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ผลบวกของอนุกรมแฟกทอเรียล

Sum of Factorial Series

สมพงษ์ ชุ่ยสุริฉาย และ มณฑกานติ เพชรอภีรักษ์

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บทคัดย่อ

ในงานวิจัยเรื่องนี้ผู้วิจัยศึกษาการหาผลบวกของอนุกรมในรูป $\sum_{n=0}^{\infty} \frac{1}{(a_n)!}$ เมื่อ $(a_n)_{n=0}^{\infty}$ เป็นลำดับเลขคณิตของจำนวนเต็มที่ไม่เป็นลบ

คำสำคัญ: ลำดับเลขคณิต แฟกทอเรียล อนุกรม

ABSTRACT

In this research, we find the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(a_n)!}$ where $(a_n)_{n=0}^{\infty}$ is an arithmetic progression of nonnegative integers.

Keywords: Arithmetic Progression, Factorial, Series

1. Introduction

This work is motivated by the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(2n)!}$ and $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ which can be evaluated by the Maclaurin series of hyperbolic cosine and hyperbolic sine functions, i.e., $\sum_{n=0}^{\infty} \frac{1}{(2n)!} = \cosh(1)$ and $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!} = \sinh(1)$ (see [4] for more details). In this paper, we give a closed form for the sum of the series $\sum_{n=0}^{\infty} \frac{1}{(a_n)!}$ where $(a_n)_{n=0}^{\infty}$ is any arithmetic progression of nonnegative integers.

2. Preliminaries

Let d be a positive integer greater than 1. We define $f_d(x) = \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$. The Absolute Convergence Test shows that the series $\sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$ is absolutely convergent for all $x \in \mathbb{R}$. Thus $f_d(x)$ is well-defined for all $x \in \mathbb{R}$. Let $\omega_d = \cos\left(\frac{2\pi}{d}\right) + i \sin\left(\frac{2\pi}{d}\right)$. The following results are known in [1].

Theorem 2.1 $1, \omega_d, \omega_d^2, \dots, \omega_d^{d-1}$ are roots of the polynomial $x^d - 1$.

Proof. It is clear by De Moivre formula that $\omega_d^d = 1$. Thus

$$(\omega_d^i)^d - 1 = (\omega_d^d)^i - 1 = 1 - 1 = 0$$

for all $i \in \{0, 1, \dots, d-1\}$. This proves that $1, \omega_d, \omega_d^2, \dots, \omega_d^{d-1}$ are roots of the polynomial $x^d - 1$. □

Theorem 2.2 $\omega_d, \omega_d^2, \dots, \omega_d^{d-1}$ are roots of the polynomial $\sum_{i=0}^{d-1} x^i$.

Proof. Since $x^d - 1 = (x-1) \left(\sum_{i=0}^{d-1} x^i \right)$ and $\omega_d^l \neq 1$ for all $l \in \{1, 2, \dots, d-1\}$, by Theorem 2.1, we obtain that $\omega_d, \omega_d^2, \dots, \omega_d^{d-1}$ are roots of $\sum_{i=0}^{d-1} x^i$. □

Let

$$A_d = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ \omega_d & \omega_d^2 & \omega_d^3 & \cdots & \omega_d^d \\ \omega_d^2 & \omega_d^4 & \omega_d^6 & \cdots & \omega_d^{2d} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \omega_d^{d-1} & \omega_d^{2(d-1)} & \omega_d^{3(d-1)} & \cdots & \omega_d^{d(d-1)} \end{pmatrix},$$

$u = (1, 1, \dots, 1)$ and $v = (1, 0, \dots, 0)$. It is worth noting that $\det(A_d) = \prod_{1 \leq i < j \leq d-1} (\omega_d^j - \omega_d^i) \neq 0$

and hence A_d is invertible.

Lemma 2.3 Let A_d , u and v be defined as above. Then $A_d u^T = d v^T$.

Proof. Let $p(x) = \sum_{i=0}^{d-1} x^i$.

By Theorem 2.2, we have that $p(\omega_d^k) = 0$ for all $k \in \{1, 2, \dots, d-1\}$. Then

$$\begin{aligned} A_d u^T &= (p(1), p(\omega_d), p(\omega_d^2), \dots, p(\omega_d^{d-1}))^T \\ &= (d, 0, 0, \dots, 0)^T \\ &= d v^T \end{aligned}$$

□

Theorem 2.4 The particular solution of the initial value problem

$$y^{(d)} - y = 0, y(0) = 1, y^{(i)}(0) = 0 \text{ for all } i \in \{1, 2, \dots, d-1\} \quad (1)$$

$$\text{is } y(x) = \frac{1}{d} \left(\sum_{i=1}^d e^{\omega_d^i x} \right).$$

Proof. The auxiliary equation is $x^d - 1 = 0$. The roots of $x^d - 1$ are $\omega_d, \omega_d^2, \dots, \omega_d^d$ and they are all distinct. Thus, the general solution of $y^{(d)} - y = 0$ is

$$y(x) = \sum_{i=1}^d c_i e^{\omega_d^i x}$$

where $c_1, c_2, \dots, c_d \in \mathbb{C}$. Then

$$y^{(i)}(x) = \sum_{j=1}^d c_j \omega_d^{ij} e^{\omega_d^j x}$$

for all $i \in \{1, 2, \dots, d-1\}$. Hence the initial conditions give the system of equations

$$\begin{aligned} \sum_{i=1}^d c_i &= 1 \\ \sum_{i=1}^d \omega_d^i c_i &= 0 \\ &\vdots \\ \sum_{i=1}^d \omega_d^{i(d-1)} c_i &= 0 \end{aligned}$$

which can be written as $A_d c^T = v^T$ where $c = (c_1, c_2, \dots, c_d)$. By Lemma 2.3, we get

$$c^T = A_d^{-1} v^T = \frac{1}{d} u^T.$$

Thus $c_i = \frac{1}{d}$ for all $i \in \{1, 2, \dots, d\}$. Therefore, the particular solution of the initial value problem is

$$y(x) = \frac{1}{d} \left(\sum_{i=1}^d e^{\omega_d^i x} \right). \quad \square$$

Theorem 2.5 $f_d(x) = \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$ satisfies problem (1).

Proof. Let $y = \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!}$. Then

$$y^{(i)} = \sum_{n=1}^{\infty} \frac{x^{dn-i}}{(dn-i)!}$$

for all $i \in \{1, 2, \dots, d\}$. In particular,

$$\begin{aligned} y^{(d)} &= \sum_{n=1}^{\infty} \frac{x^{dn-d}}{(dn-d)!} \\ &= \sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!} \\ &= y. \end{aligned}$$

It is clear that $y(0) = 1$ and $y^{(i)}(0) = 0$ for all $i \in \{1, 2, \dots, d-1\}$.

Thus $f_d(x)$ satisfies (1). $\square$

3. Sum of Factorial Series

The following theorems are needed for our main results.

Theorem 3.1 For any positive integer d ,

$$\sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!} = \frac{1}{d} \left(\sum_{i=1}^d e^{\omega_d^i x} \right)$$

Proof. For $d=1$, $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$.

For $d>1$, we get by Theorem 2.4, Theorem 2.5 and the uniqueness of the solution of the initial value problem (1) [5], that

$$\sum_{n=0}^{\infty} \frac{x^{dn}}{(dn)!} = \frac{1}{d} \left(\sum_{i=1}^d e^{\omega_d^i x} \right). \quad \square$$

Theorem 3.2 For any positive integer $d>1$, and $1 \leq i \leq d$,

$$\sum_{n=1}^{\infty} \frac{x^{dn-i}}{(dn-i)!} = \frac{1}{d} \left(\sum_{j=1}^d \omega_d^{ij} e^{\omega_d^j x} \right).$$

Proof. The desired result follows from

$$\sum_{n=1}^{\infty} \frac{x^{dn-i}}{(dn-i)!} = y^{(i)}(x) = \frac{1}{d} \left(\sum_{j=1}^d \omega_d^{ij} e^{\omega_d^j x} \right). \quad \square$$

Let $(a_n)_{n=0}^{\infty}$ be an arithmetic progression of nonnegative integers with common difference $d>0$. Then $a_n = dn+r$ where $r \geq 0$. We consider two cases:

1. $0 \leq r < d$, and
2. $r \geq d$. Write $r = kd+s$, where $k \geq 1$ and $0 \leq s \leq d-1$. Then $a_n = d(n+k)+s$.

Theorem 3.3 The following results hold:

1. If $a_n = dn+r$, where $0 \leq r < d$, then

$$\sum_{n=0}^{\infty} \frac{1}{(a_n)!} = \frac{1}{d} \left(\sum_{i=1}^d \omega_d^{i(d-r)} e^{\omega_d^i} \right).$$

2. If $a_n = d(n+k)+s$, where $k \geq 1$ and $0 \leq s \leq d-1$, then

$$\sum_{n=0}^{\infty} \frac{1}{(a_n)!} = \frac{1}{d} \left(\sum_{i=1}^d \omega_d^{i(d-s)} e^{\omega_d^i} \right) - \sum_{i=0}^{k-1} \frac{1}{(di+s)!}.$$

Proof.

1. The statement follows directly from Theorem 3.2.

2. We have

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1}{(a_n)!} &= \sum_{n=0}^{\infty} \frac{1}{(dn+s)!} - \sum_{i=0}^{k-1} \frac{1}{(di+s)!} \\ &= \frac{1}{d} \left(\sum_{i=1}^d \omega_d^{i(d-s)} e^{\omega_d^i} \right) - \sum_{i=0}^{k-1} \frac{1}{(di+s)!}.\end{aligned}$$

□

4. Examples

We give examples when $d = 3$. By Theorem 3.3, we have

1. $\sum_{n=0}^{\infty} \frac{1}{(3n)!} = \frac{1}{3} \left[e + 2e^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) \right]$
2. $\sum_{n=0}^{\infty} \frac{1}{(3n+1)!} = \frac{1}{3} \left[e - e^{-\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right) \right]$
3. $\sum_{n=0}^{\infty} \frac{1}{(3n+2)!} = \frac{1}{3} \left[e - e^{-\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right) \right]$
4. $\sum_{n=0}^{\infty} \frac{1}{(3n+3)!} = \frac{1}{3} \left[e + 2e^{-\frac{1}{2}} \cos\left(\frac{\sqrt{3}}{2}\right) \right] - \frac{1}{0!}$
5. $\sum_{n=0}^{\infty} \frac{1}{(3n+4)!} = \frac{1}{3} \left[e - e^{-\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) - \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right) \right] - \frac{1}{1!}$
6. $\sum_{n=0}^{\infty} \frac{1}{(3n+5)!} = \frac{1}{3} \left[e - e^{-\frac{1}{2}} \left(\cos\left(\frac{\sqrt{3}}{2}\right) + \sqrt{3} \sin\left(\frac{\sqrt{3}}{2}\right) \right) \right] - \frac{1}{2!}$

Note that Examples 1 - 3 are guaranteed in [2] using a different method as well as in [3].

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References

- [1] Dummit, D. S. and Foote, R. M. (2004). *Abstract Algebra* (3rd ed.). New York: John Wiley and Sons.
- [2] Griffiths, M. (2015). Irrationals sums from reciprocals of factorials. *The Mathematical Gazette*, 99 (545), 331 - 335.
- [3] Mangulis, V. (1965). *Handbook of Series for Scientists and Engineers*. New York: Academic Press..
- [4] Markley N. G. (2004). *Principle of Differential Equations, Pure and Applied Mathematics*. New York: Wiley-Interscience Series of Texts, monographs, and Tracts.
- [5] Stewart, J. (1999). *Calculus* (4th ed.). California: Brooks/Cole Publishing Company press.
