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Original Article

Analytical and numerical solutions of average run length integral equations for an EWMA control chart over a long memory SARFIMA process

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Abstract

The efficiency of a process, especially a long memory seasonal autoregressive fractional integral moving average (SARFIMA) process, has commonly been measured through the quality control chart. In this paper, a generalized long memory SARFIMA process of the exponentially weighted moving average (EWMA) control chart is carried out and shown. Also, analytical and numerical average run length (ARL) were designed to measure the efficiency of the EWMA control. Existence and uniqueness by the fixed point theory are proven for the analytical ARL. Error and convergence of numerical integration equations are also given for the numerical ARL. The findings indicated that the analytical ARL was evaluated more quickly and accurately than the numerical ARL. As a result, the analytical ARL is an alternative for measuring the efficiency of an EWMA control chart over a long memory SARFIMA process.

Keywords: average run length, EWMA control chart, SARFIMA process, composite midpoint rule, integral equation

1. Introduction

In the production process, the aim and expectation of a manufacturer is to produce quality products without defections and variations of their attributes. Statistical technique is one of the alternative approaches to analyze such process, which is called statistical process control (SPC). SPC is used in quality control to improve a process through a control chart. Control charts have been introduced by several researchers. Shewhart (1931) proposed the first control chart called the Shewhart Control Chart, based on the assumption that observations are normal distributions with a constant mean and variance, and independent, identically distributed errors, even though real observations do not bear out these assumptions. That is, one does not see specific normal distributions in real observations. Subsequently, the Cumulative Sum (CUSUM) Control Chart was introduced by Page (1954). The CUSUM control chart is more generalized than the Shewhart control chart because it can detect false signals through collection and summation of the information of any distributed real observations. Roberts (1959) propounded an effective chart, or an exponentially weighted moving average (EWMA) control chart, which can detect small size shifts in the mean of a process faster than the CUSUM control chart. Since then, the EWMA control chart has been continually developed, such as the EWMA sign, new EWMA, Fuzzy EWMA, etc., by several researchers (Abbasi, 2012; Sevil *et al.*, 2014; Zhang *et al.*, 2014; Yang & Cheng, 2011). Statistical observations, in general, are cross-section observations collected from a population in a given timeframe; and time series observations collected from a population in chronological order. Time series observations, forecasting observations, represent real observations, which include trend, season, and autocorrelation. In other words, time series observations fluctuate, with mean and variance not constant over a given

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timeframe. Several researchers have studied time series processes, autoregressive (AR), moving average (MA), autoregressive integrated moving average (ARIMA), and autoregressive fractionally integrated moving average (AFIMA) process, and applied these to forecasting models with real observations in several disciplines, such as medical science, finance, etc. Li *et al.* (2012) also studied the ARIMA process of the Box-Jenkins model (Box &Jenkins,1970) to fit the number of incidences of hemorrhagic fever with renal syndrome (HFRS) in China. Authors found that the AR and ARIMA process can describe such observations, enabling prediction and prevention. Furthermore, one of the other time series processes that can describe the characteristics of real observation movements is ARFIMA. Alireza and Ahmad (2009) forecasted the stock price index of Tehran (TSIP) with the ARFIMA process. In spite of the fact that the AFRIMA process can describe fluctuations in real observations, real observations sometimes present periodical behavior that varies seasonally. Thus, several researchers have conducted studies with the seasonal autoregressive fractionally integrated moving average (SAFIMA) process. For example, Sakha bakhsh (2013) applied consumption of petroleum products in the U.S. to model the SARFIMA (seasonal autoregressive fractional integral moving average) model with the specification of evaluation error using the decomposition method. Aye *et al.* (2015) employed aggregate retail sales of South Africa to establish a linear and non-linear model, 23 single models and three combined models. Two of the 26 models were the ARFIMA and SARFIMA models. Results showed that the combined nonlinear models were better than the single linear model. The pivotal point in the establishment of the ARFIMA and SARFIMA model was the estimation method for the fractional differencing parameter. That is, the AFRIMA and SARFIMA processes allow the fractional differencing parameter to be a fraction of a whole number instead of integer in the ARIMA process and the autocorrelation function (ACF) of the ARFIMA and SARFIMA process decays to zero on the exponential function. Meanwhile, ACF of the ARIMA process decays to zero on a geometric function (Hosking, 1981). Because the time series presents the long memory properties and periodical behavior with the season through the ACF, the long memory characteristic in the time series is captured by the fractional differencing parameter denoted by d for the ARFIMA process and D for the SARFIMA process. The ARFIMA and SARFIMA processes are long memory properties if there exists a nonzero $d, D \in (0, 0.5)$. For a short and intermediate memory property, $d, D = 0$ and $d, D \in (-0.5, 0.0)$, respectively (Barbara *et al.*, 2006). Several authors have used various methods for estimating d, D and their properties starting from theoretical derivation to implementation. Bisognin and Lopes (2009) gave proof of the properties of a long memory SARFIMA process, such as the spectral density function, the seasonal frequency, the long memory characteristics, etc. Reisen *et al.* (2014) used a semi-parametric method based on a Monte Carlo simulation for estimating parameter D of SARFIMA with two seasonal periods of long memory real observations of PM10 concentrations and hourly demand for electricity. Likewise, Richard (1996) proposed the semi-parametric method for estimating parameter D with long memory volatility for real observations including interest rates and exchange rates with respect to both the frequency and time domain of the SARFIMA model. Based on the Markov Chain Monte Carlo (MCMC) method, Ndongo *et al.* (2010) presented the classical Whittle method. Meanwhile, Mostafaei and Sakhabakhsh (2011), Sakhabakhsh, Yarmohammadi (2012) and Mostafaei (2012) used the conditional sum of squares method for estimating parameter d of the SARIMA process and D of the SARFIMA process to model energy field observations. The results indicated that energy field observations exhibit non-stationary and periodic behavior and the SARFIMA model shows a better performance in terms of fitting these observations than the SARIMA model.

To be successful in the execution of an EWMA control chart in real observations, a quality controller requires a statistic to determine whether the process is reasonable and controllable under an EWMA control chart or not. The average run length (ARL) is an indicator of the standard measurements in the performance of an efficient quality control chart. With the ARL, it is possible to predict the number of observations sampled from the process until the control chart can detect the false signals in an out-of-control process and it can then be categorized into two stages: in-control stage ARL (ARL_0) and out-of stage ARL (ARL_1). Three common methods for evaluating the ARL are as follows. First, the Monte Carlo (MC) method uses simulation for evaluating ARL until a sufficiently large number of the iteration is generated so that the approximated ARL converges to the exact ARL (Hawkins & Olwell, 1998). Second, the Markov Chain Approach (MCA) method developed the transition probability of the run on the partition of the range between the upper control limit (UCL) and lower control limit (LCL) (Champ & Rigdon, 1991). Third, the numerical integral equation (NIE) method was derived from the transformation of the integral equation whose solution represents the ARL in system of linear equations (Areepong & Sukparungsee, 2010). Several researchers have used the same means. Phanyaem *et al.* (2013) used observations from the ARMA process with exponential distribution white noise for evaluating the ARL using the NIE method and making a comparison of the ARL between the CUSUM and the EWMA control chart. Meanwhile, Piyaphon *et al.* (2014) expressed the explicit formula of the ARL of an EWMA control chart when the observations follow the Seasonal Autoregressive and Moving Average ($SARMA_S$) process with a seasonal period, S. However, as seen in the above literature review, the interesting point is that ARL evaluation for the efficiency of quality control in a time series process has rarely been proposed, especially with regard to applications in a SARFIMA process with exponential white noise under an EWMA control chart. Therefore, the three main goals of this research are to approximate the ARL, or numerical ARL designed by the NIE method with the composite midpoint rule; derive explicit formula of the ARL, or analytical ARL, of the EWMA control chart over the SARFIMA process for long-memory observations with exponential distribution white noise; and make a comparison between the numerical ARL and analytical ARL. The rest of the paper is organized as follows. In the second section, the characteristics of the EWMA control chart and generalized SARFIMA process are described and expanded upon, respectively. In the third section, the derivation of the analytical ARL is presented and the existence and uniqueness of the analytical ARL are also proved by the functional analysis concept. In the fourth section, the numerical method for numerical ARL is demonstrated. Furthermore, the error analysis and convergence of the numerical ARL are analyzed. The numerical results of the ARL and accuracy measurements are illustrated in the fifth section. Finally, the discussion and conclusions will be summarized in the sixth section.

2. EWMA Control Chart and Generalized Long Memory SARFIMA Process

An EWMA statistic depending upon a sequence of random variables, $X_t; t=1,2,3,...,n$ generated from the SARFIMA process that is expanded in the generalized form with process mean μ and variance σ^2 is defined as

$$Z_t=(1-\lambda)Z_{t-1}+\lambda X_t \;\; ; t=1,2,3,...,n; \;\; Z_0=\mu \tag{1}$$

where λ is a smoothing parameter or weighting parameter satisfying $0<\lambda\le 1$. An EWMA control chart with an initial mean process, μ_0 and width factor, K of the control chart consists of an asymptotical control limit as follows:

Upper control limit: $UCL_{EWMA}=\mu_0+K\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]},$

Center Line: $CL_{EWMA}=\mu_0,$

Lower control limit: $LCL_{EWMA}=\mu_0-K\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]}.$

The ARFIMA process is a generalized ARIMA process from an integer to a fractional differencing parameter. In practice, however, the process involves the seasonal behavior of the ARFIMA process. The SARFIMA process is a seasonal long memory process denoted by SARFIMA$(P,D,Q)_S$ with the periodic season order S, if there exists a nonzero differencing parameter, $D\in(0,\,0.5)$. The original SARFIMA $(P,D,Q)_S$ model, which can be explained in terms of a seasonal differencing binomial expansion

$$(1-B^S)^D=1-DB-\frac{1}{2!}D(1-D)B^{2S}-\frac{1}{3!}D(1-D)(2-D)B^{3S}-....$$

can be defined as

$$\Phi(B^S)(1-B^S)^D X_t=\mu+\Theta(B^S)\varepsilon_t \tag{2}$$

where $\Phi(B^S)=(1-\Phi_1B^S-\Phi_2B^{2S}-...-\Phi_PB^{PS}); |\Phi_i|<1$ is the autoregressive polynomial of degree P for $i=1,2,...,P$ and $\Theta(B^S)=(1-\Theta_1B^S-\Theta_2B^{2S}-...-\Theta_QB^{QS}); |\Theta_i|<1$ is the autoregressive polynomial of degree Q for $i=1,2,...,Q$. The white noise process, ξ_t, assumed with exponential distribution, i.e. $f(\xi_t)=\frac{1}{\beta}\exp(-\frac{\xi_t}{\beta})$ with shift parameter $\beta>0$. In a control state, it is assumed that the shift parameter β known as $\beta=\beta_0$ could be changed to an out-of-control state $\beta=\beta_0(1+\delta)$ with shift size, δ. The initial value is assigned as $X_{t-S}, X_{t-2S},..., X_{t-(P+1)S}, X_{t-(P+2)S}, X_{t-(P+3)S},..., X_{t-(P+D)S}=1$. The generalization of SARFIMA in recursive form is carried out by applying binomial expansion as:

$$\begin{aligned}X_t&=\mu+\xi_t-\Theta_1\xi_{t-S}-\Theta_2\xi_{t-2S}-...-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\Phi_2X_{t-2S}+...+\Phi_PX_{t-PS})\\&+D(X_{t-S}-\Phi_1X_{t-2S}-\Phi_2X_{t-3S}-...-\Phi_PX_{t-(P+1)S})-\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\Phi_2X_{t-4S}-...-\Phi_PX_{t-(P+2)S})\\&+\frac{D(D-1)(D-2)}{3!}(X_{t-3S}-\Phi_1X_{t-4S}-\Phi_2X_{t-5S}-...-\Phi_PX_{t-(P+3)S})-...+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\Phi_2X_{t-(D+2)S}-...-\Phi_PX_{t-(D+P)S}).\end{aligned}$$

3. Derivation of Analytical ARL with A Generalized Long Memory SARFIMA Process under an EWMA Control Chart with Existence and Uniqueness

The integral equation of the second kind is one of several types of integral equations with applications in various fields such as mechanics, electromagnetic theory, chemical kinetics, fluid dynamics, and mathematical biology (Kiusalaas, 2005; Nadir, 2014; Zang, Chen & Nie, 2014;). In this section, an analytical integral equation as the analytical ARL of the EWMA control chart with initial value u over a long memory SARFIMA process is proposed. Also, the existence and uniqueness of the analytical ARL based on Fixed Point theory in a functional analysis are proved as follows:

3.1 Derivation of analytical ARL

This research concentrated on the upper control limit side starting at 0 as the center line and ending at $b>0$, i.e. $UCL_{EWMA}=b$ and $CL_{EWMA}=0.$ Let $L(u)$ be the ARL of an EWMA control chart based on the method of Champ and Rigdon (1991). The following integral equation is the analytical ARL satisfying the EWMA control chart of a generalized long memory SARFIMA process as

$$L(u)=1+\int_0^b L\left((1-\lambda)u+\lambda\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}\right)f(y)dy \tag{3}$$

By changing the variable technique, Equation (3) becomes

$$L(u)=1+\frac{1}{\lambda}\int_0^b L(y)f\left(\frac{y-(1-\lambda)u}{\lambda}-\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}\right)dy$$

$$=1+\frac{1}{\lambda\beta}\int_0^b L(y)\left[\exp\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{1}{\beta}\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}-\frac{y}{\lambda\beta}\right)\right]dy \tag{4}$$

Let $A(u)=\exp\left(\frac{(1-\lambda)u}{\lambda\beta}+\frac{1}{\beta}\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}\right);\ 0\le u\le b$

Thus, $L(u)=1+\frac{A(u)}{\lambda\beta}\int_0^b L(y)\exp\left(\frac{-y}{\lambda\beta}\right)dy,\quad 0\le u\le b.$

Let $$q=\int_0^b L(y)\exp\left(\frac{-y}{\lambda\beta}\right)dy,\ \text{so}\ L(u)=1+\frac{A(u)}{\lambda\beta}q. \tag{5}$$

$$=\int_0^b\left(1+\frac{A(y)}{\lambda\beta}q\right)\exp\left(\frac{-y}{\lambda\beta}\right)dy$$

$$=\frac{q}{\lambda\beta}\int_0^b\exp\left[\frac{(1-\lambda)y}{\lambda\beta}+\frac{1}{\beta}\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}-\frac{y}{\lambda\beta}\right]dy$$

$$+\int_0^b\exp\left[\frac{-y}{\lambda\beta}\right]dy$$

That is, $q=\frac{q}{\lambda}\{1-\exp(-\frac{b}{\beta})\}\exp\left[\frac{1}{\beta}\left\{\begin{array}{l}\mu+\xi_t-\Theta_1\xi_{t-S}-\ldots-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+\ldots+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-\ldots-\Phi_PX_{t-(P+1)S})\\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-\ldots-\Phi_PX_{t-(P+2)S})+\ldots+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-\ldots-\Phi_PX_{t-(D+P)S})\end{array}\right\}\right]$

$$+\lambda\beta\left[1-\exp\left(\frac{-b}{\lambda\beta}\right)\right]$$

Thus,

$$q=\lambda\beta\{1-\exp(-\frac{b}{\lambda\beta})\}\Big/[1-\frac{1}{\lambda}\{1-\exp(-\frac{b}{\beta})\}\exp(\frac{1}{\beta}\begin{Bmatrix}\mu+\xi_t-\Theta_1\xi_{t-S}-...-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+...+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-...-\Phi_PX_{t-(P+1)S})\\-\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-...-\Phi_PX_{t-(P+2)S})+...+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-...-\Phi_PX_{t-(D+P)S})\end{Bmatrix})].$$

Substitute q into Equation (5), then the analytical ARL as

$$L(u)\ =\ 1+\frac{A(u)}{\lambda\beta}q=1-\frac{Nom(\beta)}{Denom(\beta)} \qquad (6)$$

where

$$Nom(\beta)=\lambda[1-\exp(\frac{-b}{\lambda\beta})]\exp\{\frac{(1-\lambda)u}{\lambda\beta}\}\text{ and}$$

$$Denom(\beta)=[1-\exp(\frac{-b}{\beta})]-\lambda\exp[\frac{-1}{\beta}\begin{Bmatrix}\mu+\xi_t-\Theta_1\xi_{t-S}-...-\Theta_Q\xi_{t-QS}+(\Phi_1X_{t-S}+...+\Phi_PX_{t-PS})+D(X_{t-S}-\Phi_1X_{t-2S}-...-\Phi_PX_{t-(P+1)S})\\-\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1X_{t-3S}-...-\Phi_PX_{t-(P+2)S})+...+(X_{t-DS}-\Phi_1X_{t-(D+1)S}-...-\Phi_PX_{t-(D+P)S})\end{Bmatrix}].$$

The ARL for the in-control state process, or ARL_0, with the setting the parameter $\beta=\beta_0,$ is as follows:

$$L_0(u)=1-\frac{Nom(\beta_0)}{Denom(\beta_0)} \qquad (7)$$

Meanwhile, with the setting of exponential parameter $\beta=\beta_1$, where $\beta_1=\beta_0(1+\delta)$, the ARL for the out-of-control state process with shift size δ can be written as follows:

$$L_1(u)=1-\frac{Nom(\beta_1)}{Denom(\beta_1)} \qquad (8)$$

3.2 Existence and uniqueness of analytical ARL

The definitions and theories based on a functional analysis that are necessary for proving the existence and uniqueness of the analytical ARL are briefly proposed as follows (Kreyszig, 1989; Rudin, 1991; Yosida, 1995):

Definition 1: Let M be a nonempty set. For all $x,y,z\in M$, $\wp:M\times M\to\mathbb{R}$ is called a distance function, or metric, on M if $\wp$ satisfies the following properties

(1) $\wp(x,y)>0$ if $x\neq y$ and $\wp(x,x)=0$
(2) $\wp(x,y)=\wp(y,x)$,
(3) $\wp(x,z)=\wp(y,x)+\wp(x,z)$
Also, a pair $(M,\wp)$ is called a metric space.

Definition 2: A sequence (x_n) of a point $x_n\in(M,\wp)$ is a Cauchy sequence if $\wp(x_m,x_n)$ tends to zero as m,n tends to infinity.

Definition 3: A metric space is complete if every Cauchy sequence has a limit $x\in M$ .In other words, $(M,\wp)$ is a complete metric space, if $\wp(x_m,x_n)$ tends to zero as m,n tends to infinity, then there exists $x\in M$ such that $\wp(x_n,x)$ tends to zero as n tends to infinity.

Definition 4: $T:M\to M$ is a contraction mapping on a complete metric space if there exists $0\le\gamma<1$ such that $\wp(T(x),T(y))\le\gamma\wp(x,y)$ for all $x,y\in M$.

Definition 5: Supremum norm, or Sup-norm, on the domain, $Dom(L)$, of the continuous function $L(u):Dom(L)\to\mathbb{R}$ is defined as: $\|L(u)\|=\sup\{|L(u)|:u\in Dom(L)\}$.

Theorem 1: Banach's Fixed Point Theorem (Sofonea *et al.*, 2006)

If $T: M \to M$ is a contraction mapping on a complete metric space $(M, \wp)$, then there is exactly one solution, or a unique fixed point, l of T such that $T(l) = l$ for all $l \in M$ .Equation (4) can be rewritten in the form of an integral equation of the second kind for a continuous function

$$k(u,y) = \frac{1}{\lambda\beta}\exp[\frac{(1-\lambda)u-y}{\lambda\beta}) + \frac{1}{\beta}\left\{\begin{matrix} \mu+\xi_t-\Theta_1\xi_{t-S}-...-\Theta_Q\xi_{t-QS}+(\Phi_1 X_{t-S}+...+\Phi_P X_{t-PS})+D(X_{t-S}-\Phi_1 X_{t-2S}-...-\Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S}-\Phi_1 X_{t-3S}-...-\Phi_P X_{t-(P+2)S})+...+(X_{t-DS}-\Phi_1 X_{t-(D+1)S}-...-\Phi_P X_{t-(D+P)S}) \end{matrix}\right\} - \frac{y}{\lambda\beta}]$$

on $[0,b]\times[0,b]; b>0$ and unknown function, or the analytical ARL, $L(u):[0,b]\to\mathbb{R}$, for a long memory SARFIMA process on the EWMA control chart as

$$L(u) - \int_0^b L(y)k(u,y)dy = 1 \tag{9}$$

Consider metric space $(M, \wp)$ with $M = C[0,b]$ which contains all of the continuous real value functions on the upper control limit side of the EWMA control chart, $L(u):[0,b]\times[0,b]\to\mathbb{R}$, with Supremum Norm, $\|L(u)\| = \sup\{|L(s)|: s\in[0,b]\}$. The mapping $T:[0,b]\to[0,b]$ is defined as

$$T(L(u)) = 1 + \int_0^b L(y)k(u,y)dy$$

Theorem 2: The analytical ARL, or $L(u)$, of an integral equation of the second kind corresponding to the long memory SARFIMA process on the EWMA control chart has an existence and a uniqueness.

Proof. (**Existence**)

Let $C([0,b])$ be a set of all the continuous functions on $[0,b]$ and $(L_n(u))_{n\geq 0}$ be a sequence of the analytical ARL on $C([0,b]$ which is defined by $L_{n+1}(u) = T(L_n(u))$ for $n \geq 0$ with any initial point $L_0(u)$ on $C([0,b]$. For $n \geq m \geq 1$, with application of the iteration, Definition 4, the triangle inequality, and the Von Neumann series: $\sum_{i=0}^{\infty}\gamma^i = \frac{1}{1-\gamma}$ converges as $|\gamma|<1$, considering the distance function

$$\begin{aligned} \wp(L_n(u), L_m(u)) &= \wp(T^n L_0(u), T^m L_0(u)) \\ &\leq \gamma^m \left[\sum_{i=0}^{n-m-1}\gamma^i\right]\wp(L_1(u), L_0(u)) \\ &\leq \gamma^m \frac{1}{1-\gamma}\wp(L_1(u), L_0(u)). \end{aligned}$$

It is obvious that $\wp(L_n(u), L_m(u))$ tends to zero as m,n tends to infinity. Thus, $(L_n(u))_{n\geq 0}$ is a Cauchy's sequence and complete sequence as $\lim_{n\to\infty} T^n(L_0(u)) \to L(u)$.

Therefore, there exists the analytical ARL, $L(u):[0,b]\to\mathbb{R}$, which is the solution of integral Equation (9).

It is sufficient to show that T is the contraction mapping to prove the uniqueness of the analytical ARL, $L(u)$. For a real value continuous function: $L_1(u),\ L_2(u) \in C[0,b]$, consider

$$\begin{aligned} \|T(L_1(u)) - T(L_2(u))\| &= \sup_{0\leq u\leq b} |\int_0^b k(u,y)[L_1(y) - L_2(y)]dy| \\ &= (b-0)|k(u,y)|\,\|L_1(y) - L_2(y)\| \end{aligned}$$

$\leq \gamma \| L_1(y) - L_2(y) \|$

where $\gamma = (b-0)| k(u,y) | < 1$

Since $\gamma < 1$, therefore, T is contraction mapping and then $L(u)$ is the fixed point of the mapping T, i.e. $T(L(u)) = L(u)$, or Equation (9)has a unique analytical ARL solution.

4. Analysis of Numerical ARL with a Generalized Long Memory SARFIMA Process on an EWMA Control Chart

As most integral equations cannot be solved with an analytical solution due to the complicatedness of integrand functions, the numerical technique is an alternative approach to find the solution to an integral equation making up the ARL defined as

$$L(u) = 1 + \frac{1}{\lambda}\int_0^b L(y) f\left(\frac{y-(1-\lambda)u}{\lambda} - \left\{\begin{matrix} \mu + \xi_t - \Theta_1\xi_{t-S} - \ldots - \Theta_Q\xi_{t-QS} + (\Phi_1 X_{t-S} + \ldots + \Phi_P X_{t-PS}) + D(X_{t-S} - \Phi_1 X_{t-2S} - \ldots - \Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S} - \Phi_1 X_{t-3S} - \ldots - \Phi_P X_{t-(P+2)S}) + \ldots + (X_{t-DS} - \Phi_1 X_{t-(D+1)S} - \ldots - \Phi_P X_{t-(D+P)S}) \end{matrix}\right\}\right) dy \quad (10)$$

In this section, NIEs accounting for the numerical ARL of an EWMA control chart over a long memory SARFIMA process are proposed. Moreover, error and convergence of the numerical ARL are demonstrated. The main methodology is described as follows.

4.1 Numerical method for evaluation of numerical ARL

The composite midpoint rule (Akinson, 1989; Kiusalaas, 2005; Matheus & Dmitry, 2008) proposed in this research is one of the methods for solving NIEs (Yalcinbas & Aynigul, 2011). The domain of the integral, $[0,b]$, is divided into m subinterval with equal length $(\frac{b}{m})$: $0 = a_1 < a_2 < \ldots < a_m = b,$ corresponding to equal weight: $w_j = \frac{b}{m} > 0$ and $m+1$ values of $L(a_0)f(a_0), L(a_1)f(a_1), \ldots, L(a_m)f(a_m).$ Namely,

$$\int_0^b L(y) f\left(\frac{y-(1-\lambda)u}{\lambda} - \left\{\begin{matrix} \mu + \xi_t - \Theta_1\xi_{t-S} - \ldots - \Theta_Q\xi_{t-QS} + (\Phi_1 X_{t-S} + \ldots + \Phi_P X_{t-PS}) + D(X_{t-S} - \Phi_1 X_{t-2S} - \ldots - \Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S} - \Phi_1 X_{t-3S} - \ldots - \Phi_P X_{t-(P+2)S}) + \ldots + (X_{t-DS} - \Phi_1 X_{t-(D+1)S} - \ldots - \Phi_P X_{t-(D+P)S}) \end{matrix}\right\}\right) dy$$

$$\approx \sum_{j=1}^{m} w_j L(a_j) f(a_j) \quad (11)$$

where $a_j = \frac{b}{m}\left(j - \frac{1}{2}\right),\ w_j = \frac{b}{m},\ j = 1,2,\ldots,m.$

Substitute Equation (11) for Equation (10) to obtain the system of m linear equations with m unknown variables as

$$L(a_i) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda} - \left\{\begin{matrix} \mu + \xi_t - \Theta_1\xi_{t-S} - \ldots - \Theta_Q\xi_{t-QS} + (\Phi_1 X_{t-S} + \ldots + \Phi_P X_{t-PS}) + D(X_{t-S} - \Phi_1 X_{t-2S} - \ldots - \Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S} - \Phi_1 X_{t-3S} - \ldots - \Phi_P X_{t-(P+2)S}) + \ldots + (X_{t-DS} - \Phi_1 X_{t-(D+1)S} - \ldots - \Phi_P X_{t-(D+P)S}) \end{matrix}\right\}\right)$$

For $i = 1,2,\ldots,m$

The above system of linear equations can be formed in the matrix notation as

$$\mathbf{L}_{m\times 1} = (\mathbf{I}_m - \mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1} \quad (12)$$

where $\mathbf{L}_{m\times 1} = \left[L(a_1), L(a_2), \ldots, L(a_m)\right]^T$, $\mathbf{1}_{m\times 1} = [1,1,\ldots,1]^T$, a unit vector, $\mathbf{R}_{m\times m}$ is a matrix with dimension $m\times m$ with element

$$r_{ij} \approx \frac{1}{\lambda} w_j f\left(\frac{a_j - (1-\lambda)a_i}{\lambda} - \left\{ \begin{array}{l} \mu + \xi_t - \Theta_1 \xi_{t-S} - \ldots - \Theta_Q \xi_{t-QS} + (\Phi_1 X_{t-S} + \ldots + \Phi_P X_{t-PS}) + D(X_{t-S} - \Phi_1 X_{t-2S} - \ldots - \Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S} - \Phi_1 X_{t-3S} - \ldots - \Phi_P X_{t-(P+2)S}) + \ldots + (X_{t-DS} - \Phi_1 X_{t-(D+1)S} - \ldots - \Phi_P X_{t-(D+P)S}) \end{array} \right\} \right),$$

$\mathbf{I}_{m\times 1} = [diag(1, 1, ..., 1)]$ is a unit diagonal vector.

For the EWMA control chart, with the initial value u substituted for a_i in $L(a_i)$, the numerical ARL can be evaluated as

$$L(u) \approx 1 + \frac{1}{\lambda}\sum_{j=1}^{m} w_j L(a_j) f\left(\frac{a_j - (1-\lambda)u}{\lambda} - \left\{ \begin{array}{l} \mu + \xi_t - \Theta_1 \xi_{t-S} - \ldots - \Theta_Q \xi_{t-QS} + (\Phi_1 X_{t-S} + \ldots + \Phi_P X_{t-PS}) + D(X_{t-S} - \Phi_1 X_{t-2S} - \ldots - \Phi_P X_{t-(P+1)S}) \\ -\frac{D(D-1)}{2!}(X_{t-2S} - \Phi_1 X_{t-3S} - \ldots - \Phi_P X_{t-(P+2)S}) + \ldots + (X_{t-DS} - \Phi_1 X_{t-(D+1)S} - \ldots - \Phi_P X_{t-(D+P)S}) \end{array} \right\} \right).$$

4.2 Asymptotic error, error bound, and convergence of the numerical ARL

The exact error (*Er*) (Akinson, 1989; Kiusalaas, 2005; Matheus & Dmitry, 2008) in an approximation of the numerical ARL of integral Equation (10), which depends on the function $L(\varsigma)f(\varsigma)$, by the composite midpoint rule, is given as

$$Er = \sum_{j=1}^{m} Er_j = \frac{b^3}{24m^3}\sum_{j=1}^{m}\left(L(\varsigma_j)f(\varsigma_j)\right)'' = \frac{b^3}{24m^2}\left(L(\varsigma)f(\varsigma)\right)'' \tag{13}$$

where $Er_j = \frac{h^3}{24}\left(L(\varsigma_j)f(\varsigma_j)\right)''$, $\varsigma_j \in (a_j - \frac{b}{2m}, a_j + \frac{b}{2m})$ for $j = 1, 2, ..., m$ and $\varsigma \in (0, b)$, and satisfies $|Er| \le \frac{M_1 b^3}{24m^2}$, where $M_1 = \max_{\varsigma\in[0,b]} |\left(L(\varsigma)f(\varsigma)\right)''|$

In order to show the convergence of the numerical ARL, the asymptotical error (Akinson, 1989) for the exact error of the Composite Midpoint Rule for the second differentiable function $L(\varsigma)f(\varsigma)$ are defined as $\lim_{m\to\infty} Er(Lf)$, i.e., $\lim_{m\to\infty} \frac{b^3}{24m^2}\left(L(\varsigma)f(\varsigma)\right)''$. It is obvious that the asymptotical error tends to zero as m tends to infinity. Therefore, the numerical ARL of a long memory SARFIMA process under an EWMA control chart is a convergence. From Equation (13), consider the term of the error bound

$$\begin{aligned} |\left(L(\varsigma)f(\varsigma)\right)''| &= |L(\varsigma)f''(\varsigma) + 2f'(\varsigma)L'(\varsigma) + f(\varsigma)L''(\varsigma)| \\ &\le |L(\varsigma)||f''(\varsigma)| + |2||f'(\varsigma)||L'(\varsigma)| + |f(\varsigma)||L''(\varsigma)| \\ &= \frac{A(u)}{\lambda^2\beta^3}\exp\left(-\frac{\varsigma}{\lambda\beta}\right)[|L(\varsigma)| + 2\lambda\beta|L'(\varsigma)| + \lambda^2\beta^2|L''(\varsigma)|] \end{aligned}$$

Thus, $\max_{\varsigma\in[0,b]} |\left(L(\varsigma)f(\varsigma)\right)''| \le \frac{A(u)}{\lambda^2\beta^3}[|L(\varsigma)| + 2\lambda\beta|L'(\varsigma)| + \lambda^2\beta^2|L''(\varsigma)|]$

As a result, the error bound for the numerical ARL was carried out as

$$M_1 = \frac{A(u)}{\lambda^2\beta^3}[|L(\varsigma)| + 2\lambda\beta|L'(\varsigma)| + \lambda^2\beta^2|L''(\varsigma)|].$$

5. Numerical Results

In this section, in comparing the numerical value of the analytical ARL and the numerical ARL as the solution of the integral equation representing the ARL of the EWMA control chart over a long memory SARFIMA process, this research desires to minimize the absolute percentage relative error (APRE) which is defined as

$$APRE = \frac{\left|L(u) - L(u)\right|}{L(u)} \times 100\% \tag{14}$$

where $L(u)$ is an analytical ARL, and $L(u)$ is a numerical ARL. The figures accounting for the ARL are shown in Tables 1-4. The figures in the parentheses in Tables 1-4 represent the computational CPU time in minutes while the APRE as the exact error is also shown in the graph in Figure 1.

Table 1. Comparison of the numerical values between the analytical and numerical ARL on SARFIMA $(1, 0.1, 2)_4$.

Shift size (δ) with $\beta_0 = 1$	Analytical ARL $L(u)$	Numerical ARL $L(u)$	*APRE* (%)
0.00	370.000444747449 (0.014)	370.000444743069 (7.037)	1.18379E-09
0.01	331.786614060609 (0.014)	331.786614056757 (13.901)	1.16099E-09
0.03	268.452923498847 (0.016)	268.452923495853 (20.718)	1.11528E-09
0.05	218.932665005619 (0.014)	218.932665003271 (27.519)	1.07246E-09
0.10	135.765481926148 (0.014)	135.765481924825 (34.337)	9.74476E-10
0.30	29.0891564514702 (0.014)	29.089156451273 (41.185)	6.77917E-10
0.40	16.0502408367698 (0.014)	16.0502408366787 (48.034)	5.67585E-10
0.50	9.72384534322514 (0.015)	9.72384534317914 (54.96)	4.73069E-10

Table 2. Comparison of the numerical values between the analytical and numerical ARL on SARFIMA $(1, 0.3, 2)_4$.

Shift size (δ) with $\beta_0 = 1$	Analytical ARL $L(u)$	Numerical ARL $L(u)$	*APRE* (%)
0.00	370.000380720424 (0.014)	370.00038071784 (6.989)	6.98374E-10
0.01	330.885692186588 (0.014)	330.885692184323 (13.712)	6.84538E-10
0.03	266.319561463609 (0.014)	266.319561461856 (20.498)	6.5823E-10
0.05	216.10241971095 (0.014)	216.102419709582 (27.3)	6.3303E-10
0.10	132.455900896147 (0.015)	132.455900895385 (34.164)	5.75275E-10
0.30	27.3876709455636 (0.014)	27.3876709454542 (40.966)	3.99445E-10
0.40	14.9333640610785 (0.014)	14.9333640610287 (47.83)	3.33483E-10
0.50	8.97520656566755 (0.014)	8.97520656564277 (54.616)	2.76096E-10

Table 3. Comparison of the numerical values between the analytical and numerical ARL on SARFIMA $(2, 0.1, 2)_{12}$.

Shift size (δ) with $\beta_0 = 1$	Analytical ARL $L(u)$	Numerical ARL $L(u)$	*APRE* (%)
0.00	370.000459420447 (0.015)	370.000459417308 (6.818)	8.48379E-10
0.01	331.217549183702 (0.016)	331.217549180947 (13.604)	8.31771E-10
0.03	267.103951517072 (0.015)	267.103951514938 (20.437)	7.98946E-10
0.05	217.141291012132 (0.015)	217.141291010463 (27.332)	7.68628E-10
0.10	133.666022227482 (0.015)	133.666022226549 (34.149)	6.98008E-10
0.30	28.0022369683228 (0.015)	28.0022369681868 (41.06)	4.85681E-10
0.40	15.3348683748081 (0.015)	15.3348683747459 (47.956)	4.05606E-10
0.50	9.24323844934806 (0.014)	9.24323844931694 (54.82)	4.72978E-10

Table 4. Comparison of the numerical values between the analytical and numerical ARL on SARFIMA $(2, 0.3, 2)_{12}$.

Shift size (δ) with $\beta_0 = 1$	Analytical ARL $L(u)$	Numerical ARL $L(u)$	*APRE* (%)
0.00	369.99998815077 (0.014)	369.999988148685 (6.864)	5.63503E-10
0.01	330.518128542793 (0.014)	330.518128540967 (13.743)	5.52477E-10
0.03	265.453049034583 (0.014)	265.453049033176 (20.622)	5.30033E-10
0.05	214.957231169933 (0.014)	214.957231168838 (27.72)	5.09405E-10
0.10	131.128316930688 (0.015)	131.128316930081 (34.663)	4.62886E-10
0.30	26.7234332275109 (0.014)	26.7234332274251 (41.589)	3.21059E-10
0.40	14.5018843590053 (0.014)	14.5018843589665 (48.5)	2.67558E-10
0.50	8.68857585076846 (0.014)	8.68857585074919 (55.301)	2.21785E-10

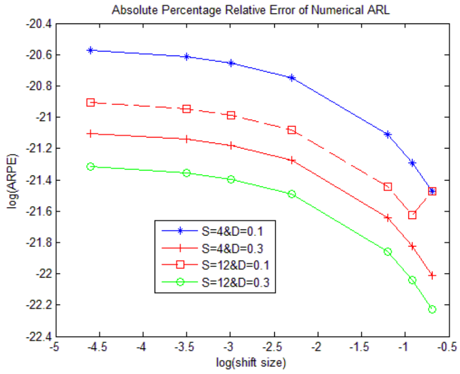

Figure 1. Error on Numerical ARL.

The figures in Tables 1-4 represent the analytical ARL evaluated from the Composite Midpoint Rule on $m = 1,000$ subintervals and its error with the shift parameter $\beta = \beta_0(1+\delta)$, where δ is shift size, starting with $ARL_0 = 370$, $u = 1$, and APRE. The different seasonal periods, S=4 for Table 1 and 2, and S=12 for Table 3 and 4, are used to compare the results. Table 1 and 2 are shown on $D = 0.1$, $b = 0.001687725$ and $D = 0.3$, $b = 0.00129642$, respectively, with $\Phi_1 = 0.1, \Theta_1 = 0.1, \Theta_2 = 0.2$. Table 3 and 4 are shown on $D = 0.1$, $b = 0.001428809$ and $D = 0.3$, $b = 0.00116401$, respectively, with $\Phi_1 = 0.1, \Phi_2 = 0.2$, $\Theta_1 = 0.1, \Theta_2 = 0.2$. In all of the figures in each table, the analytical ARL is approximately the same value as the numerical ARL. However, the CPU time of the analytical ARL is significantly lower than the CPU time of the numerical ARL. While the CPU time of the analytical ARL is steady between 0.014 to 0.016 minutes, the CPU time of the numerical ARL rises as the shift size increases.

Figure 1 demonstrates the error of the numerical ARL. As seen in the graph in figure 1, one of the differences between the fractional differencing parameter (D) and the seasonal period (S) is that S is fixed and D increases from 0.1 to 0.3, but the error decreases. In comparison, D is fixed and there is an increase in the seasonal period from S=4 to S=12, but the error still decreases.

6. Discussion and Conclusions

In this paper, the analytical and numerical ARL have been proposed, while the existence and uniqueness of the analytical ARL have been proved. Also, the error and convergence of the numerical ARL have been analyzed and shown. The findings indicated that the analytical ARL and numerical ARL have a satisfactory performance. In this research, the analytical ARL is an alternative approach for an EWMA control chart over a long memory SARFIMA process due to faster computation than the numerical ARL and greater accuracy. Furthermore, the numerical ARL approaches the analytical ARL as D increases with fixed S, or S increases with fixed D. Notice that the seasonal period and fractional differencing parameter that effect the minimizing of error are the significant factors in evaluating the ARL of an EWMA control chart over a long memory SARFIMA process. As discussed previously, the analytical ARL is a very flexible method, but it has certain limitations. In case of a complicated integrand of the integral equation, the analytical ARL cannot be directly integrated. In other cases, when two or more characteristics of a variable are considered in a quality control chart, a system of integral equations representing their ARL is possibly more suitable than analytical ARL. For these two example cases, the numerical method for the numerical ARL is preferable. Moreover, the application of real data with long memory such as production data, financial data, environmental data, etc. should be applied to the model. These problems should be addressed in future research with a focus on novel development.

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