geq 0$ whenever $e_0 > 0$.

(ii) \mathcal{E}^* is conditionally stable, that is \mathcal{E}^* is asymptotically stable for $\tau \in [0, \tau_c)$ whenever $e_0 < 0$.

Next, we analyzed the bifurcation of model (6). The time delay τ is chosen as a bifurcation parameter. To show that there exists a Hopf bifurcation at $\tau = \tau_c$, it needs to verify that

$$\frac{d\text{Re}(\lambda(\tau))}{d\tau} \Big|_{\tau=\tau_c} > 0. \tag{45}$$

Let $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ be the root of (29), so that $\alpha(\tau_c) = 0$ and $\omega(\tau_c) = \omega_0$ are satisfied when $\tau = \tau_c$. Substituting $\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$ into (29) and differentiating both sides of the resulting equation with respect to τ , this obtained

$$\begin{aligned} (4\lambda^3 + 3a_3\lambda^2 + 2a_2\lambda + a_1) \frac{d\lambda}{d\tau} - (b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)e^{-\lambda\tau} \left(\lambda + \tau \frac{d\lambda}{d\tau} \right) \\ + (3b_3\lambda^2 + 2b_2\lambda + b_1) \frac{d\lambda}{d\tau} = 0. \end{aligned} \tag{46}$$

Hence

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{3\lambda^4 + 2a_3\lambda^3 + a_2\lambda^2 - a_0}{-\lambda^2(\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0)} + \frac{2b_3\lambda^3 + b_2\lambda^2 - b_0}{\lambda^2(b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0)} - \frac{\tau}{\lambda}. \tag{47}$$

Therefore,

$$\text{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1}_{\lambda=i\omega_0} = \frac{3\omega_0^8 + 2e_3\omega_0^6 + e_2\omega_0^4 - e_0}{\omega_0^2[(b_0 - b_2\omega_0)^2 + (b_1\omega_0 - b_3\omega_0^3)^2]}. \tag{48}$$

Therefore,

$$\text{sign} \left\{ \frac{d\alpha(\tau)}{d\tau} \right\}_{\tau=\tau_c} = \text{sign} \left\{ \frac{d\alpha(\tau)}{d\tau} \right\}_{\tau=\tau_c}^{-1} = \text{sign} \left\{ \frac{3\omega_0^8 + 2e_3\omega_0^6 + e_2\omega_0^4 - e_0}{\omega_0^2[(b_0 - b_2\omega_0)^2 + (b_1\omega_0 - b_3\omega_0^3)^2]} \right\}. \tag{49}$$

Obvious that the transversality condition (45) is satisfied when $e_0 < 0$ and $e_2, e_3 > 0$. According to Routh's Theorem, the root of characteristic equation (29) crosses from left to right on the imaginary axis as τ continuously varies from a value less than τ_c to one greater than τ_c . Therefore, the conditions for Hopf bifurcation are satisfied at $\tau = \tau_c$.

Observe that $e_0 < 0$ whenever $2N^* > K, R_0 > 1$ and $R_0 > \frac{3a^* + 2\mu}{a^*}$ whereas e_2 and e_3 are positive when the condition (41)-(43) are satisfy. Thus, from these results, Lemma 5 and 6, the following theorem is established.

Theorem 8: For the model (6), the following results hold:

- (i) If $2N^* > K$, $R_0 > \frac{3a^* + 2\mu}{a^*}$ and conditions (41)-(43) are satisfied, then the endemic equilibrium \mathcal{E}^* of the model (6) is asymptotically stable for $\tau \in [0, \tau_c)$ and it is unstable when $\tau > \tau_c$.
- (ii) If all conditions as stated in (i) hold then the model (6) undergoes a Hopf bifurcation at the endemic equilibrium \mathcal{E}^* when $\tau = \tau_c$.

5. Direction of Hopf Bifurcation

In the previous section, we show that the model (6) undergoes a Hopf bifurcation and the periodic solutions will appear. The periodic solution bifurcate from endemic equilibrium \mathcal{E}^* at the critical values τ_c . The direction, stability and periods of these periodic solutions are determined by using the normal theory and the center manifold theorem as pointed by Hassard *et al.* (1981).

Let $u_1(t) = S(t) - S^*$, $u_2(t) = E(t) - E^*$, $u_3(t) = I(t) - I^*$, $u_4(t) = R(t) - R^*$, $x_i(t) = u_i(\tau t)$, $i = 1, 2, 3, 4$; $\tau = \tau_c + \mu$ where τ_c is defined by (44) and $\mu \in \mathbb{R}$, thus system (6) can be written as a functional differential equation in $C = C([-1, 0], \mathbb{R}^4)$ as

$$\dot{x} = L_\mu(x_t) + f(\mu, x_t), \tag{50}$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^T \in \mathbb{R}^4$ and $L_\mu : C \rightarrow \mathbb{R}^4, f : \mathbb{R} \times C \rightarrow \mathbb{R}^4$ are given respective by

$$L_\mu(\phi) = (\tau_c + \mu) M U(0) + (\tau_c + \mu) N U(-1), \tag{51}$$

where

$$M = \begin{bmatrix} -(a^* + \mu) & -a^* & -a^* & -a^* \\ 0 & -k_1 & 0 & 0 \\ 0 & \sigma & -k_2 & 0 \\ 0 & \kappa & \gamma & -\mu \end{bmatrix}, N = \begin{bmatrix} -m_1 & -m_2 & -m_3 & -m_4 \\ m_1 & m_2 & m_3 & m_4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, U(a) = \begin{bmatrix} \phi_1(a) \\ \phi_2(a) \\ \phi_3(a) \\ \phi_4(a) \end{bmatrix}$$

$$m_1 = q^* \left(1 - \frac{S^*}{N^*} \right), m_2 = (\beta_1 - q^*) \frac{S^*}{N^*}, m_3 = (\beta_2 - q^*) \frac{S^*}{N^*}, m_4 = -q^* \frac{S^*}{N^*}.$$

$$f(\mu, \phi) = (\tau_c + \mu) \begin{bmatrix} F_1 - F_2 \\ F_2 \\ 0 \\ 0 \end{bmatrix},$$

where

$$F_1 = n_1\phi_1^2(0) + n_2\phi_2^2(0) + n_3\phi_3^2(0) + n_4\phi_4^2(0) + n_5\phi_1(0)\phi_2(0) + n_6\phi_1(0)\phi_3(0) + n_7\phi_1(0)\phi_4(0) + n_8\phi_2(0)\phi_3(0) + n_9\phi_2(0)\phi_4(0) + n_{10}\phi_3(0)\phi_4(0),$$

$$F_2 = l_1\phi_1^2(-1) + l_2\phi_2^2(-1) + l_3\phi_3^2(-1) + l_4\phi_4^2(-1) + l_5\phi_1(-1)\phi_2(-1) + l_6\phi_1(-1)\phi_3(-1) + l_7\phi_1(-1)\phi_4(-1) + l_8\phi_2(-1)\phi_3(-1) + l_9\phi_2(-1)\phi_4(-1) + l_{10}\phi_3(-1)\phi_4(-1),$$

$$\begin{aligned}
 n_1 = n_2 = n_3 = n_4 &= -\frac{r}{K}, n_5 = n_6 = n_7 = n_8 = n_9 = n_{10} = -2\frac{r}{K}, l_1 = -\frac{q^*}{N^*} \left(1 - \frac{S^*}{N^*} \right), \\
 l_2 &= -(\beta_1 - q^*) \frac{S^*}{N^{*2}}, l_3 = -(\beta_2 - q^*) \frac{S^*}{N^{*2}}, l_4 = q^* \frac{S^*}{N^{*2}}, l_5 = (\beta_1 - q^*) \frac{1}{N^*} - (\beta_1 - 2q^*) \frac{S^*}{N^{*2}}, \\
 l_6 &= (\beta_2 - q^*) \frac{1}{N^*} - (\beta_2 - 2q^*) \frac{S^*}{N^{*2}}, l_7 = -\frac{q^*}{N^*} + 2q^* \frac{S^*}{N^{*2}}, l_8 = -(\beta_1 + \beta_2 - 2q^*) \frac{S^*}{N^{*2}}, \\
 l_9 &= -(\beta_1 - 2q^*) \frac{S^*}{N^{*2}}, l_{10} = -(\beta_2 - 2q^*) \frac{S^*}{N^{*2}}.
 \end{aligned}$$

By the Riesz representation theorem, there exist a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [-1, 0]$, such that

$$L_\mu(\phi) = \int_{-1}^0 d\eta(\theta, \mu)\phi(\theta), \quad \text{for } \phi \in C. \tag{52}$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_c + \mu)M \delta(\theta) - (\tau_c + \mu)N \delta(\theta + 1), \tag{53}$$

where $\delta(\theta)$ is Dirac delta function.

For $\phi \in C^1([-1, 0], R^4)$, define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu)\phi(s), & \theta = 0, \end{cases} \tag{54}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \phi), & \theta = 0. \end{cases} \tag{55}$$

Then system (50) is equivalent to

$$\dot{x}_t = A(\mu)x_t + R(\mu)x_t, \tag{56}$$

where

$$x_t(\theta) = x(t + \theta) \quad \text{for } \theta \in [-1, 0]. \tag{57}$$

For $\psi \in C^1([0, 1], (R^4)^*)$, define

$$A^*\psi(s) = \begin{cases} -\frac{d\psi(s)}{ds}, & s \in (0, 1] \\ \int_{-1}^0 d\eta^T(t, 0)\psi(-t), & s = 0, \end{cases} \tag{58}$$

and bilinear inner product

$$\langle \psi(s), \phi(\theta) \rangle = \bar{\psi}(0)\phi(0) - \int_{\theta=-1}^0 \int_{\xi=0}^\theta \bar{\psi}(\xi - \theta) d\eta(\theta)\phi(\xi) d\xi, \tag{59}$$

where $\eta(\theta) = \eta(\theta, 0)$. $A(0)$ and A^* are adjoint operators. By discussion in Section 4, we know that $\pm i\omega_0\tau_c$ are eigenvalues of $A(0)$. Thus, they are also eigenvalues of A^* . We need to compute eigenvector of $A(0)$ and A^* which corresponding to eigenvalues $i\omega_0\tau_c$ and $-i\omega_0\tau_c$, respectively. Suppose $v(\theta) = (1, v_1, v_2, v_3)^T e^{i\omega_0\tau_c\theta}$ is the eigenvector of $A(0)$ corresponding to $i\omega_0\tau_c$, then $A(0)v(0) = i\omega_0\tau_c v(0)$. It follows from the definition of $A(0)$, (51), (52) and (53), we have

$$\tau_c M v(0) + \tau_c N v(-1) = i\omega_0 \tau_c v(0). \tag{60}$$

For $v(-1) = v(0)e^{-i\omega_0\tau_c}$, then we obtain

$$\begin{aligned} v_1 &= -\frac{(\mu + i\omega_0)(\mu + i\omega_0 + a^*)(k_2 + i\omega_0)}{(a^* + k_1 + i\omega_0)(k_2 + i\omega_0)(\mu + i\omega_0) + a^* \sigma(\mu + i\omega_0) + a^* \kappa(k_2 + i\omega_0) + a^* \sigma\gamma}, \\ v_2 &= -\frac{\sigma(\mu + i\omega_0)(\mu + i\omega_0 + a^*)}{(a^* + k_1 + i\omega_0)(k_2 + i\omega_0)(\mu + i\omega_0) + a^* \sigma(\mu + i\omega_0) + a^* \kappa(k_2 + i\omega_0) + a^* \sigma\gamma}, \\ v_3 &= -\frac{(\mu + i\omega_0 + a^*)(\kappa(k_2 + i\omega_0) + \sigma\gamma)}{(a^* + k_1 + i\omega_0)(k_2 + i\omega_0)(\mu + i\omega_0) + a^* \sigma(\mu + i\omega_0) + a^* \kappa(k_2 + i\omega_0) + a^* \sigma\gamma}. \end{aligned}$$

Similarly, we can obtain the eigenvector $v^*(s) = D(1, v_1^*, v_2^*, v_3^*)e^{i\omega_0\tau_c s}$ of A^* corresponding to $-i\omega_0\tau_c$. It follows from the definition of A^* , (51), (52) and (53), we obtain

$$\begin{aligned} v_1^* &= \frac{\mu + a^* - i\omega_0 + m_1 e^{i\omega_0\tau_c}}{m_1 e^{i\omega_0\tau_c}}, \quad v_3^* = \frac{(i\omega_0 - \mu - a^*)m_4 + a^* m_1}{m_1(i\omega_0 - \mu)}, \\ v_2^* &= \frac{(i\omega_0 - \gamma - \mu)a^* m_1 + (i\omega_0 - \mu - a^*)(i\omega_0 - \mu)m_3 - \gamma m_4}{(i\omega_0 - k_2)(i\omega_0 - \mu)m_1}. \end{aligned} \tag{61}$$

In order to assure $\langle v^*(s), v(\theta) \rangle = 1$, the value of \bar{D} is determined. By (59), we have

$$\begin{aligned} \langle v^*(s), v(\theta) \rangle &= \bar{D}(1, \bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*)(1, v_1, v_2, v_3)^T \\ &\quad - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{D}(1, \bar{v}_1^*, \bar{v}_2^*, \bar{v}_3^*)e^{-i\omega_0\tau_c(\xi-\theta)} d\eta(\theta)(1, v_1, v_2, v_3)^T e^{i\omega_0\tau_c\xi} d\xi, \\ &= \bar{D} \left\{ 1 + v_1 \bar{v}_1^* + v_2 \bar{v}_2^* + v_3 \bar{v}_3^* + \tau_c (\bar{v}_1^* - 1)(m_1 + m_2 v_1 + m_3 v_2 + m_4 v_3) e^{-i\omega_0\tau_c} \right\}. \end{aligned}$$

Therefore, we can choose \bar{D} as

$$\bar{D} = \frac{1}{1 + v_1 \bar{v}_1^* + v_2 \bar{v}_2^* + v_3 \bar{v}_3^* + \tau_c (\bar{v}_1^* - 1)(m_1 + m_2 v_1 + m_3 v_2 + m_4 v_3) e^{-i\omega_0\tau_c}} \tag{62}$$

In the following, we use the ideas of Hassard *et al.* (1981) to compute the coordinates describing center manifold C_0 at $\mu = 0$.

Define

$$z(t) = \langle v^*, x_t \rangle, \quad W(t, \theta) = x_t - z(t)v(\theta) - \bar{z}(t)\bar{v}(\theta) = x_t - 2\text{Re}\{z(t)v(\theta)\}. \tag{63}$$

On the center manifold C_0 , we have

$$W(t, \theta) = W(z(t), \bar{z}(t), \theta) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \dots, \tag{64}$$

where z and \bar{z} are local coordinates for center manifold C_0 in the direction of v^* and \bar{v}^* . Note that W is real if x_t is real.

We only consider the real solutions. For the solution $x_t \in C_0$ of (56), since $\mu = 0$ and (50), we have

$$\begin{aligned} \dot{z}(t) &= i\omega_0 \tau_c z(t) + \bar{v}^*(0) f(0, W(z(t), \bar{z}(t), 0)) + 2\text{Re}\{z(t), v(\theta)\}, \\ &= i\omega_0 \tau_c z(t) + \bar{v}^*(0) f_0(z, \bar{z}), \\ &= i\omega_0 \tau_c z(t) + g(z, \bar{z}). \end{aligned} \tag{65}$$

where

$$g(z, \bar{z}) = \bar{v}^*(0)f_0(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11}z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} + \dots \tag{66}$$

From (63), we have $x_t = W(z, \bar{z}, \theta) + z\nu + \bar{z}\bar{\nu}$. Thus,

$$\begin{aligned} x_{nt}(0) &= z\nu_{n-1} + \bar{z}\bar{\nu}_{n-1} + W_{20}^{(n)}(0) \frac{z^2}{2} + W_{11}^{(n)}(0)z\bar{z} + W_{02}^{(n)}(0) \frac{\bar{z}^2}{2} + \dots, \\ x_{nt}(-1) &= z\nu_{n-1}e^{-i\omega_0\tau_c} + \bar{z}\bar{\nu}_{n-1}e^{i\omega_0\tau_c} + W_{20}^{(n)}(-1) \frac{z^2}{2} + W_{11}^{(n)}(-1)z\bar{z} + W_{02}^{(n)}(-1) \frac{\bar{z}^2}{2} + \dots, \end{aligned} \tag{67}$$

for $n = 1, 2, 3, 4$ and $\nu_0 = \bar{\nu}_0 = 1$.

It follows that

$$g(z, \bar{z}) = \bar{v}^*(0)f_0(z, \bar{z}) = \bar{v}^*(0)f(0, x_t) = \bar{D}\tau_c(F_1 + (\bar{v}_1^* - 1)F_2). \tag{68}$$

where

$$\begin{aligned} F_1 &= n_1x_{1t}^2(0) + n_2x_{2t}^2(0) + n_3x_{3t}^2(0) + n_4x_{4t}^2(0) + n_5x_{1t}(0)x_{2t}(0) + n_6x_{1t}(0)x_{3t}(0) \\ &\quad + n_7x_{1t}(0)x_{4t}(0) + n_8x_{2t}(0)x_{3t}(0) + n_9x_{2t}(0)x_{4t}(0) + n_{10}x_{3t}(0)x_{4t}(0), \\ F_2 &= l_1x_{1t}^2(-1) + l_2x_{2t}^2(-1) + l_3x_{3t}^2(-1) + l_4x_{4t}^2(-1) + l_5x_{1t}(-1)x_{2t}(-1) + l_6x_{1t}(-1)x_{3t}(-1) \\ &\quad + l_7x_{1t}(-1)x_{4t}(-1) + l_8x_{2t}(-1)x_{3t}(-1) + l_9x_{2t}(-1)x_{4t}(-1) + l_{10}x_{3t}(-1)x_{4t}(-1). \end{aligned}$$

Comparing the coefficients with (66), we have

$$\begin{aligned} g_{20} &= 2\bar{D}\tau_c \left\{ (n_1 + n_2\nu_1^2 + n_3\nu_2^2 + n_4\nu_3^2 + n_5\nu_1 + n_6\nu_2 + n_7\nu_3 + n_8\nu_1\nu_2 + n_9\nu_1\nu_3 + n_{10}\nu_2\nu_3) + \right. \\ &\quad \left. (\bar{v}_1^* - 1)e^{-2i\omega_0\tau_c} (l_1 + l_2\nu_1^2 + l_3\nu_2^2 + l_4\nu_3^2 + l_5\nu_1 + l_6\nu_2 + l_7\nu_3 + l_8\nu_1\nu_2 + l_9\nu_1\nu_3 + l_{10}\nu_2\nu_3) \right\}, \\ g_{02} &= 2\bar{D}\tau_c \left\{ (n_1 + n_2\bar{\nu}_1^2 + n_3\bar{\nu}_2^2 + n_4\bar{\nu}_3^2 + n_5\bar{\nu}_1 + n_6\bar{\nu}_2 + n_7\bar{\nu}_3 + n_8\bar{\nu}_1\bar{\nu}_2 + n_9\bar{\nu}_1\bar{\nu}_3 + n_{10}\bar{\nu}_2\bar{\nu}_3) + \right. \\ &\quad \left. (\bar{v}_1^* - 1)e^{2i\omega_0\tau_c} (l_1 + l_2\bar{\nu}_1^2 + l_3\bar{\nu}_2^2 + l_4\bar{\nu}_3^2 + l_5\bar{\nu}_1 + l_6\bar{\nu}_2 + l_7\bar{\nu}_3 + l_8\bar{\nu}_1\bar{\nu}_2 + l_9\bar{\nu}_1\bar{\nu}_3 + l_{10}\bar{\nu}_2\bar{\nu}_3) \right\}, \\ g_{11} &= \bar{D}\tau_c \left\{ (2n_1 + 2n_2\nu_1\bar{\nu}_1 + 2n_3\nu_2\bar{\nu}_2 + 2n_4\nu_3\bar{\nu}_3 + n_5(\nu_1 + \bar{\nu}_1) + n_6(\nu_2 + \bar{\nu}_2) + n_7(\nu_3 + \bar{\nu}_3) \right. \\ &\quad + n_8(\nu_1\bar{\nu}_2 + \bar{\nu}_1\nu_2) + n_9(\nu_1\bar{\nu}_3 + \bar{\nu}_1\nu_3) + n_{10}(\nu_2\bar{\nu}_3 + \bar{\nu}_2\nu_3) + (\bar{v}_1^* - 1)(2l_1 + 2l_2\nu_1\bar{\nu}_1 \\ &\quad + 2l_3\nu_2\bar{\nu}_2 + 2l_4\nu_3\bar{\nu}_3 + l_5(\nu_1 + \bar{\nu}_1) + l_6(\nu_2 + \bar{\nu}_2) + l_7(\nu_3 + \bar{\nu}_3) + l_8(\nu_1\bar{\nu}_2 + \bar{\nu}_1\nu_2) \\ &\quad \left. + l_9(\nu_1\bar{\nu}_3 + \bar{\nu}_1\nu_3) + l_{10}(\nu_2\bar{\nu}_3 + \bar{\nu}_2\nu_3) \right\}, \\ g_{21} &= 2\bar{D}\tau_c \left\{ (2n_1 + n_5\nu_1 + n_6\nu_2 + n_7\nu_3)W_{11}^{(1)}(0) + (2n_2\nu_1 + n_5 + n_8\nu_2 + n_9\nu_3)W_{11}^{(2)}(0) \right. \\ &\quad + (2n_3\nu_2 + n_6 + n_8\nu_1 + n_{10}\nu_3)W_{11}^{(3)}(0) + (2n_4\nu_3 + n_7 + n_9\nu_1 + n_{10}\nu_2)W_{11}^{(4)}(0) \\ &\quad + \left(n_1 + \frac{1}{2}(n_5\bar{\nu}_1 + n_6\bar{\nu}_2 + n_7\bar{\nu}_3) \right) W_{20}^{(1)}(0) + \left(n_2\bar{\nu}_1 + \frac{1}{2}(n_5 + n_8\bar{\nu}_2 + n_9\bar{\nu}_3) \right) W_{20}^{(2)}(0) \\ &\quad + \left(n_3\bar{\nu}_2 + \frac{1}{2}(n_6 + n_8\bar{\nu}_1 + n_{10}\bar{\nu}_3) \right) W_{20}^{(3)}(0) + \left(n_4\bar{\nu}_3 + \frac{1}{2}(n_7 + n_9\bar{\nu}_1 + n_{10}\bar{\nu}_2) \right) W_{20}^{(4)}(0) \\ &\quad + (\bar{v}_1^* - 1)e^{i\omega_0\tau_c} \left(\left(l_1 + \frac{1}{2}(l_5\bar{\nu}_1 + l_6\bar{\nu}_2 + l_7\bar{\nu}_3) \right) W_{20}^{(1)}(-1) \right. \\ &\quad + \left(l_2\bar{\nu}_1 + \frac{1}{2}(l_5 + l_8\bar{\nu}_2 + l_9\bar{\nu}_3) \right) W_{20}^{(2)}(-1) + \left(l_3\bar{\nu}_2 + \frac{1}{2}(l_6 + l_8\bar{\nu}_1 + l_{10}\bar{\nu}_3) \right) W_{20}^{(3)}(-1) \\ &\quad \left. + \left(l_4\bar{\nu}_3 + \frac{1}{2}(l_7 + l_9\bar{\nu}_1 + l_{10}\bar{\nu}_2) \right) W_{20}^{(4)}(-1) \right) \\ &\quad + (\bar{v}_1^* - 1)e^{-i\omega_0\tau_c} \left((2l_1 + l_5\nu_1 + l_6\nu_2 + l_7\nu_3)W_{11}^{(1)}(-1) + \right. \\ &\quad + (2l_2\nu_1 + l_5 + l_8\nu_2 + l_9\nu_3)W_{11}^{(2)}(-1) + (2l_3\nu_2 + l_6 + l_8\nu_1 + l_{10}\nu_3)W_{11}^{(3)}(-1) \\ &\quad \left. + (2l_4\nu_3 + l_7 + l_9\nu_1 + l_{10}\nu_2)W_{11}^{(4)}(-1) \right) \left. \right\} \tag{69} \end{aligned}$$

In order to determine g_{21} , we need to compute $W_{20}(\theta)$ and $W_{11}(\theta)$. From (56) and (63), we have

$$\begin{aligned} \dot{W} &= \dot{x}_t - \dot{z}v - \dot{\bar{z}}\bar{v}, \\ &= \begin{cases} A(0)W - 2\text{Re}\{\bar{v}^*(0)f_0(z, \bar{z})v(\theta)\}, & \theta \in [-1, 0), \\ A(0)W - 2\text{Re}\{\bar{v}^*(0)f_0(z, \bar{z})v(0)\} + f_0(z, \bar{z}), & \theta = 0, \end{cases} \\ &= A(0)W + H(z, \bar{z}, \theta), \end{aligned} \tag{70}$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + \dots \tag{71}$$

Substituting the series (64) and (71) into (70) and comparing the coefficients, we have

$$(A(0) - 2i\omega_0\tau_c I)W_{20}(\theta) = -H_{20}(\theta), \quad A(0)W_{11}(\theta) = -H_{11}(\theta). \tag{72}$$

By (70), we know that for $\theta \in [-1, 0)$,

$$H(z, \bar{z}, \theta) = -\bar{v}^*(0)f_0(z, \bar{z})v(\theta) - v^*(0)\bar{f}_0(z, \bar{z})\bar{v}(\theta) = -g(z, \bar{z})v(\theta) - \bar{g}(z, \bar{z})\bar{v}(\theta) \tag{73}$$

Comparing the coefficients with (71), this gives

$$H_{20}(\theta) = -g_{20}v(\theta) - \bar{g}_{02}\bar{v}(\theta), \quad H_{11}(\theta) = -g_{11}v(\theta) - \bar{g}_{11}\bar{v}(\theta). \tag{74}$$

From (72), (74), and the definition of $A(0)$, we have

$$\dot{W}_{20}(\theta) = 2i\omega_0\tau_c W_{20}(\theta) + g_{20}v(\theta) + \bar{g}_{02}\bar{v}(\theta), \quad \dot{W}_{11}(\theta) = g_{11}v(\theta) - \bar{g}_{11}\bar{v}(\theta). \tag{75}$$

Noticing $v(\theta) = v(0)e^{i\omega_0\tau_c\theta}$, we have

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_0\tau_c} v(0)e^{i\omega_0\tau_c\theta} + \frac{i\bar{g}_{02}}{3\omega_0\tau_c} \bar{v}(0)e^{-i\omega_0\tau_c\theta} + E_1 e^{2i\omega_0\tau_c\theta}, \tag{76}$$

where $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}, E_1^{(4)})^T \in R^4$ is a constant vector. Similarly, we have

$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_0\tau_c} v(0)e^{i\omega_0\tau_c\theta} + \frac{i\bar{g}_{11}}{\omega_0\tau_c} \bar{v}(0)e^{-i\omega_0\tau_c\theta} + E_2, \tag{77}$$

where $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}, E_2^{(4)})^T \in R^4$ is a constant vector.

In the following, we will find out E_1 and E_2 . From the definition of $A(0)$ and (72), we have

$$\int_{-1}^0 d\eta(\theta)W_{20}(\theta) = 2i\omega_0\tau_c W_{20}(0) - H_{20}(0), \tag{78}$$

$$\int_{-1}^0 d\eta(\theta)W_{11}(\theta) = -H_{11}(0), \tag{79}$$

where $\eta(\theta) = \eta(0, \theta)$. By (70), we know that when $\theta = 0$

$$\begin{aligned} H(z, \bar{z}, 0) &= -\bar{v}^*(0)f_0(z, \bar{z})v(0) - v^*(0)\bar{f}_0(z, \bar{z})\bar{v}(0) + f_0(z, \bar{z}) \\ &= -g(z, \bar{z})v(0) - \bar{g}(z, \bar{z})\bar{v}(0) + f_0(z, \bar{z}). \end{aligned}$$

Applying with (71), this give

$$H_{20}(0) = -g_{20}v(0) - \bar{g}_{02}\bar{v}(0) + \tau_c(G_1, G_2, 0, 0)^T, \tag{80}$$

$$H_{11}(0) = -g_{11}v(0) - \bar{g}_{11}\bar{v}(0) + \tau_c(G_3, G_4, 0, 0)^T, \tag{81}$$

where

$$\begin{aligned} G_1 &= 2\left\{ (n_1 + n_2v_1^2 + n_3v_2^2 + n_4v_3^2 + n_5v_1 + n_6v_2 + n_7v_3 + n_8v_1v_2 + n_9v_1v_3 + n_{10}v_2v_3) \right. \\ &\quad \left. - e^{-2i\omega_0\tau_c} (l_1 + l_2v_1^2 + l_3v_2^2 + l_4v_3^2 + l_5v_1 + l_6v_2 + l_7v_3 + l_8v_1v_2 + l_9v_1v_3 + l_{10}v_2v_3) \right\}, \\ G_2 &= 2e^{-2i\omega_0\tau_c} (l_1 + l_2v_1^2 + l_3v_2^2 + l_4v_3^2 + l_5v_1 + l_6v_2 + l_7v_3 + l_8v_1v_2 + l_9v_1v_3 + l_{10}v_2v_3), \\ G_3 &= 2(n_1 - l_1) + 2(n_2 - l_2)v_1\bar{v}_1 + 2(n_3 - l_3)v_2\bar{v}_2 + 2(n_4 - l_4)v_3\bar{v}_3 + (n_5 - l_5)(v_1 + \bar{v}_1) \\ &\quad + (n_6 - l_6)(v_2 + \bar{v}_2) + (n_7 - l_7)(v_3 + \bar{v}_3) + (n_8 - l_8)(v_1\bar{v}_2 + \bar{v}_1v_2) \\ &\quad + (n_9 - l_9)(v_1\bar{v}_3 + \bar{v}_1v_3) + (n_{10} - l_{10})(v_2\bar{v}_3 + \bar{v}_2v_3), \\ G_4 &= 2l_1 + 2l_2v_1\bar{v}_1 + 2l_3v_2\bar{v}_2 + 2l_4v_3\bar{v}_3 + l_5(v_1 + \bar{v}_1) + l_6(v_2 + \bar{v}_2) + l_7(v_3 + \bar{v}_3) \\ &\quad + l_8(v_1\bar{v}_2 + \bar{v}_1v_2) + l_9(v_1\bar{v}_3 + \bar{v}_1v_3) + l_{10}(v_2\bar{v}_3 + \bar{v}_2v_3). \end{aligned}$$

Since $i\omega_0\tau_c$ is the eigenvalue of $A(0)$ and $v(0)$ is the corresponding eigenvector, we obtain

$$\begin{aligned} \left(i\omega_0\tau_c I - \int_{-1}^0 e^{i\theta\omega_0\tau_c} d\eta(\theta) \right) v(0) &= 0, \\ \left(-i\omega_0\tau_c I - \int_{-1}^0 e^{-i\theta\omega_0\tau_c} d\eta(\theta) \right) \bar{v}(0) &= 0. \end{aligned} \tag{82}$$

Substituting (76) and (80) into (78), this yields

$$\left(2i\omega_0\tau_c I - \int_{-1}^0 e^{2i\theta\omega_0\tau_c} d\eta(\theta) \right) E_1 = \tau_c(G_1, G_2, 0, 0) \tag{83}$$

Then, it follows that

$$\begin{aligned} E_1^{(1)} &= \frac{(2i\omega_0 + k_1)G_1 - (G_1 + G_2)C_4 + m_1a^*G_2(1 + C_1 + C_2)e^{-2i\omega_0\tau_c}}{(2i\omega_0 + k_1)C_3 - (2i\omega_0 + a^* + \mu)C_4 + m_1a^*(1 + C_1 + C_2)e^{-2i\omega_0\tau_c}}, \\ E_1^{(2)} &= \frac{G_1m_1e^{-2i\omega_0\tau_c} + G_2C_3}{(2i\omega_0 + k_1)C_3 - (2i\omega_0 + a^* + \mu)C_4 + m_1a^*(1 + C_1 + C_2)e^{-2i\omega_0\tau_c}}, \\ E_1^{(3)} &= \frac{C_1(G_1m_1e^{-2i\omega_0\tau_c} + G_2C_3)}{(2i\omega_0 + k_1)C_3 - (2i\omega_0 + a^* + \mu)C_4 + m_1a^*(1 + C_1 + C_2)e^{-2i\omega_0\tau_c}}, \\ E_1^{(4)} &= \frac{C_2(G_1m_1e^{-2i\omega_0\tau_c} + G_2C_3)}{(2i\omega_0 + k_1)C_3 - (2i\omega_0 + a^* + \mu)C_4 + m_1a^*(1 + C_1 + C_2)e^{-2i\omega_0\tau_c}}, \end{aligned} \tag{84}$$

where $C_1 = \frac{\sigma}{2i\omega_0 + k_2}$, $C_2 = \frac{\kappa(2i\omega_0 + k_2) + \sigma\gamma}{(2i\omega_0 + \mu)(2i\omega_0 + k_2)}$, $C_3 = 2i\omega_0 + a^* + \mu + m_1e^{-2i\omega_0\tau_c}$ and $C_4 = (m_2 + m_3C_1 + m_4C_2)e^{-2i\omega_0\tau_c}$.

Similarly, substitute (77) and (81) into (79), we obtain

$$\int_{-1}^0 d\eta(\theta)E_2 = -\tau_c(G_3, G_4, 0, 0)^T \tag{85}$$

It follows that

$$\begin{aligned} E_2^{(1)} &= \frac{(G_3 + G_4)D_1 + (k_1k_2 - \sigma\alpha)a^*G_4 - \mu k_1k_2G_3}{(a^* + \mu)D_1 + \mu k_1k_2D_2 - (k_1k_2 - \sigma\alpha)m_1a^*}, \\ E_2^{(2)} &= \frac{\mu k_2(D_2G_4 - m_1G_3)}{(a^* + \mu)D_1 + \mu k_1k_2D_2 - (k_1k_2 - \sigma\alpha)m_1a^*}, \\ E_2^{(3)} &= \frac{\mu\sigma(D_2G_4 - m_1G_3)}{(a^* + \mu)D_1 + \mu k_1k_2D_2 - (k_1k_2 - \sigma\alpha)m_1a^*}, \\ E_2^{(4)} &= \frac{(\kappa k_2 + \sigma\gamma)(D_2G_4 - m_1G_3)}{(a^* + \mu)D_1 + \mu k_1k_2D_2 - (k_1k_2 - \sigma\alpha)m_1a^*}, \end{aligned} \tag{86}$$

where $D_1 = \mu k_2 m_2 + \mu \sigma m_3 + (\kappa k_2 + \sigma \gamma) m_4$ and $D_2 = -(a^* + \mu + m_1)$. Therefore, we can determined $W_{20}(\theta)$ and $W_{11}(\theta)$ from (76) and (77), respectively. Furthermore, all g_{ij} have been expressed in terms of parameters, and we can compute the following values:

$$\left. \begin{aligned} c_1(0) &= \frac{i}{2\tau_c \omega_0} \left(g_{11} g_{20} - 2 |g_{11}|^2 - \frac{|g_{02}|^2}{3} + \frac{g_{21}}{2} \right), \\ \mu_2 &= \frac{-\operatorname{Re}\{c_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_c)\}}, \\ \beta_2 &= 2\operatorname{Re}\{c_1(0)\}, \\ T_2 &= \frac{-\operatorname{Im}\{c_1(0)\} + \mu_2 \operatorname{Im}\lambda'(\tau_c)}{\tau_c \omega_0}, \end{aligned} \right\} \tag{87}$$

From the conclusion in Hassard *et al.* (1981), the main results in this section are concluded in the following theorem.

Theorem 9: The model (6), when $\tau = \tau_c$, the direction and the stability of periodic solution of Hopf bifurcation is determined by (87). Then

- (i) the sign of μ_2 determines the direction of the Hopf bifurcation: if $\mu_2 > 0 (\mu_2 < 0)$, then the Hopf bifurcation is supercritical (subcritical) and the bifurcation periodic solutions exist for $\tau > \tau_c (\tau < \tau_c)$;
- (ii) the sign of β_2 determines the stability of the bifurcating periodic solution: the bifurcation periodic solutions are stable (unstable) if $\beta_2 < 0 (\beta_2 > 0)$;
- (iii) the sign of T_2 determines the period of the bifurcating periodic solutions: the period increase (decrease) if $T_2 > 0 (T_2 < 0)$.

6. Numerical Simulation

To see the behavior of the model (6), the numerical solutions of the model (6) are simulated by using MATLAB programming with RK4 method. Parameters values use in simulating the model (6) are based on the influenza H1N1 parameters as shown in Table 1.

Table 1. Parameters are used in numerical simulation (unit week where applicable).

Parameter	Description	Value	Source
r	Birth rate	0.5	Assumed
K	Carrying capacity	10,000	Assumed
β	Contact rate	7.29	Assumed
β_E	Ability to cause infection by exposed individuals	0.21	Assumed
β_I	Ability to cause infection by infectious individuals	0.84	Assumed
μ	Natural mortality rate	0.000263	[1]
σ	Mean duration of latency	2.32	[2]
γ	Recovery rate of clinical ill	1.4	[2]
\mathcal{K}	Recovery rate in latent period	1.3	[3]
α	Disease induced mortality rate	0.065	[4]

[1] Index Mundi (2015). [2] Pourbohloul-Brunham *et al.* (2009); Yang-Longini *et al.* (2009); Tuite-Fisman *et al.* (2010). [3] Massad *et al.* (2007). [4] Longini Jr, Ackerman, and Elveback (1978).

To study the stability of the disease-free equilibrium, \mathcal{E}_0 , we choose $\beta = 2$ which yields $R_0 = 0.85$. Applying with Theorem 4, the disease-free equilibrium of the model (6) is absolutely stable for all $\tau \geq 0$. These results are shown by simulating the numerical solutions with various τ as shown in Figure 1. Observe that all solutions converge to disease-free equilibrium, \mathcal{E}_0 with various τ .

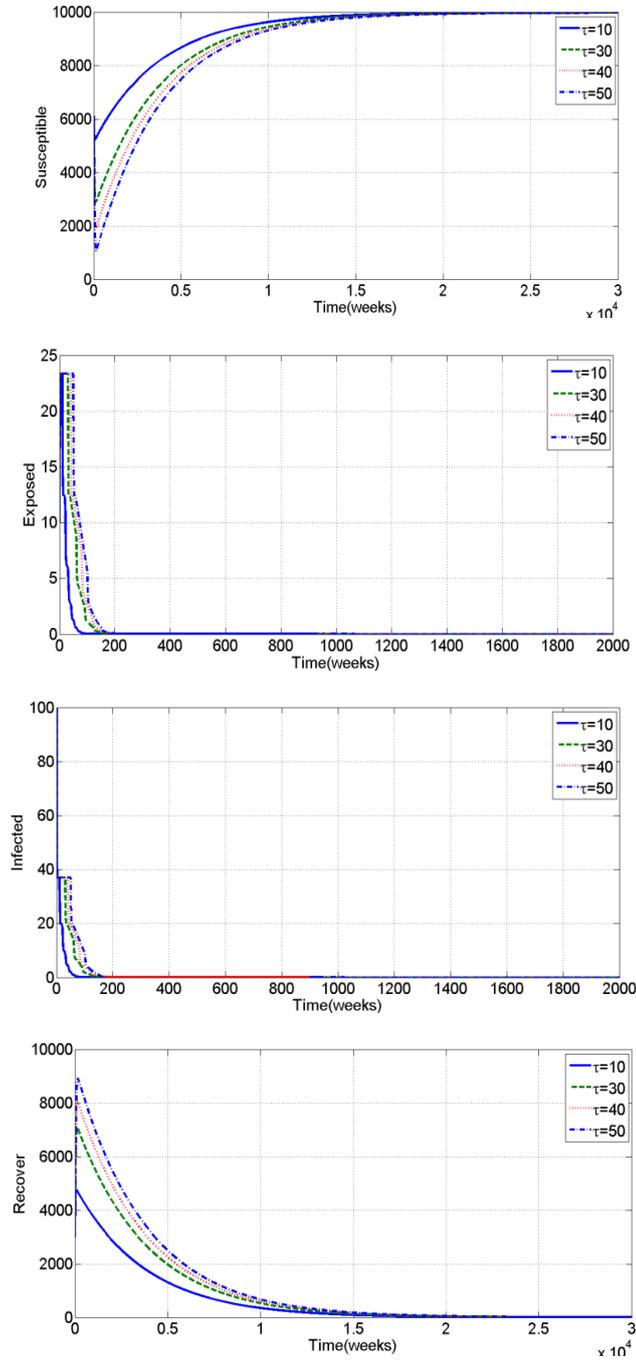


Figure 1. Numerical solutions of the model (6) with various τ and initial condition: $S(0) = 3000, E(0) = 10, I(0) = 100$ and $R(0) = 3000$. Parameter values used are as in Table 1 with $\beta = 2$ which yields $R_0 = 0.85$.

By choosing $\beta = 5$ which yields $R_0 = 2.12$, the model (6) has a unique endemic equilibrium, \mathcal{E}^* as guaranteed by Theorem 3. Further $\beta = 5$, corresponds to the conditions in Theorem 7(i). Thus, \mathcal{E}^* is absolutely stable for all $\tau \geq 0$. Numerical solutions for the case $\beta = 5$ are simulated with $\tau = 10, 38.55, 45$ as show in Figure 2, 3 and 4, respectively. It is seen that all numerical solutions converge to \mathcal{E}^* . These represents that the endemic equilibrium, \mathcal{E}^* is absolutely stable for all $\tau \geq 0$.

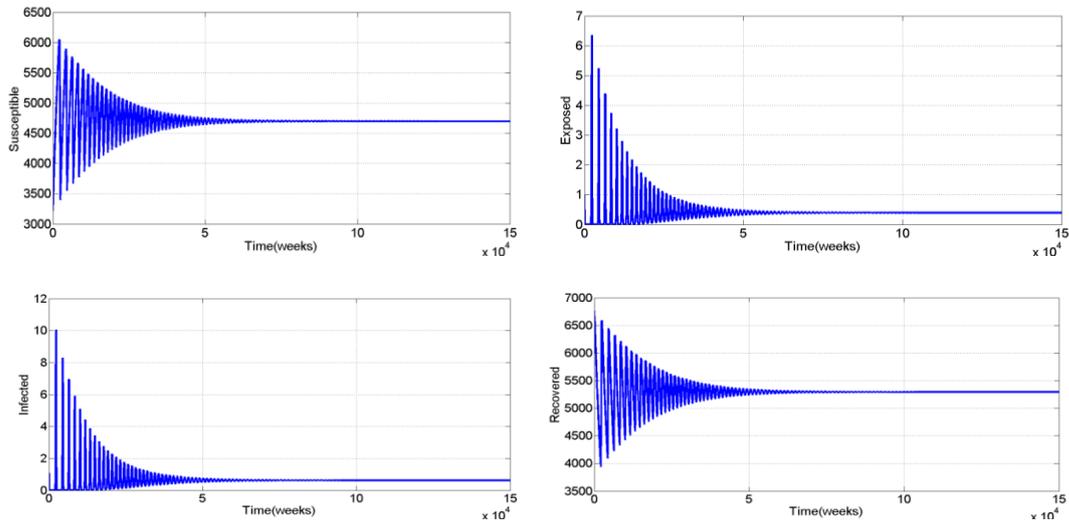


Figure 2. Numerical solutions of the model (6) for $\tau = 10 < \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 5$ which yields $R_0 = 2.12$.

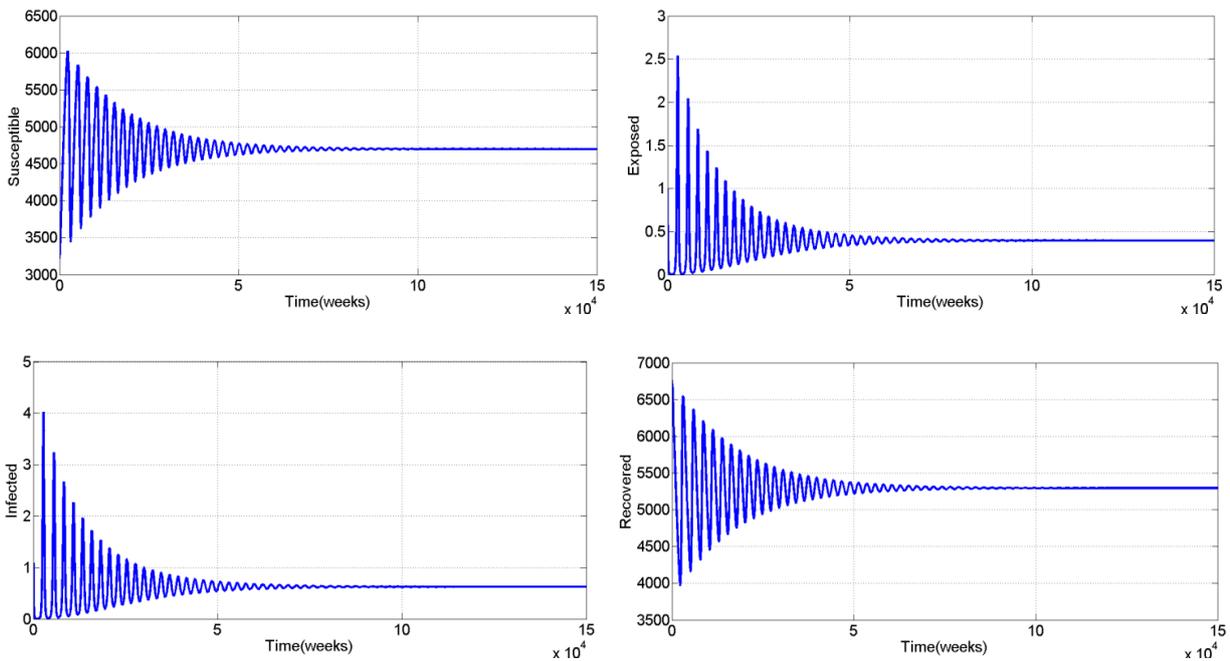


Figure 3. Numerical solutions of the model (6) for $\tau = 38.55 \approx \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 5$ which yields $R_0 = 2.12$.

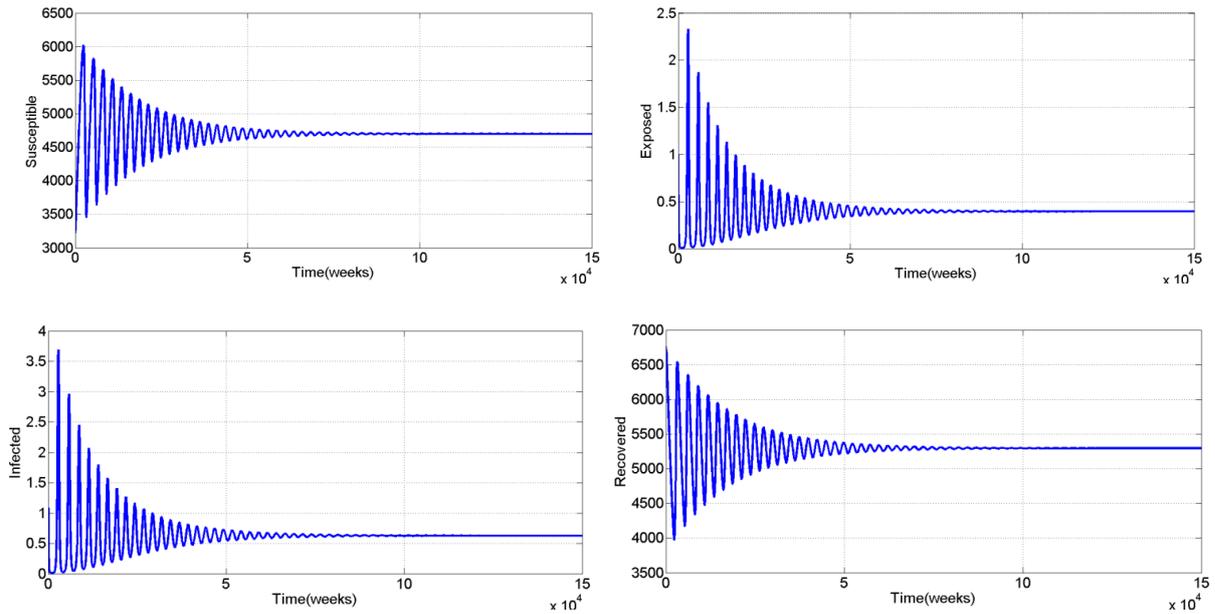


Figure 4. Numerical solutions of the model (6) for $\tau = 45 > \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 5$ which yields $R_0 = 2.12$.

Next, we considered the dynamic of the model (6) with $\beta = 7.29$ this yields $R_0 = 3.1$ and there exist a unique endemic equilibrium, \mathcal{E}^* as guaranteed by Theorem 3. Further, the conditions in Theorem 8(i) are satisfied and $\tau_c \approx 38.55$ is calculated by using (44). Figure 5 shows that for the case $\tau = 5$, the numerical solutions of the model (6) converge to endemic equilibrium when time increase. On the other hand, if we choose $\tau = 45$, the numerical solutions diverge from the endemic

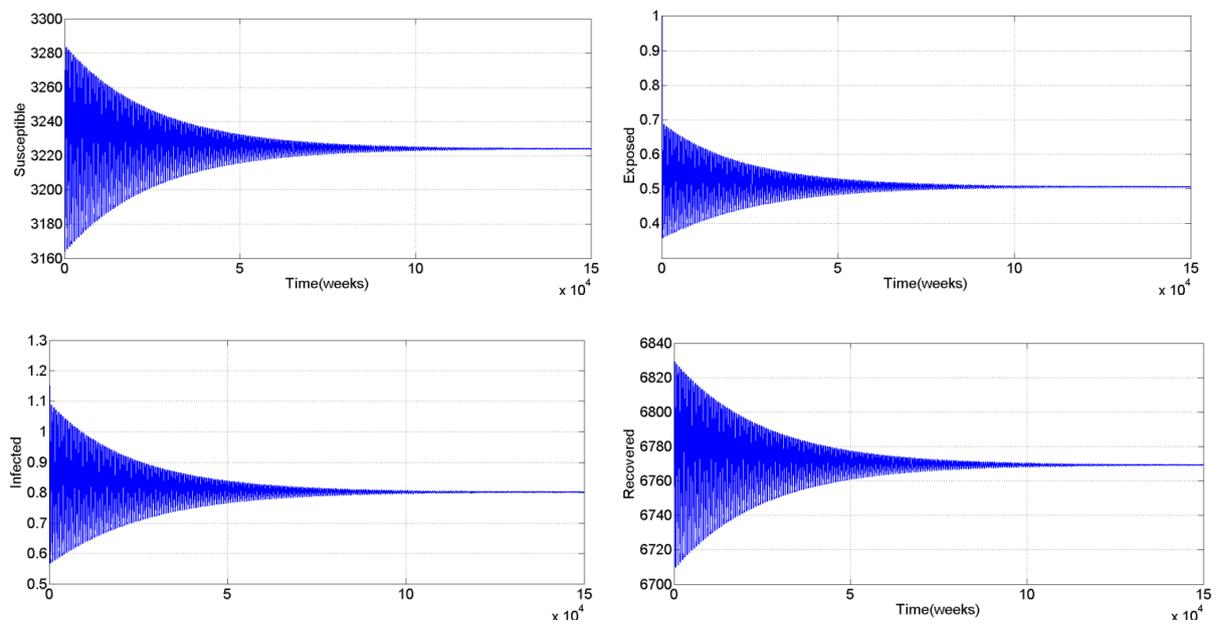


Figure 5. Numerical solutions of the model (6) for $\tau = 5 < \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 7.29$ which yields $R_0 = 3.1$.

equilibrium as shown in Figure 7. This means \mathcal{E}^* is asymptotically stable when $\tau < \tau_c$ and unstable when $\tau > \tau_c$ as guaranteed by Theorem 8(i). For the case $\tau \approx 38.55$, the numerical solutions of the model (6) are periodic as shown in Figure 6. From these results, it concludes that the model (6) undergoes a Hopf bifurcation at \mathcal{E}^* as guaranteed by Theorem 8(ii).

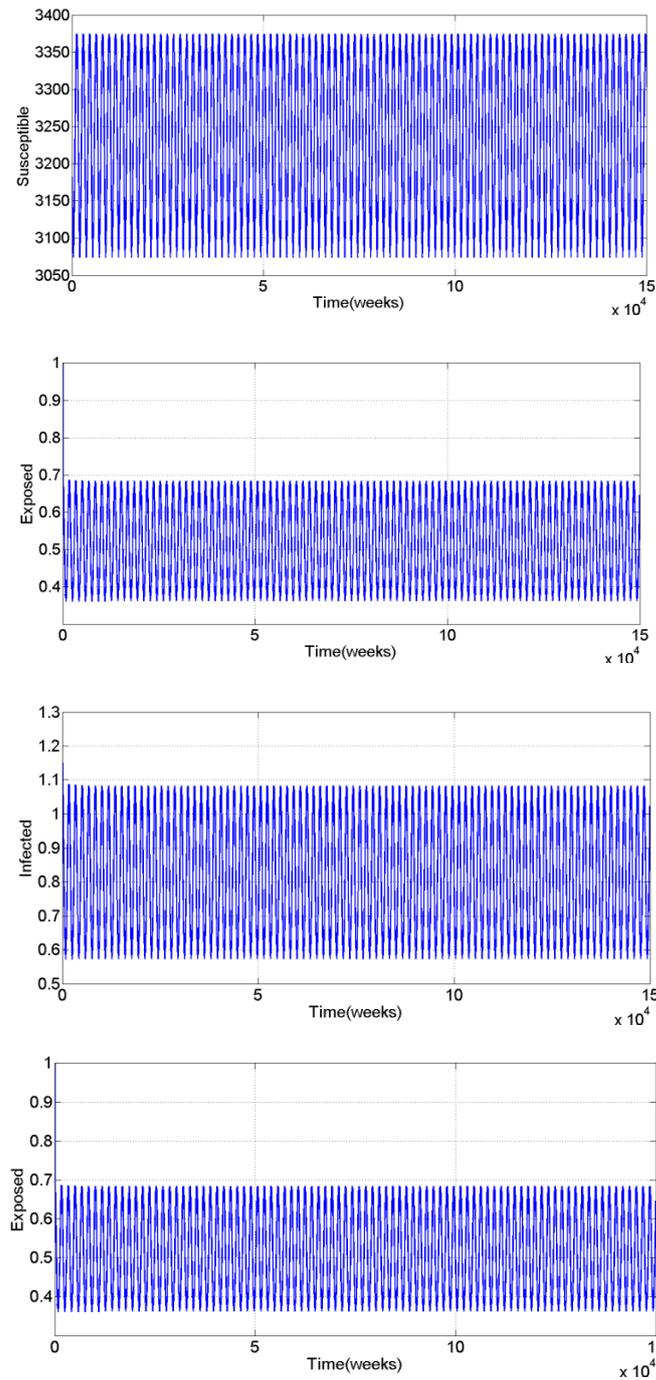


Figure 6. Numerical solutions of the model (6) for $\tau = 38.55 \approx \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 7.29$ which yields $R_0 = 3.1$.

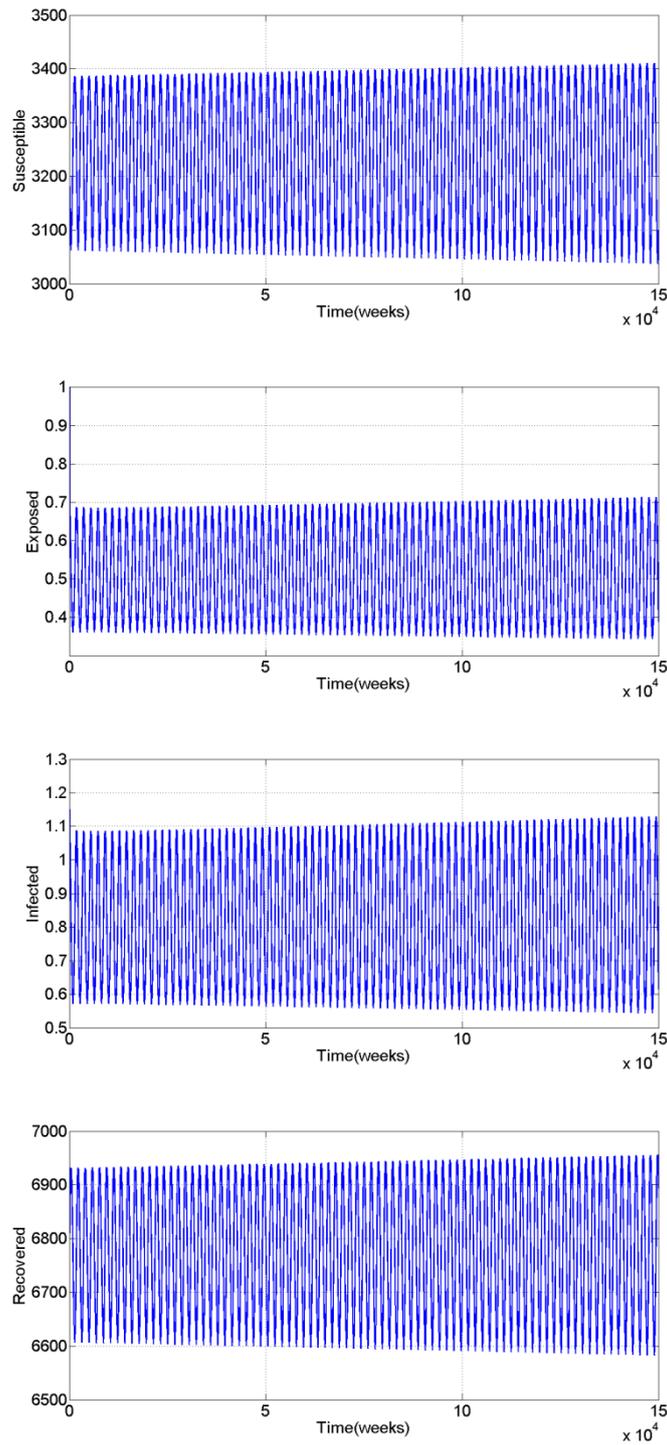


Figure 7. Numerical solutions of the model (6) for $\tau = 45 > \tau_c$ with initial condition: $S(0) = 3223, E(0) = 1, I(0) = 1$ and $R(0) = 6768$. Parameter values used are as in Table 1 with $\beta = 7.29$ which yields $R_0 = 3.1$.

Finally, the direction of Hopf bifurcation and the other properties of bifurcating periodic solution are discussed. By using $\beta = 7.29$, we obtain $\omega_0 = 0.00037$ and $\tau_n = 38.55$. Furthermore, we can calculate the following values:

$c_1(0) = 2.52 \times 10^{-10} - 1.08 \times 10^{-7}i$, $\mu_2 = -1.497 \times 10^{-3}$, $\beta_2 = 5.031 \times 10^{-10}$ and $T_2 = 2.541 \times 10^{-6}$. By Theorem 9, we conclude that Hopf bifurcation of the model (6) occurring at $\tau_c = 38.55$ is subcritical, the periodic solution exists when τ crosses τ_c from left to right, the periodic solution is unstable and the period increase.

7. Conclusions

In this paper, we study the dynamic behavior of *SEIR* delay model with logistic growth. The stability of each equilibrium point was analyzed by choosing time delay τ as a bifurcation parameter. The main of this study are summarized below:

(i) Disease-free equilibrium, \mathcal{E}_0 of the model (6) is locally asymptotically stable for all $\tau \geq 0$ when $R_0 \leq 1$ (Theorem 4).

(ii) The model (6) has a unique endemic equilibrium, \mathcal{E}^* when $R_0 > 1$ (Theorem 3) and \mathcal{E}^* is locally asymptotically stable for all $\tau \geq 0$ when $2N^* > K$, conditions (41)-(43) are satisfied and $e_0 > 0$ (Theorem 7(i)).

(iii) The unique endemic equilibrium, \mathcal{E}^* is conditionally stable when $2N^* > K$, $R_0 > \frac{3a^* + 2\mu}{a^*}$ and condition (41)-(43) are satisfied and the dynamical behavior of the model (6) undergoes a Hopf bifurcation when $\tau = \tau_c$ (Theorem 8).

Furthermore, the stability, direction and period of the periodic solution are determined by using the method based on the normal form theory and the center manifold reduction. Finally, the numerical solutions of the model (6) are simulated to verify the theoretical results.

Acknowledgements

A. Sirijampa is grateful to King Mongkut's University of Technology North Bangkok for the financial support during his study Doctor of Philosophy. This research is (partially) supported by Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Research Promotion and National Research University Project of Thailand, Office of the Higher Education Commission (under NRU-CSEC Project). Finally, the authors would like to express their gratitude to the anonymous referees for very helpful suggestions and comments which led to improvements of the original manuscript.

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