EXTENSION OF THE dRGT MASSIVE GRAVITY FOR COSMOLOGICAL ASPECTS

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Thesis
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EXTENSION OF THE *dRGT* MASSIVE GRAVITY FOR COSMOLOGICAL ASPECTS

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ABSTRACT

dRGT Massive gravity is recently one of the most promising candidates for the massive gravity theories which is successful in introducing mass to graviton without introducing the Boulware-Deser ghost instability. However, as far as the isotropic and homogeneous universe is concerned, dRGT Massive gravity is still not a good model for the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime, which represents the geometry of such a universe. Here the further modifications of the dRGT Massive gravity are considered and analyzed in various ways such as the ghost instability investigation, degree of freedom counting, and also the corresponding cosmic evolutions.

KEY WORDS : GRAVITON / dRGT MASSIVE GRAVITY / COSMOLOGICAL SOLUTION / GHOST

134 pages

การขยายทฤษฎีแมสสีฟกราวิตีแบบ *dRGT* ในมุมมองของจักรวาลวิทยา
EXTENSION OF THE *dRGT* MASSIVE GRAVITY FOR COSMOLOGICAL ASPECTS

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บทคัดย่อ

ทฤษฎีแมสสีฟกราวิตีแบบ dRGT เป็นทฤษฎีแมสสีฟกราวิตีที่น่าสนใจมากที่สุดแบบหนึ่ง โดยทฤษฎีนี้ประสบความสำเร็จในการบรรยายอนุภาคกราวิตอนแบบมีมวลในขณะที่ทฤษฎีนี้สามารถหลีกเลี่ยงความไม่เสถียรแบบ Boulware-Deser ได้ ถึงกระนั้น เราไม่สามารถใช้ทฤษฎีแมสสีฟกราวิตีแบบ dRGT สำหรับการบรรยายกาลอวกาศแบบ Friedmann-Lemaître-Robertson-Walker (FLRW) ซึ่งเป็นกาลอวกาศที่สอดคล้องกับเอกภพที่มีความไอโซทรอปีและความเอกพันธ์ได้อย่างเหมาะสม ในงานวิจัยชิ้นนี้เราจะทำการศึกษาทฤษฎีแมสสีฟกราวิตีที่ได้รับการดัดแปลง การศึกษาจะทำในหลายๆแง่มุม เช่น การตรวจสอบความเสถียร จำนวนองศาอิสระ และการวิวัฒน์ของเอกภพภายใต้ทฤษฎีนี้

134 pages

CONTENTS

CONTENTS (cont.)

LIST OF TABLES

LIST OF FIGURES

CHAPTER I
INTRODUCTION

Gravitation is one of the most important concepts in the world of physics. As one of the four physical forces in nature; namely gravity, electromagnetism, strong force, and weak force, gravity plays a crucial role in every physical system, though at times gravity is neglected due to its weak magnitude compared with other forces. The theory of gravitation started its story by Newton who proposed the theory of two-body gravitational interaction. Such a theory coincides well with nature so far for centuries and through it people have invented various useful innovations for mankinds. Despite the great uses, the situation changed when one of cornerstones of physics had been introduced by Einstein; the special relativity. In particular, the Newtonian gravity lacks an equal treatment between space and time while both are on an equal footing in the framework of special relativity. To formulate a gravity theory which also embraces special relativity, Einstein came up with a new gravity theory in 1916 formally known as general relativity.

General relativity (GR) is a theory which describes a system with gravitation by introducing a concept of relations between a geometry of spacetime and matters that exist in such a spacetime. Particularly, general relativity describes gravity as not a force formulated in a flat spacetime but rather a manifestation of a curved spacetime. Such a curved geometry is influenced by matters and energies in the spacetime. In this framework, the gravitational force exerting on a particle as we know it is just a consequence of that particle free-falling along a shortest path in the curved spacetime, known as a geodesic. Such a consideration not only incorporates the scale at which the Newtonian gravity works but also can give a correction to phenomina in the solar system scale; the well-known success of general relativity is to be able to correct the former predicted orbit of Mercury around the sun.

There are many attempts on general relativity to describe our universe. Many observational data instructs that our universe is expanding with acceleration [1, 2, 3, 4] (see Ref. [5] for a review on the observations). This fact leads to the simplest modification of general relativity in which a cosmological constant Λ is introduced as an object that drives the accelerating expansion. Unfortunately, the existence of this cosmological constant raises another problem. Namely, cosmological observations suggest that there should exist the energy density $\rho \sim 10^{-47}\ GeV^4$ in order to drive the expansion while the interpretation of the constant as a vacuum energy from quantum field theory suggest a completely different value, $\rho \sim 10^{71}\ GeV^4$.

Though it serves as an accurate gravity theory in, for example, a solar system, the fact about the cosmological constant may lead us to suspect whether the original general relativity is a complete gravitational theory and it needs to be modified when cosmology is taken into account. One possible principle for modification is an infrared modification of the theory, namely, to weaken the gravitational strength on the large scale. One may consider the generalization of the Einstein-Hilbert action from the generic Ricci scalar to be a function of it, or the so-called $f(R)$ theory [6] (see also [7] for the review). On the other hand, by taking the large-scale modification into account, one can seek for alternative gravity theory which directly relates to weakening the strength of the force at a large scale. In the field theory point of view, general relativity is a field theory of propagating massless spin-2 degrees of freedom called a graviton. The theory possesses a gauge symmetry geometrically corresponding to the well-known general covariance; the physics of the theory is invariant in any coordinate. Possessing such a symmetry, an excitation of the tensor field becomes massless and thus the corresponding force is a long-range one (the gravitational potential in the weak-field limit is proportional to $\frac{1}{r}$). Thus, the straightforward thought of weakening the gravitational strength would be to add nonzero mass to the graviton so it becomes massive (this inevitably breaks the general covariance but later we will see the process of restoring it). Consequently, adding mass to the graviton invokes the Yukawa factor in the gravitational potential of the form $\frac{e^{-mr}}{r}$ where m is the mass, causing the gravitational interaction to be modified at the scale corresponding to the graviton mass while preserving predictions inside such a scale to be those of general relativity (see [8] for a review on this topic).

Adding mass to the graviton defines the theory of massive gravity. The idea has been developing since the first attempt in 1939 when Fierz and Pauli added a mass term to the linearized version of general relativity [9]. Due to this extra term, there arised the problematic discontinuity in the theory, known as van Dam-Veltman-Zakharov (vDVZ) discontinuity, which states that the limit graviton mass approaching zero of the massive gravity cannot exactly recover the original GR [10, 11]. Later on, a nonlinear theory of massive gravity was developed with hopes that it may cure the discontinuity problem. The idea came from the existence of the scale Vainshtein claimed that non-linear effects become important inside this scale [12]. Meanwhile, Boulware and Deser investigated the massive gravity and found there exists a ghost in a generic nonlinear massive gravity [13]. Finally, in 2010, de Rham, Gabadadze, and Tolley developed the complete theory of dRGT massive gravity without the Boulware-Deser ghost by the particular construction on the interaction terms in the action [14, 15]. This is one of the best hopes so far on massive gravity.

Still, dRGT theory has some problems when being used to describe our universe. The present universe, as we believe it, is based on two cosmological principles, namely the isotropy and homogeneity, although these are valid only on quite large scale of universe; approximately 100 Mpc [16]. Moreover, such universe with those properties can be well represented by the Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime. Under the FLRW ansatz, the dRGT theory yields incorrect number of degrees of freedom which is two instead of five [17]. Moreover, one of the other three vanishing degrees of freedom turns out to be an instability, rendering the solution to be unstable [18]. There are several ways to fix this problem. One can depart from the isotropic consideration of the universe by introducing a small anisotropy to the spacetime while others can add more degrees of freedom coupled to the massive graviton. Furthermore, one can consider in extra-dimensional aspects of the massive gravity, which give a more concrete and motivated explanation to the addition of extra degree of freedom approach.

This work focuses on various modifications of the dRGT massive gravity. Each model is carefully investigated on its solutions and the corresponding dynamics. Moreover, some of the cosmological implications are also studied in each of the models in order to investigate the validity of the models.

CHAPTER II
LITERATURE REVIEWS

Even though the cosmological constant problem is one of the motivations of the massive gravity theory, the idea of introducing the mass to the graviton was studied since 1939, fews decades before the birth of quantum field theory, by Fierz and Pauli [9]. By adding a mass (self-interaction) term into the linearized general relativity, they succeeded in constructing the linear theory of massive gravity.

Several decades later, in 1970, van Dam, Veltman, and Zakharov [10, 11] studied the linear theory of massive gravity coupled to a source. They discovered a discontinuity between the linear massive gravity theory and the linearized general relativity. In particular, the predictions made by the linear massive gravity do not coincide with those from the linearized general relativity even when the graviton mass is approaching zero. This is commonly known as the van Dam-Veltman-Zakharov (vDVZ) discontinuity, named after the discoverers. It was found that when the Fierz-Pauli action is considered in a limit that the mass of graviton goes to zero, since there are two degrees of freedom for tensor mode, two for vector mode, and one for scalar mode, the scalar degree of freedom still couples to the trace of the source, i.e. the energy-momentum tensor. This causes the difference between the linear massive gravity and linearized general relativity and thanks to such a coupling the discontinuity occurs.

A few years later, a nonlinear version of massive gravity was studied by Vainshtein [12] and he claimed that there exists a Vainshtein radius r_v which corresponds to the availability of the linear theory. Particularly, through the linear theory we can trust predictions outside the region characterized by this radius while the nonlinearity becomes more effective inside. Moreover, the graviton mass determines the fate of this radius; the smaller the mass, the greater the radius will be. This discovery also provides a possibility to fix the vDVZ discontinuity in such a way that the nonlinearity will correct those predictions made by only the linear theory, via the Vainshtein mechanism, to match with those predicted by general relativity.

Same year as the Vainshtein's discovery, Boulware and Deser [13] showed that a generic class of nonlinear theory of massive gravity possesses a ghost instability. It appears that instead of having five degrees of freedom for the massive gravity, there is the sixth ghostly degree of freedom due to the wrong sign of its own kinetic term. This is known as a Boulware-Deser (BD) ghost. Such a ghostly degree of freedom brings up a unphysical excitation mode with negative energy. In other words, in the hamiltonian formalism, this is the result of the hamiltonian being a linear function of a canonical momentum. Such a hamiltonian can be as negative as it could and no ground state can be defined.

Several decades later, in 2010, de Rham, Gabadadze, and Tolley [14, 15] constructed a nonlinear theory of massive gravity that does not suffer from BD ghost, the so-called dRGT massive gravity. In lagrangian mechanics language, the theory involves summing suitable interaction terms. Each interaction term is constructed through a particular building block tensor from a reference metric, which acts like a lagrange multiplier to give constraints to the theory. Furthermore, there is no vDVZ discontinuity because the theory allow the existance of the Vainshtein mechanism, which screens additional contributions apart from those in general relativity within a certain scale.

After the remarkable discovery, there were many attempts to test this dRGT theory. In 2011, D' Amico *et al.* performed cosmological studies on the proposed ghost-free massive gravity with the Minkowski reference metric [19]. They found that the model does not allow the existence of a nontrivial isotropic and homogeneous solution. Particularly, such a solution is prohibited due to the specific construction to avoid the BD ghost. In the same year, however, it was found by Gumrukcuoglu, Lin, and Mukohyama [20] that there exists the FLRW solutions for the dRGT massive gravity whose geometry is an open three-space. Moreover, the solutions are self-expanding which is driven by a constant, acting as the dark energy, in terms of the graviton mass and the free parameter of the dRGT theory. However, they claimed that the FLRW solutions can only be approximated as a flat FLRW universe while the exact solution for the flat FLRW universe is not allowed.

Following from the last work, Gumrukcuoglu, Lin, and Mukohyama [17] has also shown that the self-accelerating cosmological solutions have vanishing vector and scalar degrees of freedom; while there are five degrees of freedom in the Fierz-Pauli linear massive gravity [8] (two for tensor, two for vector, and one for scalar), and the theory must satisfy the requirement for the massive spin-2 particle to have so many degrees of freedom [19], it appears that there are only two propagating tensor degrees of freedom when the FLRW geometry is applied. Moreover, De Felice, Gumrukcuoglu, and Mukohyama [18] considered the dRGT massive gravity with the anisotropy spacetime in the FLRW limit or, in other words, the isotropy limit. They found that in the FLRW limit, there still is a ghost instability in the theory which is not a BD ghost. These results conclude the failure of the dRGT massive gravity towards the cosmological principles. This fact invokes a question : why the nonlinear massive gravity forbids the FLRW universe while the general relativity embraces it.

In order to fix such problems, there are suggestions from Ref. [21] that one should leave from the isotropic property or modify the theory by introducing a new degree of freedom. For the former case, it is proven that considering the anisotropy universe recovers the vanishing degrees of freedom and also cure the ghost instability [22, 23]. For the latter case, to add a new degree of freedom, one can introduce a quasi-dilaton scalar and its corresponding interaction into the theory [21, 24] or a Dirac-Born-Infeld scalar along with the interaction which is motivated from the higher-dimensional point of view [25, 26, 27]. The quasi-dilaton massive gravity theory serves as a consistent theory since it allows a nontrivial cosmological solution as well as all of the degrees of freedom of massive graviton are in active and healthy [28, 29]. Alternatively, to avoid an inability to provide the meaningful cosmological solution, the authors in Ref. [19] suggest that one may choose to take the graviton mass as a varying function of an additional rolling scalar field. This consideration leads to a new model of a ghost-free massive gravity theory known as a mass-varying massive gravity [30, 31, 32]. Unfortunately, by allowing the graviton mass to be a function of the scalar field, the theory suffers from the vanishing graviton mass at late-time which renders the massive gravity meaningless in the late-time acceleration scenario. With hopes to obtain a more meaningful theory, a new class of mass-varying massive gravity was proposed in which the graviton mass is governed by a rolling k-essence field in such a way that not only the field itself but also the kinetic term of the k-essence field are involved in the graviton mass function [33]. The proposed model allows a nontrivial cosmology where it is possible to have a self-accelerating solution influenced by the massive gravity sector.

CHAPTER III
BACKGROUND KNOWLEDGE

Modified gravities, like massive gravity, obviously involves modifying the Einstein's general relativity. Usually, the modifications take place in the lagrangian of the theory since upon tackling the modified theory with tools in quantum field theory, the essence of such modifications can be extracted meaningfully. Thus, it is important to study not only the general relativity but also the field theory, where we can understand how the lagrangian of field works. This section is devoted for clarifying the methods and analysis frequently used in the classical field theory and modified gravity.

3.1 General Relativity

General relativity describes the gravity by relating geometric curvature of spacetime with matter and energy contents. The curvature of spacetime is represented via the geometric measurement structure known as a ``metric tensor'', usually denoted by $g_{\mu\nu}$. This tensor specifies how one measure a distance between two points residing in given spacetime. Given two infinitesimally adjacent points, the distance between those points, formally known as a line element, is given in terms of the metric tensor by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu. \tag{3.1}$$

Here, the Einstein's convention of summation is applied; the same (upper and lower) indices imply summation over possible values of those indices. At times, we will experience a situation where an inverse of the metric, denoted by $g^{\mu\nu}$, is considered. $g^{\mu\nu}$ is defined to satisfy the following condition,

$$g^{\mu\rho} g_{\rho\nu} = \delta^\mu_\nu. \tag{3.2}$$

Having the metric tensor and its inverse, there are particular procedures involving ``raising" or ``lowering" the indices of a tensor. For example, given a vector V^μ, we can find its corresponding ``co-vector" by lowering the index of V^μ as follows,

$$V_\mu = g_{\mu\nu} V^\nu. \tag{3.3}$$

Although they look different, both V^μ and V_μ correspond to the same geometrical object, says **V**. The only different is that each of them is represented on different bases; one on a vector basis and another on a 1-form basis. This is also true for other general-rank tensors. For example, consider the following two-indices tensors,

$$T^\mu_\nu \equiv g^{\mu\rho} T_{\rho\nu} \equiv T^{\mu\rho} g_{\rho\nu}. \tag{3.4}$$

Each of these tensors corresponds to the same object, says **T**. Roughly speaking, $T^{\mu\nu}$ is a representation of **T** on a vector/vector basis, $T_{\mu\nu}$ is on a 1-form/1-form basis, and T^μ_ν is on a vector/1-form basis.

The concepts of geometry are introduced not only by the metric tensor but also other quantities involving the metric tensor. Since general relativity involves considerations on a manifold with nontrivial curvature, one may relate two vectors on different points of spacetime through a ``connection" which is formally called a Christoffel symbol, or Christoffel connection, given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right). \tag{3.5}$$

The reason to its existence is that since the geometry in consideration is curved, the concepts of vector defined in the Euclidean geometry as the arrow that points from one to another coordinate point do not make sense anymore. Thus, a vector is redefined as an object with magnitude and direction which resides on a particular point in the manifold, without the sense of arrow pointing from one to another point in spacetime. The connection then serves us as a ``connection" from vectors (or other tensors) defined on one point to those on another point. In other words, the connection allows us to define the concept of derivative on the manifold as a covariant derivative defined as follows,

$$\nabla_\mu A^\nu \equiv \partial_\mu A^\nu + \Gamma^\nu_{\mu\rho} A^\rho. \tag{3.6}$$

The covariant derivative also works on the other kinds of tensor, for example,

$$\begin{aligned} \nabla_\mu B^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\ldots} &\equiv \partial_\mu B^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\ldots} \\ &+ \Gamma^{\alpha_1}_{\mu\rho} B^{\rho\alpha_2\cdots}{}_{\beta_1\beta_2\ldots} + \Gamma^{\alpha_2}_{\mu\rho} B^{\alpha_1\rho\cdots}{}_{\beta_1\beta_2\ldots} \\ &- \Gamma^{\rho}_{\mu\beta_1} B^{\alpha_1\alpha_2\cdots}{}_{\rho\beta_2\ldots} + \Gamma^{\rho}_{\mu\beta_2} B^{\alpha_1\alpha_2\cdots}{}_{\beta_1\rho\ldots}. \end{aligned} \tag{3.7}$$

The curvature does not only affect how we define a notion of vector and tensor but also deforms some elementary geometrical properties that we know in the Euclidean space. The famous example is triangles drawn in spacetimes with different curvatures. Particularly, a triangle living in a surface of a basketball, for example, can have the sum of its three inner angles greater than 180 degrees while one living in a surface of a saddle has the sum less than 180 degrees. In our sense, it is clear to use the triangle concept to identify one kind of curvature from another. However, we use a quite different concept to describe the curvature using mathematical symbols. We consider a change in a particular quantity transported along a closed curve in a specific geometry. The change turns out to be characterized via the Riemann curvature tensor, or just Riemann tensor, defined as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\sigma\nu} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\sigma\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}. \tag{3.8}$$

Note that, by imposing the definition of the Christoffel symbol, the Riemann curvature tensor involves second derivatives (and square of first derivatives) which naturally denote a curvature in the Euclidean sense. In the Einstein's general relativity, the Riemann tensor is introduced via its corresponding Ricci tensor. By contracting the indices, one can find the Ricci tensor to be

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\rho\mu}. \tag{3.9}$$

Contracting the remaining indices of the Ricci tensor, we obtain its corresponding Ricci scalar as follows,

$$R = R^{\mu}_{\ \mu} = g^{\mu\nu} R_{\mu\nu}. \tag{3.10}$$

These quantities, especially the Ricci tensor and the Ricci scalar, are frequently used in general relativity. Despite of the complicated structure, there are useful identities that greatly simplify calculations involving these quantities. The first two indices of Riemann tensor are anti-symmetric as well as the last two indices,

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} = -R_{\mu\nu\sigma\rho}. \tag{3.11}$$

On the other hand, the first pair of indices and the last pair are symmetric under the following swapping,

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}. \tag{3.12}$$

Moreover, there is another useful identity which involves a covariant derivative as follows,

$$\nabla_{\tau} R_{\mu\nu\rho\sigma} + \nabla_{\rho} R_{\mu\nu\sigma\tau} + \nabla_{\sigma} R_{\mu\nu\tau\rho} = 0. \tag{3.13}$$

This identity is known as the Bianchi identity. Through the previous identities, this Bianchi can be expressed in its alternative form as

$$\nabla^{\mu} G_{\mu\nu} \equiv \nabla^{\mu} \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0, \tag{3.14}$$

where $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ is formally known as the Einstein tensor. This version of Bianchi identity corresponds to a conservation laws of energy and momentum. In general relativity, energies and momentums of matters in spacetime are described through a geometrical object called energy-momentum tensor, usually denoted by $T_{\mu\nu}$. The conservation laws then appear in this language as

$$\nabla_\mu T^\mu_\nu = 0. \tag{3.15}$$

One may note that this conservation law and the Bianchi identity in Eq. (3.14) share some resemblances. One may expect a proportionality between these two quantities as follows,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu}, \tag{3.16}$$

where the proportional constant has already been determined; G is the Newtonian gravitational constant, and $M_p \equiv \frac{1}{\sqrt{8\pi G}}$ is the reduced Planck mass (here we use a unit in which $\hbar = 1$). Though such an anticipation is not trivially obvious, and there were series of procedure when this equation was being developed, this is exactly the famous Einstein's field equation, a core equation governing dynamics of metric tensor due to the presence of energies and momentums of matters. One can see that the Bianchi identity in Eq. (3.14) confirms the conservation of energy and momentum. This equation equates two kinds of quantities together; one consists of geometric quantities describing the curvature of spacetime while another one is a tensor determining energies and momentums in the spacetime. In one way, the energy-momentum tensor tells the spacetime how to curve according to its form. In general relativity (and beyond), many situations involve a specific class of the energy-momentum tensor which can be fully specified only by an energy density and a pressure of matter. Such matter is known as a perfect fluid. By definition, the energy-momentum tensor corresponding to a perfect fluid can be expressed as

$$T_{\mu\nu} = (\rho + p) U_\mu U_\nu + p g_{\mu\nu}, \tag{3.17}$$

where ρ denotes an energy density of the fluid, p is its pressure and U^μ is a 4-velocity of the fluid. By the term ``perfect fluid'', it suggests that particles in the fluid of consideration do not interact, in every way, with one another. The examples of such fluid, which turn out to be very important forms of matter in cosmology, are a dust and radiation.

A dust is a terminology for a nonrelativistic matter, hence a pressureless matter, while radiation is consist of photons as relativistic particles. One way to classify these fluids is to consider their equations of state which usually are in the following form,

$$p = p(\rho). \tag{3.18}$$

Equivalently, we can define a parameter corresponding to the equation of state as

$$w \equiv \frac{p}{\rho}. \tag{3.19}$$

This parameter is commonly known as an equation of state parameter. In this language, the equation of state parameter of the dust is zero because of its pressureless property while the radiation whose pressure is one-third of its energy density satisfying $w = \frac{1}{3}$. At times, one may encounter a system in which a perfect fluid is considered in its rest frame. The energy-momentum tensor in this case is simply

$$T^{\mu}_{\ \nu} = \text{diag}\left(-\rho, p, p, p\right). \tag{3.20}$$

The field equation in Eq. (3.16) gives rise to various kinds of solution depending on their symmetry properties. In black hole physics, for example, one can obtain the Schwarzschild solution as a static and spherically symmetric solution to Eq. (3.16) in empty space while one has the Kerr solution as a rotating and axisymmetric vacuum solution. Moreover, we can add contributions from electrodynamics, via the energy-momentum tensor, to obtain solutions incorporating, for instance, electric charge such as the Reissner-Nordström solution which is a static and spherically symmetric kind of solution. Furthermore, we can encounter an even more complex solution, the Kerr-Newman solution which is a charged-version of the Kerr solution [34, 35]. Furthermore, general relativity has also proven itself quite useful in the aspect of cosmology, which will be briefly discussed in the following calculations.

To see the cosmological interpretation of general relativity (with Λ), one can consider the universe satisfying the cosmological principles; the universe whose space is isotropic and homogeneous. The spacetime that possess such properties is represented by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric,

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij}(x^k) dx^i dx^j, \tag{3.21}$$

where N is a lapse function denoting a time parameterization and a is a scale factor determining the overall spatial scale of the universe, and Ω_{ij} is a metric describing the geometry of 3-space which is given by

$$\Omega_{ij}(\varphi^k) = \delta_{ij} + \frac{\kappa \delta_{il} \delta_{jm} \varphi^l \varphi^m}{1 - \kappa \delta_{pq} \varphi^p \varphi^q}, \tag{3.22}$$

where κ indicates the 3-space curvature; the positive value for a closed geometry, negative for a hyperbolic one, and zero for a flat space. Since we are considering the solution with isotropy and homogeneity, the energy-momentum tensor must respect the cosmological principles as well; namely being isotropic and homogeneous. Thus, the energy-momentum tensor of interest must be in its rest frame which implies $U^\mu = (U^0, 0, 0, 0)$ and consequently, assuming a perfect fluid, takes the form,

$$T^\mu_\nu = \text{diag}\left(-\rho(t), p(t), p(t), p(t)\right). \tag{3.23}$$

If the FLRW solution is considered, the corresponding Einstein's equation in Eq. (3.16) will be decomposed into two independent equations as follows,

$$M_p^2 \left(3H^2 + 3\frac{\kappa}{a^2}\right) = \rho, \tag{3.24}$$

$$M_p^2 \left(\frac{2\dot{H}}{N} + 3H^2 + \frac{\kappa}{a^2}\right) = -p, \tag{3.25}$$

where $H \equiv \frac{\dot{a}}{Na}$ and the dot represents the derivative with respect to the time t. The equation in Eq. (3.24) is called the first Friedmann equation or just Friedmann equation while (3.25) is known as the second Friedmann equation or acceleration equation. The Friedmann equation tells us how fast the spacetime expand due to the matter contents while the acceleration equation determines the acceleration of such an expansion. Since we obtained these equations from the Einstein field equation, these equations inevitably obey the Bianchi identity. The Bianchi identity implies the following equation,

$$\dot{\rho} + 3HN\left(\rho + p\right) = 0. \tag{3.26}$$

This equation is exactly the conservation of energy-momentum tensor, namely $\nabla_\mu T^\mu_\nu = 0$, in the FLRW geometry. Note that ρ can be regarded as the total energy density of the system. Usually, in the standard cosmology each matter content satisfies Eq. (3.26) separately, namely

$$\dot{\rho}_i + 3HN\left(\rho_i + p_i\right) = 0. \tag{3.27}$$

Such a situation happens if there is no interaction among different matters. We will see in Chapter VII an example of the case where the interaction exists which corresponds to nonzero contribution on the right-hand side of Eq. (3.27).

One can notice that Eq. (3.24) and Eq. (3.25) are differential equations. Since we have only two equations but with three unknown functions, namely $a(t), \rho(t), p(t)$, we need more information, or an another equation, in order to completely determine the corresponding dynamics. One may think that one can include Eq. (3.26) (or all of Eq. (3.27)) in order to obtain solutions to those functions. However, Eq. (3.26) cannot fulfil such a purpose since Eq. (3.26) is not mutually independent from Eq. (3.24) and Eq. (3.25); they are all related through the Bianchi identity. Thus, to fully determine the forms of those functions, an another additional equation is needed which turns out to be the equation of state about which we previously discussed in Eq. (3.18) or equivalently the equation of state parameter in Eq. (3.19). With such an equation, accompanied with Eq. (3.24) and Eq. (3.25), one can fully determine the dynamics of the cosmological solution properly.

The acceleration equation in Eq. (3.25) can appears in its alternative but more reasonable form. As the name suggests, by using the following relation,

$$\dot{H} = \frac{1}{a}\frac{d}{dt}\left(\frac{\dot{a}}{N}\right) - HN, \tag{3.28}$$

it can be rewritten in term of an acceleration of the scale factor in the following form,

$$\begin{aligned}\frac{d^2a}{ad\tilde{t}^2} &= -\frac{1}{M_p^2}\left(p+\frac{\rho}{3}\right),\\ &= -\frac{\rho}{M_p^2}\left(w+\frac{1}{3}\right),\end{aligned}\tag{3.29}$$

where the equations is reparameterized by a time $\tilde{t}$ satisfying $d\tilde{t} = Ndt$. For ordinary matters, such as dust and radiation, the acceleration of the scale factor is always negative which otherwise implies a deceleration of the universe. Though it is reasonable since gravitating objects, as we know them, attract one another, the observations reveal that the universe is in fact expanding with acceleration [1, 2, 3, 4]. It appears that to have an accelerating expansion, where $\frac{d^2a}{ad\tilde{t}^2} > 0$, one can fill the universe with a specific matter having the following equation of state parameter,

$$w < -\frac{1}{3}.\tag{3.30}$$

However, as far as we know there is no such matter which satisfies $w < -\frac{1}{3}$. Since the observations suggest otherwise, one may choose to modify general relativity to fix this disagreement. Here comes a very simple yet promising modification of general relativity; general relativity with a cosmological constant, or commonly known as ΛCDM model.

ΛCDM (or Λ-cold-dark-matter) model is the general relativity with the existence of a cosmological constant represented by Λ (see [36] for more reviews on ΛCDM model). The model describes a universe in which Λ is associated to the dark energy whereas nonrelativistic particles are associated with the dark matter. This model is the very first model which is able to describe the cosmic expansion with acceleration. In this model, the Einstein's field equation is slightly modified as follows,

$$G^{eff}_{\mu\nu} \equiv G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} = \frac{1}{M_p^2}T_{\mu\nu},\tag{3.31}$$

where Λ represents the cosmological constant. Depending on the sign of Λ, there exist two kinds of solutions. For a positive Λ, one obtains a de Sitter solution while others have an anti-de Sitter solution for the negative Λ, though the anti-de Sitter solution does not have a meaningful cosmology. Note that introducing the cosmological constant term on the left-hand side indicates that the geometrical nature of general relativity is modified. In particular, a spacetime governed by the modified Einstein's equation is warped differently under the influence of the energy-momentum tensor compared with one in general relativity. Mathematically speaking, the geometrical part of the Einstein's field equation is characterized by the effective Einstein tensor $G^{eff}_{\mu\nu}$. One may argue that, however, the modified Einstein's equation can always be rearranged so that the cosmological constant is incorperated in the energy-momentum tensor, rendering it to be an ``effective" energy-momentum tensor as follows,

$$G_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} - \Lambda g_{\mu\nu} \equiv \frac{1}{M_p^2} T^{eff}_{\mu\nu}. \tag{3.32}$$

In spite of the simple rearrangement, this equation has a slight different meaning from the previous equation. The geometrical part in Eq. (3.32) is exactly that in general relativity while the equation is rather sourced by the effective energy-momentum tensor $T^{eff}_{\mu\nu}$ than the actual energy-momentum tensor $T_{\mu\nu}$. This modified Einstein's equation has proven itself to be quite useful in cosmology since, in the effective energy-momentum tensor picture, the cosmological constant Λ serves similarly as perfect fluid with negative pressure producing an outward push in the system of interest. This will become clearer in the following calculations where the cosmology of general relativity is discussed.

In the ΛCDM model, it is the constant Λ that drives the expansion with acceleration of the universe. One can see such an expansion behavior by considering Eq. (3.24) and Eq. (3.25) in the absence of other matter, or in other words $\rho = p = 0$. We have

$$M_p^2 \left(3H^2 + 3\frac{\kappa}{a^2}\right) = M_p^2 \Lambda, \tag{3.33}$$

$$M_p^2 \left(\frac{2\dot{H}}{N} + 3H^2 + \frac{\kappa}{a^2}\right) = M_p^2 \Lambda. \tag{3.34}$$

From the equations above, we can see some important consequences. In the case of the flat FLRW universe where $\kappa = 0$, we may require Λ to be positive, otherwise we would have negative H^2. The positive Λ gives us a constant H and then, from the definition of H, we will have an exponentially expansion of the scale factor; $a \propto e^{Ht}$, provided that the spatial curvature is flat. This is known as the de Sitter universe. Such a universe expands so rapidly due to the positive cosmological constant, as we claimed before. We can see that the negative Λ yields a negative value of H^2, in the flat FLRW case, which is not consistent, thus an anti-de Sitter universe does not exist. By treating Λ as an additional part of the energy-momentum tensor, the corresponding energy density ρ_Λ and pressure p_Λ can be read out instantly as

$$\rho_\Lambda = M_p^2 \Lambda, \tag{3.35}$$

$$p_\Lambda = -M_p^2 \Lambda. \tag{3.36}$$

Note that the pressure of the cosmological constant is negative and equal to $-\rho_\Lambda$ which reflects a simple implication that if one naively treated Λ as a real matter, due to its negative pressure this matter repeled from each other while other positive-pressure matter attracts one another in a gravitational field. Though this naive idea may be rather an illustration than an actual physical phenomenon, it is still useful to adopt such an illustration as a mnemonic. Obviously from Eq. (3.35) and Eq. (3.36), we obtain the equation of state parameter corresponding to Λ as

$$w_\Lambda = \frac{p_\Lambda}{\rho_\Lambda} = -1. \tag{3.37}$$

This equation of state parameter agrees with the condition for the accelerating expansion in Eq. (3.30) which simply implies that the cosmological constant is capable of driving the cosmic accelerating expansion. Actually, this model is of interest in cosmology such that it fits well with most of observations via introducing only one parameter to modify general relativity. Despite being compatible to the observations, the introduction of Λ poses another issue to the model; from where or through which process this cosmological constant originates. Due to its equation of state, we can find that the corresponding energy density being constant in time,

$$\frac{d\rho_\Lambda}{dt} = 0. \tag{3.38}$$

Since Λ is constant in time, no matter how dense Λ initially was, it will be that dense during the entire evolution of the universe. In other words, the energy due to Λ fills in the spacetime more and more as the spacetime grows. One may have an idea of this energy being vacuum energy since the more vacuum there is, the more vacuum energy is filled. However, the vacuum energy estimated from quantum field theory has its value drastically different from one obtained from cosmological observations; the observed energy density for Λ is around $10^{-47}GeV^4$ while quantum field theory predicts the value of around $10^{71}GeV^4$ --- a more than hundred order of magnitude difference. This huge difference poses another theoretically dangerous issue in physics, known as the cosmological constant problem. As a result, physicists are accordingly motivated to seek for a better model, or a better modified gravity, that both suits the universe more properly and satisfies the predictions made from quantum field theorists.

Before we leave this section of introductory general relativity, there is another important property which is worth mentioned. It is the concept of a general covariance, an important characteristic of the general relativity. The general covariance states that the law of physics must be the same in every single coordinate system. In particular, the law of physics must be invariant under any coordinate transformation. Mathematically speaking, under some coordinate transformation from x to $X(x)$, the metric tensor transforms as

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(X) = \frac{\partial x^\alpha}{\partial X^\mu}\frac{\partial x^\beta}{\partial X^\nu} g_{\alpha\beta}(x). \tag{3.39}$$

Since the metric tensor is a dynamical field in general relativity, the theory must be the same under the change in Eq. (3.39). To be exact, the transformation in Eq. (3.39) leaves the Einstein's field equation in Eq. (3.16) covariant, meaning that it is the form of the equation that is invariant but not the values of each variables. Consequently, in a field theory language, this property introduces the concept of the gauge invariance of the theory of massless spin-2 particle where the gauge parameters correspond to the coordinate transformations.

Though in this section we only discuss about general relativity via the Einstein's field equation in Eq. (3.16), we can also see general relativity through its lagrangian formalism. However, since general relativity determines the dynamics of a metric, a tensorial function of spacetime, then the field theory inevitably comes into the play.

3.2 Classical Field Theory

Suppose that we have a system whose configuration is described by dynamical fields $\phi^a(x^\mu)$ parametrized by spacetime coordinates $x^\mu \in (t, x^1, x^2, x^3)$ where the latin letters represent all possible dynamical fields and the greek letters run from 0 to 3 denoting the indices of the spacetime coordinates. Let the system corresponds to the action of the form

$$S[\phi^a(x^\mu)] = \int d^4x\, \mathcal{L}(\phi^a(x^\mu), \partial_\nu \phi^a(x^\mu)), \tag{3.40}$$

while $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ is the partial derivative along the x^μ. The $\mathcal{L}$ is said to be a lagrangian density, commonly known as just a ``lagrangian", of the system. Applying the least action principle, one can find equations of motion of each dynamical fields through the following Euler-Lagrange equation,

$$\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \right) = 0. \tag{3.41}$$

In the field theory point of view, general relativity is described via the Einstein-Hilbert action of the form

$$S_{GR} = \frac{M_p^2}{2} \left(\int d^4x \sqrt{-g}\, R \right) + S_{Matter}, \tag{3.42}$$

where $g \equiv \det\ g_{\mu\nu}$. S_{Matter} denotes the matter contents which distort the geometry and gives rise to the energy-momentum tensor $T_{\mu\nu}$. From a variation of S_{Matter} with respect to $g^{\mu\nu}$, the energy-momentum tensor is defined as follows,

$$T_{\mu\nu} \equiv \frac{-2}{\sqrt{-g}}\frac{\delta S_{Matter}}{\delta g^{\mu\nu}}. \tag{3.43}$$

Through the Euler-Lagrange equation, we can obtain the equation of motion which is exactly the Einstein's field equation in Eq. (3.16). For the ΛCDM model, the corresponding action is that of the general relativity with an additional extension as follows,

$$S_{GR} = \frac{M_p^2}{2}\left(\int d^4x\sqrt{-g}(R-2\Lambda)\right) + S_{Matter}. \tag{3.44}$$

By using the Euler-Lagrange equation, the modified Einstein's equation in Eq. (3.31) can be obtained readily.

3.3 Linearized Gravity

One may see that to realize general relativity, one must deal with the nonlinear equation of motion, which is usually not able to be solved analytically. Fortunately, when considering a system in which the gravitational field is weak such as that in the solar system, we can realize the equation in a more suitable way. For a weak field approximation, instead of brute calculating for $g_{\mu\nu}$, we can treat the metric to be composed of two parts; the (Minkowskian) background and its fluctuation as follows,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{3.45}$$

where $|h_{\mu\nu}| \ll |\eta_{\mu\nu}|$. According to the weak field approximation, the Einstein's field equation can be simplified to a linear differential equation, or it can be ``linearized" as we shall see in the following calculation.

In the language of the linearized gravity, instead of the metric $g_{\mu\nu}$, the dynamical field in consideration is its fluctuation around some background metric, the Minkowski metric in this case. This means that the fluctuation $h_{\mu\nu}$ is considered as a field propagating in Minkowski spacetime. In other words, raising or lowering any index of tensors are performed through the Minkowski metric $\eta_{\mu\nu}$. When the fluctuation in

Eq. (3.45) is introduced, it is always useful to consider its corresponding inverse metric $g^{\mu\nu}$, up to the second order of perturbation, as

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + h^{\mu\rho}h_{\rho}^{\ \nu} + \mathcal{O}(3), \tag{3.46}$$

where

$$h^{\mu\nu} = \eta^{\mu\rho}\eta^{\nu\sigma}h_{\rho\sigma}, \tag{3.47}$$

$$h_{\mu}^{\ \nu} = \eta^{\rho\nu}h_{\mu\rho}. \tag{3.48}$$

To compute the geometrical quantities, in order to construct the Einstein's field equation in a linearized gravity language, we can start from the Christoffel connection in Eq. (3.5). Up to the second order of perturbation, the Christoffel connection reads

$$\begin{aligned}\Gamma^{\rho}_{\mu\nu} = &\frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) \\ &- \frac{1}{2}h^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right) + \mathcal{O}(3).\end{aligned} \tag{3.49}$$

Note that in this weak field approximation, the Christoffel connection contains no zeroth order term, which is not so surprising since the connection for the background Minkowski spacetime actually vanishes. For conveniences, we introduce

$$(\Gamma^{(1)})^{\rho}_{\mu\nu} \equiv \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right), \tag{3.50}$$

$$(\Gamma^{(2)})^{\rho}_{\mu\nu} \equiv -\frac{1}{2}h^{\rho\lambda}\left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right). \tag{3.51}$$

These quantities represent the contributions in the Christoffel connection with respect to their order of perturbation. In particular, $(\Gamma^{(1)})^{\rho}_{\mu\nu}$ is of first order and $(\Gamma^{(2)})^{\rho}_{\mu\nu}$ is of second order. The Riemann tensor can also be found in this language as, up to the second order of perturbation,

$$
\begin{aligned}
R^{\rho}{}_{\sigma\mu\nu} &= \partial_\mu(\Gamma^{(1)})^\rho_{\sigma\nu} - \partial_\nu(\Gamma^{(1)})^\rho_{\sigma\mu} \\
&\quad + \partial_\mu(\Gamma^{(2)})^\rho_{\sigma\nu} - \partial_\nu(\Gamma^{(2)})^\rho_{\sigma\mu} + (\Gamma^{(1)})^\rho_{\lambda\mu}(\Gamma^{(1)})^\lambda_{\sigma\nu} - (\Gamma^{(1)})^\rho_{\lambda\nu}(\Gamma^{(1)})^\lambda_{\sigma\mu} \\
&\quad + \mathcal{O}(3). \qquad (3.52)
\end{aligned}
$$

By considering their order of perturbation, the first line corresponds only the first order quantities while the second order terms and so on are expressed on the second line. The associated second order Ricci tensor and the Ricci scalar are

$$
\begin{aligned}
R_{\mu\nu} &= \partial_\rho(\Gamma^{(1)})^\rho_{\mu\nu} - \partial_\nu(\Gamma^{(1)})^\rho_{\mu\rho} \\
&\quad + \partial_\rho(\Gamma^{(2)})^\rho_{\mu\nu} - \partial_\nu(\Gamma^{(2)})^\rho_{\mu\rho} + (\Gamma^{(1)})^\rho_{\lambda\rho}(\Gamma^{(1)})^\lambda_{\mu\nu} - (\Gamma^{(1)})^\rho_{\lambda\nu}(\Gamma^{(1)})^\lambda_{\mu\rho} \\
&\quad + \mathcal{O}(3), \qquad (3.53) \\
R &= g^{\mu\nu}R_{\mu\nu} = \left(\eta^{\mu\nu} - h^{\mu\nu}\right)R_{\mu\nu}. \qquad (3.54)
\end{aligned}
$$

In Eq. (3.54), since $R_{\mu\nu}$ is of at least first order in $h_{\mu\nu}$, it is sufficient to expand $g^{\mu\nu}$ only up to first order to obtain a second order of R. Evaluating these quantities, we are now be able to compute the Einstein tensor $G_{\mu\nu}$ in this language. Since we are going to consider the Einstein's field equation being linear in $h_{\mu\nu}$, we can keep only the first order contributions from the $G_{\mu\nu}$ as

$$
\begin{aligned}
G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R, \\
&= \left(\partial_\rho(\Gamma^{(1)})^\rho_{\mu\nu} - \partial_\nu(\Gamma^{(1)})^\rho_{\mu\rho}\right) - \frac{1}{2}\eta_{\mu\nu}\eta^{\alpha\beta}\left(\partial_\rho(\Gamma^{(1)})^\rho_{\alpha\beta} - \partial_\beta(\Gamma^{(1)})^\rho_{\alpha\rho}\right) \\
&\quad + \mathcal{O}(2). \qquad (3.55)
\end{aligned}
$$

Upon using the first order Christoffel connection in Eq. (3.50), the linearized Einstein's field equation can be expressed as

$$
\begin{aligned}
G_{\mu\nu} &= -\frac{1}{2}\left(\Box h_{\mu\nu} - \partial_\lambda\partial_\mu h^\lambda{}_\nu - \partial_\lambda\partial_\nu h^\lambda{}_\mu + \eta_{\mu\nu}\partial_\lambda\partial_\sigma h^{\lambda\sigma} + \partial_\mu\partial_\nu h - \eta_{\mu\nu}\Box h\right) \\
&= \frac{1}{M_p^2}T_{\mu\nu}. \qquad (3.56)
\end{aligned}
$$

One can see that Eq. (3.56) is a linear differential equation of $h_{\mu\nu}$. This equation of motion is very useful for a study of small-scale structures as in our solar system, or for some other gravitating systems whose gravitational strength is weak. Moreover, one of the solutions to this linearized Einstein's field equation is able to describe a gravitational wave which is a small geometric fluctuation in given spacetime, in this case the flat Minkowski spacetime. Such a solution turns out to be useful in various fields of research both in astrophysics and in cosmology.

The linearized equation in Eq. (3.56) can also be found from the lagrangian formalism of general relativity. In particular, one can find a corresponding action of the linearized gravity which yield an equation of motion being the linearized Einstein equation in Eq (3.56). By treating $h_{\mu\nu}$ as a dynamical field, we expect to obtain the action of second order of perturbation so that the associated equation of motion is a linear differential equation of $h_{\mu\nu}$. This can be done by performing the previous perturbative procedures on the Einstein-Hilbert action in Eq. (3.42) up to the second order of perturbation. We first consider the Einstein-Hilbert action, without the source term, from Eq. (3.42),

$$S_{GR} = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\,R.$$

By noting that from Eq. (3.54), the Ricci scalar is of at least first order in $h_{\mu\nu}$, we can expand the action up to the second order. Schematically, the action is expressed as

$$\begin{aligned} S^{(2)} &= \frac{M_p^2}{2}\int d^4x\sqrt{-g}^{(1)}R^{(1)} + \sqrt{-g}^{(0)}R^{(2)}, && (3.57)\\ &= \frac{M_p^2}{2}\int d^4x\sqrt{-g}^{(1)}\eta^{\mu\nu}R^{(1)}_{\mu\nu} + \sqrt{-g}^{(0)}\eta^{\mu\nu}R^{(2)}_{\mu\nu} - \sqrt{-g}^{(0)}h^{\mu\nu}R^{(1)}_{\mu\nu}, && (3.58)\end{aligned}$$

where the superscripts denote the order of perturbation. Moreover, we also have

$$\sqrt{-g} = 1 + \frac{1}{2}h + \mathcal{O}(2), \tag{3.59}$$

which, according to the previous terminology, readily implies $\sqrt{-g}^{(1)} = \frac{1}{2}h$. After some manipulations, we can arrive at the action of linearized gravity as

$$S^{(2)} = \frac{M_p^2}{2}\int d^4x\Big[-\frac{1}{4}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \frac{1}{2}\partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \frac{1}{2}\partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{4}\partial_\lambda h\partial^\lambda h\Big]. \tag{3.60}$$

This action is consistent with Eq. (3.56) since the equation of motion of Eq. (3.60) is exactly the linearized Einstein's field equation in Eq. (3.56). Note that one can possibly formulate the perturbative theory of general relativity on some non-Minkowskian background $g^{(0)}_{\mu\nu}$. For example, one can consider the cosmological perturbation in which the dynamical field is a fluctuation around the Friedmann-Lemaître-Robertson-Walker metric.

One may expect to deduce the general covariance of the general relativity in the linearized gravity language. By the weak field approximation, the general coordinate transformation from x to $X(x)$ in Eq. (3.39) can be realized as an infinitesimal coordinate transformation from x to $x - \xi$ done to the perturbation field $h_{\mu\nu}$. The resulting transformation can be expressed as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu\xi_\nu - \partial_\nu\xi_\mu. \tag{3.61}$$

One can prove that the transformation in Eq. (3.61) leaves the linearized Einstein's field equation and also the linearized gravity action invariant up to the total derivatives in the action.

In the quantum field theory point of view, the action of linearized gravity is exactly the action governing a massless spin-2 particle commonly known as a graviton. Hence, it is well known that general relativity is a theory corresponding to a massless graviton as the gravitational mediator. We will see in the next chapter a gravitational theory where the corresponding graviton has nonzero mass, the so-called massive gravity. Moreover, the first model of the massive gravity theory will be discussed in which the model is formulated as an extension of the action of linearized gravity we previously discussed about, which was introduced for the first time by Fierz and Pauli.

CHAPTER IV
FIERZ-PAULI MASSIVE GRAVITY

The first model of massive gravity was formulated from the linearized gravity by Fierz and Pauli in 1939 [9]. They added interaction terms which are of second order of the fluctuation field $h_{\mu\nu}$ with a parameter of the interaction interpreted as a square of graviton mass. In four dimensions, the resulting action is expressed in a flat geometry background as

$$S = \int d^4x \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h \right.$$
$$\left. -\frac{1}{2}m_g^2 \left(h_{\mu\nu}h^{\mu\nu} - h^2 \right) \right], \tag{4.1}$$

where $h \equiv \eta^{\mu\nu}h_{\mu\nu}$ and raising or lowering indices are done with the Minkowskian metric $\eta_{\mu\nu}$. Note that this action contains only the gravity sector. One can include a source by adding a lagrangian of the source into the action in Eq. (4.1) yet for now we will investigate the source-free action first. The terms with derivatives come from the linearized gravity action whereas the interaction terms modify the gravity sector where the graviton becomes massive with the mass m_g. Note that the action contains all possible second order terms with appropriate coefficients. In particular, coefficients and signs in front of the derivative terms are required by the linearized gravity action while those of the interaction terms are specified by the requirement of theoretical consistency. The minus sign in front of the m_g^2 ensures the real-valued graviton mass whereas the sign between $h_{\mu\nu}h^{\mu\nu}$ and h^2 ensures the absent of the instability formally known as a Boulware-Deser (BD) ghost, obviously named after the founders (though this BD ghost appears as a terminology for a ghost in nonlinear theory of massive gravity, the concept of avoiding its existence is the same for the linear theory). At first glance, it may seem to be able to recover the general relativity by just setting the mass m_g to be zero. However, we will see in the following sections that this action actually reduces to something else which is closely related to the general relativity.

4.1 Equations of Motion

To deduce the dynamics of the theory, one can find an equation of motion of the Fierz-Pauli action in Eq. (4.1) via the Euler-Lagrange equation. The source-free equation of motion is as follows,

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{\ \nu} - \partial_\lambda \partial_\nu h^\lambda_{\ \mu} + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h$$
$$-m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = 0. \qquad (4.2)$$

In the limit $m_g \to 0$, this equation of motion reduces exactly to the linearized Einstein's field equations. Due to the presence of the nonzero m_g, the equation of motion loses the general covariance. In particular, the nonderivative terms are not covariant under the guage transformation $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$ which make the entire action not covariant under such a gauge transformation. Despite being a linear differential equation, the equation of motion in Eq. (4.2) is quite complicated. Let us simplify it by considering its divergence. After applying ∂^μ to Eq. (4.2), we have

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = 0. \qquad (4.3)$$

Using Eq. (4.3), the equation of motion in Eq. (4.2) reads

$$\Box h_{\mu\nu} - \partial_\mu \partial_\nu h - m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = 0. \qquad (4.4)$$

This equation can be further simplified by the use of its trace. Contracting $\eta^{\mu\nu}$ to Eq. (4.4), we have

$$h = 0, \qquad (4.5)$$

which indicates that $h_{\mu\nu}$ is a traceless tensor. Finally, we have an equation of motion with a set of constraints as follows,

$$\Box h_{\mu\nu} - m_g^2 h_{\mu\nu} = 0, \tag{4.6a}$$

$$\partial^\mu h_{\mu\nu} = 0, \tag{4.6b}$$

$$h = 0. \tag{4.6c}$$

This set of equations is equivalent to the equation of motion in Eq. (4.2). Eq. (4.6a) is nothing but the Klein-Gordon equation whose solution is well known in quantum field theory. We can see from Eq. (4.6b) and Eq. (4.6c) that the field $h_{\mu\nu}$ is transverse and traceless respectively. We can also deduce the number of degrees of freedom by investigating this set of equations. In four dimensions, a symmetric rank-2 tensor, like $h_{\mu\nu}$, has ten independent elements. The transverse condition in Eq. (4.6b) gives four constraints which reduces the number of independent elements into six and the tranceless condition in Eq. (4.6c) constraints one of the remaining independent elements. As a result, only five entries of the $h_{\mu\nu}$ are independent variables. These five independent variables correspond to five propagating degrees of freedom. This is an exact number of degrees of freedom of the massive graviton in four dimensions.

This result cannot be applied to the massless case, since in such a limit the existence of the general covariance makes the situation different from one in the massive gravity. In the exactly massless case $m_g = 0$, we have the gauge transformation $h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$. In such a case, the structure of general relativity renders h_{00} and h_{0i} nondynamical, leaving six independent components to propagate freely. Moreover, there are four ways to transform $h_{\mu\nu}$ by using the gauge transformation and we can still get the same equation of motion. This freedom to transform $h_{\mu\nu}$ covariantly renders four of remaining six degrees of freedom nondynamical, resulting only two propagating degrees of freedom, which also correspond to two polarizations of the gravitational wave solution in general relativity.

In terms of their corresponding degrees of freedom, massive gravity propagates more degrees of freedom than general relativity. This fact signals a discontinuity in the mass parameter. Particularly, we can still have a theory of linear massive gravity in a limit $m_g \to 0$ where there is no general covariance and the theory propagates five degrees of freedom. However, when the graviton mass is exactly zero, the theory then becomes that of general relativity which possesses the general covariance and has two active degrees of freedom. It appears that some properties in massive gravity abruptly change when the graviton mass vanishes completely. In other words, somehow the massless limit of massive gravity does not approach a theory of general relativity, in terms of the disappearance/reappearance of degrees of freedom and general covariance. These points correspond to the well known discontinuities which exist in this kind of massive gravity pointed out by van Dam, Veltmann, and Zakharov. We will see more illustrations of such discontinuities in the following section.

4.2 vDVZ Discontinuity Visualization

As pointed out briefly in the end of the previous section, the linear massive gravity possesses the problem on the discontinuities in many senses, namely, the jump of the number of the degrees of freedom and the sudden reappearance of the general covariance when $m_g = 0$. These are parts of what is known as van Dam-Veltmann-Zakharov (vDVZ) discontinuity. To see this discontinuity in a more realistic way, let us consider the gravitational lensing, a famous phenomenon in general relativity.

Gravitational lensing is a phenomenon where an object has a gravitational attraction with light as if the object is a lens to the light. In general relativity, this situation is possible since not only a material object but also any form of energy, including a photon, gravitates in a gravitational field. Since the gravitational lensing is caused by an object that lies in a vicinity of such phenomenon, the source terms in the equation of motion must be involved. In the following calculations, we will see the differences in the gravitational lensing predictions between those made by Fierz-Pauli massive gravity and general relativity. The equation of motion sourced by the energy-momentum tensor

$T_{\mu\nu}$ reads

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{\ \nu} - \partial_\lambda \partial_\nu h^\lambda_{\ \mu} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} \\ - m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = -\kappa T_{\mu\nu}, \tag{4.7}$$

where $\kappa \equiv M_p^{-1} \equiv \sqrt{8\pi G}$ is a normalization factor and G is the usual Newtonian gravitational constant. Note that the normalization is introduced so that the equation of motion is in agreement with one in general relativity with a redefinition of the field $h_{\mu\nu}$ as $h'_{\mu\nu} = \frac{2}{M_p} h_{\mu\nu}$. We can perform the same procedures as we have done for the case of the source-free equation of motion in the previous section. The divergence of this equation of motion can be found, after applying ∂^μ to Eq. (4.7), as

$$\partial^\mu h_{\mu\nu} - \partial_\nu h = -\kappa \partial^\mu T_{\mu\nu}. \tag{4.8}$$

To simplify Eq. (4.7), we use Eq. (4.8) then we obtain

$$\Box h_{\mu\nu} - \partial_\mu \partial_\nu h - m_g^2 \left(h_{\mu\nu} - \eta_{\mu\nu} h \right) = -\kappa T_{\mu\nu} + \frac{\kappa}{m_g^2} \big(\partial^\lambda \partial_\mu T_{\nu\lambda} + \partial^\lambda \partial_\nu T_{\mu\lambda} \\ - \eta_{\mu\nu} \partial^\rho \partial^\sigma T_{\rho\sigma} \big). \tag{4.9}$$

By multiplying $\eta^{\mu\nu}$ to Eq. (4.9), we obtain the trace of $h_{\mu\nu}$ as follows,

$$h = -\frac{\kappa}{3m_g^2} T - \frac{2\kappa}{3m_g^4} \partial^\rho \partial^\sigma T_{\rho\sigma}, \tag{4.10}$$

where $T \equiv \eta^{\mu\nu} T_{\mu\nu}$ denotes the trace of the energy-momentum tensor. After substituting the trace h into Eq. (4.8) and Eq. (4.9), we have

$$\begin{aligned}\left(\Box - m_g^2\right) h_{\mu\nu} &= -\kappa \left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m_g^2}\right) T\right] \\ &\quad + \frac{\kappa}{m_g^2}\left[\partial^\lambda \partial_\mu T_{\nu\lambda} + \partial^\lambda \partial_\nu T_{\mu\lambda} - \frac{1}{3}\left(\eta_{\mu\nu} + 2\frac{\partial_\mu \partial_\nu}{m_g^2}\right) \partial^\rho \partial^\sigma T_{\rho\sigma}\right], \quad (4.11a)\\ \partial^\mu h_{\mu\nu} &= -\frac{\kappa}{3m_g^2}\partial_\nu T + \frac{\kappa}{m_g^2}\partial^\mu T_{\mu\nu} - \frac{2\kappa}{3m_g^4}\partial_\nu \partial^\rho \partial^\sigma T_{\rho\sigma}, \quad (4.11b)\\ h &= -\frac{\kappa}{3m_g^2}T - \frac{2\kappa}{3m_g^4}\partial^\rho \partial^\sigma T_{\rho\sigma}. \quad (4.11c)\end{aligned}$$

Note that conservation of the source is not yet assumed here, since in general relativity the general covariance or the gauge symmetry automatically implies the conservation of the energy-momentum tensor while such a situation is not applicable in massive gravity. However, in the actual situation, we assume the conserved source in the equation of motion. Since the conservation satisfies $\partial_\mu T^\mu_{\ \nu} = 0$ in the Minkowski background, Eq. (4.11a), Eq. (4.11b), and Eq. (4.11c) accordingly read

$$\left(\Box - m_g^2\right) h_{\mu\nu} = -\kappa \left[T_{\mu\nu} - \frac{1}{3}\left(\eta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{m_g^2}\right) T\right]. \quad (4.12a)$$

$$\partial^\mu h_{\mu\nu} = -\frac{\kappa}{3m_g^2}\partial_\nu T, \quad (4.12b)$$

$$h = -\frac{\kappa}{3m_g^2}T. \quad (4.12c)$$

Now we have the equation of motion that is sourced by the conserved energy-momentum tensor, we can predict the gravitational lensing in the massive gravity. To go further, we follow the method of Green's function to solve the equation of motion. The solution thus reads

$$h_{\mu\nu}(x) = \kappa \int \frac{d^4 p}{(2\pi)^4}\, e^{ip_\lambda x^\lambda} \frac{1}{p_\rho p^\rho + m_g^2}\left[T_{\mu\nu}(p) - \frac{1}{3}\left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m_g^2}\right) T(p)\right], \quad (4.13)$$

where the Fourier transform of the source is defined as

$$T_{\mu\nu}(p) = \int d^4x \; e^{-ip_\lambda x^\lambda} T_{\mu\nu}(x), \tag{4.14}$$

$$T_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \; e^{ip_\lambda x^\lambda} T_{\mu\nu}(p). \tag{4.15}$$

Suppose that there is a point source of mass M sitting at rest at the origin where $\vec{x} = 0$. The corresponding energy-momentum tensor is

$$T_{\mu\nu}(x) = M\delta_{0\mu}\delta_{0\nu}\delta^{(3)}(x). \tag{4.16}$$

We can find the corresponding Fourier transform of the source as follows,

$$T_{\mu\nu}(p) = 2\pi M\delta^0_\mu\delta^0_\nu\delta(p^0), \tag{4.17}$$

$$T(p) = -2\pi M\delta(p^0), \tag{4.18}$$

where we have used the following definition of the delta function,

$$\delta^{(4)}(p) = \int \frac{d^4x}{(2\pi)^4} \; e^{ip_\lambda x^\lambda}. \tag{4.19}$$

With such a source, the solution for $h_{\mu\nu}$ in Eq. (4.13) is expressed as

$$h_{00}(x) = \frac{2}{3}\kappa M \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \frac{1}{\vec{p}^2 + m_g^2}, \tag{4.20a}$$

$$h_{0i}(x) = 0, \tag{4.20b}$$

$$h_{ij}(x) = \frac{1}{3}\kappa M \int \frac{d^3p}{(2\pi)^3} \; e^{i\vec{p}\cdot\vec{x}} \frac{1}{\vec{p}^2 + m_g^2}\left(\delta_{ij} + \frac{p_i p_j}{m_g^2}\right). \tag{4.20c}$$

The previous integrals can be evaluated by using the following formulae,

$$\int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \frac{1}{\vec{p}^2 + m_g^2} = \frac{e^{-m_g r}}{4\pi r}, \tag{4.21}$$

$$\begin{aligned} \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \frac{1}{\vec{p}^2 + m_g^2} p_i p_j &= -\partial_i \partial_j \int \frac{d^3p}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}} \frac{1}{\vec{p}^2 + m_g^2} \\ &= -\partial_i \partial_j \left(\frac{e^{-m_g r}}{4\pi r} \right) \end{aligned} \tag{4.22}$$

where $r \equiv \sqrt{\vec{x}\cdot\vec{x}}$. Finally, we obtain a solution for $h_{\mu\nu}$ as

$$h_{00}(x) = \frac{2}{3}\kappa M \frac{e^{-m_g r}}{4\pi r}, \tag{4.23a}$$

$$h_{0i}(x) = 0, \tag{4.23b}$$

$$h_{ij}(x) = \frac{1}{3}\kappa M \left(\frac{e^{-m_g r}}{4\pi r}\delta_{ij} - \frac{1}{m_g^2}\partial_i \partial_j \frac{e^{-m_g r}}{4\pi r} \right). \tag{4.23c}$$

We can read out an important physical implication from this form of h_{00}. By comparing it with that in linearized gravity, the (00) component of a metric perturbation $\delta g_{\mu\nu}$ is associated with a Newtonian gravitational potential by the raletion $\delta g_{00} = -2\phi$. The component δg_{00} is related to h_{00} through the relation $\delta g_{00} = \frac{2}{M_p} h_{00}$. This means that the Newtonian gravitational potential predicted from the Fierz-Pauli massive gravity has a profile of $\frac{e^{-m_g r}}{r}$ while that from linearized general relativity is of the inverse-r form. The factor $e^{-m_g r}$, known as the Yukawa factor, essentially reduces the strength of the gravitational pull especially at large scale (the scale of about $\frac{1}{m_g}$). As opposed to that in general relativity, the weakening of the gravitational force could possibly allow a system to self-expand especially at the scale of $\frac{1}{m_g}$. As a result, this characteristics of the Fierz-Pauli massive gravity becomes a key idea to explain the accelerating expansion of the universe at large scale.

Beside the remarkable property, from the above equations, we have seen the first sign of an inconsistency between the massive gravity in a massless limit and general relativity. In particular, Eq. (4.23c) diverges in such a limit, indicating that the massless limit of this theory requires more delicate procedures, which can be done by the use of the Stückelberg fields. We will discuss later about this issue in the next section.

Since now we have a solution which is sourced by a point mass M, let us review the same kind of solution but in the general relativity language. In other words, we consider the linearized Einstein's equation in Eq. (4.7) with ``exactly $m_g = 0$",

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{\ \nu} - \partial_\lambda \partial_\nu h^\lambda_{\ \mu} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} = -\kappa T_{\mu\nu}. \tag{4.24}$$

The reason for the ``emphases" is that there are quite differences between massless limit and exactly vanishing mass. To set $m_g = 0$ means we are working in the general relativity point of view in which, for example, the gauge symmetry kicks in. Consider the trace of the Eq. (4.24), we have

$$-\Box h + \partial_\mu \partial_\nu h^{\mu\nu} = -\frac{1}{2} \kappa T. \tag{4.25}$$

The trace in Eq. (4.25) simplifies Eq. (4.24) as follows,

$$\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{\ \nu} - \partial_\lambda \partial_\nu h^\lambda_{\ \mu} + \partial_\mu \partial_\nu h = -\kappa T_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \kappa T. \tag{4.26}$$

This equation can be expressed in the same form as in Eq. (4.11a) by choosing the Lorentz gauge,

$$\partial^\mu h_{\mu\nu} - \frac{1}{2} \partial_\nu h = 0. \tag{4.27}$$

Eq. (4.26) consequently becomes

$$\Box h_{\mu\nu} = -\kappa T_{\mu\nu} + \frac{1}{2} \eta_{\mu\nu} \kappa T, \tag{4.28}$$

whose solution is

$$h_{\mu\nu}(x) = \kappa \int \frac{d^4 p}{(2\pi)^4} \, e^{i p_\lambda x^\lambda} \frac{1}{p_\lambda p^\lambda} \left[T_{\mu\nu}(p) - \frac{1}{2} \eta_{\mu\nu} T(p) \right]. \tag{4.29}$$

Under the influence of the source in Eq. (4.17) and Eq. (4.18), Eq. (4.29) becomes

$$h_{00}(x) = \frac{1}{2}\kappa M \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\frac{1}{\vec{p}^2} = \frac{1}{2}\kappa M \frac{1}{4\pi r}, \tag{4.30a}$$

$$h_{0i}(x) = 0, \tag{4.30b}$$

$$h_{ij}(x) = \frac{1}{2}\kappa M \int \frac{d^3\vec{p}}{(2\pi)^3}\, e^{i\vec{p}\cdot\vec{x}}\frac{1}{\vec{p}^2} = \frac{1}{2}\kappa M \frac{1}{4\pi r}. \tag{4.30c}$$

At this level, one may notice the difference between both of the solutions; one from Fierz-Pauli massive gravity and another from the linearized general relativity. In particular, the massless limit of the solution in Eq. (4.23a) differs from one in Eq. (4.30a) by a constant factor and Eq. (4.23c) diverges in such a limit when Eq. (4.30c) does not. To get a clearer picture, we may consider their consequences on the gravitational lensing. To do that, we suppose that the massless limit of the massive gravity reduces exactly to general relativity and possesses properties that belong to general relativity, e.g. gauge symmetry. Consequently, the massless limit of a solution obtained from the massive gravity induces a deflection of light in the same way as that in general relativity. In practice, we assume that the limit $m_g \to 0$ allows us to gauge-transform Eq. (4.23a)-(4.23c) as we like. The purpose of introducing such ideas is to do the gauge transformation in order to remove the divergence in Eq. (4.23c) when the massless limit is applied. By the use of the gauge transformation, the problematic $\frac{1}{m_g^2}$ term in Eq. (4.23c) can be eliminated by choosing gauge parameters satisfying

$$\xi_i = -\frac{1}{3}\frac{\kappa M}{m_g^2}\partial_i \frac{e^{-mr}}{4\pi r}. \tag{4.31}$$

After being gauge-transformed, Eq. (4.23a)-(4.23c) can be rewritten in the massless limit as

$$h_{00}(x) = \frac{2}{3}\kappa M \frac{1}{4\pi r}, \tag{4.32a}$$

$$h_{0i}(x) = 0, \tag{4.32b}$$

$$h_{ij}(x) = \frac{1}{3}\kappa M \left(\frac{1}{4\pi r}\delta_{ij}\right). \tag{4.32c}$$

It is known in the linearized general relativity that the (00) component of a fluctuation $\delta g_{\mu\nu}$ corresponds to a Newtonian gravitational potential ϕ by the relation $\delta g_{00} = -2\phi$ and we can read a parameter γ from the (ii) component by the relation $\psi = \gamma\phi$ where $g_{ij} = -2\psi\delta_{ij}$. This γ is known as a parameterized post-Newtonian parameter which equals one in the case of general relativity. In this context, the fluctuation $h_{\mu\nu}$ is related to $\delta g_{\mu\nu}$ by

$$\delta g_{\mu\nu} = \frac{2}{M_p} h_{\mu\nu}. \tag{4.33}$$

The normalization factor $2/M_p$ is introduced so that the corresponding action coincides with the linearization of the Einstein-Hilbert action in general relativity. The gravitational potential ϕ and the parameter γ give rise to a deflection of light with an angle

$$\hat{\alpha} = \left|(1+\gamma)\int_{-\infty}^{\infty} \vec{\nabla}\phi \cdot \hat{b}\, dx\right|, \tag{4.34}$$

where $\hat{b}$ is a unit vector along which an impact parameter b is measured (the derivation of the deflection angle are in Appendix A). Note that the deflection angle in Eq. (4.34) corresponds to the photon that comes from $x = -\infty$ to $x = \infty$. With the help of Eq. (4.34), the deflection angles corresponding to both the massive gravity and the general relativity are

$$\alpha^{m_g\to 0} = \frac{4GM}{b}, \qquad \alpha^{m_g=0} = \frac{4GM}{b}. \tag{4.35}$$

One may see that the angles for both cases are the same and they do not indicate any discontinuity at first glance. However, let us have a look at the gravitational potential for each case,

$$\phi^{m_g\to 0} = -\frac{4}{3}\frac{GM}{r}, \qquad \phi^{m_g=0} = -\frac{GM}{r}. \tag{4.36}$$

If one use the gravitational potential derived from the massive gravity to measure the Newtonian constant G for a stellar object like the sun, he will measure the Newtonian constant G differently from others who use the prediction from general relativity by the factor $3/4$;

$$G_{Massive} = \frac{3}{4} G_{GR}. \tag{4.37}$$

If we adopt such a situation, the deflection angles in Eq. (4.35) will be different from each other; the angle predicted from the massive gravity will be 25% smaller from the angle predicted from general relativity. This is one of the vDVZ discontinuities that exist in the very first model of linear massive gravity.

The visualization of the vDVZ discontinuity given above does not explain to us exactly why such discontinuity emerges. In order to see its origin, we have to be more careful on a massless limit of the Fierz-Pauli massive gravity. In particular, the massless limit we considered previously, by straightforwardly setting $m_g \to 0$, reduces the massive gravity, which has five degrees of freedom but no gauge symmetry, into general relativity, which propagates two degrees of freedom and has gauge symmetry. In these senses, such a massless limit is discontinuous. If the limit is to be taken properly, it should be smooth; it should preserve both degrees of freedom and the gauge properties of the theory. Thus, to see a more rigorous picture of the vDVZ discontinuity, the massless limit of the theory must be refined. Fortunately, there is a process which reintroduce the gauge symmetry to the massive gravity known as a Stückelberg trick. By this trick, we can apply the massless limit to the massive gravity while preserving the degrees of freedom and equipping the gauge symmetry to the massive gravity. Thanks to the Stückelberg trick, not only the linear massive gravity but also other nonlinear models of massive gravity can be constructed and equipped with general covariance.

4.3 Stückelberg Trick

The Stückelberg trick is to restore a general covariance to the massive gravity so that the massless limit of the massive gravity can be considered properly. The essence of this trick is to introduce additional fields representing the gauge parameters and to equip the fields to the theory in the same way as the gauge symmetry in general relativity.

Let us consider the Fierz-Pauli massive gravity in Eq. (4.1) with source determined by $T_{\mu\nu}$,

$$S = \int d^4x \left[-\frac{1}{2}\partial_\lambda h_{\mu\nu}\partial^\lambda h^{\mu\nu} + \partial_\mu h_{\nu\lambda}\partial^\nu h^{\mu\lambda} - \partial_\mu h^{\mu\nu}\partial_\nu h + \frac{1}{2}\partial_\lambda h \partial^\lambda h + \kappa h_{\mu\nu}T^{\mu\nu} \right.$$
$$\left. -\frac{1}{2}m_g^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right)\right].$$

In the case $m_g = 0$, the infinitesimal coordinate transformation $x \to x - \xi$ induces the gauge symmetry by which under the transformation

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \tag{4.38}$$

the $m_g = 0$ action is covariant, where in the context of gauge transformation ξ_μ is a gauge parameter. To equip this gauge symmetry into the action in Eq. (4.1), let us introduce an extra field as follows,

$$A_\mu = A_\mu(x). \tag{4.39}$$

We then define a tensor $\tilde{h}_{\mu\nu}$ corresponding to $h_{\mu\nu}$ as,

$$\bar{h}_{\mu\nu} \equiv h_{\mu\nu} - \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{4.40}$$

To equip the field A_μ as the gauge parameter, we ``replace" the field $h_{\mu\nu}$ in Eq. (4.1) by the field $\bar{h}_{\mu\nu}$. In particular the derivative terms remain in the same form under the replacement since the replacement looks exactly like the gauge transformation in Eq. (4.38). More importantly, the terms involving the graviton mass are affected as follows,

$$\begin{aligned} S &= \int d^4x \left[\mathcal{L}_{m_g=0} - \frac{1}{2}m_g^2\left(\bar{h}_{\mu\nu}\bar{h}^{\mu\nu} - \bar{h}^2\right) + \kappa \bar{h}_{\mu\nu}T^{\mu\nu}\right], \\ &= \int d^4x \left[\mathcal{L}_{m_g=0} - \frac{1}{2}m_g^2\left(h_{\mu\nu}h^{\mu\nu} - h^2\right) - \frac{1}{2}m_g^2 F_{\mu\nu}F^{\mu\nu} \right. \\ &\qquad \left. + m_g^2 h_{\mu\nu}\left(\partial^\mu A^\nu + \partial^\nu A^\mu\right) - 2m_g^2\partial_\mu A^\mu h + \kappa h_{\mu\nu}T^{\mu\nu} + 2\kappa A_\nu \partial_\mu T^{\mu\nu}\right], \end{aligned} \tag{4.41}$$

where the integration by parts is used on the last term, $\mathcal{L}_{m_g=0}$ represents the linearized general relativity terms, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. The last term can be simplified in the case of conserved source. At this point, the gauge symmetry is already equipped in the theory and the corresponding gauge transformation is

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - \xi_\mu. \tag{4.42}$$

We can see that under such a gauge transformation, $\bar{h}_{\mu\nu}$ is invariant and the action after the replacement is invariant accordingly. After rescaling the field $A_\mu \to \frac{1}{m_g} A_\mu$, we have

$$\begin{aligned} S = \int d^4x \bigg[&\mathcal{L}_{m_g=0} - \frac{1}{2} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\ &+ m_g h_{\mu\nu} \left(\partial^\mu A^\nu + \partial^\nu A^\mu \right) - 2 m_g \partial_\mu A^\mu h + \kappa h_{\mu\nu} T^{\mu\nu} + \frac{2\kappa}{m_g} A_\nu \partial_\mu T^{\mu\nu} \bigg], \end{aligned} \tag{4.43}$$

where the gauge transformation is

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - m_g \xi_\mu. \tag{4.44}$$

By fixing the gauge $A_\mu = 0$, we recover the Fierz-Pauli action in Eq. (4.1). This indicates that the absence of the gauge symmetry earlier in the Fierz-Pauli action is only because the action is already gauge-fixed. Thus, the action in Eq. (4.43) is an actual massive gravity action equipped with gauge symmetry. Though the gauge symmetry is equipped via the introduction of A_μ, we still have the problem of discontinuity in the number of degrees of freedom when the massless limit is considered. If the massless limit $m_g \to 0$ is taken into account, provided that the source $T_{\mu\nu}$ is conserved, the action in Eq. (4.43) then reads

$$S = \int d^4x \left[\mathcal{L}_{m_g=0} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} + \kappa h_{\mu\nu} T^{\mu\nu} \right], \tag{4.45}$$

and the resulting gauge transformation is

$$h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu. \tag{4.46}$$

From the analysis previously, we know that the propagating degrees of freedom of the lagrangian $\mathcal{L}_{m_g=0}$ in Eq. (4.45) correspond to those in general relativity in which there are only two active degrees of freedom. For the lagrangian $-\frac{1}{2}F_{\mu\nu}F^{\mu\nu}$, since $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and A_μ is a vector field, A_μ generally has four entries in four-dimensional spacetime. Due to the unique structure of $F_{\mu\nu}$, a kinetic term of A_0, namely $(\partial_0 A_0)^2$, vanishes identically. This fact renders A_0 nondynamical. Furthermore, $F_{\mu\nu}$ is invariant under the transformation $A_\mu \to A_\mu - \partial_\mu \Lambda$ where Λ is an arbitrary function of spacetime coordinate. In other words, there exists another gauge symmetry corresponding to the field A_μ and this gauge symmetry allows us to choose a particular gauge which put an additional constraint to A_μ. Due to the facts above, only two entries of A_μ are allowed to propagate independently which coincide with the degrees of freedom of a gauge vector field. As a result, only four degrees of freedom propagate; two for massless graviton and another two for massless gauge field. Note that if the source is not conserved, then in such a limit it should shrink by at least a power of m_g [37]. Otherwise, the field A_μ will be strongly coupled to the divergence of the source and then the limit does not exist. At this point, the massless limit still does not preserve the number of the degrees of freedom since we lose one as we have seen in the above analysis. Moreover, we still miss another gauge symmetry for A_μ which must exist due to the structure of $F_{\mu\nu}$. The solution to these points is applying the trick again but this time on the field A_μ. We replace A_μ in Eq. (4.41) by a field $\bar{A}_\mu$ which is defined as

$$\bar{A}_\mu \equiv A_\mu - \partial_\mu \phi. \tag{4.47}$$

As in the previous calculation, the term $F_{\mu\nu}F^{\mu\nu}$ is invariant under this replacement since it looks exactly like the gauge transformation of the gauge field. After the replacement, the action becomes

$$
\begin{aligned}
S = \int d^4x \bigg[& \mathcal{L}_{m=0} - \frac{1}{2} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) - \frac{1}{2} m_g^2 F_{\mu\nu} F^{\mu\nu} \\
& + m_g^2 h_{\mu\nu} \left(\partial^\mu \bar{A}^\nu + \partial^\nu \bar{A}^\mu \right) - 2 m_g^2 \partial_\mu \bar{A}^\mu h + \kappa h_{\mu\nu} T^{\mu\nu} + 2\kappa \bar{A}_\nu \partial_\mu T^{\mu\nu} \bigg], \\
= \int d^4x \bigg[& \mathcal{L}_{m=0} - \frac{1}{2} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) - \frac{1}{2} m_g^2 F_{\mu\nu} F^{\mu\nu} \\
& + m_g^2 h_{\mu\nu} \left(\partial^\mu A^\nu + \partial^\nu A^\mu \right) - 2 m_g^2 \partial_\mu A^\mu h - 2 m_g^2 h_{\mu\nu} \partial^\mu \partial^\nu \phi \\
& + 2 m_g^2 \partial_\mu \partial^\mu \phi h + \kappa h_{\mu\nu} T^{\mu\nu} + 2\kappa A_\nu \partial_\mu T^{\mu\nu} + 2\kappa \phi \partial_\mu \partial_\nu T^{\mu\nu} \bigg],
\end{aligned} \tag{4.48}
$$

where the corresponding gauge transformation is

$$
h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - \xi_\mu - \partial_\mu \Lambda, \qquad \phi \to \phi - \Lambda. \tag{4.49}
$$

Again, the integration by parts is used to simplify the couple between the scalar ϕ and the source. By rescaling $A_\mu \to \frac{1}{m_g} A_\mu$, $\phi \to \frac{1}{m_g^2}\phi$, the action in Eq. (4.48) reads

$$
\begin{aligned}
S = \int d^4x \bigg[& \mathcal{L}_{m=0} - \frac{1}{2} m_g^2 \left(h_{\mu\nu} h^{\mu\nu} - h^2 \right) - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \\
& + m_g h_{\mu\nu} \left(\partial^\mu A^\nu + \partial^\nu A^\mu \right) - 2 m_g \partial_\mu A^\mu h - 2 h_{\mu\nu} \partial^\mu \partial^\nu \phi \\
& + 2 \partial_\mu \partial^\mu \phi h + \kappa h_{\mu\nu} T^{\mu\nu} + \frac{2\kappa}{m_g} A_\nu \partial_\mu T^{\mu\nu} + \frac{2\kappa}{m_g^2} \phi \partial_\mu \partial_\nu T^{\mu\nu} \bigg],
\end{aligned} \tag{4.50}
$$

where the gauge transformation becomes

$$
h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \quad A_\mu \to A_\mu - m_g \xi_\mu - \partial_\mu \Lambda, \quad \phi \to \phi - m_g \Lambda. \tag{4.51}
$$

Thus, by assuming conserved source, the massless limit of the action in Eq. (4.50) reads

$$
S = \int d^4x \left[\mathcal{L}_{m=0} - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} - 2 h_{\mu\nu} \partial^\mu \partial^\nu \phi + 2 \partial_\mu \partial^\mu \phi h + \kappa h_{\mu\nu} T^{\mu\nu} \right], \tag{4.52}
$$

and the gauge transformation in such a limit is

$$
h_{\mu\nu} \to h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu, \qquad A_\mu \to A_\mu - \partial_\mu \Lambda, \qquad \phi \to \phi. \tag{4.53}
$$

Again, if the source is not conserved, the divergence of the source should shrink fast enough, e.g. shrinks by at least a second order of m_g, or the limit does not exist because of the strong couplings. The massless limit $m_g \rightarrow 0$ now becomes exactly what we want, in such a way that the limit should preserve the degrees of freedom and the gauge symmetry. In particular, the massless limit action in Eq. (4.52) propagates five degrees of freedom; two from massless graviton, two from A_μ, and another one from the scalar ϕ. Moreover, the gauge symmetry is still in active in such a limit as shown in Eq. (4.53). In addition, the additional gauge symmetry on A_μ which was not presented in the previous analysis is already included in terms of the gauge parameter Λ.

This Stückelberg trick has proved itself useful in refining the massless limit of the Fierz-Pauli massive gravity. Not only that, it reveals all of five degrees of freedom of massive graviton (as shown in Eq. (4.52)) where the gauge symmetry is still preserved. Moreover, having all of the degrees of freedom exposed in the massless limit shows us interactions between the tensor degrees of freedom and the scalar degree of freedom as in Eq. (4.52). This is the sign to the vDVZ discontinuity since such terms should vanish if the massive gravity was reducible to general relativity. To see a clearer picture, we find the massless-limit equation of motion of $h_{\mu\nu}$ as follows

$$\begin{aligned}\Box h_{\mu\nu} - \partial_\lambda \partial_\mu h^\lambda_{\ \nu} - \partial_\lambda \partial_\nu h^\lambda_{\ \mu} + \partial_\mu \partial_\nu h - \eta_{\mu\nu} \Box h + \eta_{\mu\nu} \partial_\lambda \partial_\sigma h^{\lambda\sigma} \\ -2\partial_\mu \partial_\nu \phi + 2\eta_{\mu\nu} \partial_\rho \partial^\rho \phi = -\kappa T_{\mu\nu}. \qquad (4.54)\end{aligned}$$

Note that this equation differs from the linearized gravity equation of motion in Eq. (3.32) since this equation involves the scalar degree of freedom as well (we do not take into account the difference between Eq. (4.54) and Eq. (3.32) by a factor of $\frac{M_p}{2}$ since the difference of this kind can be easily amended by a field redefinition $h'_{\mu\nu} = \frac{2}{M_p} h_{\mu\nu}$). To decouple the scalar from the equation of motion, we need another equation which comes from varying the action in Eq. (4.52) with respect to ϕ yielding

$$\partial^\mu \partial^\nu h_{\mu\nu} - \Box h = 0. \qquad (4.55)$$

With a help of Eq. (4.55), we can simplify the trace of Eq. (4.54) and obtain an equation of motion of ϕ as follows,

$$-2\Box h + 2\partial_\mu\partial_\nu h^{\mu\nu} + 6\Box\phi = -\kappa T, \tag{4.56}$$

$$\Box\phi = -\frac{\kappa}{6}T. \tag{4.57}$$

Eq. (4.57) explicitly shows how the scalar degree of freedom in the massive gravity is involved in the situation. The equation of motion of ϕ is sourced by the trace T of the energy-momentum tensor $T_{\mu\nu}$ and consequently contributes its nonvanishing effect via Eq. (4.54). This shows us that not only the tensor part mediates the gravity but the scalar degree of freedom also determines the gravitation as well (where the vector part does not) which is a totally different story compared with the linearized general relativity. Moreover, thanks to this additional mediator, the gravity is modified from the general relativity which reflects the existence of vDVZ discontinuity in the Fierz-Pauli massive gravity theory. In addition, one can choose to reach this result through another approach via diagonalizing the action in Eq. (4.52) by applying the following (linearized) conformal transformation (see also [8]),

$$h_{\mu\nu} \to h_{\mu\nu} + \pi\eta_{\mu\nu}, \tag{4.58}$$

where $\pi \equiv \frac{\phi}{2}$. The action is diagonalized accordingly and takes the form,

$$S = \int d^4x \; \mathcal{L}_{m=0} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - 3\partial_\mu\phi\partial^\mu\phi + \kappa h_{\mu\nu}T^{\mu\nu} + \frac{1}{2}\kappa\phi T. \tag{4.59}$$

This transformed action confirms the previous result on the discontinuity being introduced via the coupling of the scalar to the trace of the source for which the coupling term $\frac{1}{2}\kappa\phi T$ is responsible. Note that this action provides the same equation of motion for ϕ as in Eq. (4.57).

Up to this point, we have seen the problematic discontinuity that exists in the linear theory of massive gravity proposed by Fierz and Pauli. The vDVZ discontinuity poses clear-cut separations between the linear massive gravity and the linearized general relativity as we can rule one out just by observational data on, for example, a light deflection around a massive star. So far general relativity fits the observations inside the scale of our solar system. At least, we can conclude that the linear massive gravity fails in such scale while the nonlinear theory may work. This was claimed long time ago by Vainshtein [12] who believes the nonlinearities can be a treatment for such discontinuity, which leads us to the next section where a viable nonlinear massive gravity are discussed.

CHAPTER V
NONLINEAR MASSIVE GRAVITY

We have seen some analyses on the linear massive gravity given by Fierz and Pauli in the previous chapter. Such model possesses a problematic vDVZ discontinuity which, claimed by Vainshtein [12], can be lifted by putting nonlinear contributions in consideration. However, the nonlinear contributions must be designed carefully to avoid an instability known as a ghost. A ghost represents a mode whose kinetic term appears with wrong sign, resulting in an unphysical excitation in the system. Pointed out by Boulware and Deser [13], this ghost appears in the theory when various class of nonlinearities are taken into account. Fortunately, de Rham, Gabadadze, and Tolley found a healthy class of nonlinearities which does not excite the ghost. This class of massive gravity, dubbed de Rham-Gabadadze-Tolley (dRGT) massive gravity [14, 15], had open a way for physicists to seek its implications on cosmology and on astrophysics. We will see in this section a rough picture of this healthy model of nonlinear massive gravity.

In order to promote the linear massive gravity to a fully nonlinear theory, one must include all orders of derivative terms in the action, which are summed up with the linearized gravity terms to be the Einstein-Hilbert action. Not only that, the interaction terms in the Fierz-Pauli thoery are appended with suitable forms of nonlinearities. Thus, in four-dimensional nonlinear massive gravity, we consider a full Einstein-Hilbert action equipped with the interaction terms as follows,

$$S = \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left(R(g) + 2m_g^2 U(g,f)\right), \tag{5.1}$$

where $R(g)$ is a Ricci scalar denoting the lagrangian density of the Einstein-Hilbert action, m_g is interpreted as graviton mass, and $U(g, f)$ is the nonlinear interaction constructed from a physical metric $g_{\mu\nu}$ and another ``reference" metric. The reference metric is another metric introduced into the theory so that nonvanishing and nonlinear interaction terms can be constructed. In particular, it is not possible to construct a nontrivial nonlinearity from only the physical metric $g_{\mu\nu}$ since any higher-order construction of $g_{\mu\nu}$ will reduce to either identity matrix or $g_{\mu\nu}$ itself due to its identity $g^{\mu\rho}g_{\rho\nu} = \delta^{\mu}_{\nu}$. By introducing the reference metric $f_{\mu\nu}$, one can include any higher-order interaction in terms of, for example, $f^{\mu\rho}g_{\rho\nu}$. To avoid the Boulware-Deser ghost, $U(g,f)$ is designed carefully as follows,

$$U(g,f) \equiv U_2(\mathcal{K}) + \alpha_3 U_3(\mathcal{K}) + \alpha_4 U_4(\mathcal{K}), \tag{5.2}$$

$$U_2(\mathcal{K}) \equiv \frac{1}{2}\left([\mathcal{K}]^2 - [\mathcal{K}^2]\right), \tag{5.2a}$$

$$U_3(\mathcal{K}) \equiv \frac{1}{3!}\left([\mathcal{K}]^3 - 3[\mathcal{K}][\mathcal{K}^2] + 2[\mathcal{K}^3]\right), \tag{5.2b}$$

$$U_4(\mathcal{K}) \equiv \frac{1}{4!}\left([\mathcal{K}]^4 - 6[\mathcal{K}]^2[\mathcal{K}^2] + 3[\mathcal{K}^2]^2 + 8[\mathcal{K}][\mathcal{K}^3] - 6[\mathcal{K}^4]\right), \tag{5.2c}$$

where the square brackets denote a trace of the argument inside them, or $[\mathcal{K}] \equiv \mathcal{K}^{\mu}_{\ \mu}$, $(\mathcal{K}^2)^{\mu}_{\ \nu} = \mathcal{K}^{\mu}_{\ \rho}\mathcal{K}^{\rho}_{\ \nu}$ and so on. Moreover, the building-block tensor $\mathcal{K}^{\mu}_{\ \nu}$ is defined as follows,

$$\mathcal{K}^{\mu}_{\ \nu} \equiv \delta^{\mu}_{\nu} - (\sqrt{g^{-1}f})^{\mu}_{\ \nu}, \tag{5.3}$$

where the square root of an arbitrary tensor A is defined such that $(\sqrt{A})^{\mu}_{\ \rho}(\sqrt{A})^{\rho}_{\ \nu} \equiv A^{\mu}_{\ \nu}$. Generally speaking, one may construct interaction with the power of higher than four which are crucial when one is interested in the theory in higher dimensions. In four dimensions, the higher order interactions vanish trivially, leaving nonvanishing interatctions up to U_4 [14, 15, 38, 39].

In the Fierz-Pauli massive gravity, one may recognize that the theory is formulated on the linearized general relativity which is based on the Minkowski metric, or more generally an arbitrary background metric. Similarly, in dRGT massive gravity we have a reference metric apart from the physical one. This reference metric is not directly the background metric but a metric for constructing nontrivial interactions. Moreover, the presence of the reference metric allows us to equip the general covariance into the theory by using the Stückelburg trick as follows,

$$f_{\mu\nu} = \partial_\mu \varphi^\rho \partial_\nu \varphi^\sigma \tilde{f}_{\rho\sigma}, \tag{5.4}$$

where φ^μ's are four Stückelburg scalar fields. Usually, the form of the reference metric $\tilde{f}_{\mu\nu}$ determines the form of the solution to this theory. For example, a spherical black hole solution are obtained when a spherical symmetric reference metric is assumed [40, 41, 42, 43] whereas an FLRW-like reference metric leads to a cosmological solution [17, 44, 27].

In the previous chapter, we have roughly seen a way to explain the accelerating expansion through the linear massive gravity. However, when the nonlinearities are in consideration, it may not be clear at first glance how the graviton mass contributes to the cosmic expansion. Thus, it is necessary to study its cosmology to see how exactly the graviton mass is responsible for such an expansion. Let us consider the FLRW physical metric as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij}(x^k) dx^i dx^j, \tag{5.5}$$

where $N(t)$ is a lapse function, $a(t)$ is a scale factor representing the spatial scale of the universe, and Ω_{ij} is the 3-dimensional spatial metric defined in Eq. (3.22). In order to respect the isotropy and homogeneity of the physical metric, let us choose the reference metric $\tilde{f}_{ab}$ to have the same FLRW form as

$$d\tilde{s}^2 = \tilde{f}_{\mu\nu}(\varphi) d\varphi^\mu d\varphi^\nu = -n(\varphi^0)^2 \left(d\varphi^0\right)^2 + \alpha(\varphi^0)^2 \Omega_{ij}(\varphi^k) d\varphi^i d\varphi^j, \tag{5.6}$$

where φ^μ`s are coordinates in the ``reference" space and φ^0 denotes the time coordinate in the reference sector. The general covariance is restored by the Stückelburg trick in Eq. (5.4) which means φ^μ's become the Stückelburg fields. Since now the general covariance is restored, the action in Eq. (5.1) is covariance under coordinate transformations. In other words, there is a gauge symmetry induced by the coordinate transformations. For simplicity, the appropriate coordinates φ^μ's can be chosen such that $\varphi^\mu = x^\mu$. As a gauge fixing, this is known as unitary gauge. Choosing such gauge, the corresponding dRGT action in Eq. (5.1) (usually known as a minisuperspace action) reads

$$\begin{aligned} S_{FLRW} = 3M_p^2 \int d^4x\sqrt{\Omega} \\ Na^3 \left[\left(-\left(\frac{\dot{a}}{aN}\right)^2 + \frac{\kappa}{a^2}\right) + m_g^2\left(F(\bar{X}) - G(\bar{X})\frac{n}{N}\right)\right], \end{aligned} \tag{5.7}$$

where Ω denotes the determinant of the tensor $\Omega_{ij}(x^k)$ as presented in Eq. (3.22), $\bar{X} \equiv \frac{\alpha}{a}$ and we have defined

$$\begin{aligned} F(\bar{X}) \equiv& \frac{1}{3}(6 + 4\alpha_3 + \alpha_4) - (3 + 3\alpha_3 + \alpha_4)\bar{X} + (1 + 2\alpha_3 + \alpha_4)\bar{X}^2 \\ &- \frac{1}{3}(\alpha_3 + \alpha_4)\bar{X}^3, \end{aligned} \tag{5.8}$$

$$G(\bar{X}) \equiv \frac{1}{3}(3 + 3\alpha_3 + \alpha_4) - (1 + 2\alpha_3 + \alpha_4)\bar{X} + (\alpha_3 + \alpha_4)\bar{X}^2 - \frac{1}{3}\alpha_4\bar{X}^3. \tag{5.9}$$

Varying this action with respect to N and a gives rise to the Friedmann equation and the acceleration equation, respectively, as follows

$$3M_p^2\left(H^2 + \frac{\kappa}{a^2}\right) = -3M_p^2 m_g^2 F, \tag{5.10}$$

$$M_p^2\left(\frac{2\dot{H}}{N} + 3H^2 + \frac{\kappa}{a^2}\right) = -3M_p^2 m_g^2\left(F - \frac{\bar{X}F_{,\bar{X}}}{3}(1-r)\right), \tag{5.11}$$

where $F_{,\bar{X}} \equiv \frac{dF}{d\bar{X}}$ and we have defined new variables as follows,

$$H \equiv \frac{\dot{a}}{aN}, \qquad r \equiv \frac{n}{N\bar{X}}. \tag{5.12}$$

From the Friedmann equation and the acceleration equation, one can obtain energy density and pressure of the fluid in the universe as

$$\rho_g = -3M_p^2 m_g^2 F, \tag{5.13}$$

$$p_g = 3M_p^2 m_g^2\left(F - \frac{\bar{X}F_{,\bar{X}}}{3}(1-r)\right), \tag{5.14}$$

whose an equation of state parameter reads

$$w_g = \frac{p_g}{\rho_g} = -1 + \frac{\bar{X} F_{,\bar{X}}}{3F}\left(1 - r\right). \tag{5.15}$$

Note that up to this point, no source is involved in the system, which means we are working with equations of motion in vacuum. However, in a general relativity language, the Friedmann equation and the acceleration equation for this model are sourced effectively by contributions from the massive graviton which are those on the right-hand side of those equations. These contributions will be shown to be responsible for the self-accelerating behavior. Since in massive gravity we have introduced the Stückelburg fields, those fields are also governed by their equations of motion. The equations of motion can be obtained by realizing that the Stückelburg fields are introduced just to equip the theory with the general covariance, which corresponds to the conservation of an effective energy-momentum tensor due to the massive gravity interactions. By evaluating the corresponding conservation equation from Eq. (3.26), one can find the Stückelburg equation of motion to be

$$F_{,\bar{X}}\left(H - \bar{X} H_\alpha\right) = 0, \tag{5.16}$$

where $H_\alpha \equiv \frac{\dot{\alpha}}{\alpha n}$. This equation provides two possible conditions by which the conservation of the energy-momentum tensor is consistent in the thoery. Each condition corresponds to each branch of the cosmological solution. One may imply from Eq. (5.15) readily that the condition $F_{,\bar{X}} = 0$ leads to a solution in which the corresponding universe is expanding due to a cosmological constant induced by the graviton mass (since the equation of state parameter is equal to -1). To investigate such behavior explicitly, let us consider

$$F_{,\bar{X}} = 0 = -\left(3 + 3\alpha_3 + \alpha_4\right) + 2\left(1 + 2\alpha_3 + \alpha_4\right)\bar{X} - \left(\alpha_3 + \alpha_4\right)\bar{X}^2. \tag{5.17}$$

This equation fixes the value of $\bar{X}$ since it is just a quadratic equation which can be solved for $\bar{X}$ in terms of the other parameters. The corresponding $\bar{X}$ is

$$\bar{X}_{\pm} = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{(\alpha_3 + \alpha_4)}. \tag{5.18}$$

Interestingly, this branch of solution implies that α can only differs from the scale factor a by a constant ($\bar{X}$). The corresponding energy density and pressure can be found from Eq. (5.13) and Eq. (5.14) to be constants as follows,

$$\begin{aligned} \rho_g &= M_p^2 \Lambda_{\pm} \\ &= -\frac{M_p^2 m_g^2}{(\alpha_3 + \alpha_4)^2} \Big[(1 + \alpha_3) \left(2 + \alpha_3 + 2\alpha_3^2 - 3\alpha_4\right) \\ &\qquad \pm 2 \left(1 + \alpha_3 + \alpha_3^2 - \alpha_4\right)^{3/2} \Big], \qquad (5.19) \\ p_g &= -\rho, \qquad (5.20) \end{aligned}$$

which clearly ensures the cosmological constant behavior satisfying $w = -1$.

Up to this point, we have not specify the curvature of the physical spacetime, denoted by κ. This means that the solution obtained here is still valid for any value of κ. Thus, the solution is also valid for any kind of curvature of the spatial hypersurfaces. The varieties of spatial curvature of the solution are caused by assuming the isometries of FLRW spacetime to the reference sector. On the other hand, if the reference metric is of the Minkowskian form, then we can only obtain a self-accelerating FLRW solution to the dRGT massive gravity whose space is characterized by open-slicing hypersurfaces [20]. Though originally formulated on the Minkowski reference metric, it is proven in Ref. [39] that any kind of reference metric does not reintroduce the BD ghost to the dRGT massive gravity theory.

Although this self-accelerating branch looks fascinating in driving the cosmic expansion in the manner of the cosmological constant, this branch actually suffers from the lack of number of degrees of freedom [17]. This flaw is due to the isotropy and homogeneity of the FLRW ansatz which somehow render the vector mode and the scalar mode nondynamical but let the tensor mode (the gravitational wave mode) propagates. Moreover, a study of FLRW limit of an anisotropic but homogeneous solution reported that those vanishing degrees of freedom are in active though one of them is always a ghost, rendering the FLRW solution to be unstable [18, 23]. These are reasons that drives physicists to seek for an extension of the massive gravity to be capable of providing a better description of the cosmic expansion.

The other branch of solution corresponds to the following condition,

$$H - \bar{X}H_\alpha = 0. \tag{5.21}$$

This branch is known as a normal branch. Unlike the previous case, this branch is rather complicated since it involves derivatives. In spite of its complexity, we can still extract some characteristics out of this branch. In order to do so, we rewrite the condition to be

$$H - \bar{X}H_\alpha = H\left(1 - \frac{1}{r}\right) - \frac{\dot{\bar{X}}}{n} = 0. \tag{5.22}$$

We can see that in case of $r = 1$, Eq. (5.22) implies $\bar{X}$ being constant. Moreover, the condition $r = 1$ sets the equation of state parameter w from Eq. (5.15) to be -1 as the constant $\bar{X}$ renders both the energy density and the pressure in Eq. (5.13) and Eq. (5.14) constants. As a result, this case leads us to a self-accelerating solution driven by an effective cosmological constant which is, however, different from the former self-accelerating branch since here $\bar{X}$ can take any value as long as $r = 1$ is satisfied. Generally, there may exist a class of solution in which $\bar{X}$ is not a constant.

This normal branch has its advantage on the number of propagating degrees of freedom since all of five degrees of freedom are in active [44]. Unfortunately, previous studies suggest that one of them is a ghost [44]. This is another reason for modifying the dRGT massive gravity since the appearance of the ghost mode unstabilizes the thoery [13]. The analyses on the appearance of the ghost is given in the next chapter where we consider an extended model of dRGT massive gravity (the calculations and results are also applicable to the original dRGT massive gravity).

One may note that Eq. (5.10), Eq. (5.11), and Eq. (5.16) are not mutually independent since they are connected through the Bianchi's identity, which then implies the conservation of energy-momentum tensor. Actually, as we have seen, the conservation of energy-momentum tensor automatically leads to the Stückelburg equation, which then confirms the Bianchi's identity. At times, obtaining the Stückelburg equation via the continuity equation on the energy-momentum tensor could be a very cumbersome calculation. An alternative approach to deal things in the Stückelburg sector is given in the Appendix B.

Even though the interactions in Eq. (5.2) are to ensure the absence of the BD ghost, this does not mean that other kinds of ghost will not appear in the theory. It has been proven that a ghost appears in the theory in various situations [18, 23, 26, 27]. Other significant problems existing in dRGT massive gravity lie on its cosmological solution. In particular, isotropy and homogeneity of the FLRW metric appear to be responsible for three vanishing degrees of freedom, leaving only two dynamical degrees of freedom [17, 18, 23, 26, 27]. This is unusual for massive gravity theories because the massive graviton must has five degrees of freedom, which can be seen in the Fierz-Pauli massive gravity. These problems encourage physicists to study beyond dRGT massive gravity. There are various ways to extend the dRGT theory; ones may add the Einstein-Hilbert term for the reference metric, known as a bi-metric gravity [45], others may introduce extra degree(s) of freedom into the theory [24, 26, 27, 28]. On the other hand, the extentions are not only introducing new objects to the theory but also include a parameterization of the graviton mass by other fields, known as a mass-varying massive gravity [30, 31, 33, 32]. In the next section, the extended dRGT massive gravity, where an extra degree of freedom is introduced via an extra dimension, is presented.

CHAPTER VI
DBI EXTENSION OF THE DRGT MASSIVE GRAVITY

In Chapter V, we have seen some drawbacks of the dRGT massive gravity, especially when its cosmological implications are in consideration. In particular, cosmological solutions to the dRGT massive gravity pose serious problems on a consistency of the theory. It was found that, in a cosmological picture, only two (tensor-mode) degrees of freedom propagate while others degrees of freedom (two for vector-mode and one for scalar-mode) vanish [17]. Furthermore, in a more realistic consideration, a study of the isotropic limit of an anisotropic (but homogeneous) solution to the dRGT massive gravity reveals that even though all five degrees of freedom are at work, one of them does propagate as a ghost mode [18, 23]. This indicates that the cosmological solution is unstable. These facts suggest a necessity of extending the dRGT massive gravity so that the problems in the theory can be lifted. There are many attempts to extend the theory. One can seek for a fully anisotropic solution to the dRGT massive gravity. The solution turns out to be free of ghost instability as well as all of the five degrees of freedom are dynamical [23]. On the other hands, one can add an extra field as an extra gravitational mediator. An example of this approach is the quasi-dilaton massive gravity in which a dilaton scalar is introduced with a particular dilaton symmetry [24, 28]. Another example is to promote the graviton mass from a constant to be a function that varies according to a scalar field, dubbed mass-varying massive gravity [30, 31, 33, 32] which will be discussed in the next chapter. Furthermore, there is a generalization in which the reference metric is also a dynamical metric apart from the physical one. Being known as a bimetric gravity, this model involves adding a kinetic term corresponding to the reference sector to the action of the dRGT massive gravity [45]. Such an aspect of modification is left as a further work.

6.1 The Model and Background Equations of Motion

In this chapter, We will investigate a model of the same concept of modification, in which an extra field is added, through a Dirac-Born-Infeld (DBI) massive gravity [25, 26, 27]. This model exists an extra scalar field which is introduced according to a domain wall moving in a five-dimensional spacetime. Moreover, the DBI scalar couples to the massive gravity in the reference metric sector in such a way that the scalar possesses a Galileon shift symmetry. In a previous study on this aspect of massive gravity, the authors investigated a structure of the DBI massive gravity corresponding to a domain wall moving in a flat Minkowskian spacetime [26]. As happened in the original dRGT massive gravity, however, the corresponding cosmological implications imply the vanishing degrees of freedom problem, hence the solution is unstable. Consequently, investigating the model in which the domain wall moves in a different geometric spacetime might be able to fix such issue. One of cosmologically reasonable choices of the reference metric would be the five-dimensional Schwarzschild-anti-de Sitter geometry [46]. Given a five-dimensional reference spacetime with coordinate $X^A = (T, \varphi^1, \varphi^2, \varphi^3, X^5)$, the Schwarzschild-anti-de Sitter metric is given as

$$\begin{aligned} ds_f^2 &= \tilde{f}_{AB} dX^A dX^B, \\ &= -f(X^5)\,(dT)^2 + \left(X^5\right)^2 \Omega_{ij}(\varphi^k) d\varphi^i d\varphi^j + \frac{1}{f(X^5)}\left(dX^5\right)^2, \end{aligned} \tag{6.1}$$

given that A, B are indices running amoung all of the spacetime indices corresponding to $T, \varphi^1, \varphi^2, \varphi^3, X^5$, i, j indicate 3-space coordinates, like $\varphi^1, \varphi^2, \varphi^3$, and

$$f(X^5) \equiv \kappa - \frac{\mu}{\left(X^5\right)^2} + \frac{\left(X^5\right)^2}{l^2}. \tag{6.2}$$

where κ is a curvature of the 3-space, μ is a parameter which has a unit of length squared, and l is a length scale determining the scale of the anti-de Sitter space. In particular, one may see a resemblance between this metric and the usual four-dimensional Schwarzschild geometry by considering the parameter μ as mass of a five-dimensional Schwarszchild black hole in the anti-de Sitter geometry, except that the r-dependence of the mass term in five-dimensional case differs from that in the four-dimensional case by a power of one. This difference is caused by the Newtonian gravitational constant defined in five-dimensional space whose unit differs from the four-dimensional Newtonian constant by a length dimension. According to the metric in Eq. (6.1), the fifth coordinate X^5 behaves as a scale factor on the 3-space in this reference spacetime which makes this form of metric reasonably interesting when the cosmology is taken into account. Since X^5 behaves as the scale factor in 3-space as well as the fifth coordinate, the fifth coordinate is then dimensionless which renders the line element in Eq. (6.1) quite peculiar in terms of units. To make it more reasonable, X^5 is rescaled to be $\lambda\phi$ where ϕ is a coordinate with length dimension while T is also rescaled accordingly to be φ^0/λ. Under such rescaling, the metric reads

$$\begin{aligned} ds_f^2 &= \tilde{f}_{AB} dX^A dX^B, \\ &= -f(\phi)\left(d\varphi^0\right)^2 + (\lambda\phi)^2\,\Omega_{ij}(\varphi^k) d\varphi^i d\varphi^j + \frac{1}{f(\phi)}\,(d\phi)^2\,, \end{aligned} \tag{6.3}$$

where

$$f(\phi) \equiv \frac{\kappa}{\lambda^2} - \frac{\mu}{\lambda^4\phi^2} + \frac{\phi^2}{l^2}. \tag{6.4}$$

Note that now an event in the five-dimensional bulk is specified by spacetime coordinate $X^A \equiv (\varphi^0, \varphi^1, \varphi^2, \varphi^3, \phi)$. Since we are working in four-dimensional massive gravity, we have to find a four-dimensional reference metric which corresponds to the five-dimensional reference metric in Eq. (6.3). Taking the cosmological principle into account, we consider a brane specified by a function $\phi = \phi(t)$. Hence, the four-dimensional metric corresponding to the brane is then a metric induced on that brane which is expressed as follows,

$$\begin{aligned} f_{\mu\nu} &= \partial_\mu X^A \partial_\nu X^B \tilde{f}_{AB}, \\ &= -f(\phi)\partial_\mu\varphi^0\partial_\nu\varphi^0 + (\lambda\phi)^2\,\Omega_{ij}(\varphi^k)\partial_\mu\varphi^i\partial_\nu\varphi^j + \frac{1}{f(\phi)}\partial_\mu\phi\partial_\nu\phi. \end{aligned} \tag{6.5}$$

This metric is a five-dimensional version of how to equip the Stückelburg fields into the massive gravity compared to the four-dimensional one in Eq. (5.4). Note that for the brane corresponding to $\phi = \phi(t)$, the reference metric in Eq. (6.5) inevitably involves a time derivative of ϕ. It may be questionable whether the BD ghost reappears in this context or not. Fortunately, it was found that the BD ghost is not presented even when a general form of the reference metric is used, even there exists derivatives in the metric [39]. Since we have introduced the DBI scalar via the reference sector, we can find its contribution to the massive gravity and the cosmological solution. For that purpose, we compute the full dRGT massive gravity action with a particular extention,

$$
\begin{aligned}
S &= S_{dRGT} + S_{DBI}, \\
&= \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left(R(g) + 2m_g^2 U(g,f)\right) - \Lambda^4\int d^4x\sqrt{-f}.
\end{aligned} \tag{6.6}
$$

where the first term is the Einstein-Hilbert lagrangian, $U(g,f)$ is the nonlinear interaction in the original dRGT theory, and the last term is the extension corresponding to the DBI scalar. In particular, this extension is a kinetic term of the DBI scalar ϕ corresponding to the brane moving in the bulk with tension of magnetude Λ. Moreover, the kinetic term is a leading order term of the entire class of lagrangian which is invariant under the Galileon shift symmetry [47, 48, 49]. Since we are interested in the cosmological picture of the model, we use the FLRW metric in Eq. (3.21) as the physical metric. Thus, by using a unitary gauge; $\varphi^\mu = x^\mu$, the action in Eq. (6.6) can be expressed accordingly as

$$
\begin{aligned}
S_{dRGT} = 3M_p^2 &\int d^4x\sqrt{\Omega} \\
&Na^3\left[\left(-\left(\frac{\dot{a}}{aN}\right)^2 + \frac{\kappa}{a^2}\right) + m_g^2\left(F(\bar{X}) - G(\bar{X})\frac{n}{N}\right)\right],
\end{aligned} \tag{6.7a}
$$

$$
S_{DBI} = -3M_p^2 m_g^2\alpha_\Lambda\int d^4x\sqrt{\Omega}\; Na^3\bar{X}^3\frac{n}{N}, \tag{6.7b}
$$

where $\Omega \equiv \det\Omega_{ij}(x^k)$, $F(\bar{X})$ and $G(\bar{X})$ are given as in Eq. (5.8) and in Eq. (5.9), and we have defined the following new parameters,

$$
\bar{X} \equiv \frac{\lambda\phi}{a}, \quad n \equiv \sqrt{f(\phi) - \frac{\dot{\phi}^2}{f(\phi)}}, \quad \alpha_\Lambda \equiv \frac{\Lambda^4}{3M_p^2 m_g^2}. \tag{6.8}
$$

By varying the action in Eq. (6.7) with respect to the lapse function N, the corresponding Friedmann equation is obtained as follows,

$$3M_p^2\left(H^2+\frac{\kappa}{a^2}\right)=\rho_g\equiv-3M_p^2m_g^2F. \tag{6.9}$$

From this Friedmann equation, contributions from the massive graviton behave as a source to the Friedmann equation exactly in the same way as those in the original model of dRGT massive gravity shown in Eq. (5.10) in Chapter V. To ensure such a claim, we vary the full action with respect to the scale factor a to obtain the corresponding acceleration equation,

$$M_p^2\left(\frac{2\dot{H}}{N}+3H^2+\frac{\kappa}{a^2}\right)=-p_g\equiv-3M_p^2m_g^2\left(F-\frac{\bar{X}F_{,\bar{X}}}{3}\left(1-r\right)\right), \tag{6.10}$$

where $r\equiv\frac{n}{N\bar{X}}$. This acceleration equation is still of the same form as Eq. (5.11) in the original dRGT theory (although here n takes a different form compared with the lapse function in the reference sector in the previous chapter, the structure of the equations of motion here is still as same as that in the pure dRGT massive gravity). As also done in the original theory, varying the action with respect to the Stückelburg fields gives equations of motion corresponding to the conservation of the effective energy-momentum tensor given by the massive gravity interactions. In this model, however, there are five Stückelburg fields, namely $\varphi^0,\varphi^1,\varphi^2,\varphi^3$, and ϕ where the first four fields should govern the conservation of the four-dimensional effective energy-momentum tensor while ϕ has its own dynamics as the moving brane. This suggests that two sets of condition must come out of varying all of the Stückelburg fields, namely the conservation and the equation of motion of ϕ. Alternatively, we can obtain those conditions via the perturbative approach presented in the Appendix B. By any means, computing for the Stückelburg equation of motion yields

$$F_{,\bar{X}}\left(H-\bar{X}H_\phi\right)=0, \tag{6.11}$$

$$J=0, \tag{6.12}$$

where

$$J \equiv \frac{a^3}{\phi n}\left[\frac{\left(G+\alpha_\Lambda \bar{X}^3\right)\left(6f^3+\phi f^2 f_{,\phi}-3\phi\dot{\phi}^2 f_{,\phi}+f\left(2\phi\ddot{\phi}-6\dot{\phi}^2\right)\right)}{2fn^2}\right.$$
$$\left.+\left(f-Nn\bar{X}\left(1+\frac{aH\phi H_\phi}{\lambda f}\right)\right)F_{,\bar{X}}\right], \tag{6.13}$$

$$H_\phi \equiv \frac{\dot{\phi}}{\phi n}, \qquad f_{,\phi} \equiv \frac{df}{d\phi}. \tag{6.14}$$

Here, the usual condition in Eq. (6.11) also rises up similarly to the original theory of dRGT massive gravity while Eq. (6.12) is the equation of motion governing the dynamics of the brane or, in other words, the DBI field ϕ.

It is worth mentioning more on the connections amoung equations of motion obtained so far. One can deduce directly the Stückelburg equation of motion in Eq. (6.11) through simple manipulation on both Eq. (6.9) and Eq. (6.10) by requiring the conservation of the energy-momentum tensor. However, the derivation through this approach will be valid even n is not specified according to Eq. (6.8) since n does not involve quantities in the physical sector explicitly. In other words, no matter how n looks like (as long as it stays in the reference sector), it is always possible to obtain such a constraint as in Eq. (6.11) given that the four-dimensional conservation of energy-momentum tensor is required. Not only that, the self-accelerating branch actually decouples n from the physical sector which can be seen in Eq. (6.10). All of these are caused by our four-dimensional setup induced from a five-dimensional one. By such a setting, we require the conservation of energy and momentum in the four-dimensional sector, in other words any matter in four dimensions cannot leak into a five-dimensional bulk. Generally, it can be shown through simple computations that if Eq. (6.11) is not assumed, then, in four dimensions, $\nabla_\mu T^{\mu\nu} \propto F_{,\bar{X}}\left(H-\bar{X}H_\phi\right)$ indicating a violation of the conservation. This suggests that one can think of the term $F_{,\bar{X}}\left(H-\bar{X}H_\phi\right)$ as a flow that conveys matter contents between the physical four-dimensional spacetime and the fifth dimension which is characterized by ϕ. In this context, to satisfy Eq. (6.11) means to cut the flow of matter from ``inside" the brane to ``outside" the brane (elsewhere in the bulk) and vise versa which can be automatically implied by the four-dimensional induced theory where the induction is done on the five-dimensional reference metric.

As usual, Eq. (6.11) can be separated into two conditions; $F_{,\bar{X}} = 0$ for a self-accelerating solution and $H - \bar{X}H_{\phi} = 0$ for a normal branch of solution. To be exact, the self-accelerating branch obeys the condition

$$F_{,\bar{X}} = 0 = -\left(3 + 3\alpha_3 + \alpha_4\right) + 2\left(1 + 2\alpha_3 + \alpha_4\right)\bar{X} - \left(\alpha_3 + \alpha_4\right)\bar{X}^2, \tag{6.15}$$

which fixes $\bar{X}$ to be a constant as,

$$\bar{X}_{\pm} = \frac{1 + 2\alpha_3 + \alpha_4 \pm \sqrt{1 + \alpha_3 + \alpha_3^2 - \alpha_4}}{\left(\alpha_3 + \alpha_4\right)}. \tag{6.16}$$

Similarly to the original dRGT theory, the effective cosmological constant can be found as

$$\rho_g = M_p^2 \Lambda_{\pm} \tag{6.17}$$

$$= -\frac{M_p^2 m_g^2}{\left(\alpha_3 + \alpha_4\right)^2}\Big[\left(1 + \alpha_3\right)\left(2 + \alpha_3 + 2\alpha_3^2 - 3\alpha_4\right) \pm 2\left(1 + \alpha_3 + \alpha_3^2 - \alpha_4\right)^{3/2}\Big]. \tag{6.18}$$

Furthermore, the pressure p_g can be found along with its equation of state parameter as

$$p_g = -\rho_g, \qquad w_g = -1, \tag{6.19}$$

which ensures the characteristics of the cosmological constant.

Note that up to this point we did not specify the value of κ which characterize the curvature of the spatial hypersurfaces. Consequently, the spatial curvature of the solution obtained here can be of any kind of slicing, as well as the solution in Chapter V. Interestingly, one may see that the dynamics of this branch, governed by the Hubble parameter H, does not depend on the DBI sector. In other words, no matter how the DBI scalar ϕ evolves, it does not affect the gravity sector since the dynamics on the gravity sector is only governed by $\bar{X}$ which is just a combination of constants.

Since in this model the DBI scalar also propagates, it is necessary to consider the equation of motion of the DBI scalar in Eq. (6.12) for the completeness of the solution. For the self-accelerating solution, using the redefinition $\psi(\phi) \equiv \dot{\phi}^2$, Eq. (6.12) implies the following equation,

$$\frac{d\psi}{d\phi} - \left(3\frac{1}{f}\frac{df}{d\phi} + \frac{6}{\phi}\right)\psi + f^2\left(\frac{1}{f}\frac{df}{d\phi} + \frac{6}{\phi}\right) = 0. \tag{6.20}$$

The solution to this equation is

$$\psi = \dot{\phi}^2 = f^2 - \left(\frac{\phi}{\phi_0}\right)^6 f^3, \tag{6.21}$$

where ϕ_0 is an integration constant. Since f is quite a complicated funtion, the solution in Eq. (6.21) cannot be solved analytically. However, we can still obtain its consequences without involving nasty computations. By the use of Eq. (6.9), we can find a relation between the lapse function and the DBI scalar as follows,

$$N^2 = \frac{3\lambda^2}{(\lambda^2\phi^2\Lambda_\pm - 3\kappa X_\pm^2)}\left(f^2 - \left(\frac{\phi}{\phi_0}\right)^6 f^3\right). \tag{6.22}$$

Furthermore, we can obtain an interval in which the DBI scalar ϕ is allowed to propagate. Requiring that $\dot{\phi}^2 > 0$, Eq. (6.12) implies the following inequality,

$$1 - \left(\frac{\phi}{\phi_0}\right)^6 f = 1 + \frac{\mu}{\lambda^4\phi_0^6}\phi^4 - \frac{\kappa}{\lambda^2\phi_0^6}\phi^6 - \frac{1}{l^2\phi_0^6}\phi^8 > 0. \tag{6.23}$$

In principle, it is possible to determine the possible interval for ϕ from this inequality. However, in the case of the flat FLRW solution where $\kappa = 0$, the inverval can be deduced easily to be

$$0 < \phi^2 < b\left[\frac{1}{2}\left(1 + \sqrt{1 + \frac{4l^2\phi_0^6}{b^4}}\right)\right]^{1/2}, \tag{6.24}$$

where $b \equiv \sqrt{\mu l^2/\lambda^4}$. This b turns out to be an interesting value in the picture of dynamical analyses. One can see that Eq. (6.21) determines how the DBI scalar ϕ changes in time due to itself (implicitly expressed as the function f) which makes Eq. (6.21) an autonomous equation. Interestingly, it can be seen from Eq. (6.21) that if ϕ evolves to some specific values, its velocity $\dot{\phi}$ vanishes and the scalar ϕ stays at that value forever. Those special values are called fixed points. In particular, there are two fixed points for Eq. (6.12), all of which involve b, namely

$$\phi^2 = b \qquad \text{and} \qquad \phi^2 = b\left[\frac{1}{2}\left(1+\sqrt{1+\frac{4l^2\phi_0^6}{b^4}}\right)\right]^{1/2}. \tag{6.25}$$

These fixed points also have their interesting physical meaning. When $\dot{\phi}^2 = 0$, it also implies that the fifth coordinate is fixed to some particular value. Such a situation render the fifth dimension meaningless as can be seen in Eq. (6.5) when $n = \sqrt{f}$ is the lapse function in the reference sector. This model thus essentially reduces to the dRGT massive gravity with FLRW reference metric presented in Chapter V.

As in Chapter V, this model also provides a normal branch which, from Eq. (6.11), corresponds to a condition

$$H - \bar{X}H_\phi = 0. \tag{6.26}$$

This branch is exactly a modified version of the normal branch which exists in the dRGT massive gravity with FLRW reference metric presented in Chapter V. For this version, however, the condition in Eq. (6.26) becomes more complicated because of how the DBI scalar ϕ is involved. To grasp some ideas out of it, we will put an assumption to continue our calculation. Let us assume that the solution satisfies $G + \alpha_\Lambda \bar{X}^3 = 0$. This condition explicitly reads

$$\begin{aligned} G + \alpha_\Lambda \bar{X}^3 = -\frac{1}{3}\Big[&-(3+3\alpha_3+\alpha_4) + 3(1+2\alpha_3+\alpha_4)\bar{X} - 3(\alpha_3+\alpha_4)\bar{X}^2 \\ &+ (\alpha_4 - 3\alpha_\Lambda)\bar{X}^3\Big] \\ = 0, & \end{aligned} \tag{6.27}$$

which implies that $\bar{X}$ is constant. Solving Eq. (6.27) for $\bar{X}$, we can find the relation between ϕ and a as follows,

$$\lambda\phi = \bar{X}a = \chi a, \tag{6.28}$$

where χ is one of the roots to Eq. (6.27). Using this relation, Eq. (6.26) reads

$$\frac{n}{N} = \chi. \tag{6.29}$$

This implies exactly $r = 1$ which is similar to the special case in Chapter V. The corresponding equation of state parameter in this setting is -1 indicating the self-accelerating expansion in the cosmological constant manner . Note that this self-accelerating class in the normal branch is not the same as the former self-accelerating branch since $\bar{X}$ can take any constant value other than $\bar{X}_{\pm}$ as long as $r = 1$ is satisfied. As a special case, here $\bar{X}$ takes a value of χ, a root to the condition $G + \alpha_\Lambda \bar{X}^3 = 0$. One may note that for a particular choice of parameter which satisfies $\alpha_4 - 3\alpha_\Lambda = 0$, Eq. (6.27) simply reduces to a quadratic equation which has two roots and it is easy to find those roots. We will come back to this parameter choice later. Not only the nature of $\bar{X}$ but also the dynamics of the DBI scalar are also different between these two kinds of self-accelerating solution. Under such conditions for this special case, since Eq. (6.12) vanishes completely, we cannot obtain an equation of motion of the DBI scalar from this equation. Fortunately, we can find how it evolves through the proportionality between ϕ and a in Eq. (6.28) and the Friedmann equation in Eq. (6.9). Assuming a spatially flat FLRW metric; $\kappa = 0$, the equation for the DBI scalar eventually reads

$$\dot{\phi}^2 = \frac{\tilde{\rho}_g \phi^2 f^2}{1 + \tilde{\rho}_g \phi^2}, \tag{6.30}$$

where

$$\tilde{\rho}_g = \frac{\rho_g}{3M_p^2 \chi^2}. \tag{6.31}$$

Since there exists only a first derivative, Eq. (6.30) can be treated as an autonomous equation from which the corresponding fixed point can be found. From Eq. (6.30), the fixed point that makes the first derivative vanished is $\phi^2 = b$ which is the same fixed point as in the former self-accelerating branch. We can also obtain a full equation governing the DBI scalar by integrating Eq. (6.30) which reads

$$\sqrt{1+c_n b}\ \tanh^{-1}\left(\sqrt{\frac{1+c_n\phi^2}{1+c_n b}}\right) - \sqrt{1-c_n b}\ \tanh^{-1}\left(\sqrt{\frac{1+c_n\phi^2}{1-c_n b}}\right) = \frac{2b\sqrt{c_n}}{l^2}t + C, \quad (6.32)$$

where C is an integration constant. We can see that the case $\phi^2 = b$ corresponds to $t \to \infty$ which is reasonable in the view of the dynamical analysis. In particular, we know that $\phi^2 = b$ is the fixed point to Eq. (6.30) such that the DBI scalar does not evolve once it reaches this value within a long enough time and the model reduces accordingly to the dRGT massive gravity with FLRW reference metric as $n = \sqrt{f}$.

6.2 Perturbations, Degrees of Freedom, and Stabilities

Up to this point, it has been shown that in this model there exist two conditions obtained from the equations of motion of the Stückelburg fields which are similar to the original dRGT theory. One is formally named in literatures as a self-accelerating branch by which it allows the FLRW universe of any kind of spatial geometry to expand with acceleration. It effectively provides a cosmological constant which originates from the existence of the graviton mass. It has been claimed that the self-accelerating branch in the dRGT massive gravity is plagued by vanishing degrees of freedom which is not supposed to happen. Another condition corresponds to what is known as a normal branch where, in some specific situations, it can also provide a self-accelerating behavior which shares some resemblances with the former self-accelerating branch. Since the model succeeds in describing the accelerating expansion, it is tempting to find out whether or not the issue of vanishing degrees of freedom is alleviated. To do such an analysis, we need to take into account a metric perturbation of the model. This analysis not only satisfies us on the purpose of counting degrees of freedom but also allows us to investigate whether the model excites a ghost instability or not. To begin such an investigation, we define a physical metric perturbation as

$$g_{\mu\nu} = g_{\mu\nu}^{(0)} + \delta g_{\mu\nu}. \tag{6.33}$$

According to this form of metric, we perform the analyses on the model through a perturbative calculation around a general background metric $g_{\mu\nu}^{(0)}$. In this case we are interested in the perturbation around a cosmological solution. In other words, we set $g_{\mu\nu}^{(0)}$ to be a flat, for simplicity, FLRW metric as in Eq. (3.21). The perturbation δg is defined accordingly as

$$\delta g_{00} \equiv -N^2 A, \tag{6.34a}$$

$$\delta g_{0i} \equiv N a B_i, \tag{6.34b}$$

$$\delta g_{ij} \equiv a^2 h_{ij}. \tag{6.34c}$$

Note that the perturbations; A, B_i, and h_{ij}, are defined separately according to whether they are temporal or spatial quantities. To be exact, each perturbation is classified due to its transformation under spatial coordinate transformations; namely A transforms as a scalar, B_i transforms as a vector, and h_{ij} transforms as a rank-2 tensor. Moreover, the perturbations are defined in this way so that they are independent of a time reparameterization N and the scale factor a. Such a definition ensures that only the geometrical deviations are encoded in the perturbations in Eq. (6.34) and not the time reparameterization or the scale factor. Not only that, these perturbations are allowed to depend on all of the spacetime coordinates. Such a dependence is rather a realistic representation because our universe does not strictly obey the cosmological principle; we have a bunch of stars, clusters of galaxies, and many objects that render the universe slightly inhomogeneous. Apart from those scattered matters, the universe is also inhomogeneous in a small scale caused by fluctuations of background temperature, known as the cosmic microwave background (CMB) (for the observational data, see Ref. [50]). Theoretically, CMB is investigated through the perturbative cosmology in which the metric perturbation plays a crucial role.

For conveniences, the perturbations in Eq. (6.34) are decomposed further into components according to their tensorial properties under spatial rotations as follows,

$$\delta g_{00} = -2N^2\Phi, \tag{6.35a}$$

$$\delta g_{0i} = Na\left(B_i^T + \partial_i B\right), \tag{6.35b}$$

$$\delta g_{ij} = a^2\left[h_{ij}^{TT} + \frac{1}{2}\left(\partial_i E_j^T + \partial_j E_i^T\right) + 2\delta_{ij}\Psi + \left(\partial_i\partial_j - \frac{1}{3}\delta_{ij}\partial_k\partial^k\right)E\right]. \tag{6.35c}$$

In such a decomposition, the perturbations are classified into three types; tensor mode, vector mode, and scalar mode. The tensor mode is represented by the 3-space metric h_{ij}^{TT} which is transverse and traceless, or, in other words, satisfies $\partial^i h_{ij}^{TT} = 0$, and $\delta^{ij} h_{ij}^{(TT)} = 0$, given that $\partial^i \equiv \delta^{ij}\partial_j$. The vector mode corresponds to two transverse vectors, namely B_i^T and E_i^T which obey $\partial^i B_i^T = 0$ and $\partial^i E_i^T = 0$. The scalar mode is determined by four scalars; Φ, B, Ψ, and E. This looks complicated at the beginning but it has its own advantages; these distinct modes actually decouple from one another in the Einstein-Hilbert action when being expanded perturbatively up to the quadratic order. Apart from the physical metric, the perturbation is done on the DBI scalar as well. The DBI scalar is expanded accordingly as follows,

$$\phi = \phi^{(0)} + \delta\phi. \tag{6.36}$$

We also stay in the unitary gauge where no perturbation on the gauge is performed. The Stückelburg fields in five dimensions are separated into two parts. Particularly, φ^μ's represent general covariance in four dimensions while ϕ propagates as an additional scalar. The first four Stückelburg fields are not perturbed but are fixed to be in the unitary gauge while the perturbation is done on the fifth Stückelburg field ϕ which in four dimensions is now interpreted as a scalar field. This means that we do the perturbations only on the fields that mediate gravity for this model, namely the physical metric and the DBI scalar. Due to the above defined perturbations, each mode can be investigated separately on the issue of degrees of freedom and stability as follow.

6.2.1 Tensor Mode

To extract information corresponding to this mode, we expand the action according to the tensor perturbation $h^{TT}_{\mu\nu}$ in Eq. (6.2) up to quadratic order where the equations of motion in Eq. (6.9) and Eq. (6.10) are taken into account. The expansion is then Fourier transformed where the transformation is done on only the spatial coordinates, namely x, y, and z. The resulting expansion then reads

$$S^{(2)}_{Tensor} = \frac{M_p^2}{8}\int d^3k dt\; Na^3\left[\frac{\dot{\tilde{h}}^{TT}_{ij}(\dot{\tilde{h}}^{*})^{ij}_{TT}}{N^2} - \left(\frac{k^2}{a^2} + M^2_{GW}\right)\tilde{h}^{TT}_{ij}(\tilde{h}^{*})^{ij}_{TT}\right], \qquad (6.37)$$

where $\tilde{h}^{TT}_{ij} = \tilde{h}(k)^{TT}_{ij}$ is a Fourier transform of h^{TT}_{ij}, the dot represents a time derivative, $\tilde{h}^{ij}_{TT} = \delta^{ik}\delta^{jl}\tilde{h}^{TT}_{kl}$, and

$$\begin{aligned} M^2_{GW} &\equiv m_g^2 A, && (6.38)\\ A &\equiv \bar{X}^2\Bigg[\left(\frac{3}{\bar{X}} - 1 - r\right) + \alpha_4\left(\left(\frac{3}{\bar{X}} - 2\right) - \left(2 - \bar{X}\right)r\right) \\ &\quad + \alpha_4\left(1 - \bar{X}\right)\left(\frac{1}{\bar{X}} - r\right)\Bigg], \\ &= -\bar{X}F_{,\bar{X}} + \bar{X}^2(1 - r)\left(1 + 2\alpha_3 + \alpha_4 - (\alpha_3 + \alpha_4)\bar{X}\right). && (6.39) \end{aligned}$$

The tensor mode action in Eq. (6.37) provides a wave solution of each Fourier mode denoted by the wave number k. The term M^2_{GW} involving the graviton mass m_g indicates the mass corresponding to this wave solution. As $m_g \to 0$, we recover the gravitational wave solution in general relativity. We can see that this action involves only the transverse and traceless tensor field $\tilde{h}^{TT}_{ij}$ which means degrees of freedom in this mode are determined only by the nature of $\tilde{h}^{TT}_{ij}$. Together with the constraints on the transverse and traceless tensor, $\tilde{h}^{TT}_{ij}$ in Eq. (6.37) can be fully determined by only 2 dynamical variables which coincides with the number of polarizations of the gravitational wave in general relativity. This is essentially similar to general relativity; the only difference is the existence of M_{GW} induced by the graviton mass.

By obtaining the action like in Eq. (6.37), we can determine whether or not the dynamical fields excite the ghost instability just by looking at the sign in front of their kinetic terms in the Fourier space. Since we are considering the second-order perturbed action around the background solution, the sign determines whether or not the solution is stable. Quantum mechanically, the solution has a well-defined ground state when the sign is plus while the sign being minus otherwise indicates that there is no ground state. By just looking at the sign of $\dot{\tilde{h}}^{TT}(\dot{\tilde{h}}^{*})_{TT}$, it can be easily implied that this mode does not excite the ghost. Moreover, to avoid another instability known as ``tachyonic instability", we must require $M_{GW}^2 > 0$ or, otherwise, the tensor mode will acquire imaginary mass. Such imaginary mass is not physical in terms of special relativity since it renders each of the Fourier mode to be superluminal (the propagation speed is faster than that of light). This can be roughly illustrated by considering one Fourier mode with the wave number k. The solution to this Fourier mode is of the following form,

$$\tilde{h}_{ij}^{TT} \propto e^{i\omega_k t}. \tag{6.40}$$

This implies the corresponding wave equation as

$$-\frac{\ddot{\tilde{h}}_{ij}^{TT}}{N^2} - \left(\frac{k^2}{a^2} + M_{GW}^2\right)\tilde{h}_{ij}^{TT} \propto \left(\frac{\omega_k^2}{N^2} - \left(\frac{k^2}{a^2} + M_{GW}^2\right)\right) e^{i\omega_k t}, \tag{6.41}$$

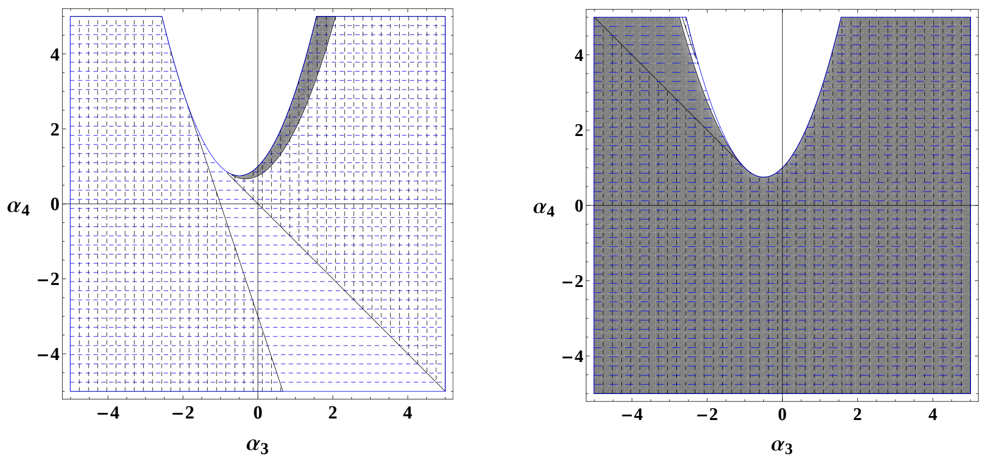
$$c_s^2 = \frac{\omega_k^2/N^2}{k^2/a^2 + M_{GW}^2}. \tag{6.42}$$

Since the null geodesic corresponds to $g_{\mu\nu}dx^\mu dx^\nu = 0$, then for a photon of energy ω_k, we have a condition

$$c^2 = 1 = \frac{\omega_k^2/N^2}{k^2/a^2}. \tag{6.43}$$

By comparing Eq. (6.42) with Eq. (6.43), we have a superluminality as $c_s^2 > 1$ in the case of $M_{GW}^2 < 0$ while $M_{GW}^2 > 0$ implies a more physical situation as $c_s^2 < 1$.

Note that up to this point, the constraint in Eq. (6.11) is not yet used, or, in other words, the solution is not yet specified to which branch it belongs. Thus, the results obtained so far should be valid for both branches. Essentially, the tensor mode of perturbation, no matter what branch it belongs to, coincides with general relativity; namely there are two degrees of freedom representing two polarizations of the gravitational wave solution and the degrees of freedom are healthy without the ghost instability and the tachyonic instability. From these results, we can deduce suitable values for the parameters of the model, namely α_3 and α_4. For the self-accelerating branch where $F_{,\bar{X}} = 0$, we require the appropriate values of so that M_{GW}^2, ρ_g, and $\bar{X}_\pm$ are all positive. The corresponding values of (α_3, α_4) are illustrated in Figure. (6.1). One can see from



(a) X_+ solution.

(b) X_- solution.

Figure 6.1 These plots show the available regions in (α_3, α_4) space satisfying the subluminal condition $M_{GW}^2 > 0$, $\rho_g > 0$, and $\bar{X}_\pm > 0$ provided that the self-accelerating branch is satisfied. Figure. (6.1a) corresponds to $\bar{X}_+ > 0$ while Figure. (6.1b) corresponds to $\bar{X}_- > 0$. The regions with horizontal blue-dashed line satisfy the subluminal condition $M_{GW}^2 > 0$ while each of $\bar{X}_\pm > 0$ is denoted by the regions with vertical black-dashed line separately in each plots. The grey regions satisfy $\rho_g > 0$. Obviously, the branch with $\bar{X}_-$ allows larger class of (α_3, α_4) than the $\bar{X}_+$ branch [27].

Figure. (6.1) that there is an obvious difference on the available regions between $\bar{X}_+$ and $\bar{X}_-$. This big difference is caused by the condition of positive energy density $\rho_g > 0$. In particular, it is the term $1 + \alpha_3 + \alpha_3^2 + \alpha_4$ that is responsible. From Eq. (6.16), we must have $1 + \alpha_3 + \alpha_3^2 + \alpha_4 > 0$ in order to have a real value of $\bar{X}$. This affects the energy density in Eq. (6.18) differently between the case of $\bar{X}_+$ and that of $\bar{X}_-$. In particular, the term $(1 + \alpha_3 + \alpha_3^2 + \alpha_4)^{3/2}$ appears with a minus sign for the branch of $\bar{X}_+$ while the term oppositely appears with a plus sign in the case of $\bar{X}_-$. Qualitatively, the case of $\bar{X}_-$ has more room to be positive according to the positive addition to the energy density

ρ_g which results in the plots shown in Figure. (6.1). Moreover, for the self-accelerating branch, r can be determined readily from the subluminal condition through Eq. (6.39). For $\bar{X}_+$ branch the subluminal condition is valid only when $r > 1$ while for $\bar{X}_-$ branch we must have $r < 1$ to satisfy $M_{GW}^2 > 0$. Obviously, $r = 1$ makes M_{GW}^2 vanishes for the self-accelerating branch, which exactly describes gravitational waves similar to general relativity. Actually, such a case is nothing but general relativity with the cosmological constant or ΛCDM model. Though the model considered here involves DBI scalar, the self-accelerating branch does not depend strongly on the scalar field which is why the results obtained so far are similar to the original dRGT massive gravity when the FLRW ansatz is taken into account [17].

Apart from what we have done on the self-accelerating branch, it is unsurprisingly complicated to do such analyses on the normal branch without any simplification. Thus, we focus on the special case mentioned earlier in the previous section. In particular, we consider the self-accelerating class in the normal branch where $\bar{X}$ is constant and $r = 1$. For simplicity, we are interested in the case $\alpha_4 - 3\alpha_\Lambda = 0$ so that Eq. (6.27) reduces to a quadratic equation and its roots can be found easily. The roots thus read

$$\bar{X} = \chi_\pm = \frac{(1 + 2\alpha_3 + \alpha_4) \pm \sqrt{1 - 2\alpha_4 - \frac{1}{3}\alpha_4^2 - \frac{4}{3}\alpha_3\alpha_4}}{2\,(\alpha_3 + \alpha_4)}. \tag{6.44}$$

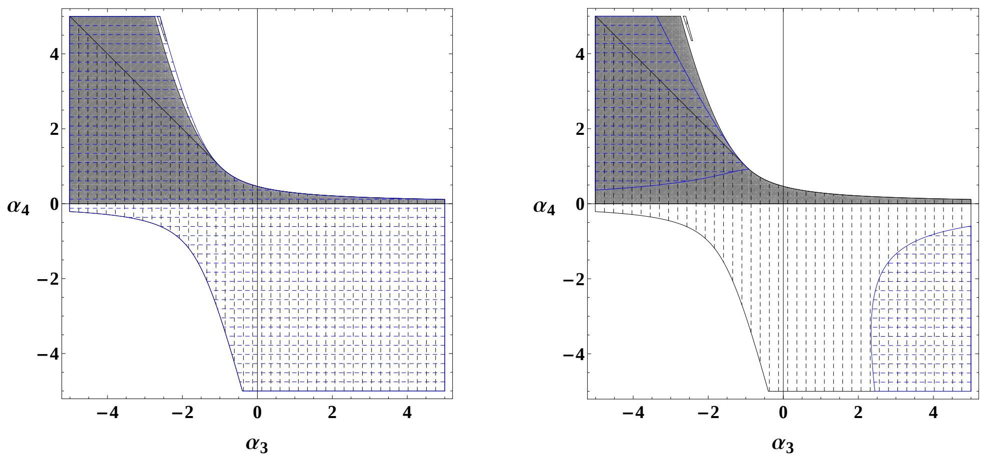
Since we have claimed that the tensor mode degree of freedom for the normal branch is not plague by a ghost instability as shown explicitly via Eq. (6.37), this solution of $\bar{X}$ is obviously healthy. However, the values of $\chi_\pm$ obtained here destines the solution whether or not there exists the superluminality. In particular, Eq. (6.39) in this branch, where $\bar{X} = \chi_\pm$ and $r = 1$, reads

$$M_{GW}^2 = -m_g^2 \bar{X} F_{,\bar{X}}, \tag{6.45}$$

$$= \mp m_g^2 \chi_\pm^2 \sqrt{1 - 2\alpha_4 - \frac{1}{3}\alpha_4^2 - \frac{4}{3}\alpha_3\alpha_4}. \tag{6.46}$$

Provided that we are considering a class of parameters by which $\bar{X} = \chi_\pm$ exists, the solution for χ_+ always suffers from the superluminality since it always causes $M_{GW}^2 < 0$.

This leaves χ_- only a possible choice of this normal branch. Furthermore, a previous study on the dRGT massive gravity corresponding to the FLRW and de Sitter reference metrics suggests the Higuchi bound on this M_{GW} by the relation $M_{GW}^2 > 2H^2$ [44]. Similarly to the former analyses, we consider the availability of the parameters by taking $\chi_- > 0$, $\rho_g > 0$, and $M_{GW}^2 > 0$ into account as well as the Higuchi bound. The available class of parameters is illustrated in Figure. (6.2).



(a) χ_- solution with $M_{GW}^2 > 0$.

(b) χ_- solution with $\frac{M_{GW}^2}{H^2} > 2$.

Figure 6.2 These plots show the available regions in (α_3, α_4) space satisfying $\rho_g > 0$, $\chi_- > 0$ and two kinds of the bounds on M_{GW} for the self-accelerating class of normal branch subjected to Eq. (6.27) and $\alpha_4 - 3\alpha_\Lambda = 0$. The subluminality condition $M_{GW}^2 > 0$ is considered together with the positivities of the energy density and χ_- in Figure. (6.2a) while the Higuchi bound is considered instead in Figure. (6.2b). The regions with horizontal blue-dashed lines in both figures correspond to the bound on M_{GW}, namely $M_{GW}^2 > 0$ and $M_{GW}^2 > 2H^2$. The regions with vertical black-dashed lines represent the condition $\chi_- > 0$ and the grey regions imply positive energy density ρ_g [27].

Up to this point, by requiring the absence of various instabilities and inconsistencies, we are able to put bounds on the parameters of the model for both branches when their self-accelerating characteristics are satisfied. The results obtained here, however, need more considerations on the other perturbation modes in order to fully determine the availability of the solutions. This leads to the next mode of perturbation which transforms under spatial rotation as a vector.

6.2.2 Vector Mode

The vector mode of perturbation is described through B_i^T and E_i^T. To perform analyses in this mode, we expand the action, in Fourier space, according to those vectors up to second order of B_i^T and E_i^T which reads

$$\begin{aligned} S^{(2)}_{Vector} = \frac{M_p^2}{2}\int d^3k dt\ Na^3 \Bigg[& \frac{k^2}{8N^2}\dot{\tilde{E}}_i^T(\dot{\tilde{E}}^*)^i_T - \frac{k^2}{2Na}\tilde{B}_i^T(\dot{\tilde{E}}^*)^i_T \\ & - \frac{k^2}{4}\left(\frac{2\dot{H}}{N} + 3H^2\right)\tilde{E}_i^T(\tilde{E}^*)^i_T + \left(\frac{k^2}{2a^2} + 3H^2\right)\tilde{B}_i^T(\tilde{B}^*)^i_T \\ & - \frac{k^2}{4}\left(\frac{p_g}{M_P^2} + \frac{1}{2}M_{GW}^2\right)\tilde{E}_i^T(\tilde{E}^*)^i_T \\ & - \left(\frac{\rho_g}{M_P^2} + \frac{m_g^2\bar{X}F_{,\bar{X}}}{(1+r)}\right)\tilde{B}_i^T(\tilde{B}^*)^i_T\Bigg], \end{aligned} \tag{6.47}$$

where the fields with tildes are their Fourier transform as before. Although there are two vector fields in the action, $\tilde{B}_i^T$ is not dynamical due to the absence of its time derivative or kinetic term. This means that $\tilde{B}_i^T$ can be found in terms of other variables in the action, in this case $\tilde{E}_i^T$, through its equation of motion. By doing so, Eq. (6.47) can be rewritten in term of only $\tilde{E}_i^T$, with helps from Eq. (6.9) and Eq. (6.10), as follows,

$$\begin{aligned} S^{(2)}_{Vector} = \frac{M_P^2}{8}\int d^3k dt\ Na^3 \Bigg[& -\frac{k^2 m_g^2 a^2 \bar{X}F_{,\bar{X}}}{N^2\left(k^2(1+r) - 2m_g^2 a^2\bar{X}F_{,\bar{X}}\right)}\dot{\tilde{E}}_i^T(\dot{\tilde{E}}^*)^i_T \\ & - \frac{k^2}{2}M_{GW}^2\tilde{E}_i^T(\tilde{E}^*)^i_T\Bigg]. \end{aligned} \tag{6.48}$$

Due to its transversal behavior, this action for E_i^T governs the dynamics of two degrees of freedom. Note that this action does not belong to any branch of solution since the constraints for each branch are not yet specified. The ghost instability can be determined readily by looking at the sign in front of the kinetic term. In particular, for the self-accelerating branch satisfying $F_{,\bar{X}} = 0$, such a condition renders the coefficient of the kinetic term vanishing, implying that $\tilde{E}_i^T$ does not propagate. This vanishing term is exactly the problem commonly known in cosmology of dRGT massive gravity. Despite the mass term involving M_{GW}^2, vector degrees of freedom does not propagate for the self-accelerating branch. On the other hand, the normal branch seems to work fine with two propagating vector degrees of freedom since there is no such restriction as $F_{,\bar{X}} = 0$ like in the self-accelerating branch. Generally, the condition to ensure the absence of the ghost is $F_{,\bar{X}} < 0$. This condition allows the vector degree of freedom to be healthy at any value of k. Since k denotes a wave number, a reciprocal of a length scale, this implies that the ghost is not excited at any length scales.

Up to now, we have investigated the action for vector perturbations and realized some properties of both branches of solution we obtained previously. For the self-accelerating branch, the vector perturbations do not propagate due to the absence of the corresponding kinetic terms. On the other hand, the normal branch has proven itself to be free of such issue as long as $F_{,\bar{X}} < 0$ is satisfied. It may be clear now for the self-accelerating branch to have an incorrect number of propagating degrees of freedom yet further investigation is needed for the normal branch which leads us to the next section in which the scalar perturbation is taken care of carefully.

6.2.3 Scalar Mode

To get a complete picture of the model, the analyses on scalar perturbations are necessary. As we have done previously, the action for scalar perturbations is expanded in Fourier space up to second order of the scalar perturbations in Eq. (6.2). The Einstein-Hilbert part thus reads

$$\begin{aligned} S^{(2)}_{Scalar,EH} = \frac{M_P^2}{2}\int d^3k dt\ Na^3 \Bigg[& \frac{k^4|\dot{\tilde{E}}|^2}{6N^2} - \frac{6|\dot{\tilde{\Psi}}|^2}{N^2} - \frac{2k^4\tilde{B}\dot{\tilde{E}}^*}{3aN} + \dot{\tilde{\Psi}}\left(\frac{12H\tilde{\Phi}^*}{N} - \frac{4k^2\tilde{B}^*}{aN}\right) \\ & + |\tilde{E}|^2\left(\frac{k^6}{18a^2} - \frac{k^4}{3}\left(\frac{2\dot{H}}{N} + 3H^2\right)\right) + |\tilde{\Psi}|^2\left(\frac{2k^2}{a^2} + 3\left(\frac{2\dot{H}}{N} + 3H^2\right)\right) \\ & + 3k^2H^2|\tilde{B}|^2 - 9H^2|\tilde{\Phi}|^2 + \frac{2k^4\tilde{\Psi}\tilde{E}^*}{3a^2} \\ & + \tilde{\Phi}\left(\frac{4k^2H\tilde{B}^*}{a} + \tilde{\Psi}^*\left(\frac{4k^2}{a^2} + 18H^2\right) + \frac{2k^4\tilde{E}^*}{3a^2}\right)\Bigg], \end{aligned} \tag{6.49}$$

where the scalar fields with tildes on top are those in Fourier space and $|Q|^2 \equiv QQ^*$. The interaction parts are expressed perturbatively, including the perturbation of the DBI scalar; $\delta\phi$, as follows,

$$\begin{aligned}
&S^{(2)}_{Scalar,dRGT} + S^{(2)}_{Scalar,DBI} \\
&\qquad = \frac{M_P^2 m_g^2}{2}\int d^3k dt\ Na^3\Bigg[3\left(\frac{G+\alpha_\Lambda \bar{X}^3}{Nn^3}\right)|\dot{\tilde{\delta\phi}}|^2 - \frac{6\phi H_\phi F_{,\bar{X}}}{Nf}\dot{\tilde{\delta\phi}}\tilde{\Psi}^* \\
&\qquad - \frac{3J_{,\phi}}{a^3N}|\tilde{\delta\phi}|^2 - \frac{k^2}{a^2Nn\bar{X}}\left((G'+3\alpha_\Lambda\bar{X}^2) - \frac{(f+Nn\bar{X})F_{,\bar{X}}}{f\bar{X}(1+r)}\right)|\tilde{\delta\phi}|^2 \\
&\qquad - \frac{k^4}{3}\left(\frac{p_g}{M_P^2m_g^2}+\frac{A}{2}\right)|\tilde{E}|^2 + 3\left(\frac{p_g}{M_P^2m_g^2}+2A\right)|\tilde{\Psi}|^2 \\
&\qquad + k^2\left(\frac{p_g}{M_P^2m_g^2} - \frac{n^2F_{,\bar{X}}}{N^2\bar{X}(1+r)}\right)|\tilde{B}|^2 + \frac{\rho_g}{M_P^2m_g^2}|\tilde{\Phi}|^2 \\
&\qquad + 3\left(\frac{f_{,\phi}(f^2+\dot{\phi}^2)F_{,\bar{X}}}{Nnf^2} - \frac{4A}{\phi}\right)\tilde{\delta\phi}\tilde{\Psi}^* + \frac{2k^2nH_\phi F_{,\bar{X}}}{N\lambda f(1+r)}\tilde{\delta\phi}\tilde{B}^* \\
&\qquad + \frac{6\lambda F_{,\bar{X}}}{a}\tilde{\delta\phi}\tilde{\Phi}^* + 18\left(F - \frac{\bar{X}F_{,\bar{X}}}{3}\right)\tilde{\Phi}\tilde{\Psi}^*\Bigg].
\end{aligned} \tag{6.50}$$

Note that neither B nor Φ is a dynamical field because of the absence of their kinetic terms in the action. Thus, these scalars can be expressed in terms of other fields in the action through their equations of motion. With helps from those equations of motion, only three scalars remain to be determined in this action, namely E, Ψ, and $\delta\phi$. Furthermore, the resulting action can be simplified by using Eq. (6.9) and Eq. (6.10). It turns out that the use of Eq. (6.9) renders the kinetic term of Ψ vanishing. Though it is apparant for B and Φ, it is not obvious that Ψ is nondynamical. To get a rough description of this, let us consider the following portion of the full scalar-mode action in Eq. (6.49) and in Eq. (6.50),

$$\begin{aligned}
S^{(2)}_{Scalar} \propto \int d^3k dt\Bigg[&-\frac{6|\dot{\tilde{\Psi}}|^2}{N^2} - \frac{2k^4\tilde{B}\dot{\tilde{E}}^*}{3aN} + \dot{\tilde{\Psi}}\left(\frac{12H\tilde{\Phi}^*}{N} - \frac{4k^2\tilde{B}^*}{aN}\right) + 3k^2H^2|\tilde{B}|^2 \\
&- 9H^2|\tilde{\Phi}|^2 + \frac{4k^2H\tilde{B}^*\tilde{\Phi}}{a} + k^2\left(\frac{p_g}{M_P^2} - \frac{m_g^2n^2F_{,\bar{X}}}{N^2\bar{X}(1+r)}\right)|\tilde{B}|^2 \\
&+ \frac{\rho_g}{M_P^2}|\tilde{\Phi}|^2 + \dots\Bigg],
\end{aligned} \tag{6.51}$$

where the ellipsis denotes the remaining terms. The reason for picking these terms is that such terms are the only ones that can contribute to the kinetic terms of $\tilde{\Psi}$. The action in

Eq. (6.51) can be simplified by the following field redefinition,

$$\bar{\Phi} \equiv H\tilde{\Phi} - \frac{k^2}{3a}\tilde{B}. \tag{6.52}$$

By the use of this redefinition, we can express Eq. (6.51) in terms of $\bar{\Phi}$ as

$$\begin{aligned} S^{(2)}_{Scalar} \propto \int d^3k dt \Bigg[&- 6\left|\frac{\dot{\tilde{\Psi}}}{N} - \bar{\Phi}\right|^2 - \frac{2k^4\tilde{B}\dot{\tilde{E}}^*}{3aN} - 3\left(1 - \frac{\rho_g}{3M_P^2H^2}\right)\left|\bar{\Phi} + \frac{k^2}{3a}\tilde{B}\right|^2 \\ &+ k^2\left(3H^2 + \frac{p_g}{M_P^2} - \frac{m_g^2 n^2 F_{,\bar{X}}}{N^2\bar{X}(1+r)} + \frac{2k^4}{3a^2}\right)|\tilde{B}|^2 + \ldots\Bigg]. \end{aligned} \tag{6.53}$$

The equation of motion of $\bar{\Phi}$ accordingly reads

$$2\left(\frac{\dot{\tilde{\Psi}}}{N} - \bar{\Phi}\right) - \left(1 - \frac{\rho_g}{3M_P^2H^2}\right)\left(\bar{\Phi} + \frac{k^2}{3a}\tilde{B}\right) = 0. \tag{6.54}$$

By using Eq. (6.9), it is obvious that the equation of motion in Eq. (6.54) yields a condition $\frac{\dot{\tilde{\Psi}}}{N} = \bar{\Phi}$. Together with Eq. (6.9), this condition implies readily the disappearance of the kinetic term of $\tilde{\Psi}$. As a consequence, $\tilde{\Psi}$ can be found in terms of other fields via its equation of motion. Finally, there are only two scalars as dynamical variables in the resulting action which reads

$$S^{(2)}_{Scalar} = \int d^3k dt \left(K_{IJ}\dot{\chi}^I\dot{\chi}^J + M_{IJ}\dot{\chi}^I\chi^J + P_{IJ}\chi^I\chi^J\right), \tag{6.55}$$

where $\chi^I = \left(\tilde{\delta\phi}, \tilde{E}\right)$. The 2×2 matrices K_{IJ}, M_{IJ}, and P_{IJ} represent the coefficients corresponding to each combination of the remaining scalars; one for the DBI scalar and another one for the scalar degree of freedom of the graviton.

Though the absences of their kinetic terms are mathematically clear, the vanishing scalar degrees of freedom, namely $\tilde{B}$, $\tilde{\Phi}$, and $\tilde{\Psi}$, deserve more explanation. Among those scalars, $\tilde{\Psi}$ is the only one rendered out of the picture by the background equation in Eq. (6.9), while $\tilde{B}$ and $\tilde{\Phi}$ cease to propagate at the lagrangian level due to the generic construction of the theory. Actually, the scalar $\tilde{\Psi}$ corresponds to the BD ghost, a common threat to the nonlinear massive gravity, which is removed by a specific structure of dRGT massive gravity, or, in this case, the background equations which set the kinetic term of $\tilde{\Psi}$ to vanish.

For the self-accelerating branch, the kinetic matrix K can be expressed as follows,

$$K_{IJ}^{(s)} = \frac{3M_p^2 m_g^2 a^3 \left(G + \alpha_\Lambda \bar{X}^3\right)}{2n^3}, \quad K_{12}^{(s)} = K_{21}^{(s)} = 0, \quad \det(K_{IJ}^{(s)}) = 0. \tag{6.56}$$

Due to the vanishing determinant and off-diagonal elements, the only propagating degree of freedom can be readily implied to be the DBI scalar. Since the DBI scalar is a field added in the theory of massive gravity, this degree of freedom does not belong to the massive gravity sector. Thus, together with the result from the vector mode, a massive graviton propagates with only two tensor degrees of freedom for the self-accelerating branch which coincides with previous studies on the dRGT massive gravity with FLRW reference metric [17]. Interestingly, an introduction of a DBI scalar does not effectively contribute to the dRGT theory when the self-accelerating branch is considered.

Due to the complexity of the calculation, we limit our study on the normal branch only to the special case of the self-accelerating class which satisfies $G + \alpha_\Lambda \bar{X}^3 = 0$ and $r = 1$. The kinetic matrix K in the action then reads

$$\begin{aligned}
K_{11}^{(n)} &= -\frac{9M_p^2 m_g^2 a \lambda^2 V^2 F_{,\bar{X}}^2}{NH^2 W}, \\
K_{12}^{(n)} = K_{21}^{(n)} &= \frac{3M_p^2 k^4 \lambda V F_{,\bar{X}}^2}{NH^2 UW}, \\
K_{22}^{(n)} &= M_p^2 \left(\frac{k^4 a^3 F_{,\bar{X}}}{4NU} - \frac{k^8 F_{,\bar{X}}^2}{m_g^2 a N H^2 U^2 W}\right), \\
\det(K_{IJ}^{(n)}) &= -\frac{9M_p^4 m_g^2 k^4 a^4 \lambda^2 V^2 F_{,\bar{X}}^3}{4N^2 H^2 UW},
\end{aligned} \tag{6.57}$$

where we have used the following definitions,

$$\begin{aligned}
U &= \left(3F_{,\bar{X}} - \frac{4k^2}{m_g^2 a^2 \bar{X}}\right), \qquad V = \left(1 + \frac{a^2 H^2}{\lambda^2 f}\right), \\
W &= \left(12A + 2F_{,\bar{X}}\left(\frac{2k^4}{m_g^2 a^4 H^2 U} + 9\bar{X}\right) + \frac{3m_g^2 \bar{X}^2 F_{,\bar{X}}^2}{H^2}\right).
\end{aligned} \tag{6.58}$$

For an arbitrary choice of parameters, it can be seen readily that the scalar degrees of freedom are at work here, namely $\tilde{E}$ which belongs to a graviton and $\tilde{\delta\phi}$ corresponding to the DBI scalar. On the ghost issue, it is quite difficult to deduce a full picture from this cumbersome matrix. Thus, we consider qualitatively only its small scale limit which corresponds to a high k limit. Under such a limit, we can write leading contributions of the quantities in Eq. (6.58) as

$$U \sim -k^2, \qquad V \sim \mathcal{O}(1), \qquad W \sim -k^2 F_{,\bar{X}}. \tag{6.59}$$

Hence, the leading terms of Eq. (6.57) read

$$\begin{aligned}
K_{11}^{(n)} &\sim \frac{F_{,\bar{X}}}{k^2}, \\
K_{12}^{(n)} &= K_{21}^{(n)} \sim F_{,\bar{X}}^2, \\
K_{22}^{(n)} &\sim 0, \\
\det(K_{IJ}^{(n)}) &\sim -F_{,\bar{X}}^2.
\end{aligned} \tag{6.60}$$

Note that the off-diagonal elements of $K_{IJ}^{(n)}$ do not vanish. In order to investigate whether or not the scalar perturbations are ghost, one way is to diagonalize this kinetic matrix and examine the signs in front of the diagonalized kinetic terms directly. Equivalently, we can just check the sign of one of the diagonal elements ($K_{11}^{(n)}$ or $K_{22}^{(n)}$) and also that of the determinant of the matrix if they are positive or not. Obviously, no matter how the sign of $F_{,\bar{X}}$ is, the determinant is always negative, in a small scale. This implies the existence of a ghost instability at the small scale among those two scalar degrees of freedom; one of them is always a ghost. Note that this results do not conclude the failure of the normal branch; only the self-accelerating class satisfying $G + \alpha_\Lambda \bar{X}^3 = 0$ is plagued by the ghost and thus considered as an unphysical solution. The result may change if other classes of normal branch are carefully investigated as we did in this work. Such a complex consideration is left as a further work.

6.3 Conclusions

After quite immense analyses, we have seen various cosmological behaviors of the dRGT theory extended by introducing a DBI scalar. With hopes for fixing the vanishing degrees of freedom and the ghost instability issues, the DBI scalar is introduced into the theory via a concept of the braneworld cosmology. The scalar is originally corresponds to a moving brane in five-dimensional reference spacetime of the Schwarzschild-anti-de Sitter form given in Eq. (6.1). To imply such a higher-dimensional quantity, a corresponding four-dimensional reference metric is introduced via an induction from the five-dimensional one as in Eq. (6.5). This induction introduces the DBI scalar in four dimensions which respect the galileon shift symmetry. Moreover, the dynamics of the DBI scalar is also governed by an extension which is a leading contribution of a general action obeying a galileon shift symmetry [47, 48, 49]. Interestingly, background equations obtained from the model are similar to those in the original dRGT theory; the DBI scalar only plays its role implicitly in the Friedmann and acceleration equations. Similar to the original one, this model can provide a self-accelerating branch of solution, characterized by $\bar{X}_{\pm}$, in which an effective cosmological constant arises naturally from the graviton mass. Remarkably, the self-accelerating branch obtained here admits any kind of spatial geometry which differs from that obtained from the original dRGT theory whose the reference metric is Minkowskian. The other branch of solution, known as a normal branch, is also an another possible solution of the model. Surprisingly, a self-accelerating class of solution, where $\bar{X}$ is a constant given by $\chi_{\pm}$ and $r = 1$, is found existing in the normal branch which also behaves in a cosmological constant manner. This self-acceleration class, however, does not share the same characteristics as in the former self-accelerating branch. The crucial difference is the availability of $\bar{X}$ for each branch. In particular, $\bar{X}$ in the former self-accelerating branch is fixed to be only $\bar{X}_{\pm}$ which are solutions to $F_{,\bar{X}} = 0$ while the latter self-accelerating class in normal branch admits any arbitrary value of $\bar{X}$ as long as $r = 1$ is satisfied. For simplicity, particular conditions $G + \alpha_\Lambda \bar{X}^3 = 0$ and $\alpha_4 - 3\alpha_\Lambda = 0$ are imposed on the self-accelerating class of normal branch to limit the possibility of $\bar{X}$ to only $\bar{X} = \chi_{\pm}$. This is quite a fascinating result since from the former study on the dRGT massive gravity, the former self-accelerating branch suffers from having an incompatible number of degrees of freedom while the normal branch tends to work better on such an issue [17]. This result can enable the normal branch to match the well-known ΛCDM model which is currently the best-fit model, no matter how mysterious and ambiguous it may be.

To satisfy the purposes of modifying the theory of massive gravity, the proposed action is expanded in terms of perturbative variables, which are defined via metric decomposition in Eq. (6.2), up to second order. Each of the degrees of freedom are considered separately according to their tensorial properties under spatial rotation, namely transforming as a tensor, a vector, or a scalar. An analysis on tensor mode of perturbation, described by a transverse-traceless tensor h_{ij}^{TT}, reveals two ghost-free tensor degrees of freedom, for both branches, which match exactly the polarizations of the gravitational wave in general relativity. In particular, there are two (ghost-free) degrees of freedom in this mode and are governed by an equation of motion similarly to the gravitational wave equation. The graviton mass contributes as the mass M_{GW} of the wave given in Eq. (6.39). Since M_{GW}^2 does not have a positive definite form, only a class of parameter setting is valid so that the tensor degrees of freedom do not give rise to a superluminal excitation. To avoid the superluminality, the action requires $M_{GW}^2 > 0$, otherwise the solution will acquire an imaginary mass and propagate with speed faster than light. By graphic representations, the possible sets of parameters are shown in Figure. (6.1) for the self-accelerating branch and in Figure. (6.2) for the normal branch, where the conditions on the positive energy density and positive $\bar{X}$ are taken into account for both branches. In Figure. (6.1), it is found that $\bar{X}_-$ solution allows more parameter setups than $\bar{X}_+$ solution does. On the other hand, the possible sets of parameter setup in the special case of normal branch, satisfying $G + \alpha_\Lambda \bar{X}^3 = 0$, which give rise to a self-accelerating evolution, characterized by $\chi_\pm$, are shown in Figure. (6.2). Such possibilities turn out to depend on which branch of $\chi_\pm$ is used to characterize the self acceleration. To be exact, it is impossible to find a parameter setting by which the χ_+ solution works. Moreover, the previous study suggests that in the case of normal branch, there is a bound $M_{GW}^2 > 2H^2$ which also restrict a parameter setting, known as the Higuchi bound. Such a bound is additionally included in the plot in Figure. (6.2) as well.

After focusing on the tensor mode, we examine a vector mode of perturbation. A corresponding action for vector mode turns out to govern only one vector perturbation, namely E_i^T which, being a transverse vector, has two degrees of freedom. Unlike the tensor mode, different branches imply different outcomes in the vector mode. For the self-accelerating branch, the condition $F_{,\bar{X}} = 0$ renders the vector degree of freedom nondynamical, indicating the absence of the vector degrees of freedom. In particular, it is $F_{,\bar{X}}$ that lies in front of the kinetic term of the vector perturbation (see Eq. (6.48)). Obviously, for $F_{,\bar{X}} = 0$ the kinetic term vanishes. However, we need $F_{,\bar{X}} < 0$ in the case of normal branch in order to avoid having ghostly degrees of freedom in the model.

Finally, the scalar mode action is carefully investigated. The action turns out to govern only two scalar perturbations, namely $\delta\phi$, a perturbation on the DBI scalar, and E, a scalar mode of graviton. In particular, B and Φ are nondynamical at the lagrangian level while Ψ, the BD ghost, is killed by the well-designed structure of the dRGT theory. For the self-accelerating branch, the scalar mode of massive graviton is rendered nondynamical due to the condition $F_{,\bar{X}} = 0$ of this branch, which, together with the vanishing vector degrees of freedom, makes the model a massive gravity theory with only two tensor degrees of freedom for its cosmological solution. Oppositely, five degrees of freedom, as there should be, are in active for the normal branch. Due to the complexity, we choose a particular class of the normal branch which represents a solution with self-accelerating expansion as in Eq. (6.27). We have shown the existence of a ghost instability residing in the sector of scalar degrees of freedom at small scale. Depending on the parameter setting, either the graviton scalar mode or the DBI scalar exhibits ghostly behaviors at the small scale. Since the ghost appears at the small scale, the model is naturally unphysical or one may consider applying this model with a particular cutoff k_{UV} at which the ghost does not appear. Also note that these results only valid for the choice of parameters satisfying Eq. (6.27). Other classes, though involves complicated calculations, may provide more physical and consistent solutions which is left as a further work.

CHAPTER VII
MASS-VARYING MASSIVE GRAVITY

We have seen an extended model of dRGT massive gravity in which an additional degree of freedom is introduced in the previous chapter. Unfortunately, the model does not succeed in fixing the remaining problems of the vanishing degrees of freedom and the existence of the ghost instability. Generally, this does not mean that adding a new degree of freedom is a dead end for the massive gravity. For example, in an extended quasi-dilaton massive gravity, all of five degrees of freedom are healthy and dynamical for the cosmological solution [28, 29]. Another remarkable example is motivated from the absence of the cosmological solution to the original dRGT massive gravity. Due to the study in Ref. [19], the original theory cannot provide a nontrivial cosmological solution given that the reference metric is Minkowskian. The authors in Ref. [19] had pointed out that such results may change if the graviton mass happens to depend on some other fields so that it no longer is a constant. This led to a generalization of massive gravity, known as a mass-varying massive gravity [30, 31, 32, 33], in which the graviton mass is promoted to a function governed by an additional scalar field. The only drawback of this model is that its self-accelerating branch of solution evolves toward a situation in which the varying graviton mass shrinks and eventually vanishes [30, 31, 32]. This means, if being applied to the cosmological solution, at last the self-accelerating expansion would be driven due to other factors but the massive graviton. Such an expansion is meaningless for the massive gravity since it does not serve as an agent of the cosmic expansion at late time. So far the mass-varying massive gravity is not successful in late-time cosmology. Here we present a new model of the mass-varying massive gravity where not only the scalar field itself but also its kinetic term govern the graviton mass function [33]. The scalar field is governed by a k-essence lagrangian, a generalization promoting the canonical kinetic term to an arbitrary function of it [51, 52, 53]. The proposed model turns out to be able to provide the cosmic acceleration characteristics at late time. Moreover, the model does give rise to the cosmological-constant-like quantity as well as an effective content capable of serving as a dark matter which is also another contribution from the graviton mass. This is a crucial point of this model since the contents in the dark sector naturally come out from only the massive gravity sector via the dependence of the graviton mass function to the kinetic term of the scalar. In other words, this model may be a possible approach to unify the dark contents together even though the k-essence lagrangian is absent.

Due to such a dark-matter-like content, it is tempting to study the model on the issue of the cosmic coincidence problem, a well-known situation of this recent universe in which the amounts of dark energy and dark matter (also with the ordinary matter) are of the same order of magnitude; the ratio of the dark energy to the matters is roughly $7 : 3$ [50]. This issue happens in various models of modified gravity. In particular, many dark energy models fail to predict such distribution; many models usually give a solution in which there is only one dominated content. Since the model does provide the ``dust-like" matter as well as the massive graviton description of the dark energy as we will see in the following sections, it is of great interest to investigate whether the model can or cannot be cleansed from the cosmic coincidence problem.

7.1 The Model and the Background Equations

In this section, we are going to investigate the proposed model which involves both a scalar field and its kinetic term in the process of promoting the graviton mass to a function. The action for the model mentioned earlier is just a combination of the dRGT massive gravity action in Eq. (5.1) but with the graviton mass function V and the lagrangian of a k-essence scalar ϕ as follows,

$$S = \int d^4x\sqrt{-g}\left[\frac{M_p^2}{2}R(g) + V(X,\phi)U(g,f) + P(X,\phi)\right], \tag{7.1}$$

where g and f represent physical and reference metrics respectively as usual, the function $P(X,\chi)$ is a lagrangian governing dynamics of the k-essence scalar ϕ which is a function of ϕ and its kinetic term $X \equiv -\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi$. The graviton mass function V also has two arguments; the field ϕ and the kinetic term X. For a special case where $V(X, \phi) = \text{const} = M_p^2 m_g^2$, the massive gravity sector in Eq. (7.1) reduces to the original dRGT action. Note that in the case that the X-dependence in the graviton mass function is removed and $P(X, \phi)$ is unity, the model recovers the very first model of mass-varying massive gravity [30, 31, 32] where the scalar propagates in the quintessence framework. As usual, $U(g, f)$ is a combination of interaction terms between the physical and the reference metric as defined in Eq. (5.2). In order to obtain the corresponding cosmological implications, we consider the physical metric g of the FLRW form as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -N(t)^2 dt^2 + a(t)^2 \Omega_{ij}(x) dx^i dx^j, \tag{7.2}$$

where the spatial metric Ω is previously defined in Eq. (3.22) and N characterizes the reparameterization of the time coordinate. To obtain a solution for an arbitrary spatial curvature (like the case in Chapter V), we choose to consider the four-dimensional reference metric of the same form as the physical one,

$$ds_f^2 = \tilde{f}_{\mu\nu} d\varphi^\mu d\varphi^\nu = -n(\varphi^0)^2 \left(d\varphi^0\right)^2 + \alpha(\varphi^0)^2 \Omega_{ij}(\varphi) d\varphi^i d\varphi^j, \tag{7.3}$$

where φ^μ is a coordinate in the reference spacetime, $n(\varphi^0)$ is a lapse function in the reference sector and $\alpha(\varphi^0)$ is a corresponding scale factor. The reference metric is then covariantized in the physical spacetime via the following Stückelburg trick,

$$f_{\mu\nu} = \partial_\mu \varphi^\rho \partial_\nu \varphi^\sigma \tilde{f}_{\rho\sigma}, \tag{7.4}$$

where φ^μ's are treated as four scalars in physical spacetime introduced to restore general covariance. Assuming unitary gauge; $\varphi^\mu = x^\mu$, plugging Eq. (7.2) and Eq. (7.4) into Eq. (7.1) gives us the following mini-superspace action,

$$\begin{aligned} S = &\int d^4x \sqrt{\Omega} \\ &N a^3 \left[3M_p^2 \left(-\left(\frac{\dot{a}}{aN}\right)^2 + \frac{\kappa}{a^2} \right) + 3V\left(F(\bar{X}) - G(\bar{X}) \frac{n}{N} \right) + P \right], \end{aligned} \tag{7.5}$$

where $F(\bar{X})$ and $G(\bar{X})$ are given as in Eq. (5.8) and in Eq. (5.9). Under the FLRW ansatz, the kinetic term becomes simply $X = \frac{\dot{\phi}^2}{2N^2}$. The Friedmann equation can be found by varying the action in Eq. (7.5) with respect to the lapse function N as follows,

$$3M_p^2\left(H^2+\frac{\kappa}{a^2}\right) = -3VF + 6XV_{,X}\left(F-G\eta\right) + \left(2XP_{,X}-P\right), \tag{7.6}$$

where we have define a new variable $\eta \equiv n/N$. Note that the terms with derivatives with respect to X appear since X also involves the physical metric, or N to be exact. By comparing to the previous massive gravity models in Chapter V and Chapter VI, this Friedmann equation coincides with ones in those models if the graviton mass function is assumed to be a constant. Note that there are terms involving the derivative of V which we will see later that these terms, with a particular assumption, behave like a dust. By varying the action in Eq. (7.5) with respect to a, one obtains the following acceleration equation,

$$M_p^2\left(\frac{2\dot{H}}{N}+3H^2+\frac{\kappa}{a^2}\right) = -3VF + VF_{,\bar{X}}\left(\bar{X}-\eta\right) - P. \tag{7.7}$$

This acceleration equation is obviously reduced to the acceleration equation in Eq. (5.11) or in Eq. (6.10) under the assumption $V = m_g^2$. In addition, these background equations, namely Eq. (7.6) and Eq. (7.7), together reduce exactly to the background equations in the k-essence model when $m_g \to 0$ [51, 52, 53]. Since now V also determines the dynamics of ϕ (through the existence of X), the equation of motion of ϕ also involves V as follows,

$$Na^3\left(3V_{,\phi}\left(F-G\eta\right)+P_{,\phi}\right) = \frac{d}{dt}\left[\left(a^3\sqrt{2X}\right)\left(3V_{,X}\left(F-G\eta\right)+P_{,X}\right)\right]. \tag{7.8}$$

Now we have three equations of motion corresponding to N, a, ϕ, we need one more equation of motion which comes from the Stückelburg sector. First, we consider the conservation of energy-momentum tensor $\nabla_\mu T^{\mu\nu} = 0$. The explicit expression of this conservation equation can be obtain via Eq. (7.6) and Eq. (7.7). By plugging Eq. (7.6) and its time derivative into Eq. (7.7), we obtain a relation

$$3HN\left[-2XP_{,X}-6XV_{,X}(F-G\eta)+VF_{,\bar{X}}\left(\bar{X}-\eta\right)\right] \\ =\frac{d}{dt}\left[-3VF+\left(2XP_{,X}+6XV_{,X}\left(F-G\eta\right)\right)-P\right]. \tag{7.9}$$

In order to have the conservation of the energy-momentum tensor, this relation must be satisfied. Through a simple calculation, it can be shown that when the graviton mass function V is absent, this conservation equation is exactly the equation of motion of the k-essence scalar. In the k-essence cosmology, an equation of motion of a k-essence scalar determines a flow of the k-essence field in a consistent way by which the energy and momentum in the system are conserved [51, 52, 53]. However, in this case an additional relation, which is exactly the Stückelburg equation of motion, is needed to ensure the conservation in Eq. (7.9). By combining Eq. (7.8) and Eq. (7.9), such a relation is then expressed as follows,

$$\frac{\dot{V}}{V}=NH\left(1-h\bar{X}\right)\frac{F_{,\bar{X}}}{G}, \tag{7.10}$$

where $h\equiv H_\alpha/H$ and H_α is the Hubble parameter in the reference sector, i.e. $H_\alpha=\frac{\dot{\alpha}}{\alpha n}$. As claimed in the previous chapters, this Stückelburg equation of motion can be computed via an alternative and perturbative approach presented in Appendix B. Note the relation in Eq. (7.10) that it is exactly the condition in the dRGT massive gravity in Eq. (5.16) when the graviton mass is constant, in other words $\dot{V}=0$. It is important to note the dependences between these background equations. These five background equations, namely Eq. (7.6), Eq. (7.7), Eq. (7.8), Eq. (7.9), and Eq. (7.10) are not mutually independent; obviously Eq. (7.10) is deduced from Eq. (7.8) and Eq. (7.9). In this sense we can use only four of them to cover all of the dynamics of the system. Though we only need to take four equations into account, the remaining equations are still complicated thanks to the graviton mass function and the k-essence sector. To simplify them, we choose the k-essence lagrangian to be only a function of X as follows,

$$P(X,\phi)=P(X), \tag{7.11}$$

and, as we have promoted V to be also a function of X and ϕ, we let V mimic the above expression as

$$V(X,\phi) = V(X), \tag{7.12}$$

Note that now the dependence on the scalar ϕ is removed from both P and V through the above assumptions. The equation of motion for ϕ in Eq. (7.8) accordingly reads

$$\frac{d}{dt}\left[\left(\frac{a^3}{\sqrt{2X}}\right)\left(6XV_{,X}\left(F - G\eta\right) + 2XP_{,X}\right)\right] = 0. \tag{7.13}$$

This equation can be treated in two separated ways; one is to directly perform a differentiation of the equation and another one is to integrate this equation. The direct differentiation of Eq. (7.13) yields

$$\frac{d}{dt}\rho_X + 3HN\rho_X = \frac{\dot{X}}{2X}\rho_X, \tag{7.14}$$

where we have define an effective energy density

$$\rho_X \equiv 2XP_{,X} + 6XV_{,X}\left(F - G\eta\right). \tag{7.15}$$

These terms appear in the right-hand side of the Friedmann equation (7.6) which is why such terms together are reasonably defined as an effective energy density. Moreover, upon defining ρ_X, Eq. (7.14) has the form of the equation of a nonconservative fluid whose energy density is ρ_X and pressure p_X is zero. The right-hand side of Eq. (7.14) denotes the nonconservation by which the corresponding matter flows accordingly to other sectors of matter. Since the corresponding pressure vanishes, Eq. (7.14) suggests that the matter of energy density ρ_X should behave like a dust. On the other hand, integrating Eq. (7.13) gives us the following relation,

$$\rho_X = \frac{\sqrt{2X}C}{a^3}, \tag{7.16}$$

where C is an integration constant. In the case that $X = \text{const}$, this relation ensures the properties of a pressureless dust that reside in the matter of energy density ρ_X. In particular, ρ_X dilutes as its volume grows when $X = \text{const}$ which is exactly the same as what the dust behaves. In addition, such a case renders Eq. (7.14) to be a fluid equation of a conserved dust.

This analyses of ρ_X provide important results. First, through the assumption of the ϕ-independent form of $V(X, \phi)$ it is possible for a massive graviton to contributes itself as a ``dust-like" matter as well as the effective dark energy. This encourages us to treat this kind of contribution as an effective dark matter. By this treatment, one can also consider the dark matter as a contribution from graviton mass which, in term of their common origins, could possibly unify the dark sector, consisting of dark matter and dark energy, together through only one quantity, the graviton mass. Because up to now we did not introduce any other kinds of matter into the system, the nonconservative flow in Eq. (7.14) suggests that such a flow should convey the matter of ρ_X to the effective dark energy sector, indicating that there should exists an interaction between them in the dark sector. This claim coincides with the key idea of interacting dark matter/dark energy models which are also widely studied [54, 55]. Furthermore, taking the early state of the universe into account, the existence of the nonconservative matter of ρ_X, thanks to the right-hand side of Eq. (7.14) interpreted as an interaction term, could possibly be consistent with an evolution of the universe in which there exists an epoch where a dust dominated as well as there exists one where the dust almost vanished. This leads us to further investigate this model whether or not it can provide a full timeline of the universe in which each matter content changes through time and end up as we see today where the dark energy is comparable to the dark matter. In other words, in the following sections we are going to find an explanation on the cosmic coincidence problem in the framework of this mass-varying massive gravity.

7.2 Dark Energy Solution

Though our target is to verify the cosmic coincidence problem in this model, the corresponding calculations are not so simple due to the complexities of the background equations. This would involves dynamical analyses of those background equations where the dynamics of each matter content is considered and thus the fixed points of the system, the stationary points through which the system evolves, are obtained. Before we consider such an aspect of calculation which will be in the very last section, at this point we can see a rough picture of the dynamics of the system through the following simple calculations. Since now we are living in an epoch believed to be a dark energy domination causing the cosmic acceleration, we consider a solution of the model whose expansion is driven by an effective cosmological constant. By taking only the massive gravity sector into account, such a situation can be acquired by a simple relation; we consider a solution in which the corresponding equation of state parameter is -1. To adopt such an idea, we define

$$\rho_g \equiv -3VF + 6XV_{,X}\left(F - G\eta\right), \tag{7.17}$$

$$p_g \equiv 3VF - VF_{,\bar{X}}\left(\bar{X} - \eta\right). \tag{7.18}$$

Note that each of ρ_g and p_g covers all the relevant terms involving the graviton mass function V. In the original dRGT massive gravity, an effective cosmological constant comes from $\rho = -3VF$ and $p = p_g$ by requiring the vanishing $F_{,\bar{X}} = 0$ so that the equation of state parameter $w = \frac{p}{\rho} = -1$ is readily satisfied. In this case we shall also adopt this kind of an equation of state parameter. First, we recall that the corresponding equation of state parameter is as follows,

$$w_g \equiv \frac{\rho_g}{p_g}. \tag{7.19}$$

By imposing the condition $w_g = -1$, we have to fulfil this following condition,

$$6XV_{,X}\left(F - G\eta\right) = VF_{,\bar{X}}\left(\bar{X} - \eta\right). \tag{7.20}$$

We can see that this condition implies the constant graviton mass in the case of the self-accelerating solution $F_{,\bar{X}}$ in the original dRGT massive gravity in Chapter V. Since in this condition the partial derivative of X is involved, we can readily integrate Eq. (7.20) with respect to X to obtain V of a power form as

$$V(X) = V_0 X^{\lambda/2}, \qquad \lambda = \frac{F_{,\bar{X}}\left(\bar{X} - \eta\right)}{3\left(F - G\eta\right)}, \tag{7.21}$$

where the divisor factor 2 is defined for conveniences. Surprisingly, his form of V coincides with a particular class of k-essence which behaves similar to a perfect fluid; such a class is governed by a lagrangian of the form $P(X) \propto X^{(1+w)/2w}$ where w is the corresponding equation of state parameter [56, 57]. Note that in general, λ can be a function which varies as time goes by. In particular, V_0 can also be a function of time in general. However, such time dependence will make Eq. (7.10) involves more cumbersome calculation. Thus, here we choose the case where V_0 is constant. The result in Eq. (7.21) is interesting since the dark energy solution to the model tends to fix the form of the graviton mass function to be of the perfect-fluid form like in the k-essence model but with even more generalized exponent. Particularly, the k-essence model possessing perfect-fluid properties corresponds to a lagrangian $P(X)$ of the power form with a constant exponent, which in this framework corresponds to its equation of state parameter. Thus, it is also tempting to choose P to be of the power form which will be in consideration in the section of dynamical analyses. To go further, we consider for simplicity dynamics satisfying the following assumption,

$$\bar{X} = \text{const}, \qquad \eta = \text{const}. \tag{7.22}$$

Through simple manipulations, these assumptions lead to the following relation,

$$h = \frac{1}{\eta}. \tag{7.23}$$

Moreover, the assumptions also render λ constant. Under these assumptions, with the value of λ in Eq. (7.21), Eq. (7.10) implies

$$\begin{aligned} \frac{\lambda \dot{X}}{2X} &= -NH\left(1-\frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G}, \\ \frac{\dot{X}}{X} &= -\frac{6\left(F-G\eta\right)}{G\eta}\frac{\dot{a}}{a}. \end{aligned} \tag{7.24}$$

Due to $\bar{X}$, η and hence F, G being constant, integrating this relation yields the following expression of X in term of the scale factor a,

$$X = C_0 a^{-\frac{6(F-G\eta)}{G\eta}}, \tag{7.25}$$

where C_0 is an integration constant. Thus, the graviton mass function V can be expressed also in term of the scale factor as follows,

$$V = V_0 X^{-\frac{\left(1-\frac{\bar{X}}{\eta}\right)\eta F_{,\bar{X}}}{6(F-G\eta)}} = V_0 C_0 a^{\left(1-\frac{\bar{X}}{\eta}\right)\frac{F_{,\bar{X}}}{G}}. \tag{7.26}$$

In addition, we can find, in principle, how the k-essence scalar evolves according to the scale factor through Eq. (7.25) for this dark energy solution as follows,

$$X = \frac{\dot{\phi}^2}{2N^2} = C_0 a^{-\frac{6(F-G\eta)}{G\eta}}. \tag{7.27}$$

The dependence of V to the scale factor in Eq. (7.26) can tell us how graviton mass behaves at late time as a scale factor grows while the equation of state parameter of an effective matter induced by the graviton mass stays at -1. In the previous model of mass-varying massive gravity, the graviton mass function unfortunately shrinks at late time due to its dependence on the inverse of (polynomials of) scale factor while the situation is quite different for the model presented here. The fate of the graviton mass depends on the exponent of the scale factor presented in Eq. (7.26). In particular, if the exponent is negative, then we have the case arising in the original mass-varying massive gravity previously mentioned. On the other hand, the positive exponent would render the graviton mass to be indefinitely large. Interestingly, the self-accelerating branch of the original dRGT massive gravity, satisfying $F_{,\bar{X}} = 0$, is recovered here in the case of vanishing exponent by which the graviton mass remain constant. This case may correspond to an evolution which is governed by the dRGT theory at late time while by the mass-varying massive gravity at an earlier time. Note that these differences between the present model and the previous mass-varying massive gravity originate from different choices of the reference metric. The original model is formulated on the Minkowskian reference metric or other kinds of constant isotropic and homogeneous metric, in contrast to the (non-constant) FLRW reference metric used in this model. In the context of a massive gravity, these differences on the results are not so surprising since in various situations it has been demonstrated that the model is sensitive to different choices of the reference metric. For example, massive gravity with Minkowskian reference metric does not admit a self-accelerating solution with arbitrary spatial geometry [19, 20] while it is possible to have the self-accelerating behavior for any kind of spatial curvature in the theory formulating on the FLRW reference metric [17, 27]. Another example is a black hole solution in massive gravity where a specific form of a reference metric is required [40, 41, 42, 43].

There is one more crucial remark on these analyses of the dark energy solution. The condition for the dark energy solution is that we need the equation of state parameter of the contributions from the graviton mass to be -1 no matter how ρ_g or p_g are. In the previous section we have seen that some of the contributions behave similarly to what dust does. This means that we can have a system whose an effective equation of state parameter is the same as that of the dark energy while among those matters there exist ``dust-like" contents introduced by the graviton mass function, namely the term $6XV_{,X}\left(F - G\eta\right)$. Such a term belongs to ρ_X which has a characteristic of an interacting dust governed by Eq. (7.14). This may be a way out for the cosmic coincidence problem since the equation of state parameter for the dark energy is observed to be closed to -1 while there is a considerable amount of dark matter whose origin is still unknown. Through this model, we may be able to explain the dark matter as a contribution from the varying graviton mass governed by the k-essence field. This leads us to the next section where we perform dynamical analyses on this model to verify those previously mentioned ideas and to investigate if the model solves the cosmic coincidence or not.

7.3 Dynamical Analyses

As mentioned in the previous section, this section is devoted for dynamical analyses of the cosmological solution to this model. The dynamical analyses are useful for a complex system whose analytic solutions may not be easy to obtain. The dynamical analysis does not actually solve the problem. Rather, it allows us to get some information of the solution without dealing with complicated procedures of differential equations. The key idea of this approach is to rewrite system of equations to be of the first derivative form. In other words, given a set of variables $x = (x^1, x^2, \dots, x^k)$ parameterized by a variable $\tilde{N}$ (usually being a time coordinate), we expect the equation of the following form,

$$\frac{dx^i}{d\tilde{N}} = f_i(x^1, x^2, \dots, x^k). \tag{7.28}$$

This set of equations is commonly known as autonomous equations. At times, though they are in first derivative form, these equations may not be obviously solvable. Rather, we look for a ``fixed point'' on which the system in consideration is stationary. For example, given a system sitting ``right at a fixed point'', without any presence of external influence the system will stay at the point forever. Mathematically, the statement corresponds to a configuration $x^i = x^i_0$ which makes all of f_i's in Eq. (7.28) vanished;

$$f_i(x^1_0, x^2_0, \dots, x^k_0) = 0. \tag{7.29}$$

Consequently, such a configuration ensures the vanishing derivatives on the left side of Eq. (7.28) and then all of x^i's cease to evolve away from the point. Note that the configuration of the system sitting right at the fixed point does not ensure a stability of the point. In other words, if there is any influence, no matter how tiny it is, altering the configuration slightly away from the fixed point, the system can either readjust itself back to the fixed point or evolve away from the point and never return. The former happens around a stable fixed point, also known as an attractor, and the latter is the case for an unstable fixed point. To determine whether a fixed point is stable or not, we may investigate it through a perturbative method. Suppose that we consider perturbations

$\delta x = (\delta x^1, \delta x^2, \ldots, \delta x^k)$ around a fixed point $x_0 = (x_0^1, x_0^2, \ldots, x_0^k)$. Then we expand Eq. (7.28) according to the defined perturbations. By keeping only the first order of perturbation, we obtain an equation for each perturbation as

$$\frac{d\delta x^i}{d\tilde{N}} = \left.\frac{\partial f_i}{\partial x^j}\right|_{x_0} \delta x^j. \tag{7.30}$$

This equation tells how each of δx^i's changes when the configuration at x_0 is perturbed by δx. For a special case where $f_i = f_i(x^i)$ (no summation is taken), it is obvious that if $\left(\frac{\partial f_i}{\partial x^i}\right)$ (no summation is taken) is negative, the change of δx^i will be in opposite direction to δx^i; particularly $\frac{d\delta x^i}{dN} < 0$ for $\delta x^i > 0$ and vise versa. Such an opposite change denotes that x_0 is a stable fixed point. On the other hand, if $\left(\frac{\partial f_i}{\partial x^i}\right)$ (no summation is taken) is positive, the fixed point is unstable; the rate of change of δx will be proportional to δx which only causes the system to evolve away exponentially from the fixed point. Generally, this is not the case for f_i can be a function of some or all of x^i's. In that case Eq. (7.30) becomes a matrix equation as follows,

$$\frac{d\delta \mathbf{x}}{d\tilde{N}} = \mathbf{J}(f_i(x_0))\delta \mathbf{x}, \tag{7.31}$$

where $\delta\mathbf{x}$ is a column matrix whose elements are $x^1, x^2, \ldots, x^k$, $\mathbf{J}(f_i(x_0))$ is a Jacobian matrix of the collection of function $f_i(x)$ evaluated at $x = x_0$. For this matrix equation, the stability may not be obvious to determine as in the simple case we have previously discussed. However, we can diagonalize $\mathbf{J}(f_i(x_0))$ so that we can simply determine the stability by only looking at the sign of the diagonal elements, provided that the matrix is not singular. This is equivalent to finding eigenvalues of the matrix $\mathbf{J}(f_i(x_0))$ and then investigating their signs. The eigenvalues together actually form a diagonalized representation of the original matrix $\mathbf{J}(f_i(x_0))$ as follows,

$$J(f_i(x_0)) \Rightarrow \text{diag}\left(\mu_1, \mu_2, \ldots, \mu_k\right). \tag{7.32}$$

The corresponding fixed point is said to be stable if all of the eigenvalues are negative while it is unstable if all of the eigenvalues are positive, otherwise the fixed point is a saddle point. This will become useful in the following analyses since we expect to obtain a stable fixed point which matches with the current status of our universe in terms of the matter contents. In other words, the desired fixed point must solve the cosmic coincidence problem and be stable during the late-time acceleration.

For the next portion of this section, we consider dynamical analyses of the mass-varying massive gravity model. Since $V(X)$ tends to be a function of the power of X in the dark energy solution considered previously, we choose to consider a k-essence field governed by the lagrangian of the same form. Such a k-essence field actually behaves as a perfect fluid in cosmology with a constant equation of state parameter [56, 57]. The lagrangian governing the k-essence field is as follows,

$$P(X) = P_0 X^{\frac{1+w}{2w}} = P_0 X^{\gamma/2}, \qquad \gamma \equiv \frac{2XP_{,X}}{P} = \frac{1+w}{w}, \tag{7.33}$$

where P_0 is a constant, and w is thus the corresponding equation of state parameter. As suggested in the dark energy solution, we also choose $V(X)$ to be of the following power form,

$$V(X) = V_0 X^{\lambda/2}, \qquad \lambda \equiv \frac{2XV_{,X}}{V}, \tag{7.34}$$

provided that the exponent λ is a constant. We will see in the very last part of this section where λ is needed to change during the evolution of our desired solution. For this moment, let us consider it as a constant for simplicity.

Through their dynamics equations, it is possible to find a fixed point which determines a stationary configuration of each of the matter content in the system. Before going any further, let us rewrite the Friedmann equation in Eq. (7.6), given that the spatial curvature is flat and $\bar{X}$ and η are constant over time (thus implying $h = 1/\eta$), as follows,

$$\begin{aligned} 1 &= \frac{-3VF}{3M_p^2H^2} + \frac{2XP_{,X} + 6XV_{,X}\left(F - G\eta\right)}{3M_p^2H^2} + \frac{-P}{3M_p^2H^2}, \\ &= x + y + z, \end{aligned} \tag{7.35}$$

where

$$x \equiv -\frac{VF}{M_p^2 H^2}, \tag{7.36}$$

$$y \equiv \frac{2XP_{,X} + 6XV_{,X}\left(F - G\eta\right)}{3M_p^2 H^2} = \frac{\rho_X}{3M_p^2 H^2}, \tag{7.37}$$

$$z \equiv -\frac{P}{3M_p^2 H^2}. \tag{7.38}$$

In this context, the energy densities on the right-hand side of the Friedmann equation are all redefined to be some kind of ``normalized" densities, commonly know as density parameters in the standard cosmology. Particularly, these kind of densities are summed up to be unity as in Eq. (7.35), which then can be considered as a constraint equation. These quantities, namely x, y, and z, are the ones we want to find their configurations corresponding to a fixed point in the framework of dynamical analysis. For conveniences, let us define a new time parameterization $\tilde{N}$ as

$$\tilde{N} \equiv \ln a. \tag{7.39}$$

In the standard cosmology, $\tilde{N}$ is known as an e-folding. Moreover, $\tilde{N}$ is related to the time coordinate t by the following relation,

$$d\tilde{N} = \frac{\dot{a}}{a}dt = HNdt. \tag{7.40}$$

Consequently, a derivative with respect to $\tilde{N}$ to a function, says A, is the following,

$$A' \equiv \frac{\dot{A}}{HN}. \tag{7.41}$$

In terms of the e-folding $\tilde{N}$, Eq. (7.10) shows us an evolution equation for X as follows,

$$\frac{X'}{X} = \frac{\dot{X}}{HNX} = \frac{2}{\lambda}\frac{F_{,\bar{X}}}{G}\left(1 - h\bar{X}\right) = -\frac{6s}{\lambda\tilde{r}}, \tag{7.42}$$

given that

$$\tilde{r} \equiv \frac{G\eta}{F}, \qquad s \equiv \frac{F_{,\bar{X}}\left(\bar{X}-\eta\right)}{3F}. \tag{7.43}$$

Not only the dynamics of X but we will also find dynamics of other density parameters x, y, and z parameterized by the e-folding $\tilde{N}$. Before going into those procedures, it is useful to rewrite the acceleration equation in Eq. (7.7) in terms of the previously defined parameters by dividing Eq. (7.7) with $3M_p^2H^2$ as follows,

$$\begin{aligned} \frac{1}{3M_p^2H^2}M_p^2\left(\frac{2\dot{H}}{N}+3H^2+\frac{\kappa}{a^2}\right) &= \frac{-3VF+VF_{,\bar{X}}\left(\bar{X}-\eta\right)-P}{3M_p^2H^2}, \\ \frac{H'}{H}=\frac{\dot{H}}{H^2N} &= \frac{3}{2}\left(-xs-y\right), \end{aligned} \tag{7.44}$$

where in the last step the constraint in Eq. (7.35) is used for the simplification. Then we compute x' for the dynamics of x which is the following autonomous equation,

$$x' = \frac{\dot{x}}{HN} = -\frac{1}{HN}\left(\frac{\dot{V}F}{M_p^2H^2}-\frac{VF}{M_p^2H^2}\frac{2\dot{H}}{H}\right). \tag{7.45}$$

By the use of Eq. (7.42) and Eq. (7.44), this equation reads

$$x' = 3x\left(y+sx-\frac{s}{\tilde{r}}\right). \tag{7.46}$$

This equation is the desired equation for the dynamical analyses for it is an equation with a first derivative of x. Through the same procedures, we can obtain an autonomous equation for y as follows,

$$y' = \frac{\dot{x}}{HN} = \frac{1}{HN}\left(\frac{\dot{\rho}_X}{3M_p^2H^2}-\frac{\rho_X}{3M_p^2H^2}\frac{2\dot{H}}{H}\right). \tag{7.47}$$

Since Eq. (7.14) determines the flow of ρ_X, we use Eq. (7.14) to simplify y' equation. The autonomous equation for y eventually reads

$$y' = 3y\left(y+sx-1-\frac{s}{\lambda\tilde{r}}\right). \tag{7.48}$$

Even though an equation for the dynamics of z can also be found, it is not necessary to find the equation since finding the equation would only give us an equation which is not mutually independent to the other equations since the z' equation can be obtained readily through x', y', and the constraint in Eq. (7.35). In other words, once x and y are specified, we can obtain z simply from the constraint equation in Eq. (7.35). However, since we have invoked an assumption of the perfect-fluid form of $P(X)$ and $V(X)$ in Eq. (7.33) and Eq. (7.34), the assumption has placed another kind of constraint amoung x, y, and z. Such constraint arises from the definition of ρ_X which reads

$$\begin{aligned}\rho_X &\equiv 2XP_{,X} + 6XV_{,X}\left(F - G\eta\right), \\ &= \gamma P + 3\lambda VF\left(1 - \tilde{r}\right). \end{aligned} \tag{7.49}$$

In terms of the density parameters, this constraint reads

$$y = -\gamma z - \lambda x\left(1 - \tilde{r}\right). \tag{7.50}$$

This constraint actually relates γ to λ when x, y, and z obtained from the previous equations are specified. Note that since we have assumed that $\bar{X}$ and η are constant, the assumption also renders $\tilde{r}$ and s constant. For more conveniences, we can find an effective equation of state parameter corresponding to the total energy density and the total pressure in the system. The effective equation of state parameter can be found via the total pressure in the right-hand side of Eq. (7.7) as follows,

$$\begin{aligned}w_{eff} &= \frac{p_{tot}}{\rho_{tot}}, \\ &= \frac{P + 3VF - VF_{,\bar{X}}\left(\bar{X} - \eta\right)}{3M_p^2H^2} = -z - x + xs, \\ &= -1 + y + xs. \end{aligned} \tag{7.51}$$

Through this effective equation of state parameter, we can determine what kind of matter dominates during a period of our consideration by just evaluating it with the configuration x and y at the period.

Now we are ready to find a fixed point of the cosmological solution of the model. As previously suggested, we set x' and y' to vanish and evaluate all of the fixed points, the possible values of x and y that satisfy the vanishing x' and y'. The fixed points are all listed in Table 7.1 along with their configurations and conditions for their stabilities of which the details will be investigated and discussed later. Here we first discuss on characteristics of each of the fixed points in the following sections.

Name	x	y	z	w_{eff}	existence	stability
(a)	0	0	1	-1	$\gamma = 0$	$\frac{s}{\tilde{r}} \geq 0$
(b)	$\frac{1}{\tilde{r}}$	0	$1 - \frac{1}{\tilde{r}}$	$-1 + \frac{s}{\tilde{r}}$	$\gamma = \lambda$	$\frac{\lambda}{1-\lambda} \leq \frac{s}{\tilde{r}} < 0$
(c)	0	$1 + \frac{s}{\lambda\tilde{r}}$	$-\frac{s}{\lambda\tilde{r}}$	$\frac{s}{\lambda\tilde{r}}$	$\gamma = 1 + \frac{\lambda\tilde{r}}{s}$	$\frac{1}{\lambda-1} < \frac{s}{\lambda\tilde{r}} < -1$
(d)	$\frac{1}{1+\lambda(\tilde{r}-1)}$	$\frac{\lambda(\tilde{r}-1)}{1+\lambda(\tilde{r}-1)}$	0	$\frac{1}{\lambda-1}$	$\lambda = \frac{s}{s-\tilde{r}}$	$0 < \lambda < 1$
(e)	$\frac{1+(\lambda-1)z_0}{1+\lambda(\tilde{r}-1)}$	$-\frac{\lambda(1-\tilde{r}(z_0+1))}{1+\lambda(\tilde{r}-1)}$	z_0	$\frac{1}{\lambda-1}$	$\lambda = \gamma = \frac{s}{s-\tilde{r}}$	$0 < \lambda < 1$

Table 7.1 Summary of the properties of the fixed points [33].

7.3.1 Fixed Point (a)

This fixed point corresponds to $(x, y) = (0, 0)$, a configuration obviously satisfies the vanishing of x' and y'. Through the constraint equation in Eq. (7.35), it can be found that $z = 1$ and Eq. (7.50) implies $\gamma = 0$. This vanishing γ indicates that $P(X)$ is only a constant which represents a cosmological constant term in the action. Thus, since $x = 0$ represents a vanishing graviton mass function, this fixed point corresponds to a trivial model which is general relativity with a cosmological constant, or the ΛCDM model. In addition, the effective equation of state parameter corresponding to this fixed point can be found as follows,

$$w_{eff}^{(a)} = -1 + y + xs = -1. \tag{7.52}$$

This $w_{eff}^{(a)}$ ensures the characteristics of the ΛCDM model in which the cosmological constant drives the cosmic expansion [36].

Though being the trivial model, for the sake of completeness we find the corresponding stability of this point. We use the method discussed previously to determine the stability. The corresponding equations according to Eq. (7.30) are the following matrix equation,

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} 3y + 6xs - 3\frac{s}{\tilde{r}} & 3x \\ 3ys & 6y + 3xs - 3 - 3\frac{s}{\lambda\tilde{r}} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.53}$$

This linear stability equation is very important for determining whether or not the fixed point in consideration is stable. We will use this stability equation many times throughout this section of dynamical analyses. Upon substituting the configuration of the fixed point (a); $(x, y) = (0, 0)$, this matrix equation is simplified as follows,

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} -3\frac{s}{\tilde{r}} & 0 \\ 0 & -3 - 3\frac{s}{\lambda\tilde{r}} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.54}$$

Because of its diagonal form, the eigenvalues of the matrix can be read out readily as

$$\mu_1 = -3\frac{s}{\tilde{r}}, \qquad \mu_2 = -3 - 3\frac{s}{\lambda\tilde{r}}. \tag{7.55}$$

To have the fixed point (a) as a stable fixed point, we require negative values for all of the eigenvalues. Hence, for positive λ the stable fixed point is implied from a condition $\frac{s}{\tilde{r}} > 0$. Note that in the case that $\frac{s}{\tilde{r}} = 0$ one of the eigenvalues vanishes. The vanishing eigenvalue(s) indicates the failure of this method to predict the stability. Particularly, the stability analysis introduced previously is a linear approximation of the full analysis. The vanishing eigenvalue can only tell us that such a linear approximation is not enough to determine the stability and the procedures involving the higher-order contributions must then be taken into account. Alternatively, one may deal with such a situation through a numerical solution of the cosmic evolution corresponding to the system. Through such an approach, one may find that, for a particular class of parameter setup, the fixed point is stable or not.

Though it provides the cosmic expansion at late time, this fixed point is not much of interest to us because it is obviously ΛCDM or, in other words, the cosmic acceleration is driven by a cosmological constant Λ and not by graviton mass (since $x = 0$). This fixed point is actually the situation happening in the previous model of mass-varying massive gravity in which the graviton mass shrinks as the universe expands for its cosmological solution and eventually vanishes at late time.

7.3.2 Fixed Point (b)

This fixed point has its configuration being $(x, y) = (x_0, 0)$. The vanishing y indicates the absence of ρ_X which is the dust-like matter arising from both the massive gravity sector and the k-essence lagrangian. We may think of this point as a massive gravity dominated period since we can always take the limit by which the k-essence is absent. By setting $x' = 0$, we can easily find x_0 through Eq. (7.46) and z_0 through Eq. (7.35) which read

$$x_0 = \frac{1}{\tilde{r}}, \qquad z_0 = 1 - \frac{1}{\tilde{r}}. \tag{7.56}$$

Here, we can see that the massive gravity fully dominates when $\tilde{r} = \frac{G\eta}{F} = 1$ which corresponds to $x_0 = 1$ and $z_0 = 0$. Moreover, we can find a relation between λ and γ of this fixed point via Eq. (7.50) to be $\lambda = \gamma$. The equation of state parameter according to this fixed point thus reads

$$w_{eff}^{(b)} = -1 + \frac{s}{\tilde{r}} = -1 + \frac{F_{,\bar{X}}\left(\bar{X} - \eta\right)}{3G\eta}. \tag{7.57}$$

This form of equation of state parameter, interestingly, confirms the claim of this point being the massive gravity dominated period since $w_{eff}^{(b)} = -1$ for $F_{,\bar{X}} = 0$ which is in agreement with the self-accelerating branch in the original dRGT massive gravity. Not only the self-accelerating branch, this fixed point also admits the normal branch existing in the original massive gravity where $\bar{X} - \eta = 0$ is satisfied (up to the assumption that $\bar{X}$ and η are constant, thus implying $h = 1/\eta$). Furthermore, depending on the model parameters, this point could have the effective equation of state parameter less than -1 which is suggested from observations that the equation of state parameter is actually slightly less than -1 [58].

In addition, both the k-essence and the massive graviton contribute themselves into the system in a similar manner for their exponents λ and γ are the same for this point and they behave as other kind of matter with nonzero pressure. This situation is quite obvious since the dust-like contributions vanish.

Regarding the stability of this fixed point, we use Eq. (7.53) with the configuration of the fixed point (b). The equation accordingly reads

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} 3\frac{s}{\tilde{r}} & \frac{3}{\tilde{r}} \\ 0 & -3+3\frac{s}{\tilde{r}}\frac{\lambda-1}{\lambda} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.58}$$

Hence, the eigenvalues of the above matrix are

$$\mu_1 = 3\frac{s}{\tilde{r}}, \qquad \mu_2 = -3+3\frac{s}{\tilde{r}}\frac{\lambda-1}{\lambda}. \tag{7.59}$$

By requiring both of the eigenvalues to be negative, we obtain readily a bound on $\frac{s}{\tilde{r}}$ as follows,

$$\frac{\lambda}{1-\lambda} < \frac{s}{\tilde{r}} < 0. \tag{7.60}$$

From the previous analyses, we can see that this fixed point is not possible to represent the recent configuration of the universe, which has a comparable amount of dark energy and dark matter, because of the absence of the dust-like sector.

7.3.3 Fixed Point (c)

As opposed to the fixed point (b), we can find a fixed point of configuration $(x, y) = (0, y_0)$. Since y_0 is the density parameter for ρ_X, the dust-like matter, this point may be able to represent a dust-dominated epoch. We can obtain the value of y_0 through Eq. (7.48) and z_0 via Eq. (7.35) as follows,

$$y_0 = 1 + \frac{s}{\lambda\tilde{r}}, \qquad z_0 = -\frac{s}{\lambda\tilde{r}}. \tag{7.61}$$

We can see that for $\frac{s}{\lambda\tilde{r}} \to 0$, y_0 dominates. Moreover, Eq. (7.50) gives an expression for γ as

$$\gamma = -\frac{y}{z} = -1 + \frac{1}{z}. \tag{7.62}$$

The effective equation of state parameter corresponding to this point reads

$$w_{eff}^{(c)} = \frac{s}{\lambda\tilde{r}} = -z, \tag{7.63}$$

which ensures that the fixed point corresponds to the dust-dominated epoch whose $w_{eff}^{(c)} \to 0$ once $\frac{s}{\lambda\tilde{r}} \to 0$ is satisfied. This existence of the dust domination is a great advantage to the model since in the course of the universe's evolution there exists a dust-dominated period for a significant duration.

As in the previous fixed points, we use Eq. (7.53) to determine the stability of the fixed point (d). Upon substituting the configuration $(x, y) = (0, y_0)$, the linear stability equation for this fixed point reads

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} 3 - 3\frac{s}{\tilde{r}}\frac{\lambda-1}{\lambda} & 0 \\ 3(1+\frac{s}{\lambda\tilde{r}})s & 3 + 3\frac{s}{\lambda\tilde{r}} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.64}$$

Thus, the corresponding eigenvalues read

$$\mu_1 = 3 - 3\frac{s}{\tilde{r}}\frac{\lambda - 1}{\lambda}, \qquad \mu_2 = 3 + 3\frac{s}{\lambda\tilde{r}}. \tag{7.65}$$

Requiring the negative eigenvalues poses a bound on $\frac{s}{\tilde{r}}$ as follows,

$$\frac{1}{\lambda - 1} < \frac{s}{\lambda\tilde{r}} < -1. \tag{7.66}$$

No matter how λ is, we always encounter the unstable fixed point if we require this fixed point to represent the dust domination; a dust-dominated period requires $w_{eff}^{(c)} = \frac{s}{\lambda\tilde{r}} = 0$ but such a condition does not reside in the stable region suggested in Eq. (7.66). Actually, this point being unstable is more useful and reasonable than the stable one. From the history of the universe, we know that there is a period mainly filled with dust (or dust-like matter) that happened at some earlier time and is not currently happening. This suggests that maybe the universe was on this kind of a fixed point at the earlier time but due to the unstability the universe adjusted its configuration away from the point and went onto some other fixed points (maybe a late-time kind of a fixed point which represents our present time). The idea may seem peculiar since such a situation possibly require some parameters to change in time in order to cover other fixed points in the timeline of the universe. We will see such a treatment in the very last section.

7.3.4 Fixed Point (d)

One more possible form of a fixed point is a nonvanishing configuration like $(x, y) = (x_0, y_0)$. Through Eq. (7.46), Eq. (7.48), and both of the constraint equations in Eq. (7.35) and Eq. (7.50), we can find the corresponding configuration to be

$$x_0 = \frac{1}{1 + \lambda\left(\tilde{r} - 1\right)}, \qquad y_0 = \frac{\lambda\left(\tilde{r} - 1\right)}{1 + \lambda\left(\tilde{r} - 1\right)}, \qquad z_0 = 0, \tag{7.67}$$

given that γ is arbitrary while $\lambda = \frac{s}{s - \tilde{r}}$. From the first glance, since there is a possibility that both of x_0 and y_0 may not be zero, this may correspond to our desire fixed point; a point which solve the cosmic coincidence problem where the amount of dark energy is of the same order as that of dark matter. As y representing the dust-like matter of energy density ρ_X, this point may be able to give a description on the dark matter at late time as well as be able to explain the cosmic coincidence problem in terms of the mass-varying massive gravity. To verify such an idea, as done previously, we compute the following effective equation of state parameter,

$$w_{eff}^{(d)} = \frac{1}{\lambda - 1}. \tag{7.68}$$

Though in the case that $\lambda = 0$ the equation of state parameter $w_{eff}^{(d)}$ is -1, such a case leads to a vanishing value of y and then obviously the dust-like matter does not exist in this manner. The stability of this fixed point can be found as usual from Eq. (7.53) as follows,

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} \frac{3\lambda\tilde{r}}{(\lambda-1)(1+\lambda(\tilde{r}-1))} & \frac{3}{1+\lambda(\tilde{r}-1)} \\ \frac{3\lambda^2\tilde{r}(\tilde{r}-1)}{(\lambda-1)(1+\lambda(\tilde{r}-1))} & \frac{3\lambda(\tilde{r}-1)}{1+\lambda(\tilde{r}-1)} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.69}$$

The above matrix is actually singular since its determinant vanishes. Since it is two dimansional, this means the matrix possesses one vanishing eigenvalue. Actually, this indicates a failure of the method of the linear stability. However, through simple manipulation, we can still get some information about the stability of this point. From Eq. (7.69), we can see that the $\delta x'$ equation only differs from the $\delta y'$ equation by a factor $\lambda\,(\tilde{r}-1)$. In this sense, the equation seems to have some kind of a ``degeneracy". The equation of two variables δx and δy can be reduced to an equation of only one variable; either δx or δy or even a function of them. Thus, if the stability on one variable is obtained, it implies readily the stability condition on another variable through the simple relation between them. Although it is not difficult to compute for the eigenvalues of the matrix, we will take another approach to obtain them. Since from Eq. (7.69) we have

$$\delta y' = \lambda\,(\tilde{r}-1)\,\delta x', \tag{7.70}$$

provided that λ and $\tilde{r}$ are constant, we can choose a relation according to the above equation as

$$\delta y = \lambda\,(\tilde{r}-1)\,\delta x. \tag{7.71}$$

Then the x' equation from Eq. (7.69) accordingly reads

$$\begin{aligned} \delta x' &= \frac{3\lambda\tilde{r}}{(\lambda-1)\,(1+\lambda\,(\tilde{r}-1))}\delta x + \frac{3}{1+\lambda\,(\tilde{r}-1)}\delta y, \\ &= \frac{3\lambda}{\lambda-1}\delta x. \end{aligned} \tag{7.72}$$

Here the change in δx is determined by δx alone. The stability is thus governed by the coefficient $\frac{3\lambda}{\lambda-1}$ which places a bound on λ as follows,

$$0 < \lambda < 1. \tag{7.73}$$

This also happen with the equation for $\delta y'$ which can be seen through using Eq. (7.70) and Eq. (7.71) on Eq. (7.72). Since this fixed point places a restriction on λ such that $\lambda = \frac{s}{s-\tilde{r}}$, the bound previously obtained also set another bound on s and $\tilde{r}$. Such a bound on λ corresponds to $w_{eff}^{(d)} \leq -1$. This is a benefit of the fixed point (d) because it can provide an explanation of the equation of state parameter being slightly less than -1 via the consideration of the stability of such a configuration. In particular, to have a stable fixed point of type (d), the universe must have $w_{eff}^{(d)} < -1$ according to the bound $0 < \lambda < 1$. Moreover, having $w_{eff}^{(d)} < -1$ can make rooms for y to have a nonzero value since the exact equality $w_{eff}^{(d)} = -1$ can only satisfy $y = 0$ for this model and thus does not coincide with the observation indicating the nonzero dark matter [50] and the effective equation of state parameter slightly less than -1 [58]. One would want λ to be so close to zero, in other words $0 < \lambda \gg 1$, to have the solution more compatible to the observational data [58]. From all of these benefits, the fixed point (d) may be a promissing configuration of our universe at its present state.

The analyses previously done on the stability equation in Eq. (7.69) are actually equivalent to finding eigenvalues of the matrix straightforwardly. It is the coefficient $\frac{3\lambda}{\lambda-1}$ that is another eigenvalue for the matrix in Eq. (7.69) apart from zero. In addition, one can explicitly show that the eigenvalues of the matrix are 0 and $\frac{3\lambda}{\lambda-1}$ via the usual characteristic equation $\det(\mathbf{M} - \lambda\mathbf{I}) = 0$.

If this fixed point were to represent the late-time status of the universe, it would serve as a great representation since this point can provide an idea of the unification of the dark sector through the concept of a massive graviton. In this fixed point we have $z = 0$ indicating the absence of the k-essence sector. If we consider the fixed point (c) as a dust-dominated period, since it also has a vanishing z, we can say that this collection of fixed points is able to describe the course of the universe without introducing the k-essence lagrangian which, depending on its form, can serve as either an another source driving the late-time expansion or an another pressureless matter (this is true even when radiation is in presence). The independence of the k-essence sector merely points out that only the massive graviton alone can introduced the effective dark matter as well as the effective dark energy, hence implying the unification of the dark sector.

7.3.5 Fixed Point (e)

This point involves an additional condition among the parameters. One can simply impose a condition $\lambda = \gamma$ and look for nonzero x and y. Under such a consideration, we obtain a very similar result to that of the fixed point (d) as follows,

$$x = \frac{1 + z_0 (\lambda - 1)}{1 + \lambda (\tilde{r} - 1)}, \qquad y = -\frac{\lambda (1 - \tilde{r} (1 - z_0))}{1 + \lambda (\tilde{r} - 1)}, \qquad z = z_0, \tag{7.74}$$

where λ is fixed to be $\frac{s}{s-\tilde{r}}$ and so is γ while z_0 can be any arbitrary value which is less than unity. Moreover, the effective equation of state parameter is the same as that for the fixed point (d); namely

$$w^{(e)}_{eff} = \frac{1}{\lambda - 1}. \tag{7.75}$$

To find a corresponding stability condition, we consider Eq. (7.53) evaluated on the configuration of this fixed point,

$$\begin{pmatrix} \delta x' \\ \delta y' \end{pmatrix} = \begin{pmatrix} \frac{3\lambda\tilde{r}(1+z_0(\lambda-1))}{(\lambda-1)(1+\lambda(\tilde{r}-1))} & \frac{3(1+z_0(\lambda-1))}{1+\lambda(\tilde{r}-1)} \\ \frac{-3\lambda^2\tilde{r}(1-\tilde{r}(1-z_0))}{(\lambda-1)(1+\lambda(\tilde{r}-1))} & \frac{-3\lambda(1-\tilde{r}(1-z_0))}{1+\lambda(\tilde{r}-1)} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}. \tag{7.76}$$

Alternatively, Eq. (7.76) can be divided into two following equations,

$$\delta x' = \frac{3 (1 + z_0 (\lambda - 1))}{1 + \lambda (\tilde{r} - 1)} \left(\frac{\lambda\tilde{r}}{\lambda - 1} \delta x + y \right), \tag{7.77}$$

$$\delta y' = \frac{-3\lambda (1 - \tilde{r} (1 - z_0))}{1 + \lambda (\tilde{r} - 1)} \left(\frac{\lambda\tilde{r}}{\lambda - 1} \delta x + y \right). \tag{7.78}$$

Regardless of the factors in the front of them, these equation are similar to one another. This is the sign of the vanishing determinant like in the case of the fixed point (d). One can convince oneself by evaluating the determinant of the matrix in Eq. (7.76) to find that it is really zero. This means that these equations possess the same kind of degeneracy as those of the fixed point (d) and we can reduce the system of two equations into only one-dimensional problem like for the previous fixed point. As happened previously for the fixed point (d), $\delta x'$ and $\delta y'$ are related through the following relation,

$$\delta y' = -\lambda \frac{1-\tilde{r}\,(1-z_0)}{1+z_0\,(\lambda-1)}\delta x'. \tag{7.79}$$

This relation allows us to choose a simple relation among δx and δy related to the one between $\delta x'$ and $\delta y'$ as follows,

$$\delta y = -\lambda \frac{1+z_0\,(\lambda-1)}{1-\tilde{r}\,(1-z_0)}\delta x, \tag{7.80}$$

which then allows us to express the $\delta x'$ equation, for example, in terms of only one variable as

$$\delta x' = \frac{3\,(1+z_0\,(\lambda-1))}{1+\lambda\,(\tilde{r}-1)}\left(\frac{\lambda\tilde{r}}{\lambda-1}\delta x - \lambda\frac{1+z_0\,(\lambda-1)}{1-\tilde{r}\,(1-z_0)}\delta x\right), \tag{7.81}$$

$$= \frac{3\lambda}{\lambda-1}\delta x. \tag{7.82}$$

Through Eq. (7.79) and Eq. (7.80), we can obtain the same form of equation for $\delta y'$ which means that the stability condition of δy is the same as that of δx. The stability condition is also the same as for the fixed point (d) which is $0 < \lambda < 1$.

Though this point has quite a similar structure as the fixed point (d), there is an obvious difference among these two point. The difference is that the fixed point (e) is valid under the assumption $\lambda = \gamma$ only while the fixed point (d) has no such restriction; λ and γ can take any values and need not to be equal to one another. Due to the assumption, z must take a nonzero value while in the case that the assumption is removed it is possible to have a configuration independent from the k-essence sector like the fixed point (d). Unlike the fixed point (d), since the fixed point (e) can involve a nonzero z, corresponding to the k-essence sector, it may not be so reasonable to claim that the unification of the dark sector via massive gravity still holds in this case.

Through the dynamical analyses, we have seen the possibility to describe each significant period in the timeline of the universe in terms of each fixed point of the FLRW solution to the proposed model. In particular, we expect the fixed point (c) to represent the dust domination while the point (d) is expected to describe the late-time cosmic expansion. The only yet biggest drawback of such consideration is the variety of the values of λ corresponding to each period mentioned above. From the beginning, λ is defined as a constant characterizing the form of the graviton mass function. Since the results highly suggest the varying λ, we may relax the assumptions that are used in the previous analyses. An obvious choice of such a relaxation is to promote λ to be a function that varies in time. This way, with appropriate adjustments, would be able to cover all of the significant periods of the universe whose λ's are different. The next section is devoted to the verificaation of such an idea and analyses of such situation are also given to test our hypothesis previously posted.

7.4 Extended Analyses

As mentioned in the previous section, though the model provides a possibility for the unification of the dark sector, the model cannot give a full and self-consistent description of the entire course of the universe due to the incompatibility of the constant λ for each of the desire fixed points. To fix that, we consider a generalization where λ is promoted to a function which varies in time so that it can change accordingly and allow an evolution through fixed points of different λ's. This λ function is defined in such a particular way that the relation $\lambda = \frac{2XV_{,X}}{V}$ still holds. With such a definition, the equations of motion appear to be the same as those in the case of constant λ. For completeness, we include another form of matter which also exists in the real universe, radiation or fluid of photons. Photon has a great deal in the cosmic evolution since in some earlier period there was an immense amount of photons in the universe, known as the radiation-dominated era.

Like the other form of matter, to include the radiation into our consideration, we have to know how radiation evolves in the system. Mathematically speaking, what form of equation is governing the dynamics of the radiation during the cosmic evolution.

As a perfect fluid, since a photon is a relativistic particle, it must has a nonzero pressure compared with the dust. It turns out that the pressure of the photon can be computed to be of one-third of its energy density or, expressed in terms of its equation of state, $p_\gamma = \frac{1}{3}\rho_\gamma$. The flow of the photon, on the FLRW geometry, can be determined through the following continuity equation,

$$\dot{\rho}_\gamma + 3HN\left(\rho_\gamma + p_\gamma\right) = \dot{\rho}_\gamma + 4HN\rho_\gamma = 0. \tag{7.83}$$

The evolution of radiation governed by Eq. (7.83) then affects the gravitation of the system and also the evolution of the scale factor a which can be determined through the following Friedmann equation and the acceleration equation, provided that the flat geometry is assumed,

$$3M_p^2H^2 = -3VF - P + \rho_X + \rho_\gamma, \tag{7.84}$$

$$\begin{aligned} M_p^2\left(\frac{2\dot{H}}{N} + 3H^2\right) &= -3VF - P + VF_{,\bar{X}}\left(\bar{X} - \eta\right) - p_\gamma, \\ &= -3VF - P + VF_{,\bar{X}}\left(\bar{X} - \eta\right) - \frac{1}{3}\rho_\gamma. \end{aligned} \tag{7.85}$$

Through these equations we can obtain a set of first-derivative autonomous equations as we obtained in the previous section. By defining the density parameter for the radiation as

$$\Omega_r \equiv \frac{\rho_\gamma}{3M_p^2H^2}, \tag{7.86}$$

we can obtain the Friedmann equation in terms of density parameters as follows,

$$1 = x + y + z + \Omega_r. \tag{7.87}$$

Moreover, the autonomous equations for x, y are expressed as follows,

$$x' = 3x\left(y + sx - \frac{s}{\tilde{r}} + \frac{4}{3}\Omega_r\right), \tag{7.88}$$

$$y' = 3y\left(y + sx - 1 - \frac{s}{\lambda\tilde{r}} + \frac{4}{3}\Omega_r\right). \tag{7.89}$$

These equations exactly reduce to the previous ones in the case of vanishing radiation; $\Omega_r = 0$. Note that λ here is now a function of time defined as

$$\lambda \equiv \lambda(t) = \frac{2XV_{,X}}{V}. \tag{7.90}$$

We will rather stick to this definition of λ than the perfect-fluid form in the previous section since such a form is not anymore valid for λ being a function. Moreover, this definition allows us to express the autonomous equations in the same way as we did in the previous section so that we can obtain the same set of fixed points, including another fixed point corresponding to the radiation-dominated epoch. Since now λ varies in time, we must also pay attention to its autonomous equation which can be found through the definition of λ above as

$$\lambda' = \frac{6s}{\tilde{r}}\left(\frac{\lambda}{2} - (1+\Gamma)\right), \tag{7.91}$$

$$\Gamma \equiv \frac{XV_{,XX}}{V_X}. \tag{7.92}$$

This autonomous equation for λ vanishes in the case of $\lambda = 2(\Gamma + 1)$ which is thus the fixed point for λ. As a special case, λ being constant in time corresponds to the system sitting at the fixed point $\lambda = 2(\Gamma + 1)$ where Γ is also constant. Regardless of the other contents, it is also possible to analyse the stability of λ alone, provided that Γ is a constant. Upon being expanded according to a perturbation $\delta\lambda$, the equation governing the stability is expressed as follows,

$$\delta\lambda' = \frac{3s}{\tilde{r}}\delta\lambda, \tag{7.93}$$

where Γ is treated to be constant from now on. The stability condition of λ can be obtained readily as $\frac{s}{\tilde{r}} < 0$. This means that we have to set the value of $\frac{s}{\tilde{r}}$ to be negative in order to have the stable fixed point or, in other words, to have an evolution toward this fixed point. Apart from that ofλ, since we have already included the radiation into the system, we can find its autonomous equation from the continuity equation of the photon and the definition of the density parameter of the radiation. Consequently from Eq. (7.83) and Eq. (7.86), the autonomous equation reads

$$\Omega_r' = 3\Omega_r \left(y + sx + \frac{4}{3} (\Omega_r - 1) \right). \tag{7.94}$$

From this equation, it is obvious that the configuration $x = 0$, $y = 0$, and $\Omega_r = 1$ is a fixed point which corresponds to the radiation-dominated period. This is the standard result in cosmology where there exists an epoch filled mainly with radiation. Due to introducing the radiation into the system, the effective equation of state parameter is modified accordingly as

$$w_{eff} = -1 + y + xs + \frac{4}{3}\Omega_r. \tag{7.95}$$

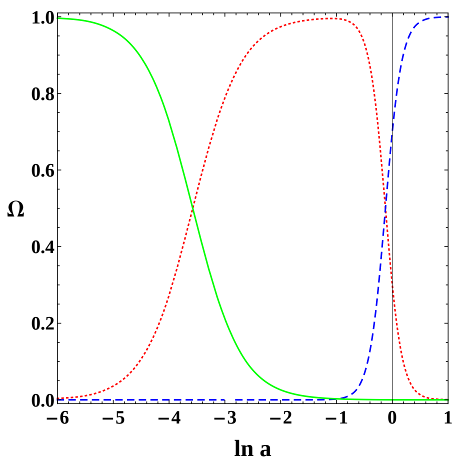
In addition, we still have the following relation between x, y, and z due to the definition of y in Eq. (7.50),

$$y = -\gamma z - \lambda x (1 - \tilde{r}),$$

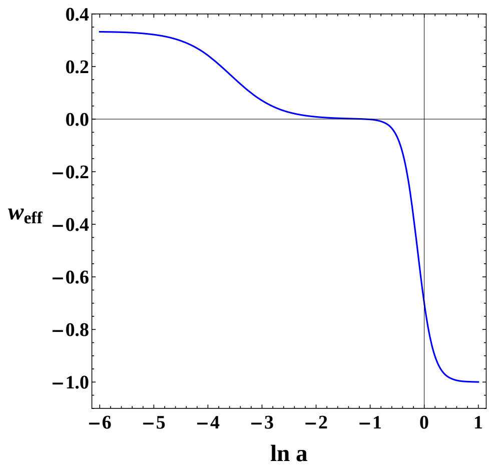
where γ is now treated as a function of time and is defined similarly to λ as $\gamma(t) \equiv \frac{2XP_{,X}}{P}$.

Up to this point, we can see that in the case $\Omega_r = 0$, the autonomous equations in this case recover the fixed points obtained in the previous section. In addition, as claimed previously, we also obtained the fixed point in which the system is filled mainly with radiation; namely the fixed point where $x = 0$, $y = 0$, and $\Omega_r = 1$. The difference from the λ-constant consideration is that the system is allowed to evolve through the various desire fixed points, such as those corresponding to the dust-dominated epoch, the late-time acceleration epoch, and even the radiation-dominated epoch. In order to investigate whether or not the evolution can be as we expected, we perform a numerical method by solving the autonomous equations numerically and then present the solution through graphic representations with various parameter settings.

In the previous section we have brought up the idea of unifying dark energy and dark matter via only a theory of massive gravity. In other words, we expect an evolution in which z vanishes for the entire course of the system while it can still give an accurate profile of the evolution. We first determine the fixed point which should represent the dust-dominated epoch. As we have seen above, the fixed point (c) is the most appropriate point since it yields the effective equation of state parameter of dust ($w_{eff} = 0$) and a dominant portion of the dust-like matter of ρ_X if we require $\frac{s}{\lambda\tilde{r}} = 0$. Moreover, z is zero under such situation. The corresponding evolution of x, y, and Ω_r is shown in Figure. (7.1) where there exist a period in which the radiation dominates, a period of mainly dust, and a period of dark energy domination successively, though it is obvious that such a situation does not alleviate the cosmic coincidence problem.



(a) Evolutions of x, y, and Ω_r.

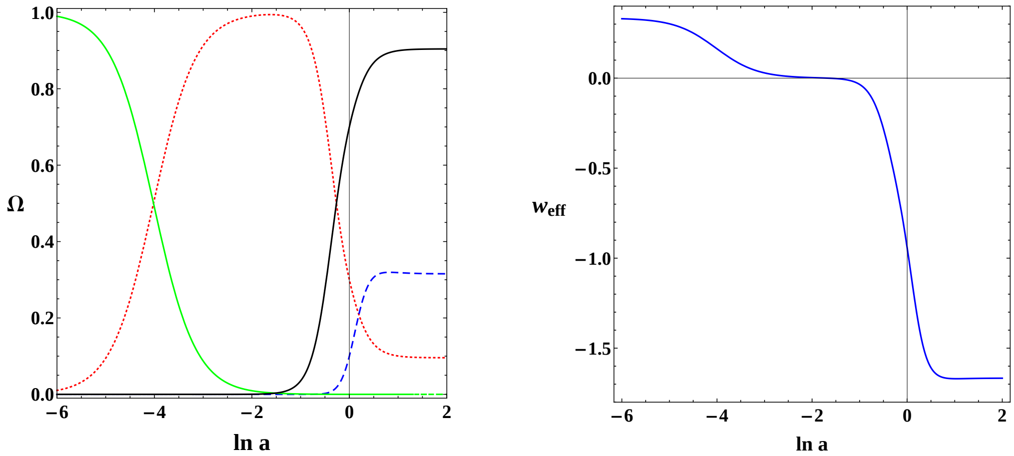


(b) w_{eff} during the evolution.

Figure 7.1 These figures show evolutions of various quantities in the case that the k-essence sector is absent. Figure. 7.1a shows the evolutions of matter contents, namely x, y, and Ω_r, parameterized by $\tilde{N} \equiv \ln a$ where the blue-dashed curve denotes that of x, the red-dotted curve indicates that of y, and the green-solid curve shows that of Ω_r. In this case, there exist a radiation-dominated epoch, a dust-dominated epoch, and a dark energy-dominated epoch. In Figure. 7.1b, the corresponding effective equation of state parameter w_{eff} during the evolution is shown which confirms the existence of the various epochs previously mentioned [33].

Now let us focus more on how to solve the cosmic coincidence problem. From the previous case, we have seen that maybe the fixed point with vanishing z may not solve the problem. Thus, we will consider the fixed point (e) for it is possible to have a nonzero z and has the same characteristics as the point (d). To determine the appropriate values of each of parameters; s, $\tilde{r}$, and Γ, we require that those parameters are assigned with specific values such that they agree with the initial condition of the dynamical variables. We also require that the initial conditions are slightly away from the exact configuration of the fixed point (e) in order to produce an entire evolution numerically (otherwise the system will sit at the fixed point indefinitely). Similarly to the point (d), the corresponding effective equation of state parameter is given by $w_{eff}^{(e)} = \frac{1}{\lambda-1}$. By requiring $w_{eff}^{(e)} = -1$ for the late-time universe, we obtain a condition on λ which is $\lambda_f \to 0$ at the point (e), given that λ_f is the value of λ at the fixed point. This condition then puts another condition on $\frac{s}{\tilde{r}}$; from $\frac{s}{\tilde{r}} = \frac{\lambda}{\lambda-1}$ we can see that the condition $\lambda_f \to 0$ fix $\frac{s}{\tilde{r}}$ to be slightly negative, in other words $\frac{s}{\tilde{r}} \to 0$ on the negative side. This condition of $\frac{s}{\tilde{r}}$ is in agreement with the stability condition for λ which requires $\frac{s}{\tilde{r}}$ to be negative. Moreover, the small value of λ is quite appropriate in the sense that according to such value λ does not drop too drastically as the system evolves and thus the dust-dominated period can exist for a considerable period of time. According to those setup, the evolution should be as we expected. However, since λ is large at earlier time and it decreases quite slowly, there is a big chance for the system to evolve towards the fixed point (b) rather than the point (e). This is because the stable region in the parameter space for the point (b) under such a circumstance is qualitatively bigger than that of (e). Moreover, λ must decrease all the way to the value of $\frac{s}{s-\tilde{r}}$ in order to satisfy the fixed point (e). The possible way to make this work is to have the effective equation of state parameter significantly less than -1 which results in a more drastic decrease in λ. Such a way can provide more chance for the system to approach the fixed point (e). This is shown in Figure. (7.2) where there exist a comparable amount of y --- the effective dark matter --- as well as $x + z$ corresponding to the effective dark energy, where $\lambda_f = 0.4$, leading to $w_{eff} = -1.67$ and the initial Λ is 1.0.

Another possibility is illustrated in Figure. (7.3) in which the assumption $\lambda = \gamma$ is satisfied for the entire evolution. Under this assumption, the description of the



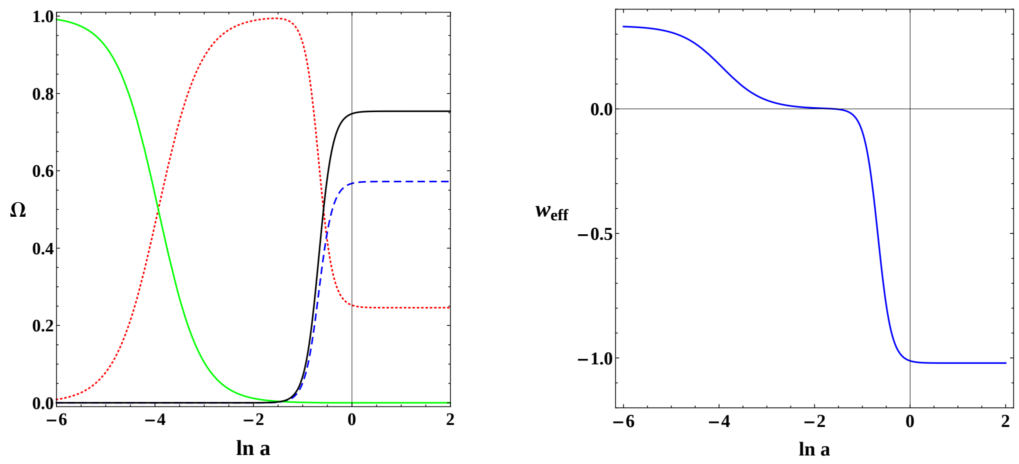
(a) Evolutions of x, y, $x+z$ and Ω_r. **(b)** w_{eff} during the evolutions.

Figure 7.2 These figures show a cosmic evolution which evolves towards the fixed point (e). Figure. 7.2a shows the evolutions of each matter content, namely x, y, $x+z$, and Ω_r, parameterized by $\tilde{N} \equiv \ln a$. The blue-dashed curve denotes the evolution of x, the red-dotted curve indicates that of y, the black-solid curve denotes that of $x+z$, and the green-solid curve shows that of Ω_r. In this case, there exist a radiation-dominated epoch, a dust-dominated epoch, and a dark energy-dominated epoch. There also is a considerable amount of dark matter y in the dark energy-dominated epoch. In Figure. 7.2b, the corresponding effective equation of state parameter w_{eff} during the evolution is shown which confirms the existence of the various epochs previously mentioned. Note that w_{eff} at late-time is significantly below -1 [33].

evolution is quite different from the previous situation. From Eq. (7.50), λ is governed by other variables through the following relation,

$$\lambda = \frac{y}{\tilde{r}x + y + \Omega_r - 1}. \tag{7.96}$$

Due to such a relation, λ is no longer a dynamical variable. Thus, $\frac{s}{\tilde{r}}$ does not have to satisfy the stability condition for λ as in the previous case. This means we can design the parameter setting so that the evolution does not go towards the fixed point (b). Such an evolution discussed above is given in Figure. (7.3) along with its effective equation of state parameter as a function of time. We can see from Figure. (7.3) that due to a particular setting of parameters, in this case $\lambda_f = 0.02$, the effective equation of state parameter



(a) Evolutions of x, y, and Ω_r. **(b)** w_{eff} during the evolutions.

Figure 7.3 These figures show a cosmic evolution which satisfies the condition $\lambda = \gamma$. Figure. 7.3a shows the evolutions of each matter content, namely x, y, $x + z$, and Ω_r, parameterized by $\tilde{N} \equiv \ln a$. The blue-dashed curve denotes the evolution of x, the red-dotted curve indicates that of y, the black-solid curve denotes that of $x + z$, and the green-solid curve shows that of Ω_r. In this case, there exist a radiation-dominated epoch, a dust-dominated epoch, and a dark energy-dominated epoch. There also is a considerable amount of dark matter y in the late-time dark energy-dominated epoch. In Figure. 7.3b, the corresponding effective equation of state parameter w_{eff} during the evolution is shown which confirms the existence of the various epochs previously mentioned. Note that w_{eff} at late-time is slightly below -1 [33].

can be designed to be slightly below -1 while at late time of the evolution there exist comparable amounts of the dark energy and the dark matter. Note that the evolution is sensitive to how the initial condition is given at the time of radiation-dominated era. Particularly, the resulting evolution changes drastically by only the slight change in the initial condition of, says, x by $\sim 10^{-16}$. This is known as the fine-tuning problem in which we have to fine-tune or adjust carefully the initial condition so that we can obtain the profile of our universe as we see today. Though this model succeeds in alleviating the cosmic coincidence problem by being able to provide the satisfactory cosmic evolution, the model suffers from the necessity of being fine-tuned in order to provide such an evolution.

7.5 Conclusion and Discussion

After a series of analyses, we have seen some particular features of a new class of mass-varying massive gravity. Apart from the previous model, this new class of mass-varying massive gravity is introduced by allowing the graviton mass to be a varying function which depends on an extra scalar field along with its kinetic term. In this work we consider such a dependence which is governed by a k-essence scalar and investigate its consequences in cosmological aspects. On the gravity sector, the FLRW ansatz is used as a physical metric governing the physical spacetime and another FLRW ansatz is considered as a reference metric in the interaction sector. To investigate effects of including the kinetic term of the scalar into the graviton mass function, we stick to a special case where the graviton mass function is a function of only the kinetic term X. Through standard procedures in cosmology, we obtained a corresponding Friedmann equation and an acceleration equation describing dynamics of the universe, along with the Stückelburg equation of motion and continuity equations of each matter content. As opposed to the previous model of the mass-varying massive gravity, this model prevents the graviton mass to shrink in time and vanish trivially when the universe approaches its late-time epoch, thanks to the dependence on the kinetic term of the scalar. We also found that among those matter contents there exists a particular contribution arising from both the graviton mass and the k-essence sector which shares some characteristics with dust. In addition to such properties, not only this contribution, denoted by an energy density ρ_X, has its equation of state parameter being zero, i.e. it is a pressureless matter, in the perfect fluid framework but also has a nontrivial interaction with other matters (see Eq. (7.14)). Because of its resemblance to the dust, it is tempting to consider a chance of such a contribution to be a promising candidate for the dark matter. Moreover, if the idea of ρ_X being the dark matter is embraced, the interaction to the other sector of ρ_X may denote the situation of the interacting dark matter/dark energy model which is widely studied as another cosmological model. Interestingly, ρ_X can even arise regardless of the presence of the k-essence lagrangian. Thus, due to the ability to provide effective dark energy of the graviton mass, the massive gravity can give rise to both of the dark quantities and hence it may be possible to unify these quantities in the dark sector together via only one concept, an existence of the massive graviton.

Since the model is able to provide a description for a dark matter, one way to assess how promissing the model is to be a dark matter candidate is to consider an evolution of the universe through a solution to the model. This leads to the last section where dynamics of the solution is carefully investigated through a method of dynamical analyses. Since the equations governing a system in consideration involve lots of differential equations. Even containing only first derivatives, it is quite difficult to solve the system of those differential equations analytically. To extract valuable information from them, the dynamical analyses focus on finding fixed points of those equations and investigate the fixed points in the cosmological aspects. From the analyses, we obtain five fixed points from the governing equations, some of which can be treated as particular periods existing in the timeline of the universe. Namely, the fixed point (a) is found to represent trivially the ΛCDM model where there is no contribution from the massive gravity. The fixed point (b) can correspond to the late-time acceleration where both the massive gravity and the k-essence scalar are presented but without the dust-like contribution. The fixed point (c) can represent a dust-dominated epoch since its equation of state parameter can be adjust to zero, the suitable value for the dust. The fixed point (d) and (e) are quite similar to one another such that both points can be designed to have their equation of state parameters equal to -1 where the dust-like contribution is also presented. The only difference is that the fixed point (d) is independent from the k-essence scalar which indicates the unification of the dark sector through only the massive graviton, the advantage of this model we mentioned previously. Though we can map each of the fixed points to various cosmic epochs, these points cannot exist in the same evolution because these points correspond to different constant values of λ's since in this aspect λ is treated as a constant throughout the evolution. This suggested us to promote λ to be a function of time so that we can cover all of the desire fixed points within one evolution. In the very last section, we have considered such a situation via numerical calculations of the autonumous equations of the system. By allowing λ to be a function, in the entire evolution according to the model there exists the fixed point (c) as a dust-dominated epoch. We then expect that the evolution would go from the fixed point (c) to the fixed point (d) which is our desire point that can represent the late-time acceleration accompanied with the unification of the dark sector. Unfortunately, from the stability analyses, such an evolution is unlikely to happen since due to the (dust-dominated) configuration of the point (c), the fixed point (b) is more stable than the point (d) and thus the system would more likely evolve to point (b), as illustrated in Figure. (7.1). We then move to the second possibility in which such a system evolves to the fixed point (e) as the late-time state. It is found that such an evolution can provide the late-time universe with comparable amounts of the dark energy and dark matter, i.e the cosmic coincidence problem can be alleviated, while the corresponding effective equation of state parameter is significantly less than -1, as illustrated in Figure. (7.2). This kind of evolution requires the condition $\lambda = \gamma$ at the time the system reaches the point (e). Moreover, if we require the condition $\lambda = \gamma$ for the entire evolution, we can construct another evolution which can also alleviate the cosmic coincidence problem, as illustrated in the Figure. (7.3). As an alternative choice, this evolution has its equation of state parameter close to -1 and thus can represent the late-time expansion. The only drawback is that the evolution is sensitive to initial conditions such that a slight change in one of the initial conditions could make the system at late-time changing drastically. We can say that this model can relieve the cosmic coincidence problem though it invokes another problem involving the fine-tuning of the initial condition, known as the fine-tune problem.

These whole analyses are under specific assumptions. One can exclude some of those assumptions to obtain a more complex system and consider such a system in a more general approach. Apart from the dynamical consideration, one can consider this model on an issue of theoretical consistencies. In particular, it is common for massive gravity to have a ghost instability, for example the BD ghost. The BD ghost can be lifted if the model of massive gravity contains a hamiltonian constraint due to the proper construction involving physical metric [13] (see also [39]). Since in this model the graviton mass is modified to be a function of the kinetic term of the scalar field, which makes it depends on the physical metric, then it is questionable whether or not there exists the BD ghost in this model. This issue is not mentioned here and is left for further work.

CHAPTER VIII
DISCUSSION AND REMAINING QUESTIONS

There has been a series of developments in massive gravity since 1939 [9] and the developments are still in progress. Although the nonlinear dRGT massive gravity has been proposed successfully [14, 15], the theory still faces a great threat in a theoretical point of view. One of the reasons why the massive gravity needs an improvement lies in its cosmological implications. When the cosmology is taken into account, dRGT massive gravity only propagates two degrees of freedom instead of five [17]. Moreover, the cosmological solution turns out to be unstable at a nonlinear level of perturbation [18, 23]. These unpleasant facts pose a threat to the theory of nonlinear massive gravity.

As in other studies, this work also seeks for a better model of massive gravity. We have been investigated possible extensions to the nonlinear dRGT massive gravity. Two aspects of extension have been investigated in this work, namely adding an extra degree of freedom to the dRGT theory and promoting the graviton mass from being a parameter to a function of some extra fields. The first kind of extension is carefully studied in the case of adding a DBI scalar field which possesses a Galileon symmetry [27]. Formulating on the five-dimensional Schwarzschild-anti-de Sitter reference metric, the proposed model admits the self-accelerating branch of solution, in which an effective cosmological constant is generated from graviton mass, as well as the normal branch whose condition also admit a self-accelerating behavior. Interestingly, the equations of motion which belong to the gravity sector are exactly in the same form as those in the pure dRGT theory which means that the DBI sector does not spoil the structure of the massivee gravity theory. In the perturbation analyses, the tensor degree of freedom is healthy in terms of its stability for both branches. On the other hand, the situations for the vector mode and scalar mode are different for each branch. For the self-accelerating branch, the vector degrees of freedom vanish according to the condition of the branch itself while for the normal branch a condition to avoid the ghost instability can be simply deduced. In the scalar mode perturbation, there is only the DBI scalar propagating for the self-accelerating branch whereas the scalar mode of the gravity sector ceases to be dynamical. Moreover, one of the scalar modes for the normal branch always exhibits a ghost instability. From these consequences, we found that the self-accelerating branch has two propagating degrees of freedom in the gravity sector which is exactly the case in the pure dRGT massive gravity, and hence the corresponding solution is unstable. Furthermore, even the normal branch has five propagating degrees of freedom in the gravity sector, it is either a gravitational scalar mode or a DBI scalar mode that exhibits the ghost instability. To sum up, we have ensured the inability for the DBI massive gravity to provide a stable and theoretically consistent cosmological solution.

The second kind of the extension to the dRGT massive gravity in this work is to treat the graviton mass to be a varying function, known as a mass-varying massive gravity [30, 31, 32, 33]. We propose a new class of the theory by allowing the graviton mass function V to depend on not only an extra scalar field but also the kinetic term of the scalar field itself [33], where the scalar field is governed by a k-essence action. By allowing the graviton mass function to depend on only the kinetic term of the scalar field, we found an extra contribution on the right-hand side of the Friedmann equation which has a potential of being an interacting dust. This dust is expected to be a candidate for the mysterious dark matter. If such an expectation were true, then the model could provide a way to unify the dark matter and the dark energy together via only one quantity, the varying graviton mass. We then investigate a possibility of this model to explain the well-known cosmic coincidence problem, a problem of coincidently comparable amounts of dark energy and dark matter. Thanks to the dynamical analyses, we can determine the possible fixed points of the system and can deduce possible courses of the cosmic evolution. From verifying such courses through numerical analyses, it is found that if one wants to embrace the unification of the dark sector by considering the case that the k-essence is absent, one may obtain a fully dark energy-dominated universe whereas the universe with a considerable amount of k-essence field allows the desire late-time phase of the universe where there exist both dark energy and dark matter. The latter case can, at least, alleviate the coincidence problem since it either predicts the effective equation of state parameter significantly less than -1 or it suffers from the fine-tuning problem, a problem of a necessity to finely tune initial condition to obtain the desire results.

As a further study, since the four dimensional theory of the DBI massive gravity cannot provide a theoretically consistent cosmological solution, it is possible to consider a full five-dimensional theory of the dRGT massive gravity or even that of the DBI massive gravity and then investigate consequences of their four-dimensional reductions. One may depart from the DBI massive gravity in order to consider adding extra degrees of freedom with other kinds of symmetry. Not only for the DBI massive gravity but there also is a couple of interesting questions left for the proposed model of the mass-varying massive gravity. An obvious one is to relax the assumptions which are put during the calculation. Such an approach may allow more possible and more complex evolutions which may be more suitable than those obtained here. Another interesting question is to investigate whether the BD ghost is reintroduced into the theory from the kinetic-dependence graviton mass function. Since the kinetic term always involves the physical metric, it is possible that such a term can spoil the ghost-free characteristics of the dRGT theory. Moreover, investigating the BD ghost issue may be able to constraint the form of the graviton mass function. These questions are left as future works.

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APPENDICES

APPENDIX A
DEFLECTION ANGLE

Gravitational lensing is one of the remarkable phenomena that exist in general relativity. When a massive object is presented, a photon that travels nearby is attracted by the gravitational field and its trajectory is deviated accordingly. In this section, a derivation of the deflection angle of the photon is given using a weak field approximation. By the weak field approximation, a fluctuation on the Minkowski metric is considered,

$$g_{\mu\nu} = \eta_{\mu\nu} + \delta g_{\mu\nu}, \tag{A.1}$$

where η represents the Minkowski metric and δg is the fluctuation. Given a spherical static source, δg in a transverse gauge is expressed as [34]

$$\delta g_{\mu\nu} = \begin{pmatrix} -2\phi & & & \\ & -2\psi & & \\ & & -2\psi & \\ & & & -2\psi \end{pmatrix}. \tag{A.2}$$

Although in Ref. [34] the transverse gauge also implies $\phi = \psi$, it is more useful to do the calculation without such condition, especially when modified gravities are taken into account. Suppose that there is a photon travelling in $+x$ direction with an impact parameter b to the source. The trajectory of the photon must satisfies the so-called geodesics equation

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = \frac{dx^\alpha}{d\lambda}\nabla_\alpha\frac{dx^\mu}{d\lambda} = 0, \tag{A.3}$$

where λ is an affine parameter which parameterizes the trajectory along the curve $x^\mu = (t, x, y, z)$. Usually, a proper time is used as the affine parameter when we consider a motion for a massive particle. However, we cannot use the proper time for the photon since it cannot be defined for the case. This explains why we have an arbitrary λ in Eq. (A.3). Since λ parameterizes the motion of the photon along a path x^μ, the derivative of x^μ with respect to λ should describe a ``velocity" of the photon, or rather its ``wave vector". Consequently, it is natural to define the wave vector of the photon as

$$k^\mu \equiv \frac{dx^\mu}{d\lambda}. \tag{A.4}$$

Thus, Eq. (A.3) can be rewritten as

$$\frac{dk^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} k^\alpha k^\beta = k^\alpha \nabla_\alpha k^\mu = 0. \tag{A.5}$$

In order to obtain a weak field approximation for the deflection angle, the wave vector k^μ which is a solution to Eq. (A.5) is decomposed perturbatively as

$$k^\mu = k^\mu_{(0)} + k^\mu_{(1)}, \tag{A.6}$$

where $k^\mu_{(0)}$ is the solution for the (Minkowskian) background while $k^\mu_{(1)}$ is a first order of perturbation which deviates from the background solution due to the presence of the fluctuation $\delta g_{\mu\nu}$. Keeping up to the first order, Eq. (A.5) reads

$$\begin{aligned} k^\alpha \nabla_\alpha k^\mu &= k^\alpha \left(\partial_\alpha k^\mu + \Gamma^\mu_{\alpha\beta} k^\beta\right), \\ &= k^\alpha_{(0)} \left(\partial_\alpha k^\mu_{(0)}\right) + k^\alpha_{(0)} \left(\partial_\alpha k^\mu_{(1)} + \Gamma^\mu_{\alpha\beta(1)} k^\beta_{(0)}\right) + k^\alpha_{(1)} \partial_\alpha k^\mu_{(0)} + \mathcal{O}(2), \end{aligned} \tag{A.7}$$

provided that the Christoffel symbol vanishes at zeroth order. Since a wave vector is also a null vector, which satisfies $k^\mu k_\mu = 0$, the wave vector for the photon travelling in $+x$ direction in the Minkowski space can be evaluated readily as

$$k^\mu_{(0)} = (k, k, 0, 0). \tag{A.8}$$

The proposed $k^{\mu}_{(0)}$ is a solution to the background (zeroth order) geodesics equation which simplify Eq. (A.7) to be of only first order,

$$k^{\alpha}_{(0)}\left(\partial_{\alpha}k^{\mu}_{(1)} + \Gamma^{\mu}_{\alpha\beta(1)}k^{\beta}_{(0)}\right) = 0, \tag{A.9}$$

where the second order and beyond are omitted. The nonzero Christoffel symbols compatible with Eq. (A.2) are

$$\Gamma^{0}_{0i(1)} = \Gamma^{i}_{00(1)} = \partial_i\phi, \tag{A.10}$$

$$\Gamma^{i}_{jk(1)} = -\left(\delta^{i}_{k}\partial_j\psi + \delta^{i}_{j}\partial_k\psi - \delta_{jk}\partial^{i}\psi\right), \tag{A.11}$$

where i, j, k denote the spatial indices $1, 2, 3$. Evaluating only the spatial components in Eq. (A.9) yields

$$\partial_1\vec{k}_{(1)} + k\vec{\nabla}\left(\phi + \psi\right) = 0, \tag{A.12}$$

where $\vec{k}_{(1)}$ is a spatial vector for $k^{\mu}_{(1)}$. From this, $\vec{k}_{(1)}$ at an arbitrary x can be computed, given that $\vec{k}_{(1)}$ vanishes at $-\infty$ and ∞, as

$$\Delta\vec{k}_{(1)} = -k\int_{-\infty}^{x}\vec{\nabla}\left(\phi + \psi\right)dx. \tag{A.13}$$

To compute the deflection angle of a photon travelling from x_i to x_f, we compute the deviation of the trajectory, i.e. $\vec{k}_{(1)}$, in the interval between x_i and x_f,

$$\Delta\vec{k}_{(1)} = \vec{k}_{(1)f} - \vec{k}_{(1)i}, \tag{A.14}$$

$$= -k\int_{x_i}^{x_f}\vec{\nabla}\left(\phi + \psi\right)dx, \tag{A.15}$$

and then we can compute the deflection angle of the photon, up to the first order, to be

$$\begin{aligned}\hat{\alpha}_{i\to f} &= \frac{\left|\Delta_{\perp}\vec{k}_{(1)}\right|}{\left|\vec{k}_0 + \Delta_{\parallel}\vec{k}_{(1)}\right|}, \\ &\approx \frac{\left|\Delta_{\perp}\vec{k}_{(1)}\right|}{k},\end{aligned} \tag{A.16}$$

where $\Delta_{\perp}$ is evaluated along a direction perpendicular to the background path and $\Delta_{\parallel}$ is along a direction parallel to the background path. By evaluating the angle for a photon travelling from $-\infty$ to ∞, Eq. (A.16) reads

$$\hat{\alpha} \approx \left|\int_{-\infty}^{\infty} \vec{\nabla}\left(\phi + \psi\right) \cdot \hat{b} dx\right|, \tag{A.17}$$

where $\hat{b}$ is a unit vector along the perpendicular direction. Usually, $\hat{b}$ denotes the direction along which an impact parameter b is measured.

APPENDIX B
STÜCKELBERG EQUATION OF MOTION

In order to equip massive gravity with general covariance, we have to introduce extra fields, known as Stückelberg fields, into the theory. As a consequence of having general covariance, the equations of motion of the Stückelberg fields can be easily found by verifying conservation law of energy and momentum. In this chapter another method of obtaining the equations of motion is presented [17]. Instead of requiring the conservation of the energy and momentum, we utilize the variational procedures of finding equations of motion on the action directly. As an example, let us consider such an approach on the dRGT massive gravity action in Eq. (5.1) as follows,

$$\begin{aligned} S &= \frac{M_p^2}{2}\int d^4x\sqrt{-g}\left(R(g)+2m_g^2U(g,f)\right), \\ &= S_{GR}+S_{int}. \end{aligned} \tag{B.1}$$

where the interaction term $U(g,f)$ is defined in Eq. (5.2), S_{GR} is the Einstein-Hilbert action for the general relativity, and S_{int} refers to the action containing the interaction term $U(g,f)$. Since the Stückelberg fields are not involved in the general relativity sector, namely the Ricci scalar term, we need to investigate only the action S_{int}. For simplicity, we consider a cosmological solution in which both the physical metric $g_{\mu\nu}$ and the reference metric $\tilde{f}_{\mu\nu}$ are of the FLRW form as follows,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -N(t)^2dt^2 + a(t)^2\Omega_{ij}(x^k)dx^idx^j, \tag{B.2}$$

$$d\tilde{s}^2 = \tilde{f}_{\mu\nu}(\varphi)d\varphi^\mu d\varphi^\nu = -n(\varphi^0)^2\left(d\varphi^0\right)^2 + \alpha(\varphi^0)^2\Omega_{ij}(\varphi^k)d\varphi^id\varphi^j, \tag{B.3}$$

where Ω_{ij} is the spatial metric defined in Eq. (3.22). For the Stückelberg fields, we focus on a variation around the unitary gauge as follows,

$$\phi^{\mu} = x^{\mu} + \pi^{\mu}, \tag{B.4}$$

where $\pi^{\mu} = \pi^{\mu}(t)$ is a perturbation around the unitary gauge. Due to the variation introduced for the Stückelberg fields, the reference metric $f_{\mu\nu} = \partial_{\mu}\phi^{\rho}\partial_{\nu}\phi^{\sigma}\tilde{f}_{\rho\sigma}$ is expanded generically as the following expression,

$$f_{\mu\nu} = \tilde{f}_{\mu\nu}(x) + \pi^{\rho}\partial_{\rho}\tilde{f}_{\mu\nu}(x) + \partial_{\mu}\pi^{\rho}\tilde{f}_{\rho\nu}(x) + \partial_{\nu}\pi^{\rho}\tilde{f}_{\mu\rho}(x) + \mathcal{O}((\pi)^2). \tag{B.5}$$

By substituting the form of $\tilde{f}_{\mu\nu}$ from Eq. (B.3) into Eq. (B.5), we obtain

$$f_{00} = -n^2\left[1 + \frac{2}{n}\partial_0\left(n\pi^0\right) + \mathcal{O}((\pi)^2)\right], \tag{B.6}$$

$$f_{0i} = f_{i0} = \alpha n\left[-\frac{n}{\alpha}D_i\pi^0 + \frac{\alpha}{n}\dot{\pi}_i + \mathcal{O}((\pi)^2)\right], \tag{B.7}$$

$$f_{ij} = \alpha^2\left[\left(1 + 2nH_{\alpha}\pi^0\right)\Omega_{ij} + D_i\pi_j + D_j\pi_i + \mathcal{O}((\pi)^2)\right], \tag{B.8}$$

where we have used the following definitions,

$$H_{\alpha} \equiv \frac{\dot{\alpha}}{\alpha n}, \quad \pi_i \equiv \Omega_{ij}\pi^j, \quad D^i \equiv \Omega^{ij}D_i, \tag{B.9}$$

where $\Omega^{ik}\Omega_{kj} = \delta^i{}_j$ and D_i denotes the covariant derivative corresponding to 3-space defined by Ω_{ij}. By using a perturbative method [17] , the building-block tensor $\mathcal{K}^{\mu}_{\ \nu} \equiv$

$\delta^\mu_\nu - (\sqrt{g^{-1}f})^\mu_\nu$ can be computed up to the first order of perturbation as follows,

$$\mathcal{K}^\mu_\nu = \mathcal{K}^{(0)\mu}{}_\nu + \mathcal{K}^{(1)\mu}{}_\nu + \mathcal{O}((\pi)^2), \tag{B.10}$$

$$\mathcal{K}^{(0)\mu}{}_\nu = \text{diag}\left(1 - \frac{n}{N}, 1 - \frac{\alpha}{a}, 1 - \frac{\alpha}{a}, 1 - \frac{\alpha}{a}\right), \tag{B.11}$$

$$\mathcal{K}^{(1)0}{}_0 = -\frac{1}{N}\partial_0\left(n\pi^0\right), \tag{B.12}$$

$$\mathcal{K}^{(1)0}{}_i = \frac{na}{N^2(1+r)}\left[-\frac{n}{\alpha}D_i\pi^0 + \frac{\alpha}{n}\dot{\pi}_i\right], \tag{B.13}$$

$$\mathcal{K}^{(1)i}{}_0 = -\frac{n}{a(1+r)}\left[-\frac{n}{\alpha}D^i\pi^0 + \frac{\alpha}{n}\dot{\pi}^i\right], \tag{B.14}$$

$$\mathcal{K}^{(1)i}{}_j = -\frac{\alpha}{2a}\left[2nH_\alpha\pi^0\delta^i{}_j + D^i\pi_j + D_j\pi^i\right], \tag{B.15}$$

where $r \equiv \frac{na}{N\alpha}$. As a consequence of keeping up to the first order, the corresponding variation of the action S_{int} thus reads

$$\delta S_{int} = -M_p^2 m_g^2 \int d^4x N a^3 \sqrt{\Omega} 3nF_{,\bar{X}}\left(H - \bar{X}H_\alpha\right)\pi^0, \tag{B.16}$$

where $F = F(\bar{X})$, and $\bar{X} \equiv \frac{\alpha}{a}$ is previously defined in Eq. (5.8). Note that the 3-space covariant derivative terms are vanishing boundary terms since there is no spatial dependence in the variation. For arbitrary variation π^0, Eq. (B.16) yields the following condition,

$$F_{,\bar{X}}\left(H - \bar{X}H_\alpha\right) = 0. \tag{B.17}$$

This condition is exactly what we have upon verifying the conservation of energy-momentum tensor in Chapter V. In other words, this is the equation of motion for the Stückelberg field π^0 whereas the other Stückelberg fields do not propagate. Similar to the previous results, two branches can be read out from Eq. (B.17); $F_{,\bar{X}} = 0$ corresponds to the self-accelerating branch while $\left(H - \bar{X}H_\alpha\right) = 0$ denotes the normal branch.

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