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Original Article

Generalized $I_{\Delta_r^s}$-statistical convergence in intuitionistic fuzzy normed linear space

Nabanita Konwar and Pradip Debnath*

Department of Mathematics, North Eastern Regional Institute of Science and Technology, Nirjuli, Arunachal Pradesh, 791109 India

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Abstract

The notion of lacunary ideal convergence in intuitionistic fuzzy normed linear space (IFNLS) was introduced in (Debnath, 2012). As a continuation of this work, in the present paper, we introduce and study the new concept of $I_{\Delta_r^s}$-statistical convergence in IFNLS. An analogous proof of the open problem discussed above with respect to $I_{\Delta_r^s}$-statistical convergence is given. Also, we suggest an open problem regarding the completeness of the space with respect to this new convergence, whose proof could open up a new area of research in nonlinear Functional Analysis in the setting of IFNLS.

Keywords: intuitionistic fuzzy normed linear space, $I_{\Delta_r^s}$-statistical convergence, Δ-convergence

1. Introduction

Zadeh (1965) introduced the fuzzy set theory in order to model certain situations where data are imprecise or vague. Later, Atanassov (1986) introduced a non-trivial extension of standard fuzzy sets namely intuitionistic fuzzy set which deals with both the degree of membership (belonging-ness) and non-membership (non-belongingness) functions of a elements within a set.

When the use of classical theories break down in some situations, fuzzy topology is considered as one of the most important and useful tools for dealing with impreciseness. In linear spaces, if the induced metric satisfies the translation invariance property, a norm can be defined there. By introducing the norm in such spaces we can get a structure of the space which is compatible with that metric or topology and this resulting structure is called a normed linear space. The idea of a fuzzy norm on a linear space was introduced by Katsaras (1984). Felbin (1992) introduced an alternative idea of a fuzzy norm whose associated metric is of Kaleva and Seikkala (1984) type. Another notion of fuzzy norm on a linear space was given by Cheng-Moderson (1994) whose associated metric is that of Kramosil-Michalek (1975) type. Again, following Cheng and Mordeson, Bag and Samanta (2003) introduced another concept of fuzzy normed linear space. In this way, there has been a systematic development of fuzzy normed linear spaces

*Corresponding author
Email address: debnath.pradip@yahoo.com

(FNLSs) and one of the important developments over FNLS is the notion of intuitionistic fuzzy normed linear space (IFNLS). With the help of fuzzy norm, Park (2004) gave the notion of an intuitionistic fuzzy metric space. Using the concept of Park (2004), again Saadati and Park (2006) introduced the notion of IFNLS.

The concept of statistical convergence was introduced by Steinhaus (1951) and Fast (1951) and later on Fridy (1985) developed the topic further. To study the convergence problems through the concept of density, the notion of statistical convergence is a very functional tool. Kostyrko *et al.* (2000) introduced a generalized notion of statistical convergence i.e. *I*-convergence, which is based on the structure of the ideal *I* of family of subsets of natural numbers *N*. Karakus *et al.* (2008) studied statistical convergence on IFNLSs. Further, Kostyrko *et al.* (2005) studied some of basic properties of *I*-convergence and defined external *I*-limit points. For some important recent work on summability methods and generalized convergence we refer to Nabiev *et al.* (2007), Savaş and Gürdal (2014), Savaş and Gürdal (2015a), Savaş and Gürdal (2015b), Yamancı and Gürdal (2013).

Kizmaz (1981) introduced the notion of difference sequence space, where the spaces $l_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$ were studied. Further, in 1995, the notion of difference sequence spaces were generalized by Et and Colak (1995) in $l_\infty(\Delta^s)$, $c(\Delta^s)$, $c_0(\Delta^s)$. Again Tripathy and Esi (2006) introduced another type of generalization of difference sequence spaces i.e. $l_\infty(\Delta_r)$, $c(\Delta_r)$, $c_0(\Delta_r)$. After this, Tripathy *et al.* (2005) generalized the above by introducing the notion of difference Δ_r^s as follows: let *r, s* be non-negative integers, then for Z a given sequence space we have,

$$Z(\Delta_r^s) = \{ x = (x_k) \in w : (\Delta_r^s x_k) \in Z\},$$ where

$\Delta_r^s x_k = (\Delta_r^s x_k) = \left(\Delta_r^{s-1} x_k - \Delta_r^{s-1} x_{k+r}\right)$ and $\Delta_r^0 x_k = x_k$ for all $k \in N$, (here and throughout the paper by *N*, we denote the set of natural numbers) which is equivalent to the binomial representation :

$$\Delta_r^s x_k = \Delta_r^s x_k = \sum_{v=0}^{s} (-1)^v \binom{s}{v} x_{k+rv} .$$

In this binomial representation, if we take *r = 1*, we get the spaces $l_\infty(\Delta^s)$, $c(\Delta^s)$, $c(\Delta^s)$; if we take *s = 1*, we get the spaces $l_\infty(\Delta_r)$, $c(\Delta_r)$, $c_0(\Delta_r)$ and if we take *r = s = 1*, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$, $c_0(\Delta)$. With the help of this new generalized difference notion i.e. the notion generalized by Tripathy *et al.* (2005), Dutta *et al.* (2014), introduced a new generalized form of statistical convergence i.e. $I_{\Delta_r^s}$-statistical convergence for real number sequences as follows: ''a sequence $x = (x_k)$ is said to be $I_{\Delta_r^s}$-statistically convergent to $L \in X$, if for every $\epsilon > 0$ and every $\delta > 0$,

$$\{n \in N : \tfrac{1}{n} |\{k \le n : || \Delta_r^s x_k - L || \ge \epsilon\}| \ge \delta\} \in I \text{''}.$$

In the current paper we use the above generalized notion of convergence of sequences in order to introduce a new generalized statistical convergence called the $I_{\Delta_r^s}$-statistical convergence on IFNLS and extend the work to obtain some important results.

2. Preliminaries

First we recall some existing definitions and examples which are related to the present work.

Definition 2.1 (Saadati & Park, 2006)

The 5-tuple $(X, \mu, v, *, \circ)$ is said to be an IFNLS if X is a linear space, $*$ is a continuous t-norm, $\circ$ is a continuous t-conorm and μ, v fuzzy sets on $X \times (0,\infty)$ satisfying the following conditions for every $x, y \in X$ and $s, t > 0$:

(a) $\mu(x, t) + \mu(x, t) \le 1$,

(b) $\mu(x, t) > 0$,

(c) $\mu(x, t) = 1$ if and only if $x = 0$,

(d) $\mu(\alpha x, t) = \mu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(e) $\mu(x, t) * \mu(y, s) \le \mu(x + y, t + s)$,

(f) $\mu(x, t) : (0,\infty) \to [0, 1]$ is continuous in t,

(g) $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,

(h) $\nu(x, t) < 1$,

(i) $\nu(x, t) = 0$ if and only if $x = 0$,

(j) $\nu(\alpha x, t) = \nu(x, \frac{t}{|\alpha|})$ for each $\alpha \neq 0$,

(k) $\nu(x, t) \circ \nu(y, s) \geq \nu(x + y, t + s)$,

(l) $\nu(x, t) : (0,\infty) \to [0, 1]$ is continuous in t,

(m) $\lim_{t\to\infty} \nu(x, t) = 0$ and $\lim_{t\to\infty} \nu(x, t) = 1$.

In this case (μ, ν) is called an intuitionistic fuzzy norm. When no confusion arises, an IFNLS will be denoted simply by X.

Definition 2.2 (Saadati & Park, 2006)

Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be convergent to $l \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\alpha \in (0, 1)$ and $t > 0$, there exists $k_0 \in N$, such that $\mu(x_k - l, t) > 1 - \alpha$ and $\nu(x_k - l, t) < \alpha$ for all $k \geq k_0$. It is denoted by (μ, ν) - $\lim x_k = l$.

Definition 2.3 (Saadati & Park, 2006)

Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = \{x_k\}$ in X is said to be a Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\alpha \in (0, 1)$ and $t > 0$, there exists $k_0 \in N$, such that $\mu(x_k - x_m, t) > 1 - \alpha$ and $\nu(x_k - x_m, t) < \alpha$ for all $k, m \geq k_0$.

Definition 2.4 (Kostyrko *et al.*, 2000)

If X is a non-empty set then a family of sets $I \subset P(X)$ is called an ideal in X if and only if

(a) $\emptyset \in I$,

(b) $A, B \in I$ implies $A \cup B \in I$,

(c) For each $A \in I$ and $B \subset A$ we have $B \in I$,

where $P(X)$ is the power set of X.

Definition 2.5 (Kostyrko *et al.*, 2000)

If X is a non-empty set then a non-empty family of sets $F \subset P(X)$ is called a filter on X if and only if

(a) $\emptyset \notin F$,

(b) $A, B \in F$ implies $A \cap B \in F$,

(c) For each $A \in F$ and $B \supset A$ we have $B \in F$.

An ideal I is called non-trivial if $I \neq \emptyset$ and $X \notin I$. A non-trivial ideal $I \subset P(X)$ is called an admissible ideal in X if and only if it contains all singletons, i.e., if it contains $\{\{x\}: x \in X\}$.

Definition 2.6 (Debnath, 2012)

Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. For $t > 0$, we define an open ball $B(x, r, t)$ with centre at $x \in X$, radius $0 < r < 1$ as $B(x, r, t) = \{y \in X : \mu(y - x, t) > 1 - r$ and $\nu(y - x, t) < r\}$.

Definition 2.7 (Steinhaus, 1951)

If K is a subset of N, the set of natural numbers, then the natural density of K, denoted by $\delta(K)$, is given by $\delta(K) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in K\}|$, whenever the limit exists, where $|A|$ denotes the cardinality of the set A.

Definition 2.8 (Karakus, 2008)

Let $(X, \mu, \nu, *, \circ)$ be an IFNLS. A sequence $x = (x_k)$ in X is said to be statistically convergent to $l \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $\epsilon > 0$ and every $t > 0$,

$\delta(\{k \in N, : \mu(x_k - l, t) \leq 1 - \epsilon$ or $\nu(x_k - l, t) \geq \epsilon\}) = 0$.

3. Main Results

In this section we are going to discuss our main results. First we define some important definitions and theorems on $I_{\Delta_r^s}$ -statistically convergence.

Definition 3.1

Let $I \subset 2^N$ and let r, s be non-negative integers and Δ be the notion of difference sequence. Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then for every $\alpha \in (0, 1)$, $\epsilon > 0$, $\delta > 0$ and $t > 0$, the sequence $x = (x_k)$ is said to be $I_{\Delta_r^s}$ -statistically convergent to $l \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, we have

$\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha$ or $\nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Here l is called the $I_{\Delta_n^s}$ - limit of the sequence $x = (x_k)$ and we write S(I, Δ_r^s) $- \lim x = l$.

Example 3.2

Consider the space of all real numbers with the usual norm i.e. $(R, |\cdot|)$ and let for all $a, b \in [0, 1]$, $a * b = ab$ and $a \circ b = \min\{a + b, 1\}$. For all $x \in R$ and every $t > 0$, let $\mu(x,t) = \frac{t}{t+|x|}$ and $\nu(x,t) = \frac{|x|}{t+|x|}$. Then $(R, \mu, \nu, *, \circ)$ is an IFNLS. We consider $I = \{A \subset N: \delta(A) = 0\}$, where $\delta(A)$ denote natural density of the set A, then I is a non-trivial admissible ideal. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} k, & if\ k = i^2,\ i \in N \\ 0, & else. \end{cases}$$

Then for every $\alpha \in (0, 1)$, $\epsilon > 0$, $\delta > 0$ and for any $t > 0$, the set $K(\alpha, t) = \{n \in N: \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1-\alpha$ or $\nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I$ will be a finite set. Since $\alpha > 0$ is fixed, when n becomes sufficiently large, the quantity $\mu(\Delta_r^s x_k, t)$ becomes greater than $1 - \alpha$ (and similarly, the quantity $\nu(\Delta_r^s x_k, t)$ becomes less than α). Hence $\delta(K(\alpha, t)) = 0$, and consequently, $K(\alpha, t) \in I$, i.e. S(I, Δ_r^s) $- \lim x = 0$.

Again,

$$\mu(\Delta_r^s x_k - 0, t) = \begin{cases} \frac{t}{t+k}, & if\ k = i^2,\ i \in N \\ 1, & else. \end{cases}$$

Then $\lim_{k\to\infty} \mu(\Delta_r^s x_k, t)$ does not exist and consequently the sequence $\{x_k\}$ is not convergent with respect to the intuitionistic fuzzy norm (μ,ν).

Lemma 3.3

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then for every $\alpha \in (0, 1)$, $\varepsilon > 0$, $\delta > 0$ and $t > 0$ the following statements are equivalent:

(a) $S(I, \Delta_r^s) - \lim x = l$,

(b) $\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha\} \geq \epsilon\}| \geq \delta\}$ *and*

$\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I$,

(c) $\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ *and* $\nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}| < \delta\} \in F(I)$,

(d) $\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha\} < \epsilon\}| < \delta\} \in F(I)$ *and*

$\{n \in N : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}| < \delta\} \in F(I)$,

(e) $I_{\Delta_r^s} - \lim \mu(\Delta_r^s x_k - l, t) = 1$ *and*

$I_{\Delta_r^s} - \lim \nu(\Delta_r^s x_k - l, t) = 0$.

Proof of the following can be obtained using similar techniques as in (Debnath, 2012).

Theorem 3.4

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. If the sequence $x = (x_k)$ is $I_{\Delta_r^s}$-statistically convergent to $l \in X$ with respect to the intuitionistic fuzzy norm (μ, ν), then S(I, Δ_r^s) $- \lim x$ is unique.

Theorem 3.5

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. If $(\mu, \nu) - \lim x = l$, then S (I, Δ_r^s) $- \lim x = l$.

Proof. Let us consider $(\mu, \nu) - \lim x = l$. Then for $\alpha \in (0, 1)$, $t > 0$, $\epsilon > 0$ and $\delta > 0$, there exists $k_0 \in$ N such that

$$\mu(x_k - l, t) > 1 - \alpha \text{ and } \nu(x_k - l, t) < \alpha,$$

for all $k \geq k_0$.

Therefore, for all $k \geq k_0$ the set,

$\{k \in N : inf\{t > 0 : \mu(x_k - l, t) \leq 1 - \alpha$ or $v(x_k - l, t) \geq \alpha\} \geq \epsilon\}$ has at most finitely many terms. Thus it follows that,

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha$ or $v(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Thus $S(I, \Delta_r^s) - lim\ x = l.$

Theorem 3.6

Let $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, v, *, \circ)$. *If* $S_{(\mu,v)} - lim\ x = l$, *then* $S(I, \Delta_r^s) - lim\ x = l$.

Proof. Let $S_{(\mu,v)} - lim\ x = l$. Then $\alpha \in (0, 1)$, $t > 0$, $\epsilon > 0$ and $\delta > 0$ there exists $n_0 \in$ N such that

$\delta(\{n \in N : \mu(x_k - l, t) \leq 1 - \alpha$ or $v(x_k - l, t) \geq \alpha\}) = 0$, for all $n \geq n_0$.

This implies,

$\delta(\{n \in N : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha$ or $v(\Delta_r^s x_k - l, t) \geq \alpha\}) = 0$, for all $n \geq n_0$.

So we have,

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha$ or $v(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Thus $S(I, \Delta_r^s) - \lim x = l.$

Theorem 3.7

Let $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, v, *, \circ)$. *Then* $S(I, \Delta_r^s) - lim\ x = l$ *if and only if there exists an increasing index sequence* $K = \{k_n\}$ *of natural numbers such that for* $k \in K$ *and* $\alpha \in (0, 1)$,

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ and $v(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}| < \delta\} \in F(I)$ and $(\mu, v) - lim_{k \in K} = l.$

Theorem 3.8

Let $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, v, *, \circ)$. *Let I be a non-trivial ideal of N. If* $x = (x_k)$ *is a* $I_{\Delta_r^s}$ *-statistically convergence in X and* $y = (y_k)$ *is a sequence in X such that* $\{n \in N : \Delta_r^s x_k \neq \Delta_r^s y_k$ *, for some* $k \leq n\} \in I$, *then y is also* $I_{\Delta_r^s}$ *-statistically convergence to the same limit.*

Proof. Let $\alpha \in (0, 1)$, $t > 0$ and $S(I, \Delta_r^s) - \lim x = l$. Then

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Now,

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s y_k - l, t) \leq 1 - \alpha\} \geq \epsilon\}| \geq \delta\} \subseteq \{n \in N : \Delta_r^s x_k \neq \Delta_r^s y_k$, *for some* $k \leq n\} \cup \{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha\} \geq \epsilon\}| \geq \delta\}.$

But both the sets in the right-hand side of the above inclusion relation belongs to I. Thus

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s y_k - l, t) \leq 1 - \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Similarly, we can prove that

$\{n \in N : \frac{1}{n}|\{k \leq n : inf\{t > 0 : v(\Delta_r^s y_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$

Therefore $S(I, \Delta_r^s) - \lim y = l.$

Hence y is $I_{\Delta_r^s}$ -statistically convergent to the same limit. □

4. Δ - Convergence in IFNLS

Here we are going to present the concept of Δ-convergence in IFNLS and establish its relation with $I_{\Delta_r^s}$ -convergence.

Definition 4.1

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then $x = (x_k)$ is said to be Δ-convergent to $l \in X$ with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $t > 0$, $\alpha \in (0, 1)$, $\epsilon > 0$ and $n \in N$ we have,

$\{k \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}$.

It is denoted by $(\mu, \nu)^{\Delta} - \lim x = l$.

Theorem 4.2

Let $x = \{x_k\}$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then $(\mu, \nu)^{\Delta} - \lim x$ is unique, if $x = \{x_k\}$ is Δ-convergent with respect to the intuitionistic fuzzy norm (μ, ν).

Proof. Let us consider $(\mu, \nu)^{\Delta} - \lim x = l_1$ and $(\mu, \nu)^{\Delta} - \lim x = l_2$ $(l_1 \neq l_2)$. Take a fixed $\alpha \in (0, 1)$, for which we choose $\gamma \in (0, 1)$ such that $(1 - \gamma) * (1 - \gamma) > 1 - \alpha$ and $\gamma \circ \gamma < \alpha$. Now, for every $t > 0$, $\epsilon > 0$, $n \in \mathrm{N}$, we have,

$\{k_1 \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l_1, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_k - l_1, t) < \alpha\} < \epsilon$.

And

$\{k_2 \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l_2, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_k - l_2, t) < \alpha\} < \epsilon$.

Consider $k = \max\{k_1, k_2\}$. Then for $k \geq n$, we will get a $p \in \mathrm{N}$ such that

$\mu(\Delta_r^s x_p - l_1, \frac{t}{2}) > \mu(\Delta_r^s x_k - l_1, \frac{t}{2}) > 1 - \gamma$ and $\mu(\Delta_r^s x_p - l_2, \frac{t}{2}) > \mu(\Delta_r^s x_k - l_2, \frac{t}{2}) > 1 - \gamma$.

Thus we have

$$\mu(l_1 - l_2, t) \geq \mu(\Delta_r^s x_p - l_1, \tfrac{t}{2}) * \mu(\Delta_r^s x_p - l_2, \tfrac{t}{2})$$
$$> (1 - \gamma) * (1 - \gamma)$$
$$> 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\mu(l_1 - l_2, t) = 1$ for all $t > 0$, which implies that $l_1 = l_2$. Similarly we can show that,

$\nu(l_1 - l_2, t) < \alpha$,

for all $t > 0$ and arbitrary $\alpha > 0$, and thus $l_1 = l_2$.

Hence $(\mu, \nu)^{\Delta} - \lim x$ is unique.

From the following theorem we can conclude that Δ-convergence is stronger than $I_{\Delta_r^s}$-convergence.

Theorem 4.3

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. If $(\mu, \nu)^{\Delta} - \lim x = l$, then $I_{\Delta_r^s} - \lim x = l$.

Proof. Let us consider $(\mu, \nu)^{\Delta} - \lim x = l$. Then for every $t > 0$, $\alpha \in (0, 1)$ and $\epsilon > 0$ we have

$\{k \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}$.

Implies, for every $\delta > 0$,

$\{n \in \mathrm{N} : \frac{1}{n} |\{k \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}| < \delta\}$.

So, we have,

$A = \{n \in \mathrm{N} : \frac{1}{n} |\{k \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha$ or $\nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \subseteq \{1, 2, ..., k - 1\}$.

As I being admissible, so we have $A \in I$. Hence $I_{\Delta_r^s} - \lim x = l$.

Theorem 4.4

Let $x = \{x_k\}$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. If $(\mu, \nu)^{\Delta} - \lim x = l$, then there exists a subsequence $\{x_{m_k}\}$ of $x = (x_k)$ such that $(\mu, \nu) - \lim x_{m_k} = l$.

Proof. Let us consider $(\mu, \nu)^{\Delta} - \lim x = l$. Then for every $t > 0$, $\alpha \in (0, 1)$, $\epsilon > 0$ and $n \in \mathrm{N}$ we have

$\{k \leq n : \inf\{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}$.

Clearly, we can select an $m_k \leq n$ such that

$\mu(\Delta_r^s x_{m_k} - l, t) > \mu(\Delta_r^s x_k - l, t) > 1 - \alpha$ and $\nu(\Delta_r^s x_{m_k} - l, t) < \nu(\Delta_r^s x_k - l, t) < \alpha$.

It follows that $(\mu, \nu) - \lim x_{m_k} = l$.

Theorem 4.5

Let I be a nontrivial ideal of N *and* $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, \nu, *, \circ)$. *If* $x = \{x_k\}$ *is* Δ-*convergent in* X *and* $y = \{y_k\}$ *is a sequence in* X *such that* $\{n \in \mathrm{N} : \Delta_r^s x_k \neq \Delta_r^s y_k$ *for some* $k \leq n\} \in I$, *then y is also* Δ-*convergent to the same limit.*

Proof. Let us consider that $x = (x_k)$ is Δ-convergence in X. For $\alpha \in (0, 1)$, $t > 0$ and $\epsilon > 0$ we have,

$$\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}.$$

This implies, for all $\delta > 0$,

$$\{n \in \mathrm{N} : \tfrac{1}{n}|\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - l, t) > 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - l, t) < \alpha\} < \epsilon\}| < \delta\} \notin I.$$

This implies,

$$\{n \in \mathrm{N} : \tfrac{1}{n}|\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in \mathrm{I}.$$

Therefore, we have

$$\{n \in \mathrm{N} : \tfrac{1}{n}|\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s y_k - l, t) \leq 1 - \varepsilon \text{ or } \nu(\Delta_r^s y_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \subseteq \{n \in \mathrm{N} : \Delta_r^s x_k \neq \Delta_r^s y_k, \text{ for some } k \leq n\} \cup \{n \in \mathrm{N} : \tfrac{1}{n}\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - l, t) \leq 1 - \alpha \text{ or } \nu(\Delta_r^s x_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\}.$$

As both the right-hand side member of the above equation are in *I*, therefore we have,

$$\{n \in \mathrm{N} : \tfrac{1}{n}|\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s y_k - l, t) \leq 1 - \alpha \text{ or } \nu(\Delta_r^s y_k - l, t) \geq \alpha\} \geq \epsilon\}| \geq \delta\} \in I.$$

Hence y is Δ-convergence at the same limit.

5. $I_{\Delta_r^s}$ -Statistically Cauchy Sequence in IFNLS

Here we introduce a new form of Cauchy sequence called $I_{\Delta_r^s}$ -statistically Cauchy sequence and find some results.

Definition 5.1

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then $x = (x_k)$ is said to be Δ-Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if,

for every $t > 0$, $\alpha \in (0, 1)$, $\epsilon > 0$, there exist $n_0, m \in \mathrm{N}$ with $m \geq k$ satisfying,

$$\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - \Delta_r^s x_m, t) > 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - \Delta_r^s x_m, t) < \alpha\} < \epsilon\}.$$

Definition 5.2

Let $x = (x_k)$ be a sequence in an IFNLS $(X, \mu, \nu, *, \circ)$. Then $x = (x_k)$ is said to be $I_{\Delta_r^s}$ -statistically Cauchy sequence with respect to the intuitionistic fuzzy norm (μ, ν) if, for every $t > 0$, $\epsilon > 0$, $\delta > 0$ and $\alpha \in (0, 1)$ there exist $m \in N$ satisfying,

$$\{n \in N : \tfrac{1}{n}|\{k \leq n : inf\ \{t > 0 : \mu(\Delta_r^s x_k - \Delta_r^s x_m, t) > 1 - \alpha \text{ and } \nu(\Delta_r^s x_k - \Delta_r^s x_m, t) < \alpha\} < \epsilon\}| < \delta\} \in F(I).$$

Definition 5.3

Let $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, \nu, *, \circ)$. *Then* $x = (x_k)$ *is said to be* $I^*_{\Delta_r^s}$ -*statistically Cauchy sequence with respect to the intuitionistic fuzzy norm* (μ, ν) *if, there exists a set* $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subset N$ *such that the set* $M' = \{n \in N : m_k \leq n\} \in F(I)$ *and the subsequence* (x_{m_k}) *of* $x = (x_k)$ *is a Cauchy sequence with respect to the intuitionistic fuzzy norm* (μ, ν).

Theorem 5.4

Let $x = (x_k)$ *be a sequence in an IFNLS* $(X, \mu, \nu, *, \circ)$. *If* $x = (x_k)$ *is*

$I_{\Delta_r^s}$-statistically convergent with respect to the intuitionistic fuzzy norm (μ, ν), then it is $I_{\Delta_r^s}$-statistically Cauchy with respect to the intuitionistic fuzzy norm (μ, ν).

Proof. Suppose that $x = (x_k)$ be a $I_{\Delta_r^s}$-statistically convergent sequence which converges to l. For a given $\alpha > 0$, choose $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1-\alpha$ and $\gamma \circ \gamma < \alpha$. Then for any $t > 0$, $\epsilon > 0$ and $\delta > 0$, we have,

$$K_\mu(\gamma, t) = \{n \in \mathrm{N} : \frac{1}{n} |\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - l, \frac{t}{2}) > 1 - \gamma\} < \epsilon\}| < \delta\}$$

and

$$K_\nu(\gamma, t) = \{n \in \mathrm{N} : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \nu(\Delta_r^s x_k - l, \frac{t}{2}) < \gamma\} < \epsilon\}| < \delta\}.$$

Then $K_\mu(\gamma, t) \in F(I)$ and $K_\nu(\gamma, t) \in F(I)$. Let $K(\gamma, t) = K_\mu(\gamma, t) \cap K_\nu(\gamma, t)$.

Then $K(\gamma, t) \in F(I)$. If $n \in K(\gamma, t)$ and we choose a fixed $m \in K(\gamma, t)$, then

$$\begin{aligned}\mu(\Delta_r^s x_k - \Delta_r^s x_m, t) &\geq \mu(\Delta_r^s x_k - l, \tfrac{t}{2}) * \mu(\Delta_r^s x_m - l, \tfrac{t}{2})\\ &> (1-\gamma) * (1-\gamma)\\ &> 1-\alpha.\end{aligned}$$

This clearly implies that

$$inf\{t > 0 : \mu(\Delta_r^s x_k - \Delta_r^s x_m, t) > 1 - \alpha\} < \epsilon .$$

which implies,

$$\frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - \Delta_r^s x_m, t) > 1 - \alpha\} < \epsilon\}| < \delta.$$

Also,

$$\begin{aligned}\nu(\Delta_r^s x_k - \Delta_r^s x_m, t) &\leq \nu(\Delta_r^s x_k - l, \tfrac{t}{2}) \circ \nu(\Delta_r^s x_m - l, \tfrac{t}{2})\\ &< \gamma \circ \gamma\\ &< \alpha.\end{aligned}$$

This implies that,

$$inf\{t > 0 : \nu(\Delta_r^s x_k - \Delta_r^s x_m, t) < \alpha\} < \epsilon ,$$

which again implies,

$$\frac{1}{n}|\{k \leq n : inf\{t > 0 : \nu(\Delta_r^s x_k - \Delta_r^s x_m, t) < \alpha\} < \epsilon\}| < \delta.$$

Therefore

$$\{n \in \mathrm{N} : \frac{1}{n}|\{k \leq n : inf\{t > 0 : \mu(\Delta_r^s x_k - \Delta_r^s x_m, t) > 1-\alpha \text{ and } \nu(\Delta_r^s x_k - \Delta_r^s x_m, t) < \alpha\} < \epsilon\}| < \delta\} \in F(I).$$

Hence $x = (x_k)$ is $I_{\Delta_r^s}$-statistically Cauchy.

Remark 5.5

*This still remains an open problem if the converse of Theorem 6.4 is true, i.e., whether every $I_{\Delta_r^s}$-statistically Cauchy sequence is $I_{\Delta_r^s}$-statistically convergent (or it becomes $I_{\Delta_r^s}$-statistically convergent under certain new conditions)**.** The completeness of IFNLS with respect to some notion of convergence would help the researchers to investigate many analogous results of classical Functional Analysis and Fixed Point Theory in the setting of an IFNLS.*

6. Conclusions

In this paper we have introduced the concept of $I_{\Delta_r^s}$-statistically convergence in IFNLS and established some new results. The extended results give us a new idea about statistical convergence in IFNLS. A new type of Cauchy sequence i.e. $I_{\Delta_r^s}$ - Cauchy sequence has also been introduced in this paper. Some existing results are generalized as well as extended and some new results are incorporated. The results obtained in this paper are more general than the corresponding results for classical and fuzzy normed spaces. The converse of Theorem 6.4 would be a very good topic for future study, because if the converse can be proved to be true, then the IFNLS becomes $I_{\Delta_r^s}$-statistically complete. A $I_{\Delta_r^s}$-statistically complete IFNLS would, in turn, open up a new area of research on fixed point theory and nonlinear functional analysis in it.

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