Chapter 1

Introduction

1.1 Background and Significance

Many important physical processes in nature are governed by partial differential equations (PDEs). For this reason, the knowledge of the mathematical character and properties of the governing equations are required. Properties of PDEs can be effectively studied by using their exact solutions. Therefore, there is interest in finding exact solutions of PDEs. In general, it is not easy to obtain exact solutions of PDEs. One of the methods for obtaining exact solutions is the group analysis method. It is well-known that the group analysis method is a powerful and direct approach to construct exact solutions of PDEs.

The research performed in the project can be formally separated in three parts. All this parts are related by the method of the study and the sequence of discoveries.

1.1.1 Fluids with internal inertia

The first part of the project deals with modeling in fluid dynamics.

Developing new technology requires developing new models in fluid dynamics. Equations of fluids with internal inertia is the new theory considered in the fluid dynamics. These equations are obtained on the base of Euler-Lagrange principle. Among fluids with internal inertia there are intensively studied two classes of models. This project is focused on group classification of a class of dispersive models [1][1]

$$\begin{gathered}\dot{\rho} + \rho\, div(u) = 0, \quad \rho\dot{u} + \nabla p = 0, \quad \dot{S} = 0, \\ p = \rho\frac{\delta W}{\delta\rho} - W = \rho\left(\frac{\partial W}{\partial\rho} - \frac{\partial}{\partial t}\left(\frac{\partial W}{\partial\dot{\rho}}\right) - div\left(\frac{\partial W}{\partial\dot{\rho}}u\right)\right) - W,\end{gathered} \tag{1.1}$$

where t is time, ∇ is the gradient operator with respect to space variables, ρ is the fluid density, u is the velocity field, $W(\rho, \dot{\rho}, S)$ is a given potential, "dot" denotes the material time derivative: $\dot{f} = \frac{df}{dt} = f_t + u\nabla f$ and $\frac{\delta W}{\delta\rho}$ denotes the variational derivative of W with respect to ρ at a fixed value of u. These models include the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski (1960) [2], Kogarko (1961) [3],Wijngaarden (1968) [4]), and the dispersive shallow water model (Green & Naghdi (1975) [5], Salmon (1998) [6]). Equations (1.1) were obtained in [1] using the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - W(\rho, \dot{\rho}, S).$$

[1] See also references therein.

This is an example of a medium behavior dependent not only on thermodynamical variables but also on their derivatives with respect to space and time. In this particular case the potential function depends on the total derivative of the density which reflects the dependence of the medium on its inertia. Another example of models where the medium behavior depends on the derivatives is constructed in [7] by assuming that the Lagrangian is of the form:

$$L = \frac{1}{2}|u|^2 - \varepsilon(\rho, |\nabla\rho|, S).$$

One of the methods for studying properties of differential equations is group analysis [8, 9, 10]. This method is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations are collected in [11, 12, 13]. Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for mathematical modeling in the present research.

An application of group analysis employs several steps. The first step is a group classification with respect to arbitrary elements. An algorithm of the group classification is applied in case where a system of differential equations has arbitrary elements in form of undefined parameters and functions. This algorithm is necessary since a specialization of the arbitrary elements can lead to an extension of admitted Lie groups. Group classification selects the functions $W(\rho, \dot{\rho}, S)$ such that the fluid dynamics equations (1.1) possess additional symmetry properties extending the kernel of admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of equations (1.1), where $W = W(\rho, \dot{\rho})$ is performed in [14] (one-dimensional case) and [15] (three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in [14, 15, 16]. Group classification of the class of models describing the behavior of a dispersive continuum with $\varepsilon = \varepsilon(\rho, |\nabla\rho|)$ was studied in [17]. It is also worth to notice that the classical gas dynamics model corresponds to $W = W(\rho, S)$ (or $\varepsilon = \varepsilon(\rho, S)$). A complete group classification of the gas dynamics equations was presented in [8]. Later, an exhausted program of studying the models appeared in the group classification of the gas dynamics equations was announced in [18]. Some results of this program were summarized in [19].

1.1.2 Application of the group analysis method to integro-differential equations

The second part of the project deals with applications of the group analysis method to integro-differential equations.

In applied mathematics and physics a special attention is given to the study of invariant solutions of integro-differential equations which are directly associated with fundamental symmetry properties of these equations. Group analysis in this case is an universal tool for obtaining complete sets of symmetries. However a direct transference of the known scheme of the group analysis method on integro-differential equations is impossible. The general algorithm for application of the group analysis to equations with nonlocal terms was proposed recently [20]. It is worth to notice that this area of the group analysis method is still developing. In the present research the group analysis method is applied to two integro-differential equations: (a) the Rudenko equation, and (b) the Boltzmann equation with sources.

The Rudenko equation

One of the most general evolution equations used in nonlinear wave physics is the following one [21, 22]:

$$\begin{array}{l} (u_x - uu_t - w_{tt})_t = u_{yy} + u_{zz} \, , \\ w = \int_0^\infty K(s) \; u(t-s) \, ds \end{array} . \tag{1.2}$$

Here the variable t is the time, and x, y, z are the spatial Cartesian coordinates. The coordinate x is distinguished as a "longitudinal" one. It coincides with a preferred orientation of the wave propagation. Other coordinates y, z are identified as "transversal" ones. They are commonly introduced in the cross-section of a wave beam.

Special cases of the equation (1.2) are well-known. In particular, if the kernel is identically zero, $K(s) \equiv 0$, the general equation (1.2) is reduced to the Khokhlov-Zabolotskaya (KZ) equation [23, 24], describing wave beams in nonlinear media:

$$(u_x - uu_t)_t = u_{yy} + u_{zz}. \tag{1.3}$$

If the kernel is the delta-function, $K = 2\delta(s)$, the model (1.2) leads to the equation

$$(u_x - uu_t - u_{tt})_t = u_{yy} + u_{zz} \tag{1.4}$$

for nonlinear beams in a dissipative medium [25, 26]. Equation (1.4) is known as the Khokhlov-Zabolotskaya-Kuznetsov (KZK) equation. It is widely used in underwater acoustics for engineering design of parametric radiating and receiving arrays [26].

If the kernel is proportional to the derivative of the delta-function, $K = 2\delta'(s)$, the integro-differential equation (1.2) becomes the Kadomtsev-Petviashvili (KP) equation

$$(u_x - uu_t - u_{ttt})_t = u_{yy} + u_{zz} \tag{1.5}$$

for nonlinear beams in a dispersive medium [27, 28]. The similar equation

$$(u_x - uu_t - u_{tttt})_t = u_{yy} + u_{zz} \tag{1.6}$$

for a scattering medium [29] follows from (1.2) when $K = 2\delta''(s)$.

There exist other models that specify or generalize the equation (1.2), e.g. by including (1.2) in a coupled systems of nonlinear equations [30, 31].

If the wave field $u = u(t, x)$ is a function of a single spatial (longitudinal) coordinate x and does not depend on transverse coordinates y, z, equation (1.2) is reduced to well-known equations for plane waves [32]. In particular, the Riemann-Hopf equation follows from (1.3), the Burgers equation follows from (1.4), and the Korteweg- de Vries equation follows from (1.5). The one-dimensional equation with fourth-order derivative for scattering medium suggested and solved in [29] follows from equation (1.6). 1D equations can be obtained by eliminating the y, z derivatives of 3D equations and the subsequent integration over dt, provided that the wave field vanishes at $t \to \pm\infty$.

A choice of the kernel as a linear combination of the delta-function and its derivatives of different orders gives a possibility to derive from (1.2) various well-known differential equations of the physics of nonlinear waves. Symmetries of such equations either have already been studied (many results obtained until 1995 are collected in [11, 12, 13]), or can be studied by the standard Lie group methods [8, 33, 9]. However, to the best of our knowledge, particular versions of the general equation (1.2) with non-degenerate kernels which maintain the integro-differential feature of the model have not been studied yet.

The exponential kernel $K = \exp(-s)$ is of particular applied interest. Equation (1.2) with such kernel describes wave beams in relaxing media. In this case the integro-differential equation is also reduced to a differential equation [21]. To derive such an equation, it is sufficient to note that the integral term w in (1.2) and the variable u for the exponential kernel are related by the following equation:

$$w_t + w = u. \tag{1.7}$$

Reduction of (1.2) to a differential equation is also possible for some more complicated kernels. For example, if $K = \exp(-s)\cos(\omega_0 s)$, then the kernel describes internal dynamics of medium with resonant inclusions. In this case, the differential relation between w and u in equation (1.2) has the form:

$$(w_t + w - u)_t + (w_t + w - u) + \omega_0^2 w = 0. \tag{1.8}$$

A special class is formed by "model" kernels which are non-zero on the finite segment, for example, within $s \in (0,\, 1]$. The simplest case is

$$K = \begin{cases} 1 & \text{if} \quad s \le 1, \\ 0 & \text{if} \quad s > 1. \end{cases}$$

For this kernel the integro-differential equation (1.2) is reduced to the difference-differential equation:

$$\begin{aligned} &(u_x - uu_t - \Delta u_t)_t = u_{yy} + u_{zz}\,, \\ &\Delta u \equiv u(t) - u(t-1). \end{aligned} \tag{1.9}$$

Note that, using the finite shift operator, one rewrites the integral term of equation (1.2) in the form

$$w(t) = \hat{L}u(t), \quad \hat{L}(\partial_t) = \int_0^\infty K(s)\, \exp(-s\partial_t)\, ds, \tag{1.10}$$

where ∂_t is the partial derivative with respect to time. The second operator $\hat{L}(\partial_t)$ is the Laplace transform of the function $K(s)$ defining the kernel of equation (1.2).

For example, if a kernel has the form of the Bessel function of zero order, then one has

$$K = J_0(s)\,, \quad \hat{L}(\partial_t) = \left(1 + \partial_t^2\right)^{-1/2}.$$

Using the tables of the Laplace transform and physical restrictions of the kernel forms, one can single out all cases when equation (1) can be reduced to a differential equation of a finite order. In the general case, decomposing the exponential function of the integrand (9) into power series, one verifies that the resulting differential equation will contain derivatives of an arbitrary order.

In the present research we use our method [34, 35] to the Rudenko equation.

The Boltzmann equation with sources

The Boltzmann kinetic equation is the basis of the classical kinetic theory of rarefied gases. Considerable interest in the study of the Boltzmann equation was always the search for exact (invariant) solutions directly associated with the fundamental properties of the equation. After the studies of the class of the local Maxwellians [36, 37, 38] new classes of invariant solutions were constructed in the 1960s in [39, 40, 41]. A decade later the BKW-solution was almost simultaneously derived in [42] and in [43]. Contrary to the Maxwellians, the Boltzmann collision integral does not vanish for this solution. The discovery of the BKW-solution stimulated a great splash of studies of exact solutions of various kinetic equations. However, the progress at that time was really limited to the construction of BKW-type solutions for different simplified models of the Boltzmann equation [44].

The Boltzmann equation is an integro-differential equation. Whereas the classical group analysis method has been developed for studying partial differential equations, the main obstacle for applying group analysis to integro-differential equations comes from presence of nonlocal integral operators. The direct group analysis for equations with nonlocal operators was worked out and successfully used in [34, 45, 20]. In particular, a complete group classification of the spatially homogeneous and isotropic Boltzmann equation without sources was obtained in [34, 35].

One of the alternative approaches for studying solutions of the Boltzmann equation, by transition to an equation for a moment generating function, was first considered in [46, 47]. The BKW-solution was obtained there. In [48], such an approach was applied to the spatially homogeneous and isotropic Boltzmann equation with sources. The author of [48] used the group analysis method for studying solutions of the equation for the generating function. However, it was not taken into account that this equation is still a nonlocal one.

In the present research we use our method [34, 35] to amend the results of [48]. A group classification of the equation for a moment generating function with respect to a source function is obtained.

1.1.3 Application of group analysis to ordinary differential equations

Many methods of solving differential equations use a change of variables that transform a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, it was attractive to transform a given differential equation into a linear equation. This problem, which is called a linearization problem, is a particular case of the equivalence problem. The equivalence problem can be formulated as follows. Let a set of invertible transformations be given. One can introduce the equivalence property according to these transformations: two differential equations are equivalent if there is a transformation of the given set which transforms one equation into another. The equivalence problem involves a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation.

The third part of the present project is focused on the study of two problems: (a) on first integrals of second-order ordinary differential equations; (b) the complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients.

Introduction to the problem of finding first integrals of second-order ordinary differential equations

We give a short review of results related with an equivalence problem for a second-order ordinary differential equation. Two types of transformations can be distinguished among the transformations used in the equivalence problem for second-order ordinary differential equations: point transformations and the generalized Sundman transformations. S.Lie [49] also noted that all second-order ODEs can be transformed to each other by means of contact transformations, thus this set of transformations cannot be applied for a classification of second-order ODEs.

Among the target equations two classes of equations can be mentioned. One set of this class was obtained by S.Lie [50]. Lie's group classification of ODEs shows that the second-order equations can possess one, two, three or eight infinitesimal symmetries. The equations with eight symmetries and only these equations can be linearized by a change of variables. Lie showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. A similar description of the equations with three symmetries were provided in [51, 52]. Another set of target classes corresponds to the Painlevé equations. Analysis of the classes of equations corresponding to the first and second Painlevé equations was done in [53, 54].

For the linearization problem one studies the classes of equations equivalent to linear equations. The first linearization problem for ordinary differential equations was solved by S.Lie [49]. He found the general form of all ordinary differential equations of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. A.M.Tresse [55] treated the equivalence problem for second order ordinary differential equations in terms of relative invariants of the equivalence group of point transformations. In [56] an infinitesimal technique for obtaining relative invariants were applied to the linearization problem.

S.Lie also noted that all second order equations can be transformed to each other by means of contact transformations, and that this is not so for third order equations.

A different approach for tackling the equivalence problem of second order ordinary differential equations was developed by E.Cartan [57]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form. The Cartan approach was further applied by S.S.Chern [58] to third order differential equations. Since none of the conditions given in [58] are implicit expressions that could be used as tests for deciding about the type of the studied equation, in a series of articles [59, 60, 61, 62, 63] the linearization problem was also considered. Linearization with respect to point transformations is studied in [59], with respect to contact transformations in [60, 61, 62, 63, 64]. The linearization problem was also investigated with respect to generalized Sundman transformations [65, 66, 67].

The linearization problem via point transformations

$$\tau = \varphi(t, x), \qquad u = \psi(t, x)$$

for a second-order equation $\ddot{x} = F(t, x, \dot{x})$ is attractive because of the simplicity of the general solution of a linear equation: a linearizable second-order ordinary differential equation is equivalent to the free particle equation $u'' = 0$. Thus, if one found linearizing transformation, then the general solution of the original equation can be relatively found easily. Note that for a linearizable equation $\ddot{x} = F(t, x, \dot{x})$ the value

$$u' = \frac{\dot{x}\psi_x + \psi_t}{\dot{x}\varphi_x + \varphi_t}$$

is a first integral of the equation. Here subscripts mean derivatives, for example, $\varphi_t = \partial\varphi/\partial t$, $\varphi_x = \partial\varphi/\partial x$ and etc... This motivated one to study equations possessing a first integral of the form

$$I = \frac{\dot{x}\tilde{A}(t,x) + \tilde{C}(t,x)}{\dot{x}\tilde{B}(t,x) + \tilde{Q}(t,x)}. \tag{1.11}$$

Notice that a second-order equation equivalent to the free particle equation via the generalized Sundman transformation also possesses a first integral of the form (1.11).

The authors of [68, 69, 70] came to the form of first integral (1.11) from the study of λ-symmetries for second-order equations which play a fundamental role. Although the equation may lack Lie point symmetries, there always exists a λ-symmetry associated to a first integral $I = I(t, x, \dot{x})$. Such a $\lambda-$symmetry can be defined in canonical form by the vector field $\mathbf{v} = \partial_x$ and the function $\lambda = -I_x/I_{\dot{x}}$. When I is of the form

$$I = C(t,x) + \frac{1}{A(t,x)\dot{x} + B(t,x)}, \quad (A \neq 0) \tag{1.12}$$

such a function λ is given by

$$\lambda(t,x,\dot{x}) = \gamma(t,x)\dot{x}^2 + \alpha(t,x)\dot{x} + \beta(t,x)\dot{x} \tag{1.13}$$

where

$$\gamma = AC_x = -a_3 \tag{1.14}$$

$$\alpha = 2BC_x - A_x/A = -a_2 - AC_t \tag{1.15}$$

$$\beta = (C_x B^2 - B_x)/A = -a_1 + A_t/A - 2BC_t. \tag{1.16}$$

In this way the study of the ODEs that admit first integrals of the form (1.12) can be seen as a problem of classification of the ODEs that admit $\mathbf{v} = \partial_x$ as $\lambda-$symmetry for some function λ of the form (1.13).

The case where $C_x = 0$:

$$I = C(t) + \frac{1}{\dot{x}A(t,x) + C(t,x)}$$

was studied in [71]. It has to be mentioned here that the case where $\tilde{B} = 0$ was completely studied in [70].

We denote by $\mathcal{B}$ the class of equations corresponding to the particular case when $\gamma = 0$ in (1.13). The equations in $\mathcal{B}$ are the ODEs of the form

$$\ddot{x} + a_2(t,x)\dot{x}^2 + a_1(t,x)\dot{x} + a_0(t,x) = 0 \tag{1.17}$$

that admit first integrals of the form (1.12) with $C_x = 0$.

A significant subclass of ODEs in $\mathcal{B}$, denoted by $\mathcal{A}$, is constituted by the equations that admit first integrals of the form $A(t,x)\dot{x} + B(t,x)$ (that is, $C = 0$ in (1.12)). By (1.15), the equations in $\mathcal{A}$ are the equations of the form (1.17) that admit $\mathbf{v} = \partial_x$ as $\lambda-$symmetry for some function $\lambda = -a_2\dot{x} + \beta$. According to the results in [70], the coefficients of the equations in $\mathcal{A}$ must satisfy either $S_1 = S_2 = 0$, where

$$S_1(t,x) = a_{1x} - 2a_{2t}, \qquad S_2(t,x) = (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1, \quad (1.18)$$

or, if $S_1 \neq 0$, $S_3 = S_4 = 0$, where

$$S_3(t,x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}), \qquad S_4(t,x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x}. \quad (1.19)$$

The equations in $\mathcal{A}$ such that $S_1 = S_2 = 0$ constitute the subclass $\mathcal{A}_1$ and they admit two functionally independent first integrals of the form $A(t,x)\dot{x} + B(t,x)$.

Several properties on the linearization through local and nonlocal transformations of the equations in $\mathcal{B}$ have been derived in [72] and [71]. All the equations in $\mathcal{A}_1$ pass the Lie test of linearization (i. e. their coefficients satisfy $L_1 = L_2 = 0$). On the contrary, none of the equations in $\mathcal{A}_2$ can be linearized through a local transformation; actually, there exist equations in $\mathcal{A}_2$ that lack Lie point symmetries (see, for example, equations (2.6) and (4.12) in [70]).

Although there exist equations (1.17) whose coefficients satisfy $L_1 = L_2 = 0$ that are not in $\mathcal{A}_1$ (see example 9 in [72]), all of them must belong to $\mathcal{B}$. It is important to remark that there are equations in $\mathcal{B}$ not linearizable through local transformations, apart from the subclass $\mathcal{A}_2$ (as the family appearing in example 2.1 in [73]). In order to linearize such type of equations one has to consider nonlocal transformations of the form

$$X = F(t,x), \qquad \mathrm{d}T = (G_1(t,x)\dot{x} + G_2(t,x))\mathrm{d}t. \quad (1.20)$$

The equations in $\mathcal{B}$ can be characterized as the unique ODEs (1.17) that can be linearized through some nonlocal transformation of the form (1.20). When $G_1(t,x) = 0$ in (1.20), the equation must belong to $\mathcal{A}_2$ and conversely. In other words, the equations in $\mathcal{A}_2$ are the unique ODEs (1.17) that can be transformed into the linear equation $X_{TT} = 0$ by means of some nonlocal transformation of the form

$$X = F(t,x), \qquad \mathrm{d}T = G(t,x)\mathrm{d}t. \quad (1.21)$$

These transformations are known in the literature as *generalized Sundman transformations* (see [65],[66], [74, 75, 76, 77, 78] and references therein). Constructive methods to determine nonlocal linearizing transformations can be derived from the algorithms that calculate the first integrals ([72], [71]). In particular, local changes of variables that linearize the equations in $\mathcal{A}_1$ can be determined by just dealing with first order ODEs. We remark that such linearizing point transformations usually appear in the literature as solutions of an involutive system of second-order partial differential equations ([79],[80]).

Complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients

Recent works by C.Wafo Soh [81] have focused on the study of systems of second-order ordinary differential equations with constant coefficients. The studies deal with symmetries of systems of linear second-order ordinary differential equations with two and three equations are considered. The goal of the present research is to study the symmetry structure of a system of n, $(n = 2, 3)$ linear second-order ordinary differential equations with constant coefficients. Since a change of the dependent and independent variables does not change the structure of the admitted Lie group, the author at first simplify the system, and then calculate the admitted Lie group of the simplified system using the standard procedure.

The paper [81] concentrated attention on the system of n equations of the form

$$\ddot{\mathbf{x}} = M\mathbf{x}, \tag{1.22}$$

where the overdot denotes differentiation with respect to t, $\mathbf{x}$ is an n-dimensional vector with complex entries, and M is an $n \times n$ matrix with complex entries.

A first simplification of system (1.22) is achieved by using the Jordan normal form J of the matrix M,

$$M = P^{-1}JP.$$

The change $\mathbf{u} = P\mathbf{x}$ reduces system (1.22) to the system

$$\ddot{\mathbf{u}} = J\mathbf{u}. \tag{1.23}$$

Next, a simplification of system (1.23) is made for coefficients corresponding to diagonal blocks of the Jordan matrix J: the authors applied scaling. This step is crucial in both papers. In fact, in case of a diagonal block of the Jordan matrix J, scaling of the dependent variables does not change the coefficients of this part of the system. Hence, one can conclude that scaling is applied to the independent variable. Because the independent variable is real valued, one can only make a real valued scaling. This allows one to reduce only one component of a diagonal coefficient: either the real or imaginary part of the eigenvalue of the Jordan matrix J. Whereas in [81] for any eigenvalue (including a complex eigenvalue) the corresponding coefficients are reduced to the real number 1. This means that the author [81] considered in the corresponding cases only real eigenvalues. Therefore, the results of [81] are not complete. For a complete study one also needs to study complex eigenvalues corresponding to diagonal Jordan blocks.

It is also worth to notice that from the paper it is unclear how the author calculated an admitted Lie algebra. In the standard procedure, the dependent and independent variables are real-valued, whereas the results of [81] are obtained for n complex-valued dependent variables. Does this mean that the authors considered $2n$ real-valued equations for calculating the admitted Lie algebra?

The goal of the present research is to correct the approach applied in [81]. The approach is illustrated by using a complete study of symmetry structures of systems of two real-valued linear second-order ordinary differential equations with constant coefficients. In application of this approach to a system with more than two equations one needs to take into account that if a real-valued matrix M has a complex eigenvalue, then the conjugate number is also an eigenvalue. Only systems of two second-order equations are considered here.

1.2 Objectives

1.2.1 Fluids with internal inertia

The research is focused on the group classification of the one-dimensional equations of fluids (1.1), where the function $W = W(\rho, \dot{\rho}, S)$ satisfies the conditions $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$.

1.2.2 Applications of group analysis to integro-differential equations

The Rudenko equation

The objective is to apply the recently developed approach of the group analysis method for the Rudenko equation. In the present research the objective is to find admitted Lie group of transformations using the explained above method.

The Boltzmann equation with sources

In the paper [48] the equation for generating function of the power moments of the Boltzmann equation solution was considered. However, this equation is still a nonlocal partial differential equation, and this property was not taken into account there. The objective of the present research is to apply the group analysis method developed recently for equations with nonlocal operators, and to make a group classification of the equation for the generating function with respect to sources.

1.2.3 Application of group analysis to ordinary differential equations

On first integrals of second-order ordinary differential equations

The objective of this part of the research is to give a criteria of the existence of a first integral of the form

$$I = \frac{\dot{x}A(t,x) + C(t,x)}{\dot{x}B(t,x) + Q(t,x)}.$$

for a second-order ordinary differential equation

$$\ddot{x} + a_3(t,x)\dot{x}^3 + 3a_2(t,x)\dot{x}^2 + 3a_1(t,x)\dot{x} + a_0(t,x) = 0.$$

Complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients

The objective is to make a complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients:

$$\ddot{\mathbf{x}} = M\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

1.3 Scope of the Work

1.3.1 Overall scope and assumptions of the group classification of fluids with internal inertia

The research is restricted by the study only the case where the function $W = W(\rho, \dot{\rho}, S)$ satisfies the conditions $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$.

1.3.2 Applications of group analysis to integro-differential equations

Overall scope and assumptions of the study of the Rudenko equation

Since it is difficult to find the general solution of the determining equations (3.10) and (3.11), the following simplification is considered. One can assume that the determining equation (3.10) is valid for any functions $u(t, x, y, z)$ and $w(t, x, y, z)$ only satisfying the first equation of (1.2). This allows to use standard procedure for solving determining equations developed for partial differential equations. After solving the determining equation (3.10), one can use the found solution for solving the integral determining equation (3.11).

Overall scope and assumptions of the study of the Boltzmann equation with sources

In the project the equation for generating function of the power moments of the Boltzmann is analyzed.

1.3.3 Application of group analysis to ordinary differential equations

Overall scope and assumptions of the study first integrals of second-order ordinary differential equations

The criteria for existence of first integrals of the form

$$I = \frac{\dot{x}A(t, x) + C(t, x)}{\dot{x}B(t, x) + Q(t, x)}.$$

for a second-order ordinary differential equation

$$\ddot{x} + a_3(t, x)\dot{x}^3 + 3a_2(t, x)\dot{x}^2 + 3a_1(t, x)\dot{x} + a_0(t, x) = 0$$

only with $L_2 = 0$ is studied.

Overall scope and assumptions of the study of the group classification of systems of two linear second-order ordinary differential equations with constant coefficients

In the research we only study systems with two linear second-order ordinary differential equations with constant coefficients of the form

$$\ddot{\mathbf{x}} = M\mathbf{x}.$$

1.4 Outcomes of the research

The outcomes of this research are 5 papers in International journals and 1 paper in Proceedings of International conference with peer reviewing:

1. **S.V.Meleshko**
Comment on "Symmetry breaking of systems of linear second-order ordinary differential equations with constant coefficients", Communications in Nonlinear Science and Numerical Simulations, 2011, **16**(9), pp.3447-3450.

2. **Ibragimov N.H., Meleshko S.V., Rudenko O.V.**
Group analysis of evolutionary integro-differential equations describing nonlinear waves: General model. J. of Phys. A: Math. and Theor., 2011, **44**(31), art.no. 315201.

3. **P.Siriwat, S.V.Meleshko**
Group classification of one-dimensional non-isentropic equations of fluids with internal inertia. Continuum Mechanics and Thermodynamics, 2012, 24:115-148 (DOI 10.1007/s00161-011-0209-6).

4. **Yu.N.Grigoriev, S.V.Meleshko**
On group classification of the spatially homogeneous and isotropic Boltzmann equation with sources. International Journal of Non-Linear Mechanics, 2012, **47**, 1014–1019.

5. **S.V.Meleshko, S.Moyo, C.Muriel, J.L.Romero, P.Guha and A.G.Choudhury**
On first integrals of second-order ordinary differential equations J. Eng. Math. (DOI 10.1007/s10665-012-9590-9)

6. **Yu.N.Grigoriev, S.V.Meleshko and A.Suriyawichitseranee**
On the equation for the power moment generating function of the Boltzmann equation. Group classification with respect to a source function. Proc. 6th Workshop "Group Analysis of Differential Equations & Integrable Systems" (Protaras, Cyprus, 2012), University of Cyprus, Nicosia, 2013, pp. 98-110

Chapter 2

Group classification of one-dimensional nonisentropic equations of fluids with internal inertia

Abstract A systematic application of the group analysis method for modeling fluids with internal inertia is presented. The equations studied include models such as the non-linear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski (1960), Kogarko (1961), Wijngaarden (1968)), and the dispersive shallow water model (Green & Naghdi (1976), Salmon (1988)). These models are obtained for special types of the potential function $W(\rho,\dot{\rho},S)$ (Gavrilyuk & Teshukov (2001)). The main feature of the present research is the study of the potential functions with $W_S \neq 0$. The group classification separates these models into 73 different classes.

The present research is focused on the group classification of the one-dimensional equations of fluids (1.1), where the function $W = W(\rho,\dot{\rho},S)$ satisfies the conditions $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$.

The report is organized as follows. The next section studies the equivalence Lie group of transformations. The equivalence transformations are applied for simplifying the function $W(\rho,\dot{\rho},S)$ in the process of the classification. In Section 3 the defining equations of the admitted Lie group are presented. Analysis of these equations separates equations (1.1) into equivalent classes. Notice that these classes are defined by the function $W(\rho,\dot{\rho},S)$. For convenience of the reader, this analysis is split into two parts. A complete study of one particular case is given in Section 4. Analysis of the other cases is similar but cumbersome. A complete study of the other cases is provided in Appendix. The result of the group classification of equations (1.1) where $W_{S\dot{\rho}\dot{\rho}} = 0$ and $W_S \neq 0$ is summarized in Table 2.1. The admitted Lie algebras are also presented in this table.

2.1 Equivalence Lie group

For finding an equivalence Lie group the algorithm described in [82, 45] is applied. This algorithm differs from the classical one [8] by assuming dependence of all coefficients from all variables including the arbitrary elements. Since the function W depends on the derivatives of the dependent variables and in order to simplify the process of finding an equivalence Lie group, new dependent variables are introduced:

$$u_3 = \dot{\rho}\ , u_4 = S.$$

Here $u_1 = \rho,\ u_2 = u,\ u_3 = \dot{\rho}$ and $u_4 = S,\ x_1 = x,\ x_2 = t$. An infinitesimal operator X^e of the equivalence Lie group is sought in the form [45]

$$X^e = \xi^i \partial_{x_i} + \zeta^{u_j} \partial_{u_j} + \zeta^W \partial_W,$$

where all the coefficients $\xi^i, \zeta^{u_j},\ (i = 1,2;\ j = 1,2,3,4)$ and ζ^W are functions of the variables $x,\ t,\ \rho,\ u,\ \dot{\rho},\ S,\ W$. Hereafter a sum over repeated indices is implied. The coefficients of the prolonged operator are obtained by using the prolongation formulae:

$$\zeta^{u_{\beta,i}} = D^e_i \zeta^{u_\beta} - u_{\beta,1} D^e_i \xi^x - u_{\beta,2} D^e_i \xi^t,\ (i = 1,2),$$

$$D^e_1 = \partial_x + u_{\beta,1}\partial_{u_\beta} + (\rho_x W_{\alpha,1} + \dot{\rho}_x W_{\alpha,2} + S_x W_{\alpha,3})\partial_{W_\alpha},$$
$$D^e_2 = \partial_t + u_{\beta,2}\partial_{u_\beta} + (\rho_t W_{\alpha,1} + \dot{\rho}_t W_{\alpha,2} + S_t W_{\alpha,3})\partial_{W_\alpha},$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $\beta = (\beta_1, \beta_2)$ are multi-indexes $(\alpha_i \geq 0)$, $(\beta_i \geq 0)$

$$(\alpha_1, \alpha_2, \alpha_3), 1 = (\alpha_1 + 1, \alpha_2, \alpha_3), \quad (\alpha_1, \alpha_2, \alpha_3), 2 = (\alpha_1, \alpha_2 + 1, \alpha_3), \quad (\alpha_1, \alpha_2, \alpha_3), 3 = (\alpha_1, \alpha_2, \alpha_3 + 1)$$

$$(\beta_1, \beta_2), 1 = (\beta_1 + 1, \beta_2), \quad (\beta_1, \beta_2), 2 = (\beta_1, \beta_2 + 1)$$

$$u_{(\beta_1,\beta_2)} = \frac{\partial^{\beta_1+\beta_2} u}{\partial x^{\beta_1} \partial t^{\beta_2}}, \quad W_{(\alpha_1,\alpha_2,\alpha_3)} = \frac{\partial^{\alpha_1+\alpha_2+\alpha_3} W}{\partial \rho^{\alpha_1} \partial \dot{\rho}^{\alpha_2} \partial S^{\alpha_3}}.$$

The conditions that W does not depend on t, x and u give

$$\zeta^{u_1}_{x_i} = 0,\ \zeta^{u_1}_u = 0,\ \ \zeta^{u_3}_{x_i} = 0,\ \zeta^{u_3}_u = 0,\ \ \zeta^{u_4}_{x_i} = 0,\ \zeta^{u_4}_u = 0,\ \zeta^W_{x_i} = 0,\ \zeta^W_{u_j} = 0,\ \ (i = 1,2).$$

Using these relations, the prolongation formulae for the coefficients ζ^{W_α} become:

$$\zeta^{W_{\alpha,i}} = \widetilde{D}^e_i \zeta^{W_\alpha} - W_{\alpha,1} \widetilde{D}^e_i \zeta^{u_1} - W_{\alpha,2} \widetilde{D}^e_i \zeta^{u_3} - W_{\alpha,3} \widetilde{D}^e_i \zeta^{u_4},\ (i = 1,2),$$

$$\widetilde{D}^e_1 = \partial_\rho + W_{\alpha,1}\partial_{W_\alpha},\ \ \widetilde{D}^e_2 = \partial_{\dot{\rho}} + W_{\alpha,2}\partial_{W_\alpha},\ \ \widetilde{D}^e_3 = \partial_S + W_{\alpha,3}\partial_{W_\alpha}.$$

For constructing the determining equations and for their solution, the symbolic computer Reduce [83] program was applied. Calculations give the following basis of generators of the equivalence Lie group

$$X^e_1 = \partial_x,\ X^e_2 = \partial_t,\ X^e_3 = t\partial_x + \partial_u,\ X^e_4 = t\partial_t + x\partial_x,$$
$$X^e_5 = t\partial_t + 2\rho\partial_\rho - u\partial_u,\ X^e_6 = \partial_W,\ X^e_7 = -u\partial_u + \rho\partial_\rho - W\partial_W + t\partial_t,$$
$$X^e_8 = \rho\varphi(S)\partial_W,\ X^e_9 = \dot{\rho} g(\rho, S)\partial_W,\ X^e_{10} = h(S)\partial_S,$$

where the functions $g(\rho, S), \varphi(S)$ and $h(S)$ are arbitrary. Here only the essential part of the operators X^e_i, $(i = 5, 6, ..., 10)$ is written.

Since the equivalence transformations corresponding to the operators X^e_5 , X^e_6, X^e_7, X^e_8,X^e_9 and X^e_{10} are applied for simplifying the function W in the process of the group classification, let us present these transformations. Because the function W depends on ρ, $\dot{\rho}$ and S only, the transformations of these variables are presented:

$$\begin{array}{rllll}
X^e_5: & \rho' = \rho e^{2a}, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = W; \\
X^e_6: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = We^{-2a}; \\
X^e_7: & \rho' = \rho e^{a}, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = W + a; \\
X^e_8: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = \rho\varphi(S)a + W; \\
X^e_9: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = S & W' = \dot{\rho} g(\rho, S)a + W \\
X^e_{10}: & \rho' = \rho, & \dot{\rho}' = \dot{\rho}, & S' = q(S, a) & W' = W;
\end{array}$$

Here a is the group parameter.

2.2 Defining equations of the admitted Lie group

An admitted generator X is sought in the form

$$X = \xi^x\partial_x + \xi^t\partial_t + \zeta^\rho\partial_\rho + \zeta^u\partial_u + \zeta^S\partial_S,$$

where the coefficients $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^S$ are functions of (x, t, ρ, u, S). Calculations showed that

$$\xi^x = (k_2t + k_3)x + k_4t + k_5, \quad \xi^t = k_2t^2 + (k_1 + 2k_3)t + k_7.$$

$$\zeta^\rho = \rho(k_8 - k_2t),\ \zeta^u = k_2(-ut + x) - u(k_1 + k_3) + k_4,$$

$$\zeta^S = \zeta^S(S),$$

$$k_2(3W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\dot\rho\dot\rho\rho}\rho + 3W_{\dot\rho\dot\rho}) = 0, \tag{2.1}$$

$$\begin{gathered} -3W_{\dot\rho\dot\rho S}\dot\rho\zeta^S + 3W_{\dot\rho\dot\rho\dot\rho}\dot\rho^2(k_1 + 2k_3 - k_8) - 3W_{\dot\rho\dot\rho\rho}\dot\rho\rho k_8 \\ +3W_{\dot\rho\dot\rho}\dot\rho(2k_3 - k_8) - \rho k_2(3W_{\dot\rho\dot\rho} + W_{\dot\rho\dot\rho\rho}\rho) = 0, \end{gathered} \tag{2.2}$$

$$k_2(W_{\rho\rho\rho\dot\rho}\dot\rho\rho - W_{\rho\rho\dot\rho}\dot\rho + 3W_{\rho\rho\dot\rho\dot\rho}\dot\rho^2 - W_{\rho\rho\rho}\rho + W_{\rho\rho}) = 0, \tag{2.3}$$

$$\begin{gathered} \zeta^S\rho(W_{\rho\rho S} - W_{\rho\rho\dot\rho S}\dot\rho) - W_{\rho\rho\rho\dot\rho}\dot\rho\rho^2k_8 - W_{\rho\rho\dot\rho}\dot\rho\rho(2k_1 + 2k_3 + k_8) \\ +W_{\rho\rho\dot\rho\dot\rho}\dot\rho^2\rho(k_1 + 2k_3 - k_8) + W_{\rho\rho\rho}\rho^2k_8 + W_{\rho\rho}\rho(2k_1 + 2k_3 + k_8) \\ +k_2\dot\rho(W_{\rho\rho\dot\rho\dot\rho}\rho^2 + 5W_{\dot\rho\dot\rho\rho}\rho + 3W_{\dot\rho\dot\rho}) = 0, \end{gathered} \tag{2.4}$$

$$k_2(W_{\rho\rho\dot\rho S}\dot\rho\rho^2 - 3W_{\rho\dot\rho S}\dot\rho\rho + 3W_{\rho\dot\rho\dot\rho S}\dot\rho^2\rho - 3W_{\dot\rho\dot\rho S}\dot\rho^2 + 3W_{\dot\rho S}\dot\rho - W_{\rho\rho S}\rho^2 + 3W_{\rho S}\rho - 3W_S) = 0, \tag{2.5}$$

$$\begin{gathered} -W_{\rho\rho\dot\rho S}\dot\rho\rho^2k_8 - 2W_{\rho\dot\rho S}\dot\rho\rho k_1 - 2W_{\rho\dot\rho S}\dot\rho\rho k_3 + W_{\rho\dot\rho S}\dot\rho\rho k_8 + W_{\rho\dot\rho\dot\rho S}\dot\rho^2\rho(2k_3 + k_1 - k_8) \\ +W_{\dot\rho\dot\rho S}\dot\rho^2(k_8 - k_1 - 2k_3) + W_{\dot\rho S}\dot\rho(2k_1 + 2k_3 - k_8) + W_{\rho\rho S}\rho^2k_8 + W_{\rho S}\rho(2k_1 + 2k_3 - k_8) \\ +W_S(k_8 - 2k_1 - 2k_3) + \dot\rho\rho k_2(W_{\rho\dot\rho\dot\rho S}\rho + 3W_{\dot\rho\dot\rho S}) + \zeta^S_S(-W_{\rho\dot\rho S}\dot\rho\rho + W_{\dot\rho S}\dot\rho + W_{\rho S}\rho - W_S) \\ +\zeta^S(-W_{\rho\dot\rho SS}\dot\rho\rho + W_{\dot\rho SS}\dot\rho + W_{\rho SS}\rho - W_{SS}) = 0, \end{gathered} \tag{2.6}$$

where k_i, $(i = 1, 2, \ldots, 8)$ are constant. The determining equations (2.1)–(2.6) define the kernel of admitted Lie algebras and its extensions. The kernel of admitted Lie algebras is determined for all functions $W(\rho, \dot\rho, S)$ and it consists of the generators

$$Y_4 = \partial_x, Y_5 = \partial_t, Y_6 = t\partial_x + \partial_u.$$

Extensions of the kernel depend on the value of the function $W(\rho, \dot\rho, S)$. They can only be operators of the form

$$k_1X_1 + k_2X_2 + k_3X_3 + k_8X_8 + \zeta^S\partial_S,$$

where

$$\begin{gathered} X_1 = t\partial_t - u\partial_u - \dot\rho\partial_{\dot\rho}, \\ X_2 = t^2\partial_t + tx\partial_x + (x - ut)\partial_u - t\rho\partial_\rho - (\rho + 3t\dot\rho)\partial_{\dot\rho}, \\ X_3 = 2t\partial_t + x\partial_x - u\partial_u - 3\dot\rho\partial_{\dot\rho} - \rho\partial_\rho, \\ X_8 = \rho\partial_\rho + \dot\rho\partial_{\dot\rho}. \end{gathered}$$

Since the function $W(\rho, \dot\rho, S)$ depends on $\dot\rho$, the term with $\partial_{\dot\rho}$ is also presented in the generators.

Relations between the constants $k_1,\ k_2,\ k_3,\ k_8$ and $\zeta^S(S)$ depend on the function $W(\rho, \dot\rho, S)$.

2.3 Case $k_2 \neq 0$

If $k_2 \neq 0$, then equation (2.1) gives

$$3W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\rho\dot\rho\dot\rho}\rho + 3W_{\dot\rho\dot\rho} = 0.$$

The general solution of this equation is

$$W(\rho, \dot\rho, \tilde S) = \rho^3 g(z, S) + \varphi_0(\rho, S), \qquad (2.7)$$

where $z = \dot\rho\rho^{-3}$. Substituting (8) into (4), one obtains

$$\rho\varphi_{0\rho\rho\rho} - \varphi_{0\rho\rho} = 0.$$

The general solution of this equation is

$$\varphi_0 = \rho^3\mu(S) + \rho I(S) + J(S), \qquad (2.8)$$

where without loss of the generality by virtue of the equivalence transformation corresponding to the operator X_8^e, it can be assumed that $I(S) = 0$. Equation (6) gives that $J' = 0$. By virtue of the equivalence transformation corresponding to X_7^e, it can also be assumed that $J = 0$. Substituting the obtained W into (2.2) and splitting it with respect to ρ, one obtains $g_{zzz} = 0$ or $g = \varphi_2(S)z^2$, where $\varphi_2 \neq 0$. Notice that the linear part of the function φ_2 is also omitted because of the equivalence transformations corresponding to the generator X_9^e. The remaining part of equation (2.2) becomes

$$\varphi_2'\zeta^S - 2\varphi_2(k_3 + k_8) = 0. \qquad (2.9)$$

If $\varphi_2' = 0$ or $\varphi_2 = q \neq 0$, then $k_3 = -k_8$ and equation (5) becomes

$$\mu'\zeta^S + 2k_1\mu = 0. \qquad (2.10)$$

For $\mu' = 0$ the function W does not depend on S. Since this case has been studied in [14], it is excluded from further study in the present research. Thus, one has to assume that $\mu' \neq 0$. From (2.10) one gets $\zeta^S = -2k_1\mu/\mu'$. Changing the entropy $\tilde S = \mu(S)$, one has

$$W(\rho, \dot\rho, \tilde S) = q\frac{\dot\rho^2}{\rho^3} + \rho^3\tilde S,$$

and the extension of the kernel is given by the generators

$$X_1 - 2\tilde S\partial_{\tilde S}, \quad X_2, \quad X_3 - X_8.$$

In the final Table 1 this is model M_1, where the tilde sign is omitted.

If $\varphi_2' \neq 0$, then from (2.2) and (2.9), one obtains

$$\zeta^S = 2\frac{\varphi_2}{\varphi_2'}(k_3 + k_8),$$

$$\mu'\varphi_2(k_3 + k_8) + \varphi_2'\mu(k_1 + k_3 + k_8) = 0. \qquad (2.11)$$

If $\mu \neq 0$ then, the last equation defines

$$k_1 = -(k_3 + k_8)(1 + \frac{\mu'\varphi_2}{\mu\varphi_2'}). \qquad (2.12)$$

Differentiation (2.12) with respect to S gives

$$(k_3 + k_8)(\frac{\mu'\varphi_2}{\mu\varphi_2'})' = 0. \tag{2.13}$$

If $(\frac{\mu'\varphi_2}{\mu\varphi_2'})' = 0$ or $\mu = q_1\varphi_2^k$, then the general solution of equations (2.1) - (2.6) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S} + q_1\rho^3\tilde{S}^k,$$

where $\tilde{S} = \varphi_2(S)$. The extension of the kernel is given by the generators

$$X_2, \quad X_3 - X_8, \quad X_8 - (k+1)X_1 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this is model M_2.

If $(\frac{\mu'\varphi_2}{\mu\varphi_2'})' \neq 0$, then the general solution of equations (2.1) - (2.6) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S} + \rho^3\mu(\tilde{S}), \quad (\mu \neq q_1\tilde{S}^k),$$

and the extension of the kernel is given by the generators

$$X_2, \quad X_3 - X_8.$$

In the final Table 1 this is model M_3.

If $\mu = 0$, then

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1, \quad X_2, \quad X_3 - X_8, \quad X_8 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this is model M_4.

Remark. The last two cases do not satisfy the restriction $W_{\dot{\rho}\dot{\rho}S} \neq 0$ announced in the title. For the case where $k_2 \neq 0$ it is not necessary to separate the study into the cases $W_{\dot{\rho}\dot{\rho}S} \neq 0$ and $W_{\dot{\rho}\dot{\rho}S} = 0$. Whereas for the analysis of the case where $k_2 = 0$, one needs to make this separation.

2.4 Results of the group classification

The result of the group classification of equations (1.1) is summarized in Table 2.1. The linear part with respect to $\dot{\rho}$ of the function $W(\rho, \dot{\rho}, S)$ is omitted. The equivalence transformation corresponding to the operator X_{10}^e is also used. This transformation allows one to simplify the dependence on entropy of the function $W(\rho, \dot{\rho}, S)$.

The first column in Table 2.1 presents the number of the extension, forms of the function $W(\rho, \dot{\rho}, S)$ are presented in the second column, extensions of the kernel of admitted Lie algebras are given in the third column, restrictions for functions and constants are in the fourth column.

Table 2.1: Group classification

	$w(\rho,\dot\rho,S)$	Extensions	Remarks
M_1	$q_0\rho^{-3}\dot\rho^2+\rho^3S$	$X_1-2S\partial_S,\quad X_2,\quad X_3-X_8$	
M_2	$\rho^{-3}\dot\rho^2S+q_1\rho^3S^k$	$X_2,\quad X_3-X_8,\quad X_8-(k+1)X_1+2S\partial_S$	
M_3	$\rho^{-3}\dot\rho^2S+\rho^3\mu(S)$	$X_2,\quad X_3-X_8$	$\mu'\neq q_1S^k$
M_4	$\rho^{-3}\dot\rho^2S$	$X_1,\quad X_2,\quad X_3-X_8,\quad X_8+2S\partial_S$	
M_5	$\phi(\rho,\dot\rho)+S$	∂_S	
M_6	$\varphi(\rho)\dot\rho^2+S$	$\partial_S,\quad X_1-2S\partial_S$	
M_7	$\varphi(\rho)\dot\rho^2+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_8	$\varphi(\rho)\dot\rho^2+\eta(\rho)S$	$X_1-2S\partial_S$	$\eta''\neq 0$
M_9	$\varphi(\rho)\dot\rho\ln\lvert\dot\rho\rvert+S$	$X_3-2S\partial_S,\quad \partial_S$	
M_{10}	$\varphi(\rho)\dot\rho\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_{11}	$\varphi(\rho)\dot\rho\ln\lvert\dot\rho\rvert+\eta(\rho)S$	$X_3-2S\partial_S$	$\eta''\neq 0$
M_{12}	$(q_1\rho+q_0)\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	$X_3-X_1,\quad \partial_S$	$q_0^2+q_1^2\neq 0$
M_{13}	$\varphi(\rho)\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	∂_S	$\varphi''\neq 0$
M_{14}	$(q_1\rho+q_0)\ln\lvert\dot\rho\rvert+h(\rho,S)$	X_3-X_1	$(q_0^2+q_1^2)h_{\rho\rho S}\neq 0$
M_{15}	$\varphi(\rho)(\ln\lvert\dot\rho\rvert+S)+\eta(\rho)+q_2S$	$X_3-X_1+\partial_S$	$\varphi''\neq 0$
M_{16}	$\varphi(\rho)\dot\rho^{k+2}+S$	$kX_3-2(k+1)X_1+2(k+2)S\partial_S,\quad \partial_S$	$k(k+1)(k+2)\neq 0$
M_{17}	$\varphi(\rho)\dot\rho^{k+2}+\eta(\rho)+S$	∂_S	$\eta''\neq 0$
M_{18}	$\varphi(\rho)\dot\rho^{k+2}+\eta(\rho)S$	$kX_3-2(k+1)X_1+2(k+2)S\partial_S$	$\eta''\neq 0$
M_{19}	$\rho^\lambda g(\dot\rho\rho^k)+\eta(\rho)+S$	∂_S	
M_{20}	$g(\dot\rho\rho^k)+q_1\rho^2+S$	$-2kX_1+(2k+1)X_3+2X_8,\quad \partial_S$	
M_{21}	$\rho g(\dot\rho\rho^k)+q_1\rho\ln\rho+S$	$(k+1)(X_3-X_1)+X_8+S\partial_S,\quad \partial_S$	
M_{22}	$\rho^\lambda g(\dot\rho\rho^k)+S$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3$ $+2X_8+2\lambda S\partial_S,\quad \partial_S$	$\lambda(\lambda-1)\neq 0$
M_{23}	$g(\dot\rho\rho^k)+S\ln\rho$	$-2kX_1+(2k+1)X_3+2X_8$	
M_{24}	$\rho g(\dot\rho\rho^k)+S\rho\ln\rho$	$(k+1)(X_3-X_1)+X_8$	
M_{25}	$\rho^\lambda(g(\dot\rho\rho^k)+S)$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3+2X_8$	$\lambda(\lambda-1)\neq 0$
M_{26}	$g(\dot\rho\rho^k)+Q(\rho S)+q_1\ln S$	$-2kX_1+(2k+1)X_3+2X_8-2S\partial_S$	
M_{27}	$\rho^\lambda g(\dot\rho\rho^k)+Q(\rho S)$	$-2(k+\lambda)X_1+(2k+\lambda+1)X_3+2X_8-2S\partial_S$	$\lambda\neq 0$
M_{28}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S$	$X_3-2S\partial_S,\quad X_8-\lambda X_1+(2\lambda+1)S\partial_S,\quad \partial_S$	
M_{29}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	$2X_8-2\lambda X_1+(2\lambda-\nu-1)X_3+2(\nu+2)S\partial_S,\quad \partial_S$	$\eta''=q_1\rho^\nu\neq 0$
M_{30}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	∂_S	$\eta''\neq q_1\rho^\nu,\quad \eta''\neq 0$
M_{31}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+\eta(\rho)S$	$X_3-2S\partial_S$	$\eta''\neq 0$
M_{32}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho\ln\rho$	$X_3-2S\partial_S,\quad -\lambda X_1+X_8+2\lambda S\partial_S$	
M_{33}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S(\rho\ln\rho+q_1)$	$X_3-2S\partial_S$	$q_1\neq 0$
M_{34}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho\ln\rho+q_2\ln S$	$2(X_8-\lambda X_1+\lambda X_3)+X_3-2S\partial_S$	$q_2\neq 0$
M_{35}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho\ln\rho+q_2S^\alpha$	$2(1-\alpha)(X_8-\lambda X_1+\lambda X_3)+X_3-2S\partial_S$	$q_2\alpha(\alpha-1)\neq 0$
M_{36}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S(\ln\rho+q_1)$	$X_3-2S\partial_S$	
M_{37}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S(\ln\rho+q_1)+\alpha S\ln S$	$-2\lambda\alpha X_1+(2\alpha\lambda+1)X_3+2\alpha X_8+2(\alpha-1)S\partial_S$	$\alpha\neq 0$
M_{38}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho^{\nu+2}$	$X_3-2S\partial_S,\quad -\lambda X_1+X_8+(2\lambda-\nu-1)S\partial_S$	$(\nu+2)(\nu+1)\neq 0$
M_{39}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S(\rho^{\nu+2}+q_1)$	$X_3-2S\partial_S$	$q_1(\nu+2)(\nu+1)\neq 0$
M_{40}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho^{\nu+2}+q_1\ln S$	$-2\lambda X_1+(2\lambda+1)X_3+2X_8-2(\nu+2)S\partial_S$	$q_1(\nu+2)(\nu+1)\neq 0$
M_{41}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+S\rho^{\nu+2}+q_2S^\alpha$	$2\lambda(\alpha-1)X_1+(\alpha(\nu+1-2\lambda)+2\lambda+1)X_3$ $+2(1-\alpha)X_8-2(\nu+2)S\partial_S$	$(\nu+2)(\nu+1)\neq 0,$ $q_2\alpha(\alpha-1)\neq 0$
M_{42}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+g(\rho S)+q_2\ln S$	$-2\lambda X_1+(2\lambda+1)X_3+2X_8-2S\partial_S$	
M_{43}	$q_0\dot\rho\rho^\lambda\ln\lvert\dot\rho\rvert+\rho^\nu g(\rho S)+q_2S^{-\nu}$	$-2\lambda X_1+(2\lambda-\nu+1)X_3+2X_8-2S\partial_S$	$\nu\neq 0$
M_{44}	$\rho^\lambda(q_1+q_0\ln(\rho^\nu\lvert\dot\rho\rvert))+S$	$-2(\lambda+\nu)X_1+(\lambda+2\nu+1)X_3$ $+2X_8+2\lambda S\partial_S,\quad \partial_S$	
M_{45}	$q_0\rho^\lambda\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	∂_S	$\eta\neq\rho^\lambda(q_1+q_2\ln\rho)$
M_{46}	$q_0\rho^\lambda\ln\lvert\dot\rho\rvert+\eta(\rho)+S$	$X_3-X_1,\quad \partial_S$	$\eta''\neq q_1\rho^{(\lambda-2)},$ $\lambda(\lambda-1)=0$
M_{47}	$q_0\rho^\lambda\ln(\lvert\dot\rho\rvert\rho^\nu)+S$	$X_3-X_1,\quad (1-\lambda)X_1+2X_8+2\lambda S\partial_S,\quad \partial_S$	$\lambda(\lambda-1)=0$
M_{48}	$q_0\ln\lvert\dot\rho\rvert+S\ln\rho+f(S)$	X_3-X_1	
M_{49}	$\rho(q_0\ln\lvert\dot\rho\rvert+S\ln\rho)+f(S)$	X_3-X_1	$f'\neq 0$
M_{50}	$\rho(q_0\ln\lvert\dot\rho\rvert+S\ln\rho)$	$X_3-X_1,\quad X_8$	
M_{51}	$q_0\ln\lvert\dot\rho\rvert+\phi(\rho S)+q_1\ln S$	$X_3-X_1,\quad X_1+2X_8-2S\partial_S$	$(z\phi(z)')''\neq 0$
M_{52}	$\rho(q_0\ln\lvert\dot\rho\rvert+\phi(\rho S))+q_1S^{-1}$	$X_3-X_1,\quad X_8-S\partial_S$	$(z\phi(z)')''\neq 0$
M_{53}	$\rho^\lambda(q_0\ln\lvert\dot\rho\rvert+\phi(\rho S))+f(S)$	X_3-X_1	$\lambda(\lambda-1)=0,$ $(Sf'+\lambda f)'\neq 0,$ $(z\phi(z)')''\neq 0$
M_{54}	$q_0\rho^\lambda(\ln\lvert\dot\rho\rvert+S)+\eta(\rho)+q_1S$	$X_3-X_1+\partial_S$	$\lambda(\lambda-1)\neq 0,$ $\eta''\neq\rho^{\lambda-2}(\nu\ln\rho+q_2)$
M_{55}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda$	$X_3-X_1+\partial_S,$ $2(\lambda-1)X_1+2X_8-(\lambda+2\nu+1)\partial_S$	$\lambda(\lambda-1)\neq 0$
M_{56}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda+q_2S$	$X_3-X_1+\partial_S$	$q_2\lambda(\lambda-1)\neq 0$
M_{57}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+S)+q_1\rho^\lambda+q_2e^{\kappa S}$	$-2(\kappa(\lambda+\nu)+\lambda)X_1+2\kappa X_8+2\lambda\partial_S$ $+(2\lambda+\kappa(\lambda+2\nu+1))X_3$	$q_2\lambda(\lambda-1)\neq 0$
M_{58}	$q_0\rho^\lambda(\ln(\lvert\dot\rho\rvert\rho^\nu)+g(\rho S))+q_2S^{-\lambda}$	$-2(\lambda+\nu)X_1+(\lambda+2\nu+1)X_3+2X_8-2S\partial_S$	$\lambda(\lambda-1)\neq 0,$ $(z^{\lambda+1}g'(z))''\neq 0$
M_{59}	$q_0\rho^\lambda\dot\rho^{k+2}+\eta(\rho)+S$	$-2(2k+\lambda+2+(k+1)\nu)X_1$ $+(k\nu+3k+2\lambda+2)X_3$ $+2(k+2)(X_8+(\nu+2)S\partial_S,\quad \partial_S$	$(k+1)(k+2)\neq 0,$ $\eta''=q_1\rho^\nu\neq 0$
M_{60}	$q_0\rho^\lambda\dot\rho^{k+2}+\eta(\rho)+S$	∂_S	$(k+1)(k+2)\neq 0,$ $\eta''\neq q_1\rho^\nu,\eta''\neq 0$
M_{61}	$q_0\rho^\lambda\dot\rho^{k+2}+S$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S,$ $(k+\lambda+1)X_3+2(k+1)X_8-2\lambda S\partial_S,\quad \partial_S$	$(k+1)(k+2)\neq 0$
M_{62}	$q_0\rho^\lambda\dot\rho^{k+2}+g(\rho S)+q_2\ln S$	$-2\lambda X_1+(k+2\lambda+2)X_3+2(k+2)(X_8-S\partial_S)$	$(k+1)(k+2)\neq 0$ $(zg'(z))''\neq 0$
M_{63}	$q_0\rho^\lambda\dot\rho^{k+2}+\rho^\nu g(\rho S)+q_2S^{-\nu}$	$-2(\nu(k+1)+\lambda)X_1+(k(\nu+1)+2\lambda+2)X_3$ $+2(k+2)(X_8-S\partial_S)$	$\nu(k+1)(k+2)\neq 0$ $(z^{\nu+1}g'(z))''\neq 0$
M_{64}	$q_0\rho^\lambda\dot\rho^{k+2}+S\eta(\rho)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$(k+1)(k+2)\neq 0,$ $\eta''\neq q_1\rho^\nu,\ \eta''\neq 0$
M_{65}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\ln(\rho S^\beta)+q_2)$	$2(k+1-\beta\lambda)X_1+(\beta(k+2\lambda+2)-k)X_3$ $+2(k+2)(\beta X_8-S\partial_S)$	$(k+1)(k+2)\neq 0$
M_{66}	$q_0\rho^\lambda\dot\rho^{k+2}+\rho\ln(\rho)S$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S,$ $2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2S\partial_S)$	$(k+1)(k+2)\neq 0$

Table 2.1. Continue

	$w(\rho,\dot\rho,S)$	Extensions	Remarks
M_{67}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho\ln(\rho)+q_2)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$q_2\neq 0$, $(k+1)(k+2)\neq 0$
M_{68}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho\ln\rho+q_2\ln S$	$2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2S\partial_S)$	$q_2\neq 0$, $(k+1)(k+2)\neq 0$
M_{69}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho\ln\rho+q_2S^\beta)$	$2(\beta(k+\lambda+1)+k+1)(X_3-X_1)$ $-(k+2)X_3+2(k+2)(\beta X_8+S\partial_S)$	$\beta q_2\neq 0$, $(k+1)(k+2)\neq 0$
M_{70}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho^\nu$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$, $(k+\lambda+1)X_3+2(k+1)X_8$ $-2(\lambda+\nu(k+1))S\partial_S)$	$\nu\neq 0$, $(k+1)(k+2)\neq 0$
M_{71}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho^\nu+q_2)$	$2(k+1)X_1-kX_3-2(k+2)S\partial_S$	$q_2\neq 0$, $(k+1)(k+2)\neq 0$
M_{72}	$q_0\rho^\lambda\dot\rho^{k+2}+S\rho^\nu+q_2\ln S$	$2\lambda(X_3-X_1)+(k+2)(X_3+2X_8-2\nu S\partial_S)$	$q_2\neq 0$, $(k+1)(k+2)\neq 0$
M_{73}	$q_0\rho^\lambda\dot\rho^{k+2}+S(\rho^\nu+q_2S^\beta)$	$2\beta(\lambda+\nu(k+1))+\nu(k+1)(X_3-X_1)$ $-(k+2)(\beta\nu-\beta+\nu)X_3$ $+2(k+2)(\beta X_8+\nu S\partial_S)$	$q_2\beta\neq 0$, $(k+1)(k+2)\neq 0$

2.5 Appendix. Case $k_2 = 0$

For further study the knowledge of $\zeta^S(S)$ plays a key role. For example, for $k_2 = 0$ equation (2.2) becomes

$$W_{\dot\rho\dot\rho S}\zeta^S = W_{\dot\rho\dot\rho\dot\rho}\dot\rho k_1 + 2k_3(W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\dot\rho\dot\rho}) - k_8(W_{\dot\rho\dot\rho\dot\rho}\dot\rho + W_{\dot\rho\dot\rho\rho}\rho + W_{\dot\rho\dot\rho}). \tag{2.14}$$

In the present research we study the case where

$$W_{\dot\rho\dot\rho S} = 0.$$

By virtue of the equivalence transformation corresponding to the generator X_9^e, the general solution of the equation $W_{\dot\rho\dot\rho S} = 0$ is

$$W(\rho,\dot\rho,\tilde S) = \phi(\rho,\dot\rho) + h(\rho,S),$$

where $\phi_{\dot\rho}h_S \neq 0$. Since for $\phi_{\dot\rho\dot\rho} = 0$ equations (1.1) are equivalent to the gas dynamics equations, it is assumed that $\phi_{\dot\rho\dot\rho}\neq 0$. Equation (2.14) reduces to

$$k_1a + k_3b - k_8c = 0, \tag{2.15}$$

where

$$a = \dot\rho\phi_{\dot\rho\dot\rho\dot\rho},\ \ b = 2(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \phi_{\dot\rho\dot\rho}),\ \ c = -(\dot\rho\phi_{\dot\rho\dot\rho\dot\rho} + \rho\phi_{\dot\rho\dot\rho\rho} + \phi_{\dot\rho\dot\rho}).$$

In the further study the following strategy is used. Notice that equation (2.4) is linear with respect to ζ^S with the coefficient $h_{\rho\rho S}$, i.e.,

$$h_{\rho\rho S}\zeta^S = A$$

with some function $A = A(\rho,\dot\rho,S)$ which is independent of ζ^S. If $h_{\rho\rho S} = 0$, then due to equivalence transformations one can also assume that

$$h(\rho,S) = \eta(\rho) + \mu(S),$$

where $\mu'\neq 0$. In this case equation (2.6) leads to

$$\zeta^S = (-2k_1\mu - 2k_3\mu + k_8\mu + c_0)/\mu',$$

where c_0 is an arbitrary constant. The admitted generator takes the form

$$X = k_1(X_1 - 2\tilde S\partial_{\tilde S}) + k_3(X_3 - 2\tilde S\partial_{\tilde S}) + k_8(X_8 + \tilde S\partial_{\tilde S}) + c_0\partial_{\tilde S}, \tag{2.16}$$

where $\tilde{S} = \mu(S)$. Remaining equations are (2.2) and (2.4). The relations between constants k_1, k_3 and k_8 depend on the functions $\eta(\rho)$ and $\phi(\rho, \dot{\rho})$. If $h_{\rho\rho S} \neq 0$, then the function ζ^S is defined by equation (2.4). In this case one needs to satisfy the system of equations (2.2), (2.6) and the condition that $\zeta^S = \zeta^S(S)$.

The analysis of the relations between the constants k_1, k_3 and k_8, follows to the algorithm developed for the gas dynamics equations [8]: the vector space $Span(V)$, where the set V consists of the vectors (a, b, c) with $\rho, \dot{\rho}$ and S are changed, is analyzed. This algorithm allows one to study all possible subalgebras without omission.

2.5.1 $\dim(Span(V)) = 3$

If the function $W(\rho, \dot{\rho}, S)$ is such that $\dim(Span(V)) = 3$, then equation (2.15) is only satisfied for

$$k_1 = 0, \quad k_3 = 0, \quad k_8 = 0.$$

In this case equations (2.4) and (2.6) become

$$\zeta^S h_{\rho\rho S} = 0, \quad \zeta^S_S(\rho h_{\rho S} - h_S) + \zeta^S(\rho h_{\rho SS} - h_{SS}) = 0.$$

Since for $\zeta^S = 0$ there are no extensions of the kernel of admitted Lie algebras, one has to consider $\zeta^S \neq 0$. The general solution of the first equation, after using the equivalence transformation corresponding to the generator X_8^e, is

$$h = \mu(S),$$

where $\mu' \neq 0$. The general solution of the second equation is $\zeta^S = c/\mu'$. Hence

$$W(\rho, \dot{\rho}, \tilde{S}) = \phi(\rho, \dot{\rho}) + \tilde{S},$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_5.

2.5.2 $\dim(Span(V)) = 2$

There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$, which is orthogonal to the set V:

$$\alpha a + \beta b + \gamma c = 0. \tag{2.17}$$

This means that the function $\phi(\rho, \dot{\rho})$ satisfies the equation

$$(\alpha + 2\beta + \gamma)\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \gamma\rho\phi_{\dot{\rho}\dot{\rho}\rho} = -(2\beta + \gamma)\phi_{\dot{\rho}\dot{\rho}}. \tag{2.18}$$

The characteristic system of this equation is

$$\frac{d\dot{\rho}}{(\alpha + 2\beta + \gamma)\dot{\rho}} = \frac{d\rho}{\gamma\rho} = \frac{d\phi_{\dot{\rho}\dot{\rho}}}{-(2\beta + \gamma)\phi_{\dot{\rho}\dot{\rho}}}. \tag{2.19}$$

Case $\gamma = 0$

Because $\phi_{\dot\rho\dot\rho} \neq 0$ and $(\alpha, \beta, \gamma) \neq 0$, one has that $\alpha + 2\beta \neq 0$. The general solution of equation (2.18) is

$$\phi_{\dot\rho\dot\rho} = \tilde\varphi(\rho)\dot\rho^k, \tag{2.20}$$

where $\tilde\varphi(\rho) \neq 0$ is an arbitrary function and $k = 2\beta/(\alpha + 2\beta)$. Since $\dim(Span(V)) = 0$ for $(\rho\tilde\varphi'/\tilde\varphi)' = 0$, one has to assume that $(\rho\tilde\varphi'/\tilde\varphi)' \neq 0$.

Substitution of (2.20) into (2.15) gives

$$k_8\tilde\varphi'\rho - \tilde\varphi(k(k_1 + 2k_3 - k_8) + 2k_3 - k_8) = 0. \tag{2.21}$$

The case $k_8 \neq 0$ leads to $(\rho\tilde\varphi'/\tilde\varphi)' = 0$. Hence, $k_8 = 0$ and equation (2.21) becomes

$$k(k_1 + 2k_3) + 2k_3 = 0. \tag{2.22}$$

Let $k = 0$. Due to equation (2.21) one gets $k_3 = 0$. Integrating (2.20), one finds $\phi = \varphi(\rho)\dot\rho^2$. Equation (2.4) becomes

$$h_{\rho\rho S}\zeta^S + 2h_{\rho\rho}k_1 = 0. \tag{2.23}$$

Assume that $h_{\rho\rho} = 0$, this means that after using the equivalence transformation corresponding to the generator X_8^e, one has that $h = \mu(S)$, where $\mu' \neq 0$. Equation (2.6) after integration gives

$$\zeta^S = -2k_1\mu/\mu' + c_0/\mu',$$

where c_0 is a constant of the integration. Thus,

$$W(\rho, \dot\rho, \tilde S) = \varphi(\rho)\dot\rho^2 + \tilde S.$$

and the extension of the kernel is given by the generators

$$\partial_{\tilde S}, \quad X_1 - 2\tilde S\partial_{\tilde S},$$

where $\tilde S = \mu(S)$. In the final Table 1 this is model M_6.

Assume that $h_{\rho\rho} \neq 0$. For the existence of an extension of the kernel, equation (2.23) implies that $h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S)$, where $\mu\eta'' \neq 0$. In this case equation (2.4) becomes

$$\mu'\zeta^S + 2k_1\mu = 0.$$

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_1 = 0$ and equation (2.6) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot\rho, \tilde S) = \varphi(\rho)\dot\rho^2 + \eta(\rho) + \tilde S,$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde S},$$

where $\tilde S = \mu_2(S)$. In the final Table 1 this is model M_7.

If $\mu' \neq 0$, then $\zeta^S = -2k_1\mu/\mu'$, and equation (2.6) gives

$$(\mu_2'/\mu')' = 0.$$

Hence, without loss of generality one can assume that $\mu_2 = 0$. Therefore,

$$W(\rho, \dot\rho, \tilde S) = \varphi(\rho)\dot\rho^2 + \eta(\rho)\tilde S.$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_8.

Remark. In the cases where $\mu' \neq 0$ one can assume that $\mu_2(S) = f(\mu(S))$. This simplifies calculations.

Let $k \neq 0$. Equation (2.22) gives

$$k_1 = -2k_3\frac{1+k}{k}.$$

The function $\phi(\rho, \dot{\rho})$ is obtained by integrating equation (2.20). The integration depends on the value of k.

Let $k = -1$, then

$$\phi = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}|. \tag{2.24}$$

Substituting (2.24) into (2.4), one obtains

$$\zeta^S h_{\rho\rho S} + 2k_3 h_{\rho\rho} = 0. \tag{2.25}$$

If $h_{\rho\rho} = 0$, then $h = \mu(S)$ with $\mu' \neq 0$, and equation (2.6) leads to

$$\zeta^S = -2k_3\mu/\mu' + c_0/\mu'.$$

Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_9.

If $h_{\rho\rho} \neq 0$, then

$$h(\rho, S) = \mu(S)\eta(\rho) + \mu_2(S), \quad (\mu\eta'' \neq 0).$$

Equation (2.4) becomes $\mu'\zeta^S + 2k_3\mu = 0$.

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_3 = 0$ and equation (2.6) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is defined by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu_2(S)$. In the final Table 1 this is model M_{10}.

If $\mu' \neq 0$, then

$$\zeta^S = -2k_3\mu(S)/\mu'.$$

Similar to the case $k = 0$, equation (2.6) gives $\mu_2 = 0$. Therefore

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}\ln|\dot{\rho}| + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{11}.

Let $k = -2$, then

$$\phi = \varphi(\rho) \ln |\dot{\rho}|. \tag{2.26}$$

Equation (2.4) becomes

$$\zeta^S h_{\rho\rho S} - k_3 \varphi'' = 0 \tag{2.27}$$

Assuming that $h_{\rho\rho S} = 0$, one has

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (2.6) leads to

$$\zeta^S = c_0/\mu'. \tag{2.28}$$

Therefore

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho) \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and (a) for $\varphi'' = 0$, one has two admitted generators

$$X_3 - X_1, \ \partial_{\tilde{S}},$$

(b) for $\varphi'' \neq 0$, there is the only admitted generator

$$\partial_{\tilde{S}}.$$

Here $\tilde{S} = \mu(S)$. In the final Table 1 case (a) is model M_{12} and case (b) is model M_{13}.

Assuming that $h_{\rho\rho S} \neq 0$, one has

$$\zeta^S = k_3 \frac{\varphi''}{h_{\rho\rho S}}.$$

Notice that here $k_3 \neq 0$, otherwise there is no an extension of the kernel of admitted Lie algebras. Hence,

$$\left(\frac{\varphi''}{h_{\rho\rho S}} \right)_\rho = 0. \tag{2.29}$$

If $\varphi'' = 0$, then equation (2.6) is also satisfied. Therefore there is the only extension

$$X_3 - X_1,$$

and

$$W(\rho, \dot{\rho}, \tilde{S}) = (q_1 \rho + q_0) \ln |\dot{\rho}| + h(\rho, S),$$

where $(q_0^2 + q_1^2) h_{\rho\rho S} \neq 0$. In the final Table 1 this is model M_{14}.

If $\varphi'' \neq 0$, then equations (2.29) and (2.6) give

$$h(\rho, S) = \varphi(\rho)\mu(S) + \eta(\rho) + q_2 \mu(S),$$

where $\mu' \neq 0$. Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)(\ln |\dot{\rho}| + \tilde{S}) + \eta(\rho) + q_2 \tilde{S}, \ \ (\varphi'' \neq 0),$$

and the extension of the kernel is

$$X_3 - X_1 + \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{15}.

Let $k(k+1)(k+2) \neq 0$ in (2.20), then

$$\phi = \varphi(\rho)\dot{\rho}^{(k+2)} \tag{2.30}$$

Substituting (2.30) into (2.4), one obtains

$$\zeta^S h_{S\rho\rho} k - 2k_3(k+2)h_{\rho\rho} = 0. \tag{2.31}$$

If $h_{\rho\rho} = 0$, then one can consider that $h = \mu(S)$, where $\mu' \neq 0$. Equation (2.6) is

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu' + c_0/\mu'.$$

In this case

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{16}.

If $h_{\rho\rho} \neq 0$, then for an existence of an extension of the kernel, equation (2.31) requires that

$$h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S),$$

where $\mu\eta'' \neq 0$. Equation (2.31) becomes

$$\zeta^S \mu' k - 2k_3(k+2)\mu = 0.$$

If $\mu' = 0$, then $\mu_2' \neq 0$, $k_3 = 0$ and equation (2.6) gives $\zeta^S = c_0/\mu_2'$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \eta(\rho) + \tilde{S}, \quad (\eta'' \neq 0).$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where $\tilde{S} = \mu_2(S)$. In the final Table 1 this is model M_{17}.

If $\mu' \neq 0$, then

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu',$$

Similar to the case $k = 0$, equation (2.6) gives $\mu_2 = 0$. Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2}\varphi(\rho) + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this is model M_{18}.

Case $\gamma \neq 0$.

In this case the general solution of (2.19) is

$$\phi = \rho^\lambda g(z), \quad (g'' \neq 0), \tag{2.32}$$

where $z = \dot{\rho}\rho^k$, $k = -(\alpha + 2\beta)/\gamma - 1$, $\lambda = 2(\beta + \alpha)/\gamma + 1$. Substituting ϕ into (2.14), one obtains

$$zg'''k_0 + g''\tilde{k_0} = 0, \tag{2.33}$$

where $k_0 = k_1 + 2k_3 - k_8(k+1)$ and $\tilde{k_0} = 2k_3 - k_8(2k + \lambda + 1)$. If $k_0 \neq 0$, then $\dim(Span(V)) \leq 1$, hence, $k_0 = 0$ and $\tilde{k_0} = 0$, which mean that

$$k_1 = -k_8(k + \lambda), \quad k_3 = k_8(2k + \lambda + 1)/2.$$

Equation (2.4) becomes

$$\zeta^S h_{S\rho\rho} + k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho}) = 0. \tag{2.34}$$

Assume that $h_{\rho\rho S} = 0$ or

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (2.4) and (2.6) become, respectively,

$$k_8(\rho\eta''' - (\lambda - 2)\eta'') = 0, \quad \zeta^S = k_8\lambda\mu/\mu' + c_0/\mu'.$$

If $\rho\eta''' - (\lambda - 2)\eta'' \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \eta(\rho) + \tilde{S},$$

and there is the only extension of the kernel of admitted Lie algebras corresponding to the generator

$$\partial_{\widetilde{S}},$$

where $\widetilde{S} = \mu(S)$. In the final Table 1 this is model M_{19}.

If $\rho\eta''' - (\lambda - 2)\eta'' = 0$ or

$$\eta = \begin{cases} q_1\rho^2, & \lambda = 0, \\ q_1\rho\ln(\rho), & \lambda = 1, \\ q_1\rho^\lambda, & \lambda(\lambda - 1) \neq 0. \end{cases}$$

Then,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras corresponding to the generators is

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8 + \lambda\widetilde{S}\partial_{\widetilde{S}}, \ \ \partial_{\widetilde{S}},$$

where $\widetilde{S} = \mu(S)$. In the final Table 1 these models correspond to M_{20}-M_{22}.

Assume that $h_{S\rho\rho} \neq 0$ in (2.34), then

$$\zeta^S = -k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho})/h_{S\rho\rho}.$$

Since $\zeta^S = \zeta^S(S)$, one has

$$\frac{-\rho h_{\rho\rho\rho} + (\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} = H(S), \tag{2.35}$$

and $\zeta^S = k_8 H(S)$.

If $H = 0$, then the general solution of (2.35) is

$$h_{\rho\rho}(\rho, S) = \mu(S)\rho^{\lambda-2}. \tag{2.36}$$

Hence,

$$h(\rho, S) = \mu(S)\eta(\rho) + \mu_2(S),$$

where $\mu'(S) \neq 0$ and

$$\eta = \begin{cases} \ln\rho, & \lambda = 0, \\ \rho\ln(\rho), & \lambda = 1, \\ \rho^\lambda, & \lambda(\lambda-1) \neq 0. \end{cases}$$

Equation (2.6) gives

$$k_8\left(\lambda\mu_2' + \mu'\left(\rho^2\eta'' - \lambda(\rho\eta' - \eta)\right)\right) = 0.$$

This equation leads to: (a) if $\lambda = 0$, then $k_8 = 0$, (b) if $\lambda \neq 0$, then $\mu_2' k_8 = 0$. Hence, an extension of the kernel of admitted Lie algebras occurs for $\lambda \neq 0$. In this case $\mu_2' = 0$, which allows one to assume that $\mu_2 = 0$. Thus,

$$W(\rho, \dot\rho, \tilde S) = \rho^\lambda g(\dot\rho\rho^k) + \tilde S\eta(\rho),$$

and the extension is given by the generator

$$-(k+\lambda)X_1 + \frac{(2k+\lambda+1)}{2}X_3 + X_8.$$

In the final Table 1 these models correspond to M_{23}-M_{25}.

If $H \neq 0$, then equation (2.35) leads to

$$h = \rho^\lambda Q + \mu_2,$$

where $\mu = \mu(S)$, $\mu_2 = f(\mu(S))$, $Q = Q(z)$, $z = \rho\mu$ and $\mu' \neq 0$. Here $H(S) = \mu/\mu' \neq 0$. Substitution of

$$W(\rho, \dot\rho, \tilde S) = \rho^\lambda g(\dot\rho\rho^k) + \rho^{-\lambda}Q(\rho\mu(S)) + f(\mu(S))$$

into (2.6) gives

$$\mu f'' + (\lambda+1)f' = 0.$$

Hence,

$$f' = c\mu^{-(\lambda+1)}.$$

Integration of this equation depends on λ:

$$\mu_2 = \begin{cases} q_1\ln\mu, & \lambda = 0, \\ q_1\mu^{-\lambda}, & \lambda \neq 0. \end{cases}$$

Thus,

$$\begin{array}{lcl} \lambda = 0 & : & W(\rho, \dot\rho, \tilde S) = g(\dot\rho\rho^k) + Q(\rho\tilde S) + q_1\ln\tilde S, \\ \lambda \neq 0 & : & W(\rho, \dot\rho, S) = \rho^\lambda(g(\dot\rho\rho^k) + Q(\rho\tilde S)). \end{array}$$

The extension of the kernel is given by the generator

$$-(k+\lambda)X_1 + \frac{(2k+\lambda+1)}{2}X_3 + X_8 - \tilde S\partial_{\tilde S},$$

where $\tilde S = \mu(S)$. In the final Table 1 these models correspond to M_{26}-M_{27}.

2.5.3 $\dim(Span(V)) = 1$

Let $\dim(Span(V)) = 1$. There exists a constant vector $(\alpha, \beta, \gamma) \neq 0$ such that

$$(a, b, c) = (\alpha, \beta, \gamma)B$$

with some function $B(\rho, \dot{\rho}, S) \neq 0$. Since $\phi_{\dot{\rho}\dot{\rho}} \neq 0$, one has $\beta - 2\alpha \neq 0$, and

$$\rho\phi_{\rho\dot{\rho}\dot{\rho}} = \lambda\phi_{\dot{\rho}\dot{\rho}}, \quad \dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} = k\phi_{\dot{\rho}\dot{\rho}},$$

where

$$\lambda = \frac{3\alpha - \beta - \gamma}{\beta - 2\alpha}, \quad k = \frac{\alpha}{\beta - 2\alpha}.$$

These relations give

$$\phi_{\dot{\rho}\dot{\rho}} = c_1\rho^{\lambda}\dot{\rho}^{k}, \tag{2.37}$$

where $c_1 \neq 0$ is constant. Equation (2.2) becomes

$$k_1 k + 2k_3(k + 1) - k_8(k + \lambda + 1) = 0. \tag{2.38}$$

Integration of (2.37) depends on the value of k. Notice that $k^2 + \lambda^2 \neq 0$, otherwise $\dim(Span(V)) = 0$.

Case $k = -1$.

Integrating (2.37), one obtains

$$\phi = q_0\rho^{\lambda}\dot{\rho}\ln|\dot{\rho}|. \tag{2.39}$$

Equation (2.38) gives

$$k_1 = -\lambda k_8,$$

and equation (2.4) becomes

$$h_{S\rho\rho}\zeta^S = -2k_3h_{\rho\rho} - k_8(\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)). \tag{2.40}$$

Assuming that $h_{S\rho\rho} = 0$ or

$$h(\rho, S) = \eta(\rho) + \mu(S), \quad (\mu' \neq 0),$$

equation (2.40) is reduced to the equation

$$\rho\eta''' k_8 - (k_8(2\lambda - 1) - 2k_3)\eta'' = 0. \tag{2.41}$$

The general solution of equation (2.6) is

$$\zeta^S = (k_8(2\lambda + 1) - 2k_3)\frac{\mu}{\mu'} + \frac{c_0}{\mu'},$$

where c_0 is an arbitrary constant.

If $\eta'' = 0$, then without loss of the generality one can assume that $\eta = 0$. Equation (2.41) is satisfied. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_8 - \lambda X_1 + (2\lambda + 1)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{28}.

If $\eta'' \neq 0$, then

$$k_3 = k_8 \left(\lambda - \frac{1}{2} - \frac{\rho \eta'''}{2\eta''} \right). \tag{2.42}$$

Because k_3 is constant, one has

$$k_8 \left(\frac{\rho \eta'''}{\eta''} \right)' = 0.$$

Assume that

$$\left(\frac{\rho \eta'''}{\eta''} \right)' = 0$$

or $\eta'' = q_1 \rho^\nu$, where ν is constant. Substituting η'' into (2.42), one gets

$$k_3 = k_8 \left(\lambda - \frac{\nu + 1}{2} \right).$$

Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1 \rho^\nu, \quad q_1 \neq 0),$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$2X_8 - 2\lambda X_1 + (2\lambda - \nu - 1) X_3 + 2(\nu + 2) \tilde{S} \partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$ and $q_1 \neq 0$. In the final Table 1 this model corresponds to M_{29}.

If

$$\left(\frac{\rho \eta'''}{\eta''} \right)' \neq 0,$$

then $k_8 = 0$,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{30}.

Assuming that $h_{S\rho\rho} \neq 0$, equation (2.40) gives

$$\zeta^S = -2k_3 \frac{h_{\rho\rho}}{h_{S\rho\rho}} - k_8 \frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}. \tag{2.43}$$

Differentiating equation (2.43) with respect to ρ, one obtains

$$2k_3 \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho + k_8 \left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}} \right)_\rho = 0. \tag{2.44}$$

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho = 0$, then $h = \eta(\rho)\mu(S) + f(\mu(S))$, and equation (2.44) becomes

$$k_8 \left(\frac{\rho \eta'''}{\eta''} \right)' = 0. \tag{2.45}$$

Here $\mu' \eta'' \neq 0$.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$. Equation (2.6) gives

$$f(\mu) = c_0\mu. \tag{2.46}$$

Changing the function η such that $\eta + c_0 \ \rightarrow \ \eta$, one obtains

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{31}.

If $\left(\dfrac{\rho\eta'''}{\eta''}\right)' = 0$, then $\eta'' = \rho^{\nu}$, where ν is constant. Further study depends on ν.

If $\nu = -1$, then

$$\eta = \rho\ln\rho. \tag{2.47}$$

Substitution of (2.47) into (2.6) gives

$$2(k_3 - \lambda k_8)(f'\mu - f) = (c_1 - k_8 f), \tag{2.48}$$

where c_1 is a constant of the integration.

Assume that $f'\mu - f = 0$, then $f = q_1\mu$, and equation (2.48) becomes

$$q_1 k_8\mu = c_1.$$

Because $\mu' \neq 0$, one obtains $q_1 k_8 = 0$ and $c_1 = 0$. If $q_1 = 0$, then

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \tilde{S}\rho\ln\rho,$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \ -\lambda X_1 + X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{32}. If $q_1 \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^{\lambda}\ln|\dot{\rho}| + \tilde{S}(\rho\ln\rho + q_1), \ \ (q_1 \neq 0),$$

and the extension is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{33}.

If $f'\mu - f \neq 0$, then

$$k_3 = \lambda k_8 + \frac{c_1 - k_8 f}{2(f'\mu - f)}. \tag{2.49}$$

Differentiating the last equation with respect to μ, one gets

$$\left(\frac{c_1 - k_8 f}{f'\mu - f}\right)' = 0$$

or

$$c_0(f'\mu - f) = c_1 - k_8 f,$$

where c_0 is constant. Notice that if $c_0 = 0$, then an extension of the kernel only occurs for $k_8 \neq 0$. This means that $f = const$ which is without loss of generality can be assumed $f = 0$, and then $f'\mu - f = 0$. Hence one has to assume that $c_0 \neq 0$. This implies that

$$f'\mu - \alpha f = q_3,$$

where $k_8 = c_0(1-\alpha)$ and $c_1 = c_0 q_3$. Notice that by virtue of the equivalence transformation corresponding to the generator X_7^e one can assume that $\alpha q_3 = 0$. We also note that for $\alpha = 1$ one obtains $k_8 = q_3 = 0$, which prohibits an extension of the kernel. Hence, $\alpha \neq 1$. The extension of the kernel of admitted Lie algebras is given by the only generator

$$2(1-\alpha)(X_8 - \lambda X_1 + \lambda X_3) + X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}\rho \ln\rho + f(\tilde{S}),$$

$\tilde{S} = \mu(S)$ and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^{\alpha}, & \alpha(\alpha - 1) \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{34} and M_{35}.

If $\nu = -2$, then $h = \mu(S)\ln\rho + f(\mu(S))$. Integrating equation (2.6), one has

$$(2k_3 - (2\lambda + 1)k_8)(\mu f' - f) - \mu k_8 = c_1, \tag{2.50}$$

where c_1 is a constant of the integration. If $f'\mu - f = 0$ or $f = q_1\mu$, then $k_8 = 0$, and $c_1 = 0$, and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho}\rho^{\lambda} \ln|\dot{\rho}| + \tilde{S}(\ln\rho + q_1). \tag{2.51}$$

The extension of the kernel in this case is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{36}. If $f'\mu - f \neq 0$, then

$$2k_3 = (2\lambda + 1)k_8 + \frac{c_1 + \mu k_8}{\mu f' - f},$$

and, hence,

$$\left(\frac{c_1 + k_8\mu}{\mu f' - f}\right)' = 0$$

or

$$c_0(\mu f' - f) = c_1 + k_8\mu,$$

where c_0 is constant. Notice that if $c_0 = 0$, then $k_8 = 0$, and there is no an extension of the kernel of admitted Lie algebras. Hence, $c_0 \neq 0$, and

$$f'\mu - f = q_3 + \alpha\mu,$$

where $k_8 = c_0\alpha$ and $c_1 = c_0 q_3$. The general solution of the last equation is

$$f = \alpha\mu\ln(\mu) + q_1\mu - q_3.$$

Thus, the extension of the kernel of admitted Lie algebras is given by the generator

$$-2\lambda\alpha X_1 + (2\alpha\lambda + 1)X_3 + 2\alpha X_8 + 2(\alpha - 1)\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho,\dot\rho,\tilde S)=q_0\dot\rho\rho^\lambda\ln|\dot\rho|+\tilde S(\ln\rho+q_1)+\alpha\tilde S\ln(\tilde S),$$

and $\tilde S=\mu(S)$. Notice also that the previous case (2.51) is included in the present case by setting $\alpha=0$. In the final Table 1 this model corresponds to M_{37}.

Let $(\nu+1)(\nu+2)\neq 0$, then $h=\rho^{\nu+2}\mu(S)+f(\mu(S))$, and equation (2.6) gives

$$(2k_3-(2\lambda-\nu-1)k_8)(\mu f'-f)+(\nu+2)fk_8=c_1. \qquad (2.52)$$

If $f'\mu-f=0$ or $\mu_2=q_1\mu$, then

$$(\nu+2)\mu q_1k_8=c_1.$$

Because $(\nu+2)\mu'\neq 0$, one obtains that $q_1k_8=0$ and $c_1=0$. If $q_1=0$, then

$$W(\rho,\dot\rho,\tilde S)=q_0\dot\rho\rho^\lambda\ln|\dot\rho|+\tilde S\rho^{\nu+2},$$

and the extension of the kernel is given by the generators

$$X_3-2\tilde S\partial_{\tilde S},\ \ -\lambda X_1+X_8+(2\lambda-\nu-1)\tilde S\partial_{\tilde S},$$

where $\tilde S=\mu(S)$. In the final Table 1 this model corresponds to M_{38}.

If $q_1\neq 0$, then $k_8=0$. Thus

$$W(\rho,\dot\rho,\tilde S)=q_0\dot\rho\rho^\lambda\ln|\dot\rho|+\tilde S(\rho^{\nu+2}+q_1),\ \ (q_1\neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3-2\tilde S\partial_{\tilde S}.$$

In the final Table 1 this model corresponds to M_{39}.

If $\mu f'-f\neq 0$, then

$$2k_3=(2\lambda-\nu-1)k_8+\frac{c_1-(\nu+2)fk_8}{\mu f'-f},$$

and, hence,

$$\frac{c_1-(\nu+2)fk_8}{\mu f'-f}=c_0,$$

where c_0 is constant. Notice that if $c_0=0$, then an extension of the kernel only occurs for $f=const$, whereas by virtue of the equivalence transformation corresponding to the generator X_7^e one can assume that $f=0$, and then $f'\mu-f=0$. Hence, $c_0\neq 0$, and

$$f'\mu-\alpha f=q_2,$$

where

$$c_1=c_0q_2,\ \ k_8=c_0\frac{1-\alpha}{\nu+2}.$$

Here, as in the previous case, one has to require that $\alpha\neq 1$. Hence,

$$W(\rho,\dot\rho,\tilde S)=q_0\dot\rho\rho^\lambda\ln|\dot\rho|+\tilde S\rho^{\nu+2}+f(\tilde S),$$

and the admitted generator is

$$2\lambda(\alpha-1)X_1+(\alpha(\nu+1-2\lambda)+2\lambda+1)X_3+2(1-\alpha)X_8-2(\nu+2)\tilde S\partial_{\tilde S},$$

where

$$f = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^{\alpha}, & \alpha(\alpha-1) \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{40} and M_{41}.

Returning to (2.44), if $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho \neq 0$, then equation (2.44) gives

$$2k_3 = -k_8 \left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda-1)}{h_{S\rho\rho}}\right)_\rho / \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho. \tag{2.53}$$

Thus,

$$\left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda-1)}{h_{S\rho\rho}}\right)_\rho / \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = const$$

or

$$\rho h_{\rho\rho\rho} - H h_{S\rho\rho} = k_0 h_{\rho\rho}$$

where k_0 is constant and $H = H(S)$ is some function. Notice that for $H = 0$ one has the contradiction $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$. Hence, $H(S) \neq 0$. The general solution of the last equation (up to an equivalence transformation) is

$$h(\rho, S) = \rho^\nu g(\rho\mu(S)) + f(\mu(S)), \tag{2.54}$$

where $\mu' \neq 0$. Equation (2.6) becomes

$$\mu f'' + (\nu+1) f' = 0. \tag{2.55}$$

Thus,

$$W(\rho, \dot\rho, \tilde{S}) = q_0 \rho^\lambda \dot\rho \ln|\dot\rho| + \rho^\nu g(\rho\tilde{S}) + f(\tilde{S}),$$

and the extension is given by the only generator

$$-2\lambda X_1 + (2\lambda - \nu + 1)X_3 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

Here $\tilde{S} = \mu(S)$, and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \nu = 0; \\ q_2 \tilde{S}^{-\nu}, & \nu \neq 0. \end{cases}$$

In the final Table 1 these models correspond to M_{42} and M_{43}.

Case $k = -2$.

Integrating (2.37), one obtains

$$\phi = q_0 \rho^\lambda \ln|\dot\rho|, \quad (q_0 \neq 0). \tag{2.56}$$

Substituting (2.56) into (2.2), one gets

$$k_1 = -k_3 + k_8 \frac{1-\lambda}{2}.$$

Equation (2.4) becomes

$$2h_{S\rho\rho}\zeta^S = 2q_0 k_3 \rho^{\lambda-2}\lambda(\lambda-1) - k_8(2\rho h_{\rho\rho\rho} - 2(\lambda-2)h_{\rho\rho} + q_0\rho^{\lambda-2}\lambda(\lambda^2-1)). \tag{2.57}$$

Assuming that $h_{S\rho\rho} = 0$ or

$$h(\rho, S) = \eta(\rho) + \mu(S), \ \ (\mu' \neq 0),$$

equation (2.6) and (2.4) are reduced to the equations, respectively,

$$\zeta^S = \frac{k_8\lambda\mu + c_0}{\mu'}, \tag{2.58}$$

$$q_0\lambda(\lambda-1)(2k_3 - k_8(\lambda+1)) = 2k_8\rho^{2-\lambda}(\eta'''\rho - \eta''(\lambda-2)), \tag{2.59}$$

where c_0 is the constant of integration.

Let $\lambda(\lambda-1) \neq 0$. Equation (2.59) gives

$$k_3 = k_8\left(\frac{\lambda+1}{2} + \frac{(\eta'''\rho - \eta''(\lambda-2))}{q_0\lambda(\lambda-1)\rho^{\lambda-2}}\right),$$

Differentiating this equation with respect to ρ, one has

$$k_8\left(\rho^{2-\lambda}(\eta'''\rho - \eta''(\lambda-2))\right)_\rho = 0.$$

If $\left(\rho^{2-\lambda}(\eta'''\rho - \eta''(\lambda-2))\right)_\rho = 0$, then $\eta'' = \rho^{\lambda-2}(\tilde{q}_1 + q_0\lambda(\lambda-1)\nu\ln(\rho))$ or, by virtue of equivalence transformations,

$$\eta = \rho^\lambda(q_1 + q_0\nu\ln(\rho)).$$

Here ν and q_1 are constant. Thus,

$$W(\rho, \dot\rho, \tilde{S}) = \rho^\lambda(q_1 + q_0\ln(\rho^\nu|\dot\rho|)) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-2(\lambda+\nu)X_1 + (\lambda+2\nu+1)X_3 + 2X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}}, \ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{44}. If $\left(\rho^{2-\lambda}(\eta'''\rho - \eta''(\lambda-2))\right)_\rho \neq 0$, then $k_8 = 0$. Thus,

$$W(\rho, \dot\rho, \tilde{S}) = q_0\rho^\lambda\ln|\dot\rho| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{45}.

Let $\lambda(\lambda-1) = 0$. Equation (2.59) becomes

$$k_8(\eta'''\rho - \eta''(\lambda-2)) = 0.$$

If $\eta'' \neq q_1\rho^{(\lambda-2)}$, then $k_8 = 0$. Thus

$$W(\rho, \dot\rho, \tilde{S}) = q_0\rho^\lambda\ln|\dot\rho| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-X_1 + X_3, \ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{46}. If $\eta'' = q_1\rho^{(\lambda-2)}$ or

$$W(\rho,\dot\rho,\tilde S) = q_0\rho^\lambda \ln(|\dot\rho|\rho^\nu) + \tilde S, \quad (\lambda(\lambda-1)=0),$$

then the extension of the kernel is given by the generators

$$-X_1 + X_3,\ (1-\lambda)X_1 + 2X_8 + 2\lambda\tilde S\partial_{\tilde S},\ \partial_{\tilde S}.$$

In the final Table 1 this model corresponds to M_{47}.

Assuming that $h_{S\rho\rho}\neq 0$ in equation (2.57), one obtains

$$2\zeta^S = q_0\lambda(\lambda-1)(2k_3 - k_8(\lambda+1))\frac{\rho^{\lambda-2}}{h_{S\rho\rho}} - 2k_8\frac{\rho h_{\rho\rho\rho} - (\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}. \tag{2.60}$$

Differentiating the last equation with respect to ρ, one gets

$$q_0\lambda(\lambda-1)(2k_3 - k_8(\lambda+1))\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 2k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho. \tag{2.61}$$

If $\lambda(\lambda-1)=0$, then equation (2.61) becomes

$$k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0.$$

Let

$$\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0,$$

then

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\lambda-2)h_{\rho\rho}, \tag{2.62}$$

where $H = H(S)$ is a function of the integration. A solution of the last equation depends on the function $H(S)$.

Assuming that $H = 0$, one has $\zeta^S = 0$,

$$h(\rho, S) = \mu(S)\rho^\lambda \ln\rho + f(\mu(S)),$$

where $\mu' \neq 0$. Equation (2.6) becomes

$$k_8(\lambda f' + q_0(\lambda-1)\rho^\lambda) = 0. \tag{2.63}$$

If $\lambda = 0$, then equation (2.63) gives $k_8 = 0$. Thus,

$$W(\rho,\dot\rho,\tilde S) = q_0\ln|\dot\rho| + \widetilde S\ln\rho + f(\widetilde S),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

If $\lambda = 1$, then equation (2.63) becomes $k_8 f' = 0$. For $f'\neq 0$ one has $k_8 = 0$,

$$W(\rho,\dot\rho,\tilde S) = \rho(q_0\ln|\dot\rho| + \widetilde S\ln\rho) + f(\widetilde S), \quad (f'\neq 0),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1 these models correspond to M_{48} and M_{49}. For $f' = 0$ one has

$$W(\rho, \dot\rho, \tilde S) = \rho(q_0 \ln|\dot\rho| + \widetilde S \ln\rho),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \quad X_8.$$

In the final Table 1 this model corresponds to M_{50}.

Assuming that $H \neq 0$ in (2.62), one obtains

$$h(\rho, S) = \rho^\lambda \phi(\rho\mu(S)) + f(\mu(S)), \tag{2.64}$$

where $\mu' \neq 0$. Substitution of $h(\rho, S)$ into (2.6) gives

$$k_8(\mu f' + \lambda f)' = 0. \tag{2.65}$$

If $(\mu f' + \lambda f)' = 0$ or

$$f = \begin{cases} q_1 \ln(\mu), & \lambda = 0, \\ q_1\mu^{-1}, & \lambda = 1, \end{cases}$$

then

$$W(\rho, \dot\rho, \tilde S) = \rho^\lambda(q_0 \ln|\dot\rho| + \phi(\rho\widetilde S)) + f(\widetilde S),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \;\; (1-\lambda)X_1 + 2X_8 - 2\tilde S\partial_{\tilde S}.$$

In the final Table 1 these models correspond to M_{51} and M_{52}. If $(\mu f' + \lambda f)' \neq 0$, then $k_8 = 0$,

$$W(\rho, \dot\rho, \tilde S) = \rho^\lambda(q_0 \ln|\dot\rho| + \phi(\rho\widetilde S)) + f(\widetilde S),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1 this model corresponds to M_{53}.

Returning to (2.61), let $\lambda(\lambda - 1) \neq 0$. Assume also that

$$\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 0,$$

which means that

$$h(\rho, S) = q_0\mu(S)\rho^\lambda + \eta(\rho) + f(\mu(S)),$$

where $\mu' \neq 0$. Then equations (2.61) becomes

$$k_8\left(\rho^{2-\lambda}(\rho\eta''' - (\lambda-2)\eta''\right)_\rho = 0.$$

If $\left(\rho^{2-\lambda}(\rho\eta''' - (\lambda-2)\eta''\right)_\rho \neq 0$, then $k_8 = 0$. Substituting into (2.6), one obtains

$$k_3 f'' = 0. \tag{2.66}$$

Since for $k_3 = 0$ there is no extension of the kernel, one has $f'' = 0$. Thus,

$$W(\rho, \dot\rho, \tilde S) = q_0\rho^\lambda(\ln|\dot\rho| + \tilde S) + \eta(\rho) + q_1\tilde S,$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3 - \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{54}.

If $\left(\rho^{2-\lambda}(\rho\eta''' - (\lambda-2)\eta'')\right)_\rho = 0$, then $\eta'' = \rho^{\lambda-2}(\widetilde{\nu}\ln\rho + \widetilde{q}_1)$, where $\widetilde{\nu}$ and $\widetilde{q}_1$ are constant. Using equivalence transformations, one finds that $\eta = \rho^\lambda(q_0\nu\ln\rho + q_1)$, where $\widetilde{\nu} = q_0\nu\lambda(\lambda-1)$ and $\widetilde{q}_1 = q_1\lambda(\lambda-1) + q_0\nu(2\lambda-1)$. In this case

$$k_1 = -(k_3 - k_8\frac{\lambda-1}{2}),\ \ \zeta^S = (2k_3 - k_8(\lambda+2\nu+1))/(2\mu'),$$

and equation (2.6) becomes

$$(2k_3 - k_8(\lambda+2\nu+1))f' - 2k_8\lambda f = \widetilde{q}_2,$$

where $\widetilde{q}_2$ is constant. The last equation can be rewritten in the form

$$\alpha f' - lf = \widetilde{q}_2,$$

where

$$k_8 = \frac{l}{2\lambda},\ \ k_3 = \frac{\alpha}{2} + \frac{l}{2\lambda}\frac{\lambda+2\nu+1}{2}.$$

Further analysis depends on the constants α and l. Notice that for the existence of an extension of the kernel of admitted Lie algebras, one needs to require that $\alpha^2 + l^2 \neq 0$. Hence, for $\alpha = 0$, one has $l \neq 0$, which means that without loss of generality one can assume that $f = 0$. In the case $f = 0$ one obtains

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda,$$

and the extension of the kernel is given by the generators

$$X_1 - X_3 - \partial_{\tilde{S}},\ \ 2(\lambda-1)X_1 + 2X_8 - (\lambda+2\nu+1)\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{55}. For $\alpha \neq 0$, one has

$$f = \begin{cases} q_2\mu, & l = 0; \\ q_2 e^{-\kappa\mu} & l \neq 0, \end{cases}$$

and

$$k_1 = -\frac{\kappa(\lambda+\nu)+\lambda}{2\lambda},\ \ k_8 = \frac{\kappa}{2\lambda},\ \ k_3 = \frac{1}{2} + \frac{\kappa}{4\lambda}(\lambda+2\nu+1),$$

where $l = \kappa\alpha$ and $q_2 \neq 0$ is constant. Thus, one obtains:

(a) for the function $f(\widetilde{S}) = q_2\widetilde{S}$:

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda + q_2\widetilde{S},\ \ (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-X_1 + X_3 + \partial_{\tilde{S}};$$

(b) for the function $f(\widetilde{S}) = q_2 e^{\kappa\widetilde{S}}$:

$$W(\rho,\dot{\rho},\tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \widetilde{S}) + q_1\rho^\lambda + q_2 e^{\kappa\widetilde{S}},\ \ (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-2(\kappa(\lambda+\nu)+\lambda)X_1+2\kappa X_8+(2\lambda+\kappa(\lambda+2\nu+1))X_3+2\lambda\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{56} and M_{57}.

Assume that $\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho \neq 0$, then from (2.61) one finds

$$k_3=k_8\left(\frac{\left(\frac{\rho h_{\rho\rho\rho}-(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{q_0\lambda(\lambda-1)\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho}+\frac{\lambda+1}{2}\right). \tag{2.67}$$

Since for $k_8=0$ there is no an extension, then

$$\frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}-\frac{(\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho}=const$$

or

$$\rho h_{\rho\rho\rho}+H(S)h_{S\rho\rho}=(\lambda-2)h_{\rho\rho}+\nu\rho^{\lambda-2}, \tag{2.68}$$

where ν is constant and $H(S)$ is some function. Notice that for $H(S)=0$ one obtains

$$h_{\rho\rho}=(\nu\ln\rho+\mu(S))\rho^{\lambda-2},$$

which leads to the contradiction

$$\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho=0.$$

Hence, one has to assume that $H(S)\neq 0$, which gives

$$h_{\rho\rho}(\rho,S)=\rho^{\lambda-2}(\widetilde{\nu}\ln\rho+\widetilde{g}(\rho\mu(S)))$$

or

$$h(\rho,S)=\rho^{\lambda}(\nu\ln\rho+g(\rho\mu(S)))+f(\mu(S)),$$

where $\mu'\neq 0$. Equation (2.6) gives $f=q_2\mu^{-\lambda}$. Thus

$$W(\rho,\dot{\rho},\tilde{S})=q_0\rho^{\lambda}\left(\ln(|\dot{\rho}|\rho^{\nu})+g(\rho\widetilde{S})\right)+q_2\widetilde{S}^{-\lambda},$$

and the extension of the kernel is given by the only generator

$$-2(\lambda+\nu)X_1+2X_8+(\lambda+2\nu+1)X_3-2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{58}.

Case $(k+1)(k+2)\neq 0$

Returning to integration of (2.37), if $(k+2)(k+1)\neq 0$, then one obtains

$$\phi=q_0\rho^{\lambda}\dot{\rho}^{k+2} \tag{2.69}$$

Substituting (2.69) into (2.2), one has

$$k_3=-k_1\frac{k}{2(k+1)}+k_8\frac{k+\lambda+1}{2(k+1)},$$

and equation (2.4) becomes

$$\zeta^S h_{S\rho\rho} + h_{\rho\rho}\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) + k_8\rho h_{\rho\rho\rho} = 0. \qquad (2.70)$$

Assuming that $h_{S\rho\rho} = 0$, one finds

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where $\mu' \neq 0$. Equation (2.70) becomes

$$\eta''\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) + k_8\rho\eta''' = 0. \qquad (2.71)$$

Let $\eta'' \neq 0$, then

$$k_1 = -k_8\frac{k+1}{k+2}\left(\frac{2k+\lambda+2}{k+1} + \frac{\rho\eta'''}{\eta''}\right).$$

Differentiating the last equation with respect to ρ, one gets

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0. \qquad (2.72)$$

If $\frac{\rho\eta'''}{\eta''} = k_0 = const$, then $\eta'' = q_1\rho^\nu$, where $\nu = k_0$. This gives that

$$W(\rho, \dot\rho, \tilde S) = q_0\rho^\lambda\dot\rho^{k+2} + \eta(\rho) + \widetilde{S}, \ \ (\eta'' = q_1\rho^\nu \neq 0),$$

and the extension of the kernel is given by the generators

$$-\frac{k+1}{k+2}\left(\frac{2k+\lambda+2}{k+1} + \nu\right)X_1 + \frac{k\nu+3k+2\lambda+2}{2(k+2)}X_3 + X_8 + (\nu+2)\tilde S\partial_{\tilde S}, \ \partial_{\tilde S},$$

where $\eta'' = q_1\rho^\nu$, $\tilde S = \mu(S)$ and $q_1 \neq 0$. In the final Table 1 this model corresponds to M_{59}, $(k^2+\lambda^2 \neq 0)$.

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$,

$$W(\rho, \dot\rho, \tilde S) = q_0\rho^\lambda\dot\rho^{k+2} + \eta(\rho) + \tilde S,$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde S}.$$

In the final Table 1 this model corresponds to M_{60}, $(k^2+\lambda^2 \neq 0)$.

Considering (2.71), let $\eta'' = 0$. Without loss of the generality one can assume that $\eta = 0$. Equation (2.6) gives

$$\zeta^S = -\frac{\mu}{\mu'}\left(k_1\frac{k+2}{k+1} + k_8\frac{\lambda}{k+1}\right) + \frac{c_0}{\mu'}.$$

Thus,

$$W(\rho, \dot\rho, \tilde S) = q_0\rho^\lambda\dot\rho^{k+2} + \tilde S,$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_1 - \frac{k}{2(k+1)}X_3 - \frac{k+2}{k+1}\tilde{S}\partial_{\tilde{S}}, \quad \frac{k+\lambda+1}{2(k+1)}X_3 + X_8 - \frac{\lambda}{k+1}\tilde{S}\partial_{\tilde{S}}, \ \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{61}, $(k^2+\lambda^2 \neq 0)$. Returning to (2.70), assume that $h_{S\rho\rho} \neq 0$. Then

$$\zeta^S = -\frac{h_{\rho\rho}}{h_{S\rho\rho}}\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) - k_8\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}.$$

Differentiating this equation with respect to ρ, one finds

$$\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho\left(k_1\frac{k+2}{k+1} + k_8\frac{2k+\lambda+2}{k+1}\right) + k_8\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0. \tag{2.73}$$

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho \neq 0$, then

$$k_1 = -k_8\frac{k+1}{k+2}\left(\frac{2k+\lambda+2}{k+1} + \frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}\right).$$

Extension of the kernel occurs only for

$$\frac{\left(\frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho} = const,$$

which means that

$$\rho h_{\rho\rho\rho} - H(S)h_{S\rho\rho} = \widetilde{\nu} h_{\rho\rho},$$

where $H(S)$ is some function and ν is constant. Notice that for $H(S) = 0$ one has

$$h_{\rho\rho}(\rho, S) = \rho^{\widetilde{\nu}}\mu(S)$$

which leads to the contradiction $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$. Hence, $H(S) \neq 0$, and then

$$h_{\rho\rho}(\rho, S) = \rho^{\widetilde{\nu}}\tilde{g}(\rho\mu(S)),$$

or

$$h(\rho, S) = \rho^\nu g(\rho\mu(S)) + f(\mu(S)),$$

where $\mu' \neq 0$ and $(z^{\nu+1}g'(z))'' \neq 0$. Equation (2.6) leads to the condition

$$\mu f' + \nu f = \widetilde{q}_2,$$

where $\widetilde{q}_2$ is constant. The general solution of the last equation depends on ν:

$$f(\mu) = \begin{cases} q_2\ln(\mu), & \nu = 0, \\ q_2\mu^{-\nu}, & \nu \neq 0. \end{cases}$$

Thus, setting $\tilde{S} = \mu(S)$, one gets

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda \dot{\rho}^{k+2} + \rho^\nu g(\rho\tilde{S}) + f(\tilde{S}),$$

and the extension of the kernel of admitted Lie algebras is defined by the generator

$$-\frac{\nu(k+1)+\lambda}{k+2}X_1 + \frac{k(\nu+1)+2\lambda+2}{2(k+2)}X_3 + X_8 - \tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{62} and M_{63}, $(k^2+\lambda^2 \neq 0)$.

If $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$, then $h(\rho, S) = \mu(S)(\eta(\rho) + f(\mu(S)))$, where $\eta''\mu' \neq 0$. Equation (2.73) becomes

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

If $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$, equation (2.6) leads to the equation

$$\mu f'' + 2f' = 0.$$

A solution of the last equation is $f(\mu) = c_1/\mu + c_0$, where c_0 and c_1 are constant. Without loss of generality, one can assume that $c_1 = c_0 = 0$. Thus,

$$k_3 = -k_1\frac{k}{2(k+1)}, \quad \zeta^S = -k_1\frac{k+2}{k+1}\frac{\mu}{\mu'},$$

and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}\eta(\rho).$$

The extension of the kernel consists of the generator

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{64}, $(k^2+\lambda^2 \neq 0)$.

If $\frac{\rho\eta'''}{\eta''} = k_0 = const$, then $\eta'' = \widetilde{q}_1\rho^{\nu-2}$, where $\nu = 2(k_0+1)$. One can choose the function $\eta(\rho)$ as follows

$$\eta = \begin{cases} \ln(\rho), & \nu = 0, \\ \rho\ln(\rho), & \nu = 1, \\ \rho^\nu, & \nu(\nu-1) \neq 0. \end{cases}$$

This reduces equation (2.6) to the equations

$$\begin{aligned} \nu = 0: &\quad a\mu f' = b + \widetilde{q}_2\mu^{-1}, \\ \nu = 1: &\quad a\mu f' + bf = \widetilde{q}_2\mu^{-1}, \\ \nu(\nu-1) \neq 0: &\quad a\mu f' + \nu bf = \widetilde{q}_2\mu^{-1}. \end{aligned} \tag{2.74}$$

where $a = k_1(k+2) + k_8(\lambda + \nu(k+1))$, $b = k_8(k+1)$ and $\tilde{q}_2$ is constant. Notice that the condition $a^2 + b^2 = 0$ leads to the relations $k_1 = 0$ and $k_8 = 0$. These conditions do not allow an extension of the kernel of admitted Lie algebras. Hence, one has to assume that $a^2 + b^2 \neq 0$.

Let us consider the case $\nu = 0$, where $\eta = \ln(\rho)$. In this case $a \neq 0$, because otherwise $b = 0$. Using equivalence transformations, the general solution of equation $(2.74)_{\nu=0}$ has the representation:

$$f = \beta \ln(\mu) + q_2,$$

where β and q_2 are constant. Substituting the representation of the function $f(\mu)$ into equation $(2.74)_{\nu=0}$, one finds that $\beta a = b$ and $\widetilde{q}_2 = 0$. Therefore,

$$k_1 = a\frac{k+1-\lambda\beta}{(k+1)(k+2)}, \quad k_3 = a\frac{\beta(k+2\lambda+2)-k}{2(k+1)(k+2)}, \quad k_8 = a\frac{\beta}{k+1},$$

and

$$W = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}\left(\ln\left(\rho\tilde{S}^\beta\right) + q_2\right),$$

where $\tilde{S} = \mu(S)$. The extension of the kernel of admitted Lie algebras is defined by the only generator

$$\frac{k+1-\beta\lambda}{k+2}X_1 + \frac{\beta(k+2\lambda+2)-k}{2(k+2)}X_3 + \beta X_8 - \tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{65}, $(k^2+\lambda^2 \neq 0)$. In other two cases $\nu = 1$ and $\nu(\nu-1) \neq 0$ one has to solve the equation

$$a\mu f' + \nu b f = \widetilde{q}_2\mu^{-1}, \quad (\nu \neq 0). \tag{2.75}$$

By virtue of equivalence transformations the function f is equivalent to the function $\tilde{f} = f - r\mu^{-1}$, where r is constant. The change $f = \tilde{f} + r\mu^{-1}$ reduces equation (2.75) to the equation

$$a\mu\tilde{f}' + \nu b\tilde{f} = (\widetilde{q}_2 + (a-\nu b)r)\mu^{-1}.$$

This means that for $a - \nu b \neq 0$ one can assume in (2.75) that $\tilde{q}_2 = 0$. Therefore the analysis of solutions of equation (2.75) is reduced to the study of solutions of either the homogeneous equation

$$a\mu f' + \nu b f = 0, \tag{2.76}$$

or the nonhomogeneous equation

$$\mu f' + f = q_2\mu^{-1}, \quad (q_2 \neq 0). \tag{2.77}$$

The function $f = 0$ is the trivial solution of equation (2.76). In this case k_1 and k_3 are arbitrary. Thus

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}\eta(\rho),$$

and the extension of the kernel consists of the generators

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad (k+\lambda+1)X_3 + 2(k+1)X_8 - 2(\lambda+\nu(k+1))\tilde{S}\partial_{\tilde{S}}).$$

Here $\tilde{S} = \mu(S)$. In the final Table 1 these models correspond to M_{66} and M_{70}, $(k^2+\lambda^2 \neq 0)$.

The only nontrivial solution of equation (2.76) has the representation

$$f(\mu) = q_2\mu^\beta, \quad (q_2 \neq 0, \quad \beta \neq -1).$$

Substituting the representation into equation (2.76), it becomes

$$\beta(k_1(k+2) + k_8(\lambda+\nu(k+1))) + k_8\nu(k+1) = 0. \tag{2.78}$$

If $\beta = 0$, then $k_8 = 0$, and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}(\eta(\rho) + q_2),$$

with the extension

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{67} and M_{71}, $(k^2 + \lambda^2 \neq 0)$. If $\beta \neq 0$, then equation (2.78) gives

$$k_1 = -k_8\frac{\beta(\lambda + \nu(k+1)) + \nu(k+1)}{\beta(k+2)}.$$

Thus,

$$k_3 = k_8\frac{\beta(k\nu + k + 2\lambda + 2) + k\nu}{2(k+2)\beta}, \quad \zeta^S = k_8\frac{\nu}{\beta}\frac{\mu}{\mu'},$$

and the potential function is

$$W(\rho, \dot{\rho}, S) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}(\eta(\rho) + q_2\tilde{S}^\beta), \ \ (q_2\beta(\beta+1) \neq 0).$$

The extension of the kernel of admitted Lie algebras is defined by the only generator

$$2\frac{\beta(\lambda + \nu(k+1)) + \nu(k+1)}{(k+2)}(X_3 - X_1) - (\beta\nu - \beta + \nu)X_3 + 2\beta X_8 + 2\nu\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{69} and M_{73}, $(k^2 + \lambda^2 \neq 0)$.

The representation of the general solution of equation (2.77) is $f = q_2\mu^{-1}\ln(\mu)$. Substituting the representation into equation (2.75), it gives

$$\widetilde{q}_2 = aq_2, \ \ a - \nu b = 0.$$

Hence,

$$k_1 = -k_8\frac{\lambda}{k+2}.$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda\dot{\rho}^{k+2} + \tilde{S}\eta(\rho) + q_2\ln(\tilde{S}), \ \ (q_2 \neq 0),$$

and the extension of the kernel is defined by the generator

$$2\lambda(X_3 - X_1) + (k+2)(X_3 + 2X_8 - 2\nu\tilde{S}\partial_{\tilde{S}}).$$

In the final Table 1 these models correspond to M_{68} and M_{72}, $(k^2 + \lambda^2 \neq 0)$.

2.5.4 $\dim(Span(V)) = 0$

In this case the vector

$$(\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}}, \ 2(\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \phi_{\dot{\rho}\dot{\rho}}), \ -(\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \rho\phi_{\dot{\rho}\dot{\rho}\rho} + \phi_{\dot{\rho}\dot{\rho}}))$$

is constant. This condition implies that

$$\phi = q_0\dot{\rho}^2.$$

Substituting ϕ into (2.2) and (2.4), one gets, respectively,

$$k_3 = \frac{1}{2}k_8,$$

$$\zeta^S h_{S\rho\rho} + 2k_1 h_{\rho\rho} + k_8(\rho h_{\rho\rho\rho} + 2h_{\rho\rho}) = 0. \tag{2.79}$$

Assume that $h_{S\rho\rho} \neq 0$, then

$$\zeta^S = -2ak_1 - k_8 b, \tag{2.80}$$

where $a = \dfrac{h_{\rho\rho}}{h_{S\rho\rho}}$, $b = \dfrac{\rho h_{\rho\rho\rho} + 2h_{\rho\rho}}{h_{S\rho\rho}}$. Differentiating (2.80) with respect to ρ, one obtains

$$2k_1 a_\rho + k_8 b_\rho = 0. \tag{2.81}$$

If $a_\rho = 0$ then, $h(\rho, S) = \eta(\rho)\mu(S) + f(\mu(S))$, where $\eta''\mu' \neq 0$. Equation (2.81) becomes

$$k_8 \left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

If $\left(\dfrac{\rho\eta'''}{\eta''}\right)' \neq 0$, then $k_8 = 0$, and equation (2.6) becomes

$$k_1 f'' = 0.$$

Since for $k_1 = 0$ there is no extension of the kernel, without loss of generality one can assume that $f = 0$. Thus,

$$W = \dot{\rho}^2 q_0 + \eta(\rho)\tilde{S},$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where $\tilde{S} = \mu(S)$. In the final Table 1 this model corresponds to M_{64}, $(k = \lambda = 0)$.

If $\left(\dfrac{\rho\eta'''}{\eta''}\right)' = 0$ or $\eta'' = \rho^{\nu-2}$. Finding the function $\eta(\rho)$ depends on the value of ν.

Let $\nu(\nu - 1) \neq 0$, then $\eta = \rho^\nu$ and equation (2.6) becomes

$$2k_1\mu f'' + \nu k_8 (\mu f'' + f') = 0. \tag{2.82}$$

If $f'' = 0$, then $f = q_1\mu$ and equation (2.82) is reduced to the equation

$$k_8 q_1 = 0.$$

Hence, if $q_1 \neq 0$, then $k_8 = 0$ and

$$W = \dot{\rho}^2 q_0 + (\rho^\nu + q_1)\tilde{S}, \quad (q_1 \neq 0),$$

the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

In the final Table 1 this model corresponds to M_{71}, $(k = \lambda = 0)$.

If $q_1 = 0$, then k_8 is arbitrary, and

$$W = \dot{\rho}^2 q_0 + \rho^\nu \tilde{S}.$$

The extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},\ X_3 + 2X_8 - 2\nu\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{70}, $(k = \lambda = 0)$.

If $f'' \neq 0$, then equation (2.82) gives that

$$\mu f'' - \beta f' = 0,\ \ (\mu \neq 0),$$

where β is constant and

$$k_1 = -\nu k_8 \frac{(\beta + 1)}{2\beta}. \tag{2.83}$$

Thus,

$$W = \dot{\rho}^2 q_0 + \rho^\nu \tilde{S} + f(\tilde{S}),$$

and the extension of the kernel is given by the generator

$$-\nu(\beta + 1)X_1 + \beta X_3 + 2\beta X_8 + 2\nu\tilde{S}\partial_{\tilde{S}},\ \ (\beta \neq 0).$$

Here

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1. \end{cases}$$

In the final Table 1 these models correspond to M_{72} and M_{73}, $(k = \lambda = 0)$.

For $\nu = 1$ one has $\eta = \rho\ln(\rho)$. Further analysis of this equation is similar to the previous case:

$$W = q_0\dot{\rho}^2 + \tilde{S}(\rho\ln\rho + q_1),\ \ (q_1 \neq 0): \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},$$
$$W = q_0\dot{\rho}^2 + \tilde{S}\rho\ln\rho: \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},\ X_3 + 2X_8 - 2\lambda\tilde{S}\partial_{\tilde{S}},$$
$$W = q_0\dot{\rho}^2 + \tilde{S}\rho\ln\rho + f(\tilde{S}): \quad -(k+1)X_1 + kX_3 + 2kX_8 + 2\tilde{S}\partial_{\tilde{S}},\ \ (k \neq 0),$$

where

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1, \end{cases}$$

and $q_1 \neq 0$. In the final Table 1 these models correspond to M_{67}, M_{66}, M_{68} and M_{69}, $(k = \lambda = 0)$, respectively.

Let $\nu = 0$, then $\eta = \ln(\rho)$, and equation (2.6) becomes

$$k_8 = 2k_1\mu f''.$$

This equation gives

$$k_1(\mu f'')' = 0.$$

Since for $k_1 = 0$ there is no extension, one has that $\mu f''$ is constant or after using equivalence transformation, one finds

$$f = \mu(\beta\ln(\mu) + q_2).$$

Thus,

$$W = q_0\dot{\rho}^2 + \tilde{S}(\ln(\rho\tilde{S}^\beta) + q_2): \qquad X_1 + \beta(X_3 + 2X_8) - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to $\mathrm{M}_{65},\ (k=\lambda=0)$.

If in equation (2.81) $a_\rho \neq 0$, then there exists a constant ν and a function $H(S)$ such that

$$b-\nu a+H(S)=0$$

or

$$\rho h_{\rho\rho\rho}+H(S)h_{S\rho\rho}=(\nu-2)h_{\rho\rho}.$$

Hence,

$$k_1=\nu k_8/2.$$

Notice that if $H=0$ then $a_\rho=0$, hence $H\neq 0$. In this case

$$h=\rho^\nu g(\rho\mu(S))+f(\mu(S)), \tag{2.84}$$

where $\mu'\neq 0$. Equation (2.6) becomes

$$\mu f''+(\nu+1)f'=0. \tag{2.85}$$

Thus,

$$W=q_0\dot{\rho}^2+\rho^\nu g(\rho\tilde{S})+f(\tilde{S}):\quad -\nu X_1+X_3+2X_8-2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to M_{62} and $\mathrm{M}_{63},\ (k=\lambda=0)$.

If $h_{S\rho\rho}=0$, then

$$h=\eta(\rho)+\mu(S),$$

where $\mu'\neq 0$, and equations (2.4) (or (2.79)) and (2.6) become, respectively,

$$2k_1\eta''+k_8(\rho\eta'''+2\eta'')=0. \tag{2.86}$$

$$(\zeta^S\mu')'=-2k_1\mu' \tag{2.87}$$

Equation (2.87) gives

$$\zeta^S=(-2k_1\mu+c_0)/\mu' \tag{2.88}$$

Hence, if $\eta''=0$, then one can assume that $\eta=0$. In this case

$$W=q_0\dot{\rho}^2+\tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1-2\tilde{S}\partial_{\tilde{S}},\quad X_3+2X_8,\quad \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to $\mathrm{M}_{61},\ (k=\lambda=0)$.

If $\eta''\neq 0$, then equation (2.86) leads to

$$k_1=-k_8\left(\frac{\rho\eta'''}{2\eta''}+1\right).$$

This gives that

$$k_8\left(\frac{\rho\eta'''}{\eta''}\right)'=0.$$

For $\rho\eta'''=\nu\eta''$ one has

$$2k_1+k_8(\nu+2)=0.$$

In this case

$$W = q_0\dot{\rho}^2 + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1\rho^\nu),$$

and the extension of the kernel is given by the generators

$$-(\nu+2)X_1 + X_3 + 2X_8 + 2(\nu+2)\tilde{S}\partial_{\tilde{S}}, \;\; \partial_{\tilde{S}}.$$

In the final Table 1 this model corresponds to M_{59}, $(k = \lambda = 0)$.

For $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$ one has the only generator $\partial_{\tilde{S}}$. In the final Table 1 this model corresponds to M_{60}, $(k = \lambda = 0)$.

Chapter 3

Group analysis of evolutionary integro-differential equations describing nonlinear waves: General model

Abstract The research deals with an evolutionary integro-differential equation describing nonlinear waves. Particular choice of the kernel in the integral leads to well-known equations such as the Khokhlov-Zabolotskaya equation, the Kadomtsev-Petviashvili equation and others. Since solutions of these equations describe many physical phenomena, analysis of the general model studied in the project is important. One of the methods for obtaining solutions differential equations is provided by the Lie group analysis. However, this method is not applicable to integro-differential equations. Therefore we discuss new approaches developed in modern group analysis and apply them to the general model considered in the present research. Reduced equations and exact solutions are also presented.

3.1 Physical statement and main physical parameters

For definiteness, a concrete physical object is considered which is most simple and, at the same time, can be adequately described by models like (1.2). Namely, we will deal with high-intensity acoustic waves. The general equation (1.2), as well as majority of the particular models (1.3)-(1.8) have been written at first for nonlinear acoustic waves.

Note that equation (1.2) is written in certain dimensionless variables in order to reduce all coefficients of the equation into unity. To discuss a physical meaning of mathematical models we rewrite equation (1.2) using initial physical notations:

$$\frac{\partial}{\partial \tau}\left[\frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3 \rho} p \frac{\partial p}{\partial \tau} - W\right] = \frac{c}{2}\Delta_\perp p, \tag{3.1}$$

$$\begin{aligned} W &= \frac{m}{2c}\frac{\partial}{\partial \tau}\int_{-\infty}^{\tau} K\left(\frac{\tau - \tau'}{t_0}\right)\frac{\partial p}{\partial \tau'}\, d\tau' \\ &= \frac{m}{2c}\frac{\partial^2}{\partial \tau^2}\int_{0}^{\infty} K\left(\frac{\xi}{t_0}\right) p\left(\vec{r},\ \tau - \xi\right)\, d\xi. \end{aligned} \tag{3.2}$$

Here x is the coordinate along the direction of wave propagation; $\Delta_\perp$ is the Laplace operator written in the coordinates y, z on the orthogonal plane, $\tau = t - x/c$ is the time in the moving system of coordinates propagating with the sound velocity c, ε is the parameter of nonlinearity, and ρ is the density of a medium. The acoustic pressure p is chosen as the wave field variable. The constant m characterizes the "force" of time delay processes, and t_0 is the typical "memorizing time" of a medium.

Let us note an important point before passing to further discussion. The natural question arises: why is the coordinate x instead of time t used as a "slow" (evolutionary) variable in equations (1.2) and (3.1)? The answer is that the difference between x and t depends only on the way of description which depends on the statement of problem and ease of analysis of results. In the case of non-wave problems (e.g description of turbulence) the problem is posed as follows. At the initial moment $t = 0$ a distribution of the velocity field in space $u(t = 0,\ x) = u_0(x)$ is given, and the solution $u(t, x)$ is sought with growing time $t > 0$. In the corresponding experiment, sensors measuring the velocity field are placed in various locations, and the measurement is made by all sensors at the same time t_1. These results determine the spatial structure of the field $u(t_1, x)$. Then similar measurements performed at t_2 give the field profile $u(t_2, x)$. Repeating the measurements we trace the field evolution with respect to time.

When propagating waves are of interest, the experiment is done in a different way. The only sensor placed at the position x_1 measures the variation of the signal with respect to time: $u(x_1, t)$. Then the sensor is moved to another position $x_2 > x_1$ and the signal $u(x_2\,,\ t)$ is measured. By moving the sensor of the vibration velocity (or the acoustic pressure) farther and farther from the source of wave, we trace the evolution of the form of the wave profile as the wave propagates. In real experiments a wave gets distorted at distances of the order of thousand wavelengths, whereas, for a good reconstruction of the wave profile within its each length λ, one has to place no fewer than ten sensors. In this case the method of the "slow time" is very inconvenient. Moreover, this method is completely inappropriate in those cases when the wave profile contains shock fronts whose extent is very small, e.g. $10^{-4}\lambda$.

But in various acoustic problems, e.g. those dealing with standing waves in a resonator, it is convenient to utilize the "slow time" instead of the "slow coordinate." It is clear that the resonator has a limited length and, by measuring the field at the lowest modes, it is quite realistic to place several sensors along the length of the resonator and perform simultaneous measurement with them at various moments of time.

Let us return to the physical model (3.1), (3.2). The integration within the limits $-\infty < \tau' < \tau$ in the first integral (3.2) means that the wave behavior at a given moment τ is determined by the values of the field variable at the preceding moments from τ to the infinitely distant past. Consequently, the kernel $K(\tau)$ describing the "memory" of a medium, must be nonzero only at positive values of its argument and tend to zero for $\tau \to +\infty$. Decreasing can be non-monotone and can look like oscillatory damping (see the example leading to formula (7)).

In order to understand how the concrete form of the kernel is related with the measured characteristics of the medium, we shall consider the simplest model of a plane wave moving in a medium without nonlinearity. In other words, let us consider the equation

$$\frac{\partial p}{\partial x} - \frac{m}{2c}\frac{\partial^2}{\partial \tau^2}\int_0^\infty K\left(\frac{\xi}{t_0}\right) p\,(x,\ \tau - \xi)\ d\xi = 0. \tag{3.3}$$

Let us establish a relation of a kernel with the dispersion law. A solution is sought in the form

$$p = \exp\left(-i\omega\, t + ik\, x\right), \quad k = k' + ik''. \tag{3.4}$$

Here k is the wave number, and k', k'' are its real and imaginary parts. Substituting (3.4) into (3.3), we find

$$\begin{aligned} k' &= -\frac{m\omega^2 t_0}{2c}\int_0^\infty K(s)\sin\left(\omega t_0\, s\right)\, ds, \\ k'' &= \frac{m\omega^2 t_0}{2c}\int_0^\infty K(s)\cos\left(\omega t_0\, s\right)\, ds. \end{aligned} \tag{3.5}$$

The first formula in (3.5) gives a frequency-dependent addition to the velocity of the wave propagation: $c(\omega) = c\left(1 - ck'(\omega)/\omega\right)$. The second one defines the absorption coefficient or the law of spatial decrease of the wave amplitude: $p_0 \exp\left(-k''x\right)$.

Evidently, the integrals (3.5) must be convergent for physically feasible kernels. The concrete form of a kernel can be reconstructed on the base of corresponding physical model, or on the base of experimental measurements.

A relaxing medium provides an important model known as the Mandelstam-Leontovich model (see [21, 84]). The kernel for this model has the exponential form (see the example leading to formula (1.7)). In this case

$$k' = -\frac{m\omega}{2c}\frac{(\omega t_0)^2}{1 + (\omega t_0)^2}, \quad k'' = \frac{m\omega}{2c}\frac{\omega t_0}{1 + (\omega t_0)^2}. \tag{3.6}$$

The frequency dependencies (3.6) of the dispersion k' and the absorption k'' were confirmed repeatedly in experiments. One could proceed in an opposite way. First establish the dependencies (3.6) as empirical generalization of measured data, and then reconstruct the kernel by means of a standard procedure. This procedure exploits the causality principle according to which two functions k' and k'' cannot be arbitrary but should be connected by relations of Kramers-Kronig's type [32].

The method of kernel reconstruction has been utilized for deriving mathematical models used in medical applications of ultrasound [85]. It is known that, within the most interesting frequency range, the absorption of the ultrasound in soft tissues behaves like $k'' \sim \omega^{2-\nu}$, $0 < \nu < 1$. It is easy to reconstruct the kernel $K(s) = s^{\nu-1}$ and verify that the corresponding absorption coefficient

$$k'' = \frac{m}{2ct_0}\Gamma(\nu)\cos\left(\frac{\pi}{2}\nu\right)(\omega t_0)^{2-\nu} \tag{3.7}$$

has the correct frequency dependence. Note that the considered power kernel has a singularity at $s = 0$ and is not integrable in semi-infinite limits. However, the convolution of this kernel with the oscillating function describing a wave provides convergence of the integral for k''. This example demonstrates a wide variety of situations which can be met in applications.

In conclusion of this section we demonstrate how one has to change variables in equation (3.1), (3.2) to reduce it to the simplest normalized form (1.2). One has to set

$$\tau \to t_0 t, \quad p \to p_0 u, \quad x \to x_0 x, \quad y \to y_0 y, \quad z \to z_0 z, \tag{3.8}$$

where the constant t_0 (the "memory" time) is defined by the structure of kernel, and the other constants are:

$$p_0 = \frac{m}{2\varepsilon}c^2\rho, \quad x_0 = \frac{2}{m}ct_0, \quad y_0 = z_0 = \frac{ct_0}{\sqrt{m}}. \tag{3.9}$$

3.2 Admitted Lie group

As for differential equations an admitted Lie group of integro-differential equation (1.2) is defined by the determining equations. These equations are integro-differential equations for the coordinates of the infinitesimal generator

$$X = \xi^t \partial_t + \xi^x \partial_x + \xi^y \partial_y + \xi^z \partial_z + \zeta^u \partial_u + \zeta^w \partial_w,$$

where the coordinates ξ^t, ξ^x, ξ^y, ξ^z, ζ^u and ζ^w are functions depending on the variables (t, x, y, z, u, w). The system (1.2) comprises a partial differential equation and an integro-differential equation. The determining equation related with the partial differential equation is obtained by the standard procedure:

$$\zeta^{u_{tx}} - u\zeta^{u_{tt}} - 2u_t\zeta^{u_t} - \zeta^{w_{ttt}} = \zeta^{u_{yy}} + \zeta^{u_{zz}}, \tag{3.10}$$

where the coefficients $\zeta^{u_{tx}}$, $\zeta^{u_{tt}}$, ζ^{u_t}, $\zeta^{w_{ttt}}$, $\zeta^{u_{yy}}$ and $\zeta^{u_{zz}}$ are the coefficients of the prolonged generator X:

$$\overline{X} = X + \zeta^{u_{tx}} \partial_{u_{tx}} + \zeta^{u_{tt}} \partial_{u_{tt}} + \zeta^{u_t} \partial_{u_t} + \zeta^{w_{ttt}} \partial_{w_{ttt}} + \zeta^{u_{yy}} \partial_{u_{yy}} + \zeta_x^{u_{zz}} \partial_{u_{zz}}.$$

The general theory of constructing determining equations for integro-differential equations can be found in [20]. Formerly the determining equation related with integro-differential equation is obtained applying the following strategy. First, one has to construct the canonical Lie-Bäcklund operator equivalent to the generator X:

$$\widetilde{X} = (\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z)\partial_u + (\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z)\partial_w.$$

Then the Lie-Bäcklund operator has to be prolonged up to the maximum order of derivatives of the equation. Finally, the determining equation is obtained by applying the prolonged Lie-Bäcklund operator to the equation, where the actions of the derivatives are considered in terms of the Frechet derivatives:

$$\psi^w(t, x, y, z) = \int_0^\infty K(s)\psi^u(t - s, x, y, z)\, ds. \tag{3.11}$$

Here

$$\psi^u(h_1) = \zeta^u(h_2) - \xi^t(h_2)u_t(h_1) - \xi^x(h_2)u_x(h_1) - \xi^y(h_2)u_y(h_1) - \xi^z(h_2)u_z(h_1),$$
$$\psi^w(h_1) = \zeta^w(h_2) - \xi^t(h_2)w_t(h_1) - \xi^x(h_2)w_x(h_1) - \xi^y(h_2)w_y(h_1) - \xi^z(h_2)w_z(h_1),$$

where for the sake of simplicity of the presentation we denoted

$$h_1 = (t, x, y, z), \quad h_2 = (t, x, y, z, u(t, x, y, z), w(t, x, y, z)).$$

The determining equations (3.10) and (3.11) have to be satisfied for any solution of equations (1.2). Notice that the determining equation (3.11) is still an integral equation.

Since it is difficult to find the general solution of the determining equations (3.10) and (3.11), the following simplification is considered. One can assume that the determining equation (3.10) is valid for any functions $u(t, x, y, z)$ and $w(t, x, y, z)$ only satisfying the first equation of (1.2). This allows to use standard procedure for solving determining equations developed for partial differential equations. After solving the determining equation (3.10), one can use the found solution for solving the integral determining equation (3.11). It has to be noticed that this way of solving the determining equations (3.10) and (3.11) can give a particular solution. In the present research this method is used.

The described method of solving the determining equations (3.10) and (3.11) will be illustrated on the one-dimensional case of equations (1.2). For the other cases final results will be presented in next sections.

3.2.1 One-dimensional case

In the one-dimensional case equations (1.2) are

$$\frac{\partial}{\partial t}(u_x - uu_t - w_{tt}) = 0, \quad w(t,x) = \int_0^\infty K(s)u(t-s,x)\,ds. \tag{3.12}$$

The admitted generator is sought in the form

$$X = \xi^t(t,x,u,w)\partial_t + \xi^x(t,x,u,w)\partial_x + \zeta^u(t,x,u,w)\partial_u + \zeta^w(t,x,u,w)\partial_w.$$

Applying the group analysis method to the first equation, one finds that

$$\begin{gathered} \xi^t = t(\xi'/2 + k_1)t + g, \quad \zeta^u = u(k_1 - \xi'/2) - t\xi''/2 - g', \\ \zeta^w = -(6\xi' w + t^3\xi''')/12 + 3wk_1 + t^2\mu + t\eta + \zeta. \end{gathered} \tag{3.13}$$

where k_1 is constant, $\xi = \xi(x)$, $\zeta = \zeta(x)$, $\eta = \eta(x)$, $\mu = \mu(x)$ and $g = g(x)$ are arbitrary functions.

Remark. There is another representation of the first equation of (3.12). This equation is obtained by integrating with respect to t and setting the arbitrary function of the integration to zero:

$$u_x - uu_t - w_{tt} = 0, \quad w(t,x) = \int_0^\infty K(s)u(t-s,x)\,ds. \tag{3.14}$$

In this case the first step in finding admitted Lie group leads to (3.13) with the particular case of the function $\mu = -\xi''/2$.

The determining equation for the second equation of (3.12)

$$w(t,x) = \int_0^\infty K(s)u(t-s,x)\,ds.$$

is the equation

$$(\widetilde{\zeta}^w - \widetilde{\xi}^t w_t - \widetilde{\xi}^x w_x)(t,x) = \int_0^\infty K(s)(\widetilde{\zeta}^u - \widetilde{\xi}^t u_t - \widetilde{\xi}^x u_x)(t-s,x)\,ds, \tag{3.15}$$

where

$$\begin{gathered} \widetilde{\xi}^t(t,x) = \xi^t(t,x,u(t,x),w(t,x)), \ \widetilde{\xi}^x(t,x) = \xi^x(t,x,u(t,x),w(t,x)), \\ \widetilde{\zeta}^w(t,x) = \zeta^w(t,x,u(t,x),w(t,x)), \ \widetilde{\zeta}^w(t,x) = \zeta^w(t,x,u(t,x),w(t,x)). \end{gathered}$$

Substituting the coefficients (3.13) into (3.15), let us satisfy equation (3.15). Notice that

$$(\widetilde{\xi}^x w_x)(t,x) = \int_0^\infty K(s)(\widetilde{\xi}^x u_x)(t-s,x)\,ds,$$

and

$$\begin{gathered} \widetilde{\xi}^t(t-s) = \widetilde{\xi}^t(t) - s\left(k_1 + \tfrac{1}{2}\xi'\right), \\ \widetilde{\zeta}^u(t-s) = \left(k_1 - \tfrac{1}{2}\xi'\right)u(t-s) - \tfrac{1}{2}(t-s)\xi'' - g'. \end{gathered}$$

Here and further, the argument x is omitted, further tilde is also omitted. The determining equation (3.15) becomes

$$\begin{gathered} -\zeta^w(t) + \int_0^\infty K(s)\left(\zeta^u(t-s) - s\left(k_1 + \tfrac{1}{2}\xi'\right)u_t(t-s)\right)\,ds \\ = (6\xi' w + t^3\xi''')/12 - 3wk_1 - t^2\mu - t\eta - \zeta + \left(k_1 - \tfrac{1}{2}\xi'\right)w - g'\int_0^\infty K(s)\,ds \\ -\tfrac{1}{2}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right) + \left(\tfrac{1}{2}\xi' + k_1\right)\int_0^\infty sK(s)u_t(t-s)\,ds \\ = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g'\int_0^\infty K(s)\,ds \\ -\tfrac{1}{2}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right) + \left(\tfrac{1}{2}\xi' + k_1\right)\int_0^\infty sK(s)u_t(t-s)\,ds. \end{gathered}$$

Let us calculate

$$\begin{array}{l}\int_0^\infty sK(s)u_t(t-s)\,ds = -\int_0^\infty sK(s)\,du(t-s)\\ = -sK(s)u(t-s)|_0^\infty + \int_0^\infty (K(s)+sK'(s))u(t-s)\,ds\\ = w + \int_0^\infty sK'(s)u(t-s)\,ds.\end{array}$$

Here it is assumed that

$$sK(s)u(t-s)|_0^\infty = 0.$$

The determining equation becomes

$$\begin{array}{c}t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g'\int_0^\infty K(s)\,ds\\ -\frac{1}{2}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right) + \left(\frac{1}{2}\xi' + k_1\right)\int_0^\infty sK(s)u_t(t-s)\,ds\\ = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g'\int_0^\infty K(s)\,ds - \frac{1}{2}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right)\\ + \left(\frac{1}{2}\xi' + k_1\right)\left(w + \int_0^\infty sK'(s)u(t-s)\,ds\right)\\ = t^3\xi'''/12 + w(\frac{1}{2}\xi' - k_1) - t^2\mu - t\eta - \zeta - g'\int_0^\infty K(s)\,ds - \frac{1}{2}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right)\\ + \left(\frac{1}{2}\xi' + k_1\right)\int_0^\infty sK'(s)u(t-s)\,ds\\ = t^3\xi'''/12 - t^2\mu - t\eta - \zeta - g'\int_0^\infty K(s)\,ds - \frac{1}{3}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right)\\ + \int_0^\infty \left(\left(\frac{1}{2}\xi' + k_1\right)sK'(s) + \left(\frac{1}{2}\xi' - k_1\right)K(s)\right)u(t-s)\,ds = 0.\end{array}$$

Since $u = 0$, $w = 0$ is a solution of equations (3.12), the determining equation has to be satisfied on this solution. Thus, one obtains

$$\begin{array}{c}-t^3\xi'''/12 + t^2\mu + t\eta + \zeta + g'\int_0^\infty K(s)\,ds\\ +\frac{1}{3}\xi''\left(t\int_0^\infty K(s)\,ds - \int_0^\infty sK(s)\,ds\right) = 0,\end{array} \tag{3.16}$$

and, hence,

$$\int_0^\infty \left(\left(\frac{1}{2}\xi' + k_1\right)sK'(s) + \left(\frac{1}{2}\xi' - k_1\right)K(s)\right)u(t-s)\,ds = 0.$$

Since $u(t,x)$ is an arbitrary, the last equation gives

$$\left(\frac{1}{2}\xi' + k_1\right)sK'(s) + \left(\frac{1}{2}\xi' - k_1\right)K(s) = 0,$$

which means that ξ' is constant, for example, $\xi = 2kx + k_0$. Thus,

$$(k + k_1)\,sK'(s) + (k - k_1)K(s) = 0,$$

or $K(s) = K_0 s^\alpha$, and

$$(k + k_1)\,\alpha + (k - k_1) = 0.$$

Equation (3.16) becomes

$$t^2\mu + t\eta + \zeta + g'\int_0^\infty K(s)\,ds = 0.$$

This equation implies

$$\mu = 0, \quad \eta = 0, \quad \zeta = -g'\int_0^\infty K(s)\,ds.$$

To prevent problems with convergency of the integral $\int_0^\infty K(s)\,ds$, one can assume that

$$g' = 0.$$

Remark. One can call the transformations corresponding to the generators

$$X = g\partial_t - g'\partial_u$$

formally admitted. These generators can be used for constructing invariant solutions.

Thus, one obtains that the Lie group corresponding to the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t,$$

$$X_3 = k^t t\partial_t + k^x x\partial_x + k^u u\partial_u + k^w w\partial_w,$$

is admitted by equations (3.12). Here

$$\begin{array}{c} k^t = k + k_1, \quad k^x = 2k, \quad k^u = k_1 - k, \quad k^w = 3k_1 - k, \\ (k + k_1)\alpha + (k - k_1) = 0. \end{array} \tag{3.17}$$

Notice also that

$$k^u = \alpha k^t, \quad k^t \neq 0.$$

Since the integro-differential equations are nonlocal, not any admitted Lie group has the property to transform a solution of integro-differential equations into a solution. However, for the transformations corresponding to the generators X_1 and X_2 it is trivial to check that these transformations possess this property. Let us also check that the scaling group corresponding to the generator X_3 maps any solution of equations (3.12) into a solution of the same equations.

The transformation corresponding to the generator X_3 is

$$t' = te^{ak^t}, \; x' = xe^{ak^x}, \; u' = ue^{ak^u}.$$

which maps a function $u(t, x)$ into the function

$$u'(t', x') = e^{ak^u} u(t'e^{-ak^t}, x'e^{-ak^x}).$$

Let us consider the transformation of the integral

$$\begin{array}{ll} & \int_0^\infty K(s')\, u'(t' - s', x')\, ds' \\ = & e^{ak^u} \int_0^\infty K(s') u((t' - s')e^{-ak^t}, x'e^{-ak^x})\, ds' \\ = & e^{ak^u} \int_0^\infty K(s') u(t'e^{-ak^t} - s'e^{-ak^t})\, ds' \\ = & e^{a(k^u + k^t)} \int_0^\infty K(s'e^{-ak^t} e^{ak^t}) u(t - s'e^{-ak^t})\, d(s'e^{-ak^t}) \\ = & e^{a(k^u + k^t)} \int_0^\infty K(se^{ak^t}) u(t - s)\, ds \\ = & e^{a(k^u + k^t)} \int_0^\infty K_0(se^{ak^t})^\alpha u(t - s)\, ds \\ = & e^{a(k^u + (\alpha+1)k^t)} \int_0^\infty K_0 s^\alpha u(t - s)\, ds \\ = & e^{a(k^u + (\alpha+1)k^t)} w(t, x) \end{array}$$

Thus, for checking one only needs to check that

$$-k^w + k^u + (\alpha + 1)k^t = 0.$$

Indeed

$$\begin{array}{c} -k^w + k^u + (\alpha + 1)k^t = \\ = -3k_1 + k + k_1 - k + (\alpha + 1)(k + k_1) \\ = k_1(-3 + 1 + \alpha + 1) + k(1 - 1 + \alpha + 1) \\ = k_1(\alpha - 1) + k(\alpha + 1) = 0. \end{array}$$

Here the condition (3.17) was used. One also needs to check the other conditions:

$$k^u - k^t - k^x = 2k^u - 2k^t = k^w - 3k^t$$

Indeed,

$$\begin{gathered} k^u - k^t - k^x = k_1 - k - (k + k_1) - 2k = -4k, \\ 2k^u - 2k^t = 2(k_1 - k) - 2(k + k_1) = -4k, \\ k^w - 3k^t = 3k_1 - k - 3(k + k_1) = -4k. \end{gathered}$$

3.2.2 Classification of subalgebras

The commutator table of the Lie algebra $L_3 = \{X_1, X_2, X_3\}$ is

	X_1	X_2	X_3
X_1	0	0	$k^x X_1$
X_2	0	0	$k^t X_2$
X_3	$-k^x X_1$	$-k^t X_2$	0

The set of automorphisms is defined by the commutators table:

$$\begin{aligned} A_1: &\quad x_1' = x_1 + a_1 k^x x_3, \\ A_2: &\quad x_2' = x_2 + a_2 k^t x_3, \\ A_3: &\quad x_1' = x_1 e^{a_3 k^x}, \;\; x_2' = x_2 e^{a_3 k^t}. \end{aligned}$$

where only changeable coordinates of the automorphisms are presented.

If $\alpha = 1$, then $k^x = 0$, and the operator X_1 composes a center of the Lie algebra. Thus the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, \; X_2 + \gamma X_1, \; X_1\},$$

where $\gamma = \pm 1$.

If $\alpha \neq 1$, then $k^x \neq 0$, and the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3, \; X_2 + \lambda X_1, \; X_1\}.$$

For $\alpha \neq 0$, using the automorphism A_3 the subalgebra $X_2 + \lambda X_1$ can be also reduced to $X_2 \pm X_1$.

3.2.3 Invariant solutions

The optimal system of subalgebras of the Lie algebra L_3 defines the complete set of representations of solutions invariant with respect to L_3.

Case $\alpha = 1$

The subalgebra $\{X_3 + \lambda X_1\}$. The generator $\{X_3 + \lambda X_1\}$ is

$$t\partial_t + u\partial_u + \lambda\partial_x.$$

It is convenient to separate on two cases (a) $\lambda = 0$, (b) $\lambda \neq 0$. The invariants are

$$\begin{aligned} \lambda = 0: &\quad u/t, x; \\ \lambda \neq 0: &\quad u/t, te^{x\lambda_o}, \;\; (\lambda_o = -1/\lambda). \end{aligned}$$

Invariant solutions have the representations

$$\lambda = 0: \quad u = t\varphi(x);$$
$$\lambda \neq 0: \quad u = t\varphi(y), \quad (y = te^{x\lambda_o}).$$

Notice that for $\lambda = 0$ the integral

$$w = \int_0^\infty K_0 s(t-s)\varphi(x)\,ds = K_0\varphi(x)\int_0^\infty s(t-s)\,ds,$$

is divergent. Hence, one has only to consider the case $\lambda \neq 0$. In this case

$$w(t,x) = K_0\int_0^\infty s(t-s)\varphi((t-s)e^{x\lambda_o})\,ds = K_0 e^{-3x\lambda_o}\int_y^{-\infty}(y-z)z\varphi(z)\,d(y-z)$$
$$= K_0 e^{-3x\lambda_o}\int_0^\infty s(y-s)\varphi(y-s)\,ds = e^{-3x\lambda_o}W(y).$$
$$w_t = e^{-2x\lambda_o}W',\ w_{tt} = e^{-x\lambda_o}W'',$$

where we used the relations

$$s = t - ze^{-x\lambda_o} = e^{-x\lambda_o}(te^{x\lambda_o} - z) = e^{-x\lambda_o}(y-z).$$

Equation (3.14) becomes

$$y\left((\lambda\varphi+1)\varphi' - \varphi^2\right) = K_0\frac{d^2}{dy^2}\left(\int_0^\infty s(y-s)\varphi(y-s)\,ds\right).$$

The subalgebra $\{X_2 + \gamma X_1\}$. The generator $\{X_2 + \gamma X_1\}$ is $\partial_t + \gamma\partial_x$. The invariants are

$$u, y = x - \gamma t.$$

An invariant solution has the representation

$$u = \varphi(y).$$

Hence, one has

$$w(t,x) = K_0\int_0^\infty s\varphi(x-\gamma(t-s))\,ds = K_0\int_y^{-\infty}(y-z)\varphi(z)\,d(y-z)$$
$$= K_0\int_0^\infty s\varphi(y-s)\,ds = W(y).$$
$$w_t = -\gamma W',\ w_{tt} = W'',$$

where we used the relations

$$s = t + \gamma(z-x) = -\gamma(x-\gamma t) + \gamma z = -\gamma(y-z).$$

Equation (3.14) becomes

$$(1+\gamma\varphi)\varphi' = K_0\frac{d^2}{dy^2}\left(\int_0^\infty s(y-s)\varphi(y-s)\,ds\right).$$

The subalgebra $\{X_1\}$. The generator $\{X_1\}$ is ∂_x. The invariants are

$$u, t.$$

An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t,x) = K_0\int_0^\infty s\varphi(t-s)\,ds.$$

Equation (3.14) becomes

$$-\varphi\varphi' = K_0\frac{d^2}{dt^2}\left(\int_0^\infty s\varphi(t-s)\,ds\right).$$

3.2.4 Case $\alpha \neq 1$.

The subalgebra $\{X_3\}$. The generator $\{X_3\}$ is

$$X_3 = k^t t\partial_t + k^x x\partial_x + k^u u\partial_u, \quad (k^x \neq 0),$$

where the constants k^t, k^x and k^u are defined by (3.17). Substituting the representation of the invariant solution

$$u = x^{k^u/(2k)}\varphi(y), \quad t = yx^{k^t/(2k)}$$

into the second equation of (3.14), one has

$$\begin{aligned} w = & \int_0^\infty K_0 s^\alpha u(t-s)\,ds = K_0 x^{k^u/(2k)} \int_0^\infty s^\alpha \varphi((t-s)x^{-k^t/(2k)})\,ds \\ = & -K_0 x^{((1+\alpha)k^t+k^u)/(2k)} \int_{tx^{-k^t/(2k)}}^{-\infty} (y-z)^\alpha \varphi(z)\,dz \\ = & K_0 x^{((1+\alpha)k^t+k^u)/(2k)} \int_{-\infty}^{y} (y-z)^\alpha \varphi(z)\,dz \\ = & K_0 x^{((1+\alpha)k^t+k^u)/(2k)} W(y) = K_0 x^{(k^t+2k^u)/(2k)} W(y), \end{aligned}$$

where we used the relations

$$s = t - zx^{k^t/(2k)} = x^{k^t/(2k)}(tx^{-k^t/(2k)} - z) = x^{k^t/(2k)}(y-z), \quad (1+\alpha)k^t + k^u = k^t + 2k^u,$$

and

$$W(y) = \int_{-\infty}^{y} (y-z)^\alpha \varphi(z)\,dz.$$

The derivatives are changed as follows

$$\frac{\partial\varphi}{\partial t} = \frac{y}{t}\varphi', \quad \frac{\partial\varphi}{\partial x} = -\frac{k^t}{2k}\frac{y}{x}\varphi',$$

$$\frac{\partial}{\partial t}\left(\frac{\partial W}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{y}{t}W'\right) = \frac{1}{t^2}\left(-yW' + y\,(yW')'\right) = \frac{y^2}{t^2}W''.$$

Thus, the first equation of (3.14) becomes

$$\begin{aligned} & u_x - uu_t - v_{tt} = u_x - uu_t - \tfrac{\partial^2}{\partial t^2}\int_0^\infty K_0 s^\alpha u(t-s)\,ds \\ = & \tfrac{k^u}{2k}\tfrac{x^{k^u/(2k)}}{x}\varphi - \tfrac{k^t}{2k}\tfrac{x^{k^u/(2k)}y}{x}\varphi' - x^{k^u/(2k)}\varphi x^{k^u/(2k)}\tfrac{y}{t}\varphi' - K_0\tfrac{x^{(k^t+2k^u)/(2k)}y^2}{t^2}W'' \\ = & \tfrac{k^u}{2k}\tfrac{x^{k^u/(2k)}}{x}\varphi - \tfrac{k^t}{2k}\tfrac{x^{k^u/(2k)}y}{x}\varphi' - x^{k^u/(2k)}\varphi x^{k^u/(2k)}\tfrac{y}{t}\varphi' - K_0 y x^{(2k^u-k^t)/(2k)}W'' \\ = & \tfrac{x^{k^u/(2k)}}{x}\left(\tfrac{k^u}{2k}\varphi - \tfrac{k^t}{2k}y\varphi' - \varphi x^{(k^x+k^u-k^t)/(2k)}\varphi' - K_0 x^{(k^u-k^t+k^x)/(2k)}W''\right) \\ = & \tfrac{x^{k^u/(2k)}}{x}\left(\tfrac{k^u}{2k}\varphi - \tfrac{k^t}{2k}y\varphi' - \varphi\varphi' - K_0 W''\right) = 0. \end{aligned}$$

Here, the following relation was used

$$k^u + k^x - k^t = k_1 - k + 2k - (k + k_1) = 0.$$

Thus, the reduced equation is

$$\alpha\varphi + ((\alpha-1)\varphi - y)\varphi' = (1-\alpha)K_0 \int_0^\infty s^\alpha \varphi''(y-s)\,ds,$$

where

$$W''(y) = \frac{d^2}{dy^2}\left(\int_{-\infty}^{y}(y-z)^\alpha\varphi(z)\,dz\right) = \frac{d^2}{dy^2}\left(\int_0^\infty s^\alpha\varphi(y-s)\,ds\right) = \int_0^\infty s^\alpha\varphi''(y-s)\,ds.$$

The subalgebra $\{X_2 + \lambda X_1\}$. The generator $\{X_2 + \lambda X_1\}$ is $\partial_t + \lambda\partial_x$. The invariants are

$$u, y = x - \lambda t.$$

An invariant solution has the representation of a traveling wave type

$$u = \varphi(y).$$

As in the previous case one only needs to study the case $\lambda \neq 0$. If $\lambda > 0$, then one has

$$\begin{gathered} w(t,x) = K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t-s))\, ds = \lambda^{-\alpha-1} K_0 \int_y^\infty (z-y)^\alpha \varphi(z)\, d(z-y) \\ = \lambda^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y+s)\, ds = W(y). \\ w_t = -\lambda W',\ w_{tt} = \lambda^2 W'', \end{gathered}$$

where

$$s = t + \frac{z-x}{\lambda} = -\frac{x - \lambda t}{\lambda} + \frac{z}{\lambda} = \frac{z-y}{\lambda}.$$

The reduced equation is

$$(1 + \lambda\varphi)\varphi' = \lambda^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s)\, ds. \tag{3.18}$$

If $\lambda < 0$, then

$$\begin{gathered} w(t,x) = K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t-s))\, ds = K_0(-\lambda)^{-\alpha-1} \int_y^{-\infty} (y-z)^\alpha \varphi(z)\, d(y-z) \\ = (-\lambda)^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y+s)\, ds = W(y). \end{gathered}$$

Equation (3.14) becomes

$$(1 + \lambda\varphi)\varphi' = (-\lambda)^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s)\, ds. \tag{3.19}$$

Combining equations (3.18) and (3.19), one has

$$(1 + \lambda\varphi)\varphi' = |\lambda|^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y+s)\, ds. \tag{3.20}$$

The subalgebra $\{X_1\}$. The generator $\{X_1\}$ is ∂_x. The invariants are u, t. An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t,x) = K_0 \int_0^\infty s^\alpha \varphi(t-s)\, ds.$$

Equation (3.14) becomes

$$-\varphi\varphi' = K_0 \left(\int_0^\infty s^\alpha \varphi''(t-s)\, ds \right).$$

3.3 Two-dimensional equation

The studied equations are

$$\frac{\partial}{\partial t}\left(u_x - upu_t - w_{tt}\right) = u_{yy}, \quad w(t,x,y) = \int_0^\infty K(s)u(t-s,x,y)\,ds.$$

The first step gives the generator

$$X = \xi^t(t,x,y,p,w)\partial_t + \xi^x(t,x,y,p,w)\partial_x + \xi^y(t,x,y,p,w)\partial_y \\ +\zeta^u(t,x,y,p,w)\partial_p + \zeta^v(t,x,y,p,w)\partial_w,$$

where the coefficients are

$$\begin{gathered}\xi^t = (2t\xi' + \xi'' y^2 + 3h' y)/6 + tk_1 + g, \quad \xi^y = \xi, \quad \xi^y = (4y\xi' + 3yk_1)/6 + h, \\ \zeta^u = ((6k_1 - 4\xi') - \xi''' y^2 - 2t\xi'' - 6g' - 3yh'')/6, \\ \zeta^v = (3k_1 - \xi')w + t^2\mu + t\eta + \zeta,\end{gathered} \tag{3.21}$$

Here k_1 is constant, $\xi = \xi(x)$, $h = h(x)$, $g = g(x)$, $\zeta = \zeta(t,x,y)$, $\eta = \eta(t,x,y)$ and $\mu = \mu(t,x,y)$ are arbitrary functions.

The determining equation for the second equation

$$w(t,x,y) = \int_0^\infty K(s)u(t-s,x,y)\,ds.$$

is the equation

$$(\zeta^v - \xi^t w_t - \xi^x w_x - \xi^y w_y)(t,x,y) = \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y)(t-s,x,y)\,ds.$$

Substituting the coefficients (3.21) into the last equation, one obtains

$$\mu = 0, \quad \eta = 0, \quad \zeta = 0, \quad g' = 0, \quad h'' = 0,$$

$$\left(\frac{1}{3}\xi' + k_1\right) sK'(s) + \left(\frac{2}{3}\xi' - k_1\right) K(s) = 0,$$

which means that ξ' is constant, for example, $\xi = 3kx + k_0$. Thus,

$$(k + k_1)\, sK'(s) + (2k - k_1)K(s) = 0,$$

or

$$K(s) = K_0 s^\alpha,$$

and

$$(k + k_1)\,\alpha + (2k - k_1) = 0.$$

Therefore, the admitted Lie algebra is defined by the generators

$$X_1 = \partial_x, \ X_2 = \partial_t, \ X_3 = y\partial_t + 2x\partial_y, \ X_4 = \partial_y, \\ X_5 = 2t(k + k_1)\partial_t + 6kx\partial_x + (4k + k_1)y\partial_y + 2(k_1 - 2k)u\partial_u + 6v(k_1 - k)\partial_w.$$

3.4 3-dimensional case

The studied equations are

$$\frac{\partial}{\partial t}(u_x - uu_t - w_{tt}) = u_{yy} + u_{zz}, \quad w(t,x,y,z) = \int_0^\infty K(s)u(t-s,x,y,z)\,ds.$$

The first step gives the generator

$$X = \xi^t(t,x,y,u,z,w)\partial_t + \xi^x(t,x,y,u,z,w)\partial_x + \xi^y(t,x,y,u,z,w)\partial_y$$
$$+\zeta^u(t,x,y,u,z,w)\partial_u + \zeta^w(t,x,y,z,u,w)\partial_w$$

with the coefficients

$$\xi^t = t(\xi' + 2k_1) + 3\xi''(z^2+y^2)/4 + (h'y + f'z)/2 + g,$$

$$\xi^x = 5\xi, \;\; \xi^y = 3\xi' y + zk_2 + yk_1 + h, \;\; \xi^z = 3\xi' z + zk_1 - yk_2 + f,$$
$$\zeta^u = 2(k_1 - 2\xi')u - 3\xi'''(z^2+y^2)/4 - \xi'' t - g' - (h''y + f''z)/2,$$
$$\zeta^w = w(6k_1 - 7\xi') + \xi''' t^3/3 + t^2\mu + t\eta + \zeta.$$

where k_1, k_2 are constant, $\xi = \xi(x)$, $h = h(x)$, $g = g(x)$, $f = f(x)$, $\zeta = \zeta(x,y,z)$, $\eta = \eta(x,y,z)$ and $\mu = \mu(x,y,z)$ are arbitrary functions. It is obvious that the generator corresponding to the rotation in the plain y and z is admitted. This generator is defined by k_2:

$$X = z\partial_y - y\partial_z.$$

Excluding this generator one obtains

$$\xi^t = t(\xi' + 2k_1) + 3\xi''(z^2+y^2)/4 + (h'y + f'z)/2 + g,$$

$$\xi^x = 5\xi, \;\; \xi^y = y(3\xi' + k_1) + h, \;\; \xi^z = z(3\xi' + k_1) + f,$$
$$\zeta^u = 2(k_1 - 2\xi')u - 3\xi'''(z^2+y^2)/4 - \xi'' t - g' - (h''y + f''z)/2,$$
$$\zeta^w = w(6k_1 - 7\xi') + \xi''' t^3/3 + t^2\mu + t\eta + \zeta.$$

The determining equation for the second equation

$$w(t,x,y,z) = \int_0^\infty K(s)u(t-s,x,y,z)\,ds$$

is the equation

$$(\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z)(t,x,y,z)$$
$$= \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z)(t-s,x,y,z)\,ds.$$

Substituting the coefficients into the last equation, one obtains

$$g = k_3, \quad h = 2k_4 x + k_5, \quad f = 2k_6 x + k_7$$

$$\xi^t = t(k + 2k_1) + k_4 y + k_6 z + k_3, \;\; \xi^x = 5(kx + k_0), \;\; \xi^y = y(3k + k_1) + 2k_4 x + k_5,$$
$$\xi^z = z(3k + k_1) + 2k_6 x + k_7, \;\; \zeta^u = 2(k_1 - 2k)u, \;\; \zeta^w = w(6k_1 - 7k).$$
$$(k + 2k_1)\, sK'(s) + 2(2k - k_1)K(s) = 0,$$

which means that

$$K(s) = K_0 s^{\alpha},$$

and

$$k(\alpha+4) + 2k_1(\alpha-1) = 0.$$

The admitted generators are

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = z\partial_y - y\partial_z, \quad X_4 = y\partial_t + 2x\partial_y,$$

$$X_5 = z\partial_t + 2x\partial_z, \quad X_6 = \partial_y, \quad X_7 = \partial_z,$$

$$\begin{array}{c} X_8 = t(k+2k_1)\partial_t + 5kx\partial_x + (3k+k_1)y\partial_y + (3k+k_1)z\partial_z \\ +2(k_1-2k)u\partial_u + w(6k_1-7k)\partial_w. \end{array}$$

3.5 Exponential kernel

There exists one known exact solution to 1D equation for an exponential kernel [21, 84]:

$$\begin{array}{c} u_x - uu_t - w_{tt} = 0, \\ w = \int_0^{\infty} \exp(-s)\; u(t-s)\, ds. \end{array}$$

One can seek for it in the form of a traveling wave:

$$u = u(t + \alpha\, x).$$

The solution has the form:

$$t + \gamma = \frac{1}{\Delta} \ln \frac{|u - \alpha + \Delta|^{1-\Delta}}{|u - \alpha - \Delta|^{1+\Delta}}.$$

Here α, Δ and γ are the constants. This solution has evident physical meaning for the parameters $\Delta > |\alpha|$, $0 < \Delta < 1$. This is shown in Fig. **??** and the solution describes the shape of a single shock front in a relaxing medium.

It is interesting to derive this solution using computed symmetries.

Since for the exponential kernel $K(s) = e^{-s}$ one has the relation (1.7):

$$w_t = u - w,$$

equations

$$u_{xt} - uu_{tt} - u_t^2 = w_{ttt} \tag{3.22}$$

and

$$u_x - uu_t = w_{tt} \tag{3.23}$$

can be reduced to the partial differential equations, respectively,

$$u_{xtt} - (1+u)u_{ttt} - 3u_t u_{tt} + u_{xt} - uu_{tt} - u_t^2 = 0, \tag{3.24}$$

and

$$u_{xt} - (1+u)u_{tt} - u_t^2 + u_x - uu_t = 0. \tag{3.25}$$

Admitted Lie groups of these equations are as follows. The admitted Lie group of equation (3.24) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = g(x)\partial_t - g'(x)\partial_u,$$

where the function $g = g(x)$ is an arbitrary function. The admitted Lie group of equation (3.25) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_t - \partial_u.$$

The commutators table of the algebra $\{X_1, X_2, X_3\}$ is

	X_1	X_2	X_3
X_1	0	0	X_2
X_2	0	0	0
X_3	$-X_2$	0	0

The generator X_2 is a center of the Lie algebra. The set of automorphisms is defined by the commutators table:

$$\begin{array}{ll} A_1: & x_2' = x_2 + a_1 x_3, \\ A_3: & x_2' = x_2 - a_3 x_1. \end{array}$$

The one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, \ X_2, \ X_1\}.$$

Representations of invariant solutions are

$$\begin{array}{rl} X_3 + \lambda X_1: & u = -\frac{x}{\lambda} + \varphi(x^2 - 2\lambda t), \quad (\lambda \neq 0), \\ X_3: & u = -\frac{t}{x} + \varphi(x), \\ X_2: & u = \varphi(x), \\ X_1: & u = \varphi(t). \end{array}$$

and the reduced equations are

$$\begin{array}{rl} X_3 + \lambda X_1: & (4\lambda^3(\varphi' + \varphi\varphi') - \lambda^2\varphi^2)' + 1 = 0, \\ X_3: & \varphi' + \frac{1}{x}\varphi = 0, \\ X_2: & \varphi' = 0, \\ X_1: & (\varphi' + \varphi\varphi' + \varphi^2/2)' = 0. \end{array}$$

The solution

$$u = u(t + \alpha x)$$

is invariant with respect to the operator $X_1 - \alpha X_2 = \partial_x - \alpha\partial_t$. The subalgebra corresponding to this operator is equivalent to the subalgebra with the generator: X_1. The reduced equation of an invariant solution corresponding to the generator X_1 is

$$\varphi' = \frac{k - \varphi^2}{2(1 + \varphi)}.$$

where k is an arbitrary constant of the integration. The general solution of this equation depends on the constant k:

$$\begin{array}{rcl} k = \alpha^2 > 0 & : & t + c_0 = \frac{1}{\alpha}\ln\left(\frac{|\varphi+\alpha|^{1-\alpha}}{|\varphi-\alpha|^{1+\alpha}}\right), \\ k = -\alpha^2 < 0 & : & t + c_0 = -\ln(\varphi^2 + \alpha^2) - \frac{2}{\alpha}\arctan(\frac{\varphi}{\alpha}), \\ k = 0 & : & t + c_0 = -2\ln(\varphi) + \frac{2}{\varphi}. \end{array}$$

3.6 Delay equation

It is desirable to derive an exact solution for any kernel which does not permit the reduction of integral equation to the differential one. Such an example exists. That is the model kernel which is nonzero on the finite segment, say, $s \in [0, 1]$. The simplest case is $K = 1,\ s \leq 1;\ K = 0,\ s > 1$. For this kernel the integral equation is reduced to the difference-differential equation

$$u_x - uu_t - \Delta u_t\,, \qquad \Delta u \equiv u(t) - u(t-1).$$

Its solution can be sought in the form of a traveling wave: $u = u(t + \alpha x)$. The reduced equation can be integrated one time. The solution is $u\,(t-1) = \frac{1}{2}\left[u^2\,(t) + 2\,(1-\alpha)\,u\,(t) - \beta^2\right]$. Here α and β are constant. This formula defines a mapping $u\,(t) \to u\,(t-1)$ which offers easy possibility to construct curves representing profiles of the wave. These profiles are shown in Fig. **??**. They display the image of a shock front in medium with constant "memory" within the segment [0, 1].

The equation

$$u_x(t,x) = (u(t,x)+1)u_t(t,x) - u_t(t-1,x)$$

is a delay differential equation. Algorithm for applying the group analysis method to delay differential equations is given in [20, 86, 45]. Calculations show that the admitted Lie group is defined by the generators

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_t - \partial_u.$$

Representations of invariant solutions are given in the previous case. The reduced equations are

$$\begin{aligned} X_3 + \lambda X_1 :&\quad 2\lambda^2(\varphi'(z)(\varphi(z)+1) + \varphi'(z+2\lambda)) = 1,\\ X_3 :&\quad \varphi' + \tfrac{1}{x}\varphi = 0,\\ X_2 :&\quad \varphi' = 0,\\ X_1 :&\quad \varphi'(t)(\varphi(t)+1) + \varphi'(t-1) = 0. \end{aligned}$$

The reduced equations can be integrated

$$\begin{aligned} X_3 + \lambda X_1 :&\quad \lambda^2(\varphi^2(z) + 2\varphi(z) + \varphi^2(z+2\lambda)) = z + c_0,\\ X_3 :&\quad \varphi = c_0/x,\\ X_2 :&\quad \varphi = c_0,\\ X_1 :&\quad \varphi^2(t) + 2\varphi(t) + 2\varphi(t-1) = c_0. \end{aligned}$$

3.7 Conclusion

The nonlinear *integro-differential* evolution equation (1.2) considered in the present research is not an exotic model. It encapsulates numerous mathematical models formulated by *differential* evolution equations and differs from them significantly not only in its form, but mostly due to its physical content meaning. Namely, any dispersion (frequency-dependent phase velocity) must be strongly connected with frequency-dependent absorption. Such connection follows from the causality principle. For example, waves having infinite velocities of propagation which are allowed by differential equations of Burgers and Korteweg-de Vries type must disappear on their way, since otherwise a cause appears at a certain point later than its effect. The causality principle is given in physical models by integral Kramers-Kronig relations. Consequently, a consistent model must contain integral terms, in other words, be represented in an integro-differential form. Though this conclusion is well known, mathematical models described by purely differential evolution equations have been widely accepted in the nonlinear wave physics due to their simplicity compared to the integro-differential models. It seems that the consistent integro-differential nonlinear models will meet more applications in future.

The present research provides a first step in application of the Lie group analysis to Equation (1.2). The approach used in this research is described in [20]. The analysis of the determining equation for the integro-differential equation allows, in particular, to single out a class of kernels used for deriving mathematical models in medical applications of ultrasound [85].

Note that for particular kernels the integro-differential equation (1.2) becomes a partial differential equation or a delay partial differential equation. In these cases the complete group classification of equation (1.2) can be obtained. In the case of partial differential equations the classical group analysis is used. For delay partial differential equations the analysis developed in [86, 45] and described in [20] is applied. A complete study of particular cases is given in the report. This provides a new result in the application of the group analysis method to partial and delay partial differential equations.

Along with admitted Lie groups, representations of exact solutions and reduced equations are constructed in the report. A complete solution and a physical interpretation of some of them is presented.

We hope that more results will be obtained in future by applying the above approach for solving concrete models of physical significance as well as for new mathematical developments. In particular, it is interesting to make the preliminary group classification of *exceptional kernels* by applying the method of an *a priory* use of symmetries [87] to the integro-differential equations of the form (1.2).

Chapter 4

On the equation for the power moment generating function of the Boltzmann equation. Group classification with respect to a source function

Abstract

An admitted Lie group of transformations for the spatially homogeneous and isotropic Boltzmann equation with sources was firstly studied by Nonnenmacher [1984]. In fact, in this paper the equation for generating function of the power moments of the Boltzmann equation solution was considered. However, this equation is still a nonlocal partial differential equation, and this property was not taken into account there. In the present research the admitted Lie group of this equation is studied using original method developed by Grigoriev and Meleshko [1986] for group analysis of equations with nonlocal operators. This method allows us to correct Nonnenmacher's approach. A group classification of the equation for the generating function with respect to sources is obtained. In the process of the group classification the algebraic method considered in Nikitin and Popovych [2001] is applied.

4.1 General Equations

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form [88]

$$\varphi_t(y,t) + \varphi(y,t)\varphi(0,t) = \int_0^1 \varphi(ys,t)\varphi(y(1-s),t)\ ds + \hat{q}(y,t). \tag{4.1}$$

Here the function $\varphi(y,t)$ is related with the Fourier transform $\tilde{\varphi}(k,t)$ of the isotropic distribution function $f(v,t)$ by the formulae

$$\varphi(k^2/2,t) = \tilde{\varphi}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)f(v,t).dv$$

Similarly, the transform of the source function $q(v,t)$ is

$$\tilde{\hat{q}}(k,t) = \frac{4\pi}{k}\int_0^\infty v\sin(kv)q(v,t)\,dv,$$

and

$$\tilde{\hat{q}}(k,t) = \hat{q}(k^2/2, t).$$

The inverse Fourier transform of $\tilde{\varphi}(k,t)$ gives the distribution function

$$f(v,t) = \frac{4\pi}{v}\int_0^\infty k\sin(kv)\tilde{\varphi}(k,t)\,dk.$$

Normalized moments of the distribution function are introduced by the formulae

$$M_n(t) = \frac{4\pi}{(2n+1)!!}\int_0^\infty f(v,t)v^{2n+2}dv, \qquad (n = 0,1,2...). \tag{4.2}$$

Following [89], one can obtain a system of equations for the moments (4.2) from (4.1). It is sufficient to substitute the expansions in power series

$$\varphi(y,t) = \sum_{n=0}^\infty (-1)^n M_n(t)\frac{y^n}{n!}, \qquad \hat{q}(y,t) = \sum_{n=0}^\infty (-1)^n q_n(t)\frac{y^n}{n!},$$

into (4.1), where

$$q_n(t) = \frac{1}{(2n+1)!!}4\pi\int_0^\infty q(v,t)v^{2n+2}dv, \qquad (n = 0,1,2...)$$

are the normalized moments of the source function. As a result, one derives the moment system considered in [48]:

$$\frac{dM_n(t)}{dt} + M_n(t)M_0(t) = \frac{1}{n+1}\sum_{k=0}^n M_k(t)M_{n-k}(t) + q_n(t). \tag{4.3}$$

For $q(v,t) = 0$ this system was derived in [43] in a very complicated way.

Let us define moment generation functions for the distribution function $f(v,t)$ and for the source function $q(v,t)$:

$$G(\omega,t) = \sum_{n=0}^\infty \omega^n M_n(t), \qquad S(\omega,t) = \sum_{n=0}^\infty \omega^n q_n(t).$$

Multiplying equations (4.3) by ω^n, and summing over all n, one obtains for $G(\omega,t)$ the equation

$$\frac{\partial^2(\omega G)}{\partial t\partial\omega} + M_0(t)\frac{\partial(\omega G)}{\partial\omega} = G^2 + \frac{\partial(\omega S)}{\partial\omega}. \tag{4.4}$$

Here the obvious relations are used

$$\sum_{n=0}^\infty (n+1)\omega^n M_n(t) = \frac{\partial(\omega G)}{\partial\omega}, \quad \sum_{n=0}^\infty (n+1)\omega^n q_n(t) = \frac{\partial(\omega S)}{\partial\omega},$$

$$\sum_{n=0}^\infty \omega^n \sum_{k=0}^n M_k(t)M_{n-k}(t) = G^2.$$

In contrast to the case of homogeneous relaxation with $q(v,t) = 0$, the gas density $M_0(t) \equiv \varphi(0,t)$ is not constant. From equation (4.3) for $n = 0$ one can obtain

$$M_0(t) = \int_0^t q_0(t')dt' + M_0(0).$$

Notice also that

$$M_0(t) = G(t, 0). \tag{4.5}$$

This is the reason why equation (4.4) has a nonlocal term. This fact was not taken into account in [48] in the process of finding an admitted Lie group. The lack of this condition can lead to incorrect admitted Lie groups. In the present research this omission is corrected.

4.2 Admitted Lie algebra of the equation for the generating function

Equation (4.4) is conveniently rewritten in the form

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \tag{4.6}$$

where $u(0) = u(t, 0)$. Here $\omega = x$, $G = u$ and $(\omega S)_\omega = g$.

As mentioned, because of the presence of the term $u(0)$, equation (4.6) differs from a partial differential equation. Hence, the classical group analysis method cannot be applied to this equation. A method that can be used for such equations with nonlocal terms was developed in [34, 45, 20]. In this section the latter method is applied for finding an admitted Lie group of equation (4.6).

The admitted generator is sought in the form

$$X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \zeta(t, x, u)\partial_u.$$

According to the algorithm [34, 45, 20], the determining equation for equation (4.6) is

$$x\psi_{tx} + \psi_t + u(0)(x\psi_x + \psi) - 2\psi u + \psi(0)(xu)_x = 0, \tag{4.7}$$

where

$$\psi(t, x) = \zeta(t, x, u(t, x)) - u_t(t, x)\tau(t, x, u(t, x)) - u_x(t, x)\xi(t, x, u(t, x)), \quad \psi(0) = \psi(t, 0).$$

After substituting the derivatives u_{tx}, u_{txx} and u_{ttx} found from equation (4.6) and its derivatives with respect to x and t into (4.7), one obtains the determining equation

$$\begin{gathered}
\zeta_{tx}x^2 + \zeta_t x + \zeta_u g x + \zeta_u u^2 x + g\xi + u^2\xi - 2ux\zeta + ux\zeta(0) \\
-x\left(g_t\tau + g_x\xi + g(\tau_t + \xi_x)\right) - \tau_t u^2 x - \xi_x u^2 x - xu_x(0)(u_x x + u)\xi(0) \\
+u(0)(\zeta_x x^2 - \zeta_u ux - u\xi + x\zeta + x\xi_x u + x\tau_t u) - \tau_x u_{tt} x^2 - x^2 u_x u_{tt}\tau_u \\
-u_t u_{xx}\xi_u x^2 - u_{xx}\xi_t x^2 + u_t u_x x(\zeta_{uu}x - \tau_{tu}x + \tau_u u(0)x - \xi_{xu}x + \xi_u) \\
+u_t(\xi_x x + \zeta_{xu}x^2 - \xi - \tau_{tx}x^2 - 2\tau_u x(g + u^2) + u(0)x(2\tau_u u - x\tau_x)) \\
+u_t^2 x(\tau_u - x\tau_{xu}) + xu_t(0)(\tau - \tau(0))(u_x x + u) + u_x^2 x^2(\xi_u u(0) - \xi_{tu}) \\
+xu_x(x(\tau_t u(0) + \zeta(0) + \zeta_{tu}) - \xi_{tx}x - \xi_t - 2\xi_u g - 2\xi_u u^2 + 2\xi_u uu(0)) \\
-u_t^2 u_x\tau_{uu}x^2 - u_t u_x^2\xi_{uu}x^2 = 0.
\end{gathered} \tag{4.8}$$

Here

$$\tau(0) = \tau(t, 0, u(t, 0)),\ \xi(0) = \xi(t, 0, u(t, 0)),\ \zeta(0) = \zeta(t, 0, u(t, 0)),$$
$$u_t(0) = u_t(t, 0),\ u_x(0) = u_x(t, 0).$$

Differentiating the determining equation (4.8) with respect to u_{tt}, u_{xx}, and then with respect to u_t and u_x, one gets

$$\tau_u = 0,\ \tau_x = 0,\ \xi_u = 0,\ \xi_t = 0.$$

Hence,

$$\tau = \tau(t),\ \xi = \xi(x),$$

and

$$\tau(0) = \tau.$$

Differentiating the determining equation with respect to u_t, and then u_x, one finds

$$\zeta_{uu} = 0$$

or

$$\zeta(t, x, u) = u\zeta_1(t, x) + \zeta_0(t, x).$$

The coefficient with $u_x u_x(0)$ in the determining equation (4.8) gives $\xi(0) = 0$. Continuing splitting the determining equation (4.8) with respect to u_t, and then u_x, one finds

$$\zeta_1(t, x) = -x^{-1}\xi(x) + \zeta_{10}(t).$$

Hence,

$$\zeta(0) = \zeta(t, 0) = u(0)(\zeta_{10}(t) - \xi'(0)) + \zeta_0(t, 0).$$

The coefficient with $u_x u(0)$ leads to the condition

$$\zeta_{10} = -\tau_t + \xi'(0).$$

Differentiating the determining equation with respect to u twice, one has

$$\xi_x = 2\frac{\xi}{x} - \xi'(0).$$

The general solution of this equation is

$$\xi = x(c_1 x + c_0).$$

Equating the coefficient with u_x to zero, one derives

$$\tau_{tt}(t) = \zeta_0(t, 0).$$

The coefficient with $u(0)$ in the determining equation (4.8) gives

$$x\zeta_{0x} + \zeta_0 = 0.$$

This equation only has one solution which is nonsingular at $x = 0$:

$$\zeta_0(t, x) = 0.$$

Hence, $\zeta_0(t, 0) = 0$, and

$$\tau = c_2 t + c_3.$$

The remaining part of the determining equation (4.8) becomes

$$g_t(c_2 t + c_3) + x g_x(c_1 x + c_0) = -2g(c_1 x + c_2). \qquad (4.9)$$

Thus, the admitted generator has the form

$$X = c_0X_0 + c_1X_1 + c_2X_2 + c_3X_3,$$

where

$$X_0 = x\partial_x,\ X_1 = x(x\partial_x - u\partial_u),\ X_2 = t\partial_t - u\partial_u,\ X_3 = \partial_t. \tag{4.10}$$

The values of the constants $c_0,\ c_1,\ c_2,\ c_3$ and relations between them depend on the function $g(t, x)$.

The trivial case of the function

$$g = 0,$$

satisfies equation (4.9), and corresponds to the case of the spatially homogeneous and isotropic Boltzmann equation without a source term. In this case, the complete group classification of the Boltzmann equation was carried out in [34, 35] using its Fourier image (4.1) with $\hat{q}(y, t) = 0$. The four–dimensional Lie algebra $L^4 = \{Y_1,\ Y_2,\ Y_3,\ Y_4\}$ spanned by the generators

$$Y_0 = y\partial_y,\quad Y_1 = y\varphi\partial_\varphi,\quad Y_2 = t\partial_t - \varphi\partial_\varphi,\quad Y_3 = \partial_t \tag{4.11}$$

defines the complete admitted Lie group G^4 of (4.1). There are direct relations between the generators (4.10) and (4.11).

Indeed, since the functions $\varphi(y, t)$ and $u(x, t)$ are related through the moments $M_n(t)$, $(n = 0, 1, 2, ...)$, it is sufficient to check that the transformations of moments defined through these functions coincide.

Let us consider the transformations corresponding to the generators Y_0 and X_0:

$$\begin{array}{rcl} Y_0 = y\partial_y & : & \bar{t} = t,\ \overline{y} = ye^a,\ \overline{\varphi} = \varphi; \\ X_0 = x\partial_x & : & \bar{t} = t,\ \overline{x} = xe^a,\ \overline{u} = u. \end{array}$$

The transformed functions are $\overline{\varphi}(\overline{y}, \bar{t}) = \varphi(\overline{y}e^{-a}, \bar{t})$ and $\overline{u}(\overline{x}, \bar{t}) = u(\overline{x}e^{-a}, \bar{t})$. The transformations of moments are, respectively:

$$\begin{aligned} \overline{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \overline{\varphi}(\overline{y}, \bar{t})}{\partial \overline{y}^n}_{|\overline{y}=0} = (-1)^n \frac{\partial^n \varphi(\overline{y}e^{-a}, \bar{t})}{\partial \overline{y}^n}_{|\overline{y}=0} = (-1)^n e^{-na} \frac{\partial^n \varphi}{\partial y^n}(0, \bar{t}) \\ &= e^{-na} M_n(\bar{t}); \\ \overline{M}_n(\bar{t}) &= n! \frac{\partial^n u(\overline{x}e^{-a}, \bar{t})}{\partial \overline{x}^n}_{|\overline{x}=0} = e^{-na} n! \frac{\partial^n u}{\partial x^n}(0, \bar{t}) = e^{-na} M_n(\bar{t}). \end{aligned}$$

Hence, one can see that the transformations of moments defined through the functions $\varphi(y, t)$ and $u(x, t)$ coincide.

The Lie groups of transformations corresponding to the generators Y_1 and X_1 are

$$\begin{array}{rcl} Y_1 = y\varphi\partial_\varphi & : & \bar{t} = t,\ \overline{y} = y,\ \overline{\varphi} = \varphi e^{ya}; \\ X_1 = x(x\partial_x - u\partial_u) & : & \bar{t} = t,\ \overline{x} = \frac{x}{1-ax},\ \overline{u} = (1 - ax)u. \end{array}$$

These transformations map the functions $\varphi(y, t)$ and $u(x, t)$ to $\overline{\varphi}(\overline{y}, \bar{t}) = e^{\overline{y}a}\varphi(\overline{y}, \bar{t})$ and $\overline{u}(\overline{x}, \bar{t}) = \dfrac{1}{1 + a\overline{x}} u(\bar{t}, \dfrac{\overline{x}}{1 + a\overline{x}})$. The transformations of moments are, respectively:

$$\begin{aligned} \overline{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \overline{\varphi}(\overline{y}, \bar{t})}{\partial \overline{y}^n}_{|\overline{y}=0} = (-1)^n \frac{\partial^n (e^{\overline{y}a}\varphi(\overline{y}, \bar{t}))}{\partial \overline{y}^n}_{|\overline{y}=0} \\ &= (-1)^n \left(\left(\frac{\partial}{\partial y} + a \right)^n \varphi \right) (0, \bar{t}); \\ \overline{M}_n(\bar{t}) &= n! \frac{\partial^n \overline{u}(\overline{x}, \bar{t})}{\partial \overline{x}^n}_{|\overline{x}=0} = n! \frac{\partial^n}{\partial \overline{x}^n} \left(\frac{1}{1 + a\overline{x}} u(\bar{t}, \frac{\overline{x}}{1 + a\overline{x}}) \right)_{|\overline{x}=0}. \end{aligned}$$

Using computer symbolic calculations with Reduce [83] one can check that these transformations of moments also coincide.

The Lie groups of transformations corresponding to the generators Y_2 and X_2 are

$$\begin{array}{lcl} Y_2 = t\partial_t - \varphi\partial_\varphi & : & \bar{t} = te^a,\ \overline{y} = y,\ \overline{\varphi} = \varphi e^{-a}; \\ X_2 = t\partial_t - u\partial_u & : & \bar{t} = te^a,\ \overline{x} = x,\ \overline{u} = ue^{-a}. \end{array}$$

These transformations map the functions $\varphi(y,t)$ and $u(x,t)$ to $\overline{\varphi}(\overline{y},\bar{t}) = e^{-a}\varphi(\overline{y},\bar{t}e^{-a})$ and $\overline{u}(\overline{x},\bar{t}) = e^{-a}u(\overline{x},\bar{t}e^{-a})$. The transformations of moments are, respectively:

$$\begin{aligned} \overline{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \overline{\varphi}(\overline{y},\bar{t})}{\partial \overline{y}^n}_{|\overline{y}=0} = (-1)^n e^{-a} \frac{\partial^n \varphi(\overline{y},\bar{t}e^{-a})}{\partial \overline{y}^n}_{|\overline{y}=0} \\ &= (-1)^n e^{-a} \frac{\partial^n \varphi}{\partial y^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a}; \\ \overline{M}_n(\bar{t}) &= n! \frac{\partial^n \overline{u}(\overline{x},\bar{t})}{\partial \overline{x}^n}_{|\overline{x}=0} = n! e^{-a} \frac{\partial^n u(\overline{x},\bar{t}e^{-a})}{\partial \overline{x}^n}_{|\overline{x}=0} \\ &= n! e^{-a} \frac{\partial^n u}{\partial x^n}(0,\bar{t}e^{-a}) = M_n(\bar{t}e^{-a})e^{-a}. \end{aligned}$$

The case where the transformations of moments corresponding to the generators $Y_3 = \partial_t$ and $X_3 = \partial_t$ coincide is trivial. These direct relations between the Lie algebras confirm correctness of our calculations.

4.3 Comparison with the results of the paper by T.F.Nonnenmacher

Let us formulate the results of [48] using the variables of the present research. The admitted generator obtained in [48] has the form

$$Z_g = \tau(t)\left(\partial_t - M_0(t)u\partial_u\right) + \alpha u\partial_u + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x, \qquad (4.12)$$

where

$$m_0(t) = \int_0^t M_0(t')\,dt', \quad \tau(t) = \left(\beta - \alpha\int_0^t e^{-m_0(t')}\,dt'\right) e^{m_0(t)},$$

α, β, γ and δ are constant. The function $g(t,x)$ has to satisfy the equation

$$\tau(t)\frac{\partial g}{\partial t} + x(x(\gamma-\delta) - \gamma)\frac{\partial g}{\partial x} = -2\left(x(\gamma-\delta) + M_0(t)\tau(t) - \alpha\right) g. \qquad (4.13)$$

Since $M_0(t)$ is unknown, comparison of our results is only possible for $g = 0$. Moreover, in contrast to equation (4.9), the source function $g(t,x)$ in (4.13) as a solution of equation (4.13) depends on the function $M_0(t)$, whereas the function $M_0(t)$ also depends on the source function. This makes equation (4.13) nonlocal and very complicated.

Comparing the operator Z_g for $g = 0$ with (4.10), one obtains that the part related with the constants γ and δ coincides with the result of the present report, whereas the part related with the constants α and β is completely different. Indeed, in this case equation (4.13) is satisfied identically, $M_0(t) = M_0(0)$, and for

$$\begin{array}{lll} M_0(0) \neq 0: & m_0(t) = tM_0(0), & \tau(t) = \beta e^{tM_0(0)} + \frac{\alpha}{M_0(0)}\left(1 - e^{tM_0(0)}\right); \\ M_0(0) = 0: & m_0(t) = 0, & \tau(t) = \beta - \alpha t. \end{array}$$

The admitted generator (4.13) becomes

$$
\begin{array}{lll}
M_0(0) \neq 0 & : & Z_0 = \left(\beta - \frac{\alpha}{M_0(0)}\right) e^{tM_0(0)} \left(\partial_t - M_0(0) u \partial_u\right) + \frac{\alpha}{M_0(0)} \partial_t \\
 & & +(\gamma - \delta) x (x\partial_x - u\partial_u) - \gamma x \partial_x; \\
M_0(0) = 0 & : & Z_0 = \beta \partial_t - \alpha (t\partial_t - u\partial_u) + (\gamma - \delta) x (x\partial_x - u\partial_u) - \gamma x \partial_x.
\end{array}
$$

One can see that the above results coincide with [48] only for $M_0(0) = 0$. The case $M_0(0) = 0$ corresponds to a gas with zero density which is not realistic. For $M_0(0) \neq 0$, the coefficient with the exponent $e^{tM_0(0)}$ plays a crucial role. This coefficient only vanishes for

$$\alpha = M_0(0)\beta. \tag{4.14}$$

In this case the admitted Lie algebra found in [48] is a proper subalgebra of the Lie algebra defined by the generators (4.10). Thus, all invariant solutions with $(\alpha, \beta, \gamma, \delta) = (M_0(0)\beta, \beta, \gamma, \delta)$ considered in [48] are particular cases of invariant solutions obtained in [34, 20]. In particular, the well-known BKW-solution is an invariant solution with respect to the generator $Y_{BKW} = c(Y_1 - Y_0) + Y_3$. In the Lie algebra (4.10) this solution is related with the generator $X_{BKW} = c(X_1 - X_0) + X_3$. Other classes of invariant solutions studied in [48] correspond to (4.14) with the particular choice $\beta = 0$.

4.4 On equivalence transformations of the equation for the generating function

For the group classification one needs to know equivalence transformations. Let us find some of them using the generators (4.10) and considering their transformations of the left hand side of equation (4.6)

$$Lu = xu_{tx} + u_t - u^2 + u(0)(xu_x + u).$$

The transformations corresponding to the generator $X_0 = x\partial_x$ map a function $u(t,x)$ into the function

$$\overline{u}(\overline{t}, \overline{x}) = u(\overline{t}, \overline{x}e^{-a}),$$

where a is the group parameter. Hence,

$$\overline{L}\overline{u} = Lu.$$

One can check that the Lie group of transformations

$$\overline{t} = t, \ \overline{x} = xe^a, \ \overline{u} = u, \ \overline{g} = g$$

is an equivalence Lie group of equation (4.6).

Similarly, one derives that the transformations corresponding to the generator $X_3 = \partial_t$ define the equivalence Lie group:

$$\overline{t} = t + a, \ \overline{x} = x, \ \overline{u} = u, \ \overline{g} = g.$$

The transformations corresponding to the generator $X_2 = t\partial_t - u\partial_u$ map a function $u(t,x)$ into the function

$$\overline{u}(\overline{t}, \overline{x}) = e^{-a} u(\overline{t}e^{-a}, \overline{x}).$$

Hence,

$$\overline{L}\overline{u} = e^{-2a}Lu.$$

One can conclude that the transformations

$$\bar{t} = t, \ \overline{x} = xe^{a}, \ \overline{u} = u, \ \overline{g} = ge^{-2a}$$

compose an equivalence Lie group of equation (4.6).

The transformations corresponding to the generator $X_1 = x(x\partial_x - u\partial_u)$ map a function $u(t,x)$ into the function

$$\overline{u}(\bar{t},\overline{x}) = \frac{1}{1+a\overline{x}}u(\bar{t}, \frac{\overline{x}}{1+a\overline{x}}).$$

Hence,

$$\overline{L}\overline{u} = (1-ax)^2 Lu$$

and the transformations

$$\bar{t} = t, \ \overline{x} = \frac{x}{1-ax}, \ \overline{u} = (1-ax)u, \ \overline{g} = (1-ax)^2 g$$

compose an equivalence Lie group of transformations.

Thus, it has been shown that the Lie group corresponding to the generators

$$X_0^e = x\partial_x, \ X_1^e = x(x\partial_x - u\partial_u - 2g\partial_g), \ X_2^e = t\partial_t - u\partial_u - 2g\partial_g, \ X_3^e = \partial_t$$

is an equivalence Lie group of equation (4.6).

4.5 Group classification

Group classification of equation (4.6) is carried out up to the equivalence transformations considered above.

Equation (4.9) can be rewritten in the form

$$c_0h_0 + c_1h_1 + c_2h_2 + c_3h_3 = 0, \tag{4.15}$$

where

$$h_0 = xg_x, \ h_1 = x(xg_x + 2g), \ h_2 = tg_t + 2g, \ h_3 = g_t. \tag{4.16}$$

One of the methods for analyzing relations between the constants c_0, c_1, c_2 and c_3 consists of employing the algorithm developed for the gas dynamics equations [8]: one analyzes the vector space $Span(V)$, where the set V consists of the vectors

$$v = (h_0, h_1, h_2, h_3)$$

with t and x are changed. This algorithm allows one to study all possible admitted Lie algebras of equation (4.6) without omission. Unfortunately, it is difficult to implement.

In [90] an algebraic algorithm for group classification was applied, which essentially reduces this study to a simpler problem. Here we follow this algorithm[1]. Observe here that because of the nonlinearity of the equivalence transformations corresponding to the generator X_1, it is difficult to select out equivalent cases with respect to these transformations, whereas the algebraic algorithm free of this complication.

[1]The authors thank the anonymous referee for pointing to the possibility of applying to the analysis of equation (4.15) the algorithm considered in [90]

First, let us study the Lie algebra L_4 composed by the generators $\{X_0, X_1, X_2, X_3\}$. The commutator table is

	X_0	X_1	X_2	X_3
X_0	0	X_1	0	0
X_1	$-X_1$	0	0	0
X_2	0	0	0	$-X_3$
X_3	0	0	X_3	0

The inner automorphisms are

$$\begin{array}{lcl} A_0 & : & \hat{x}_1 = x_1 e^a, \\ A_1 & : & \hat{x}_1 = x_1 + a x_0, \\ A_2 & : & \hat{x}_3 = x_3 e^a, \\ A_3 & : & \hat{x}_3 = x_3 + a x_2, \end{array}$$

where only the changed coordinates are presented.

Second, one can notice that the results of using the equivalence transformations corresponding to the generators $X_0^e, X_1^e, X_2^e, X_3^e$ are similar to changing coordinates of a generator X with regarding to the basis change. These changes are similar to the inner automorphisms. Indeed, the coefficients of the generator X are changed according to the relation [8]:

$$X = (X\overline{t})\partial_{\overline{t}} + (X\overline{x})\partial_{\overline{x}} + (X\overline{u})\partial_{\overline{u}}.$$

Any generator X can be expressed as a linear combination of the basis generators:

$$\hat{x}_0\hat{X}_0 + \hat{x}_1\hat{X}_1 + \hat{x}_2\hat{X}_2 + \hat{x}_3\hat{X}_3 = x_0 X_0 + x_1 X_1 + x_2 X_2 + x_3 X_3, \tag{4.17}$$

where

$$\hat{X}_0 = \overline{x}\partial_{\overline{x}},\ \hat{X}_1 = \overline{x}(\overline{x}\partial_{\overline{x}} - \overline{u}\partial_{\overline{u}}),\ \hat{X}_2 = \overline{t}\partial_{\overline{t}} - \overline{u}\partial_{\overline{u}},\ \hat{X}_3 = \partial_{\overline{t}}.$$

Using the invariance of a generator with respect to a change of the variables, the basis generators X_i, $(i = 0, 1, 2, 3)$ and $\hat{X}_i$, $(i = 0, 1, 2, 3)$ in corresponding equivalence transformations are related as follows

$$\begin{array}{lcl} X_0^e & : & X_0 = \hat{X}_0, X_1 = e^{-a}\hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3; \\ X_1^e & : & X_0 = \hat{X}_0 + a\hat{X}_1, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = \hat{X}_3; \\ X_2^e & : & X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2, X_3 = e^a\hat{X}_3; \\ X_3^e & : & X_0 = \hat{X}_0, X_1 = \hat{X}_1, X_2 = \hat{X}_2 - a\hat{X}_3, X_3 = \hat{X}_3. \end{array}$$

Substituting these relations into the identity (4.17), one obtains that the coordinates of the generator X in the basis $B = \{X_0, X_1, X_2, X_3\}$ and in the basis $\hat{B} = \{\hat{X}_0, \hat{X}_1, \hat{X}_2, \hat{X}_3\}$ are related similar to the changes defined by the inner automorphisms.

This observation allows us to use an optimal system of subalgebras of the Lie algebra L_4 for studying equation (4.15). The construction of such an optimal system is not difficult. Moreover, it is simplified if one notices that $L_4 = F_1 \bigoplus F_2$, where $F_1 = \{X_0, X_1\}$ and $F_2 = \{X_2, X_3\}$ are ideals of the Lie algebra L_4. This decomposition gives possibility to apply a two-step algorithm [18, 91]. The result of construction of an optimal system of subalgebras is presented in Table 1.

For obtaining functions $g(t, x)$ using the optimal system of subalgebras one needs to substitute the constants c_i corresponding to the basis generators of a subalgebra into equation (4.15), and solve the system of equations thus obtained. The result of group classification is presented in Table 2, where α and k are constant and the function Φ is an arbitrary function of its argument.

Table 4.1: Optimal system of subalgebras

	Algebra		Algebra
1.	$\{X_2 + \alpha X_0,\ X_3 + \beta X_0,\ X_1\}$	11.	$\{\ X_2,\ X_3\}$
2.	$\{\ X_0,\ X_1,\ X_3\}$	12.	$\{\ X_3 + \alpha X_0,\ X_1\}$
3.	$\{\ X_0,\ X_1,\ X_2\}$	13.	$\{\ X_0,\ X_1\}$
4.	$\{\ X_0,\ X_2,\ X_3\}$	14.	$\{\ X_2 + \alpha X_0\}$
5.	$\{\ X_2,\ X_3\}$	15.	$\{\ X_2 + X_1\}$
6.	$\{\ X_2 - X_0,\ X_1 +\ X_3\}$	16.	$\{\ X_2\}$
7.	$\{\ \alpha X_2 - 2X_0,\ X_3\}$	17.	$\{\ X_3 + \alpha X_0\}$
8.	$\{\ X_2 + X_1,\ X_3\}$	18.	$\{\ X_3 + X_1\}$
9.	$\{\ X_0,\ X_2\}$	19.	$\{\ X_3\}$
10.	$\{\ X_2 + \alpha X_0,\ X_1\}$	20.	$\{\ X_0\}$
		21.	$\{\ X_1\}$

Table 4.2: Group classification

No.	Function	Admitted generators	Condition
1.	$g = kx^{-2}$	$\{X_2 + X_0,\ X_3,\ X_1\}$	
2.	$g = kx^2(xt+1)^{-4}$	$\{\ X_2 - X_0,\ X_1 +\ X_3\}$	
3.	$g = kx^{\alpha}$	$\{\ \alpha X_2 - 2X_0,\ X_3\}$	$\alpha \neq -2$
4.	$g = kx^{-2}e^{2x^{-1}}$	$\{\ X_2 + X_1,\ X_3\}$	
5.	$g = kt^{-2}$	$\{\ X_0,\ X_2\}$	
6.	$g = kx^{-2}t^{2(\alpha-1)}$	$\{\ X_2 + \alpha X_0,\ X_1\}$	$\alpha \neq 1$
7.	$g = kx^{-2}e^{2\alpha t}$	$\{\ X_3 + \alpha X_0,\ X_1\}$	$\alpha \neq 0$
8.	$g = t^{-2}\Phi(xt^{-\alpha})$	$\{\ X_2 + \alpha X_0\}$	
9.	$g = x^{-2}e^{2x^{-1}}\Phi(te^{x^{-1}})$	$\{\ X_2 + X_1\}$	
10.	$g = \Phi(xe^{-\alpha t})$	$\{\ X_3 + \alpha X_0\}$	
11.	$g = x^{-2}\Phi(t + x^{-1})$	$\{\ X_3 + X_1\}$	
12.	$g = \Phi(x)$	$\{\ X_3\}$	
13.	$g = \Phi(t)$	$\{\ X_0\}$	
14.	$g = x^{-2}\Phi(t)$	$\{\ X_1\}$	

Chapter 5

On first integrals of second-order ordinary differential equations

Abstract

Here we discuss first integrals of a particular representation associated with second-order ordinary differential equations. The linearization problem is a particular case of the equivalence problem together with a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation. The relationship between the integral form, the associated equations, equivalence transformations and some examples are considered as part of the discussion illustrating some important aspects and properties.

5.1 Invariants of a class of second-order equations

We recall some known properties of a second-order equation,

$$\ddot{x} + a_3(t,x)\dot{x}^3 + 3a_2(t,x)\dot{x}^2 + 3a_1(t,x)\dot{x} + a_0(t,x) = 0. \tag{5.1}$$

This form of equation is conserved with respect to any change of the independent and dependent variables

$$\tau = \varphi(t,x), \qquad u = \psi(t,x). \tag{5.2}$$

In fact, derivatives are changed by the formulae

$$\begin{aligned} u' = \;& g(t,x,\dot{x}) = \frac{D_t\psi}{D_t\varphi} = \frac{\psi_t + \dot{x}\psi_x}{\varphi_t + \dot{x}\varphi_x}, \\ u'' = \;& P(t,x,\dot{x},\ddot{x}) = \frac{D_t g}{D_t\varphi} = \frac{g_t + \dot{x}g_x + \ddot{x}g_{\dot{x}}}{\varphi_t + \dot{x}\varphi_x} \\ = \;& (\varphi_t + \dot{x}\varphi_x)^{-3}(\ddot{x}(\varphi_t\psi_x - \varphi_x\psi_t) + \dot{x}^3(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x) \\ & +\dot{x}^2(\varphi_t\psi_{xx} - \varphi_{xx}\psi_t + 2(\varphi_x\psi_{tx} - \varphi_{tx}\psi_x)) \\ & +\dot{x}(\varphi_x\psi_{tt} - \varphi_{tt}\psi_x + 2(\varphi_t\psi_{tx} - \varphi_{tx}\psi_t)) + \varphi_t\psi_{tt} - \varphi_{tt}\psi_t). \end{aligned} \tag{5.3}$$

Here D_t is the operator of the total derivative with respect to t, and

$$\Delta = \varphi_t\psi_x - \varphi_x\psi_t \neq 0.$$

Since the Jacobian of the change of variables $\Delta \neq 0$, the equation

$$u'' + b_3(\tau,u)u'^3 + 3b_2(\tau,u)u'^2 + 3b_1(\tau,u)u' + b_0(\tau,u) = 0 \tag{5.4}$$

becomes (5.1), where

$$
\begin{aligned}
a_1 &= \Delta^{-1}\left(\varphi_x\psi_{xx} - \varphi_{xx}\psi_x + \varphi_x^3 b_0 + 3\varphi_x^2\psi_x b_1 + 3\varphi_x\psi_x^2 b_2 + \psi_x^3 b_3\right), \\
a_2 &= \Delta^{-1}\left(3^{-1}(\varphi_t\psi_{xx} - \varphi_{xx}\psi_t + 2(\varphi_x\psi_{tx} - \varphi_{tx}\psi_x)) + \varphi_t\varphi_x^2 b_0\right. \\
&\quad \left.+\varphi_x(2\varphi_t\psi_x + \varphi_x\psi_t)b_1 + (\varphi_t\psi_x^2 + 2\varphi_x\psi_t\psi_x)b_2 + \psi_t\psi_x^2 b_3\right), \\
a_3 &= \Delta^{-1}\left(3^{-1}(\varphi_x\psi_{tt} - \varphi_{tt}\psi_x + 2(\varphi_t\psi_{tx} - \varphi_{tx}\psi_t)) + \varphi_t^2\varphi_x b_0\right. \\
&\quad \left.+(\varphi_t^2\psi_x + 2\varphi_t\varphi_x\psi_t)b_1 + (2\varphi_t\psi_t\psi_x + \varphi_x\psi_t^2)b_2 + \psi_t^2\psi_x b_3\right), \\
a_0 &= \Delta^{-1}\left(\varphi_t\psi_{tt} - \varphi_{tt}\psi_t + \varphi_t^3 b_0 + 3\varphi_t^2\psi_t b_1 + 3\varphi_t\psi_t^2 b_2 + \psi_t^3 b_3\right).
\end{aligned}
\tag{5.5}
$$

Two quantities play a major role in the study of equations (5.4):

$$
\begin{aligned}
L_1 &= -\frac{\partial \Pi_{11}}{\partial u} + \frac{\partial \Pi_{12}}{\partial \tau} - b_0\Pi_{22} - b_2\Pi_{11} + 2b_1\Pi_{12}, \\
L_2 &= -\frac{\partial \Pi_{12}}{\partial u} + \frac{\partial \Pi_{22}}{\partial \tau} - b_3\Pi_{11} - b_1\Pi_{22} + 2b_2\Pi_{12},
\end{aligned}
$$

where

$$
\Pi_{11} = 2(b_1^2 - b_2 b_0) + b_{1\tau} - b_{0u}, \qquad \Pi_{22} = 2(b_2^2 - 3b_1 b_3) + b_{3\tau} - b_{2u},
$$
$$
\Pi_{12} = b_2 b_1 - b_3 b_0 + b_{2\tau} - b_{1u}.
$$

Under point transformation (5.2) these components are transformed as follows [92]:

$$
\widetilde{L}_1 = \Delta(L_1\varphi_t + L_2\psi_t), \quad \widetilde{L}_2 = \Delta(L_1\varphi_x + L_2\psi_x). \tag{5.6}
$$

Here tilde means that a value corresponds to system (5.1): the coefficients b_i are exchanged with a_i, the variables τ and u are exchanged with t and x, respectively.

S.Lie [49] showed that any equation with $L_1 = 0$ and $L_2 = 0$ is equivalent to the equation $u'' = 0$. R.Liouville [92] also found other relative invariants, for example,

$$
\begin{aligned}
v_5 = \;& L_2(L_1 L_{2\tau} - L_2 L_{1\tau}) + L_1(L_2 L_{1u} - L_1 L_{2u}) - \\
& b_3 L_1^3 + 3b_2 L_1^2 L_2 - 3b_1 L_1 L_2^2 + b_0 L_2^3,
\end{aligned}
$$

and

$$
w_1 = L_1^{-4}\left(-L_1^3(\Pi_{12}L_1 - \Pi_{11}L_2) + R_1(L_1^2)_\tau - L_1^2 R_{1\tau} + L_1 R_1(b_1 L_1 - b_0 L_2)\right),
$$

where

$$
R_1 = L_1 L_{2\tau} - L_2 L_{1\tau} + b_2 L_1^2 - 2b_1 L_1 L_2 + b_0 L_2^2.
$$

Notice that for the Painlevé equations $L_1 \neq 0$ and $L_2 = 0$, $v_5 = 0$ and $w_1 = 0$.

Remark 5.1.1. Without loss of generality one can assume that $L_1 \neq 0$ and $L_2 = 0$, otherwise a change of the dependent and independent variables such that the functions $\varphi(t,x)$ and $\psi(t,x)$ satisfy the equation

$$
\varphi_y L_1 + \psi_y L_2 = 0
$$

leads to this case. For the sake of simplicity we study equations with $L_1 \neq 0$ and $L_2 = 0$.

5.2 General difficulties of the equivalence problem

Despite the fact that the criteria for linearizability can be simply checked, there are certain difficulties for finding the linearizing transformation. Let us consider a second-order ordinary differential equation

$$y'' + b(x,y)y'^2 + c(x,y)y' + d(x,y) = 0, \tag{5.7}$$

where the coefficients satisfy the conditions

$$c_y = 2b_x, \quad d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{5.8}$$

The transformation

$$t = \varphi(x), \qquad u = \psi(x,y) \tag{5.9}$$

mapping equation (5.7) into the equation $u'' = 0$ is found from the compatible conditions

$$\psi_{yy} = \psi_y b, \ 2\psi_{xy} = \varphi_x^{-1}\psi_y\varphi_{xx} + c\psi_y, \qquad \psi_{xx} = \varphi_x^{-1}\psi_x\varphi_{xx} + \psi_y d, \tag{5.10}$$

$$\frac{2\varphi'\varphi''' - 3\varphi''^{\,2}}{\varphi'^{\,2}} = H, \tag{5.11}$$

where $H = 4(d_y + bd) - (2c_x + c^2)$. Notice that by virtue of the second equation of (5.8) the function $H = H(x)$. To solve the system (5.10), (5.11), one has to firstly solve equation (5.11). The change $\varphi' = g^{-2}$ reduces equation (5.11) into the equation

$$g'' + \frac{1}{4}Hg = 0. \tag{5.12}$$

It is well-known that the Riccati substitution

$$g' = gv$$

reduces equation (5.12) into the Riccati equation

$$v' + v^2 + \frac{1}{4}H = 0.$$

Thus, in order to solve equation (5.11) one has to be able to solve the Riccati equation, which is not solvable in the general case.

The example presented above shows that the solution of the linearization problem is only theoretical: in many applications it becomes impossible to find the linearizing transformation. A similar problem is also encountered in finding the intermediate integral.

5.3 Existence of the First Integral

The existence of the first integral of the form:

$$I = A(t,x) + \frac{1}{B(t,x)\dot{x} + Q(t,x)}, \quad (B \neq 0), \tag{5.13}$$

of a second-order equation requires that the necessary form of the equation is (5.1), where the coefficients are related by the equations

$$\begin{gathered} a_0 = (Q_t - A_t Q^2)/B, \qquad a_1 = (B_t + Q_x - 2A_t BQ - A_x Q^2)/(3B), \\ a_2 = (B_x - A_t B^2 - 2A_x BQ)/(3B), \qquad a_3 = -A_x B. \end{gathered} \tag{5.14}$$

The sufficient conditions for existence of an intermediate integral of the form (5.13) are obtained if one considers (5.14) as equations for the functions $A(t,x)$, $B(t,x)$ and $Q(t,x)$ with given coefficients $a_i(t,x)$, $(i=0,1,2,3)$.

Equations (5.14) give

$$A_t = B^{-1}G, \qquad A_x = -B^{-1}a_3, \qquad Q_t = a_0 B + B^{-1}GQ^2,$$
$$B_t = -Q_x + 2GQ + 3a_1 B - a_3 B^{-1}Q^2, \tag{5.15}$$

where

$$G = B^{-1}(B_x - 3a_2 B + 2a_3 Q).$$

The function $G(t,x)$ is introduced for simplicity of calculations.

The equations $(A_x)_t = (A_t)_x$ and $(B_x)_t = (B_t)_x$ give

$$G_x = -B^{-1}Q_x a_3 - a_{3t} + 3a_1a_3 + 3a_2G - B^{-2}a_3^2Q^2 + G^2,$$

$$\begin{aligned} Q_{xx} \;=\; & B^{-2}(Q_xB(3a_2B - 4a_3Q + 3BG) - G_tB^3 + B^3(3a_{1x} - 3a_{2t} + 2a_0a_3) \\ & +(6a_2a_3 - a_{3x})BQ^2 - 4a_3^2Q^3 + 4a_3BGQ^2). \end{aligned} \tag{5.16}$$

The equation $(Q_{xx})_t = (Q_t)_{xx}$ becomes

$$\begin{aligned} G_{tt} \;=\; & B^{-4}(4G_tQ_xB^3 - 3G_tB^4a_1 + 4G_tB^2a_3Q^2 - 2Q_x^2B^2G - 2Ga_3^2Q^4 \\ & +3Q_xB^3(2a_{2t} - a_{1x} - a_0a_3) - 4Q_xBGa_3Q^2 + B^4G^2a_0 + B^4G(a_{0x} + 3a_0a_2) \\ & +B^4(a_{0t}a_3 + a_{1tx} + 3a_{1x}a_1 - 2a_{2tt} - 6a_{2t}a_1 + a_{3t}a_0 + 3a_0a_1a_3 - \lambda_1) \\ & -3B^2a_3Q^2(a_{1x} - 2a_{2t} + a_0a_3)). \end{aligned} \tag{5.17}$$

The equation $(G_{tt})_x - (G_x)_{tt} = 0$ leads to $S = 0$, where

$$\begin{aligned} S \;\equiv\; & 12G_tQ_xB^3G - 6G_t^2B^4 - 6Q_x^2B^2G^2 + 12G_tB^2Ga_3Q^2 - 12Q_xBG^2a_3Q^2 \\ & +12(a_{1x} - 2a_{2t} + a_0a_3)B^2(G_tB^2 - Q_xBG - Ga_3Q^2) - 6G^2a_3^2Q^4 + 3B^4G\lambda_1 \\ & -B^4(\lambda_{1x} - 3a_2\lambda_1 + 6(a_{1x} - 2a_{2t} + a_0a_3)^2). \end{aligned}$$

Furthermore the equations

$$S_x - 6(G + a_2)S = 0, \qquad B^2S_t - 6(Q_xB - B^2a_1 + a_3Q^2)S = 0$$

are

$$Ba_3Q_x = 3B^2G\mu_1 - 5B^2G^2 + B^2(3a_1a_3 - \mu_2) - a_3^2Q^2, \tag{5.18}$$

$$\begin{aligned} G_t \;=\; & (15B^2\lambda_1)^{-1}(6Q_xB\lambda_1(5G - \mu_1) - 3B^2G(\lambda_{1t} + 6a_1\lambda_1) + 6a_3\lambda_1Q^2(5G - \mu_1) \\ & +B^2(\lambda_1\mu_{1t} + 12a_{1x}\lambda_1 - 24a_{2t}\lambda_1 + \lambda_{1t}\mu_1 + 12a_0a_3\lambda_1 + 6a_1\lambda_1\mu_1)), \end{aligned} \tag{5.19}$$

where all coefficients μ_i, $(i = 1, 2, ..., 7)$ are presented in the Appendix.

For further analysis one needs to consider two cases: a) $a_3 \neq 0$ and b)[1] $a_3 = 0$. It is also worth noting that because of the relative invariant v_5 the property for a_3 not to be equal to zero is an invariant property of the point transformations conserving $L_2 = 0$.

[1]This case has been studied in the literature.

5.3.1 Case $a_3 \neq 0$

Let $a_3 \neq 0$, then equation (5.19) gives

$$Q_x = (Ba_3)^{-1}(3B^2G\mu_1 - 5B^2G^2 + B^2(3a_1a_3 - \mu_2) - a_3^2Q^2).$$

Thus, all first-order derivatives of the unknown functions $A(t,x)$, $B(t,x)$, $Q(t,x)$ and $G(t,x)$ are found:

$$\begin{gathered} A_t = B^{-1}G, \qquad A_x = -B^{-1}a_3, \;\; Q_t = Ba_0 + B^{-1}GQ^2, \\ Q_x = (-5B^2G^2 + 3B^2G\mu_1 + B^2(3a_1a_3 - \mu_2) - a_3^2Q^2)/(Ba_3), \\ B_t = (5BG^2 - 3BG\mu_1 + B\mu_2 + 2Ga_3Q)/a_3, \qquad B_x = BG + 3Ba_2 - 2a_3Q, \\ G_t = (-10G^3 + 8G^2\mu_1 - G\mu_3 + \lambda_1\mu_4)/a_3, \qquad G_x = 6G^2 + 3G(a_2 - \mu_1) - a_{3t} + \mu_2. \end{gathered} \tag{5.20}$$

The overdetermined system (5.20) is compatible if the conditions

$$\begin{aligned} (A_t)_x - (A_x)_t = 0, \qquad (B_t)_x - (B_x)_t = 0, \\ (Q_x)_t - (Q_t)_x = 0, \qquad (G_t)_x - (G_x)_t = 0 \end{aligned} \tag{5.21}$$

are satisfied. Notice also that by virtue of (5.20), equations (5.15) are satisfied. Hence, it is not necessary to substitute the first-order derivatives into the intermediate equations (5.16) and (5.17).

The conditions (5.21) are reduced to the equations

$$H \equiv 12G^3a_3 - G^2\mu_5 - G\mu_6 - \mu_7 = 0, \tag{5.22}$$

$$75G^4 - 80\mu_1G^3 + 5q_2G^2 - q_1G - q_0 = 0, \tag{5.23}$$

where coefficients q_i, $(i = 0, 1, 2)$ are presented in the Appendix. Let us also add to this set of equations the following equations:

$$H_x = 0, \qquad H_t = 0. \tag{5.24}$$

The equation (5.22) is a polynomial equation of third degree with respect to G. Excluding from equations (5.23) and (5.24) the value

$$G^3 = (G^2\mu_5 + G\mu_6 + \mu_7)/(12a_3),$$

equations (5.23) and (5.24) become

$$5\alpha_1G^2 + \beta_1G + \gamma_1 = 0, \qquad \alpha_2G^2 + \beta_2G + \gamma_2 = 0, \qquad 25\alpha_3G^2 + \beta_3G + 25\gamma_3 = 0, \tag{5.25}$$

where all coefficients α_i, β_i and γ_i, $(i = 1, 2, 3)$ are presented in Appendix.

In solving equation (5.22) with respect to G, one has to also satisfy the conditions $G_t = (G)_t$ and $G_x = (G)_x$. Satisfying these conditions is equivalent to satisfying equations (5.24). Thus, further study is just an algebraic study of equations (5.22) and (5.25). This study depends on the coefficients α_i, β_i, $(i = 1, 2, 3)$.

For example, assume that $\alpha_1 \neq 0$. From the first equation of (5.25) one finds G^2. Substituting G^2 into (5.22) and the remaining equations of (5.25), one obtains linear equations with respect to G. One needs to study resolving these linear equations with respect G. This depends on the coefficients of these equations. Notice that one does not need to differentiate equations any more.

5.4 Case $G = 0$

Let us consider the case $G = 0$ without restrictions for[2] λ_2. Then

$$\begin{array}{cc} A_t = 0, \quad A_x = -a_3/B, & Q_t = a_0 B, \\ B_t = (-Q_x B + 3a_1 B^2 - a_3 Q^2)/B, & B_x = 3a_2 B - 2a_3 Q. \end{array} \tag{5.26}$$

The equations $(A_x)_t - (A_t)_x = 0$ and $(B_x)_t - (B_t)_x = 0$ give

$$Q_x a_3 B = -a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2, \tag{5.27}$$

$$\begin{array}{c} Q_{xx} B^2 + Q_x B(2a_3 Q - 3a_2 B) + B^3(3a_{2t} - 3a_{1x} - 2a_0 a_3) \\ +BQ^2(a_{3x} - 6a_2 a_3) + 2B^2 Q(3a_1 a_3 - a_{3t}) + 2a_3^2 Q^3 = 0. \end{array} \tag{5.28}$$

5.4.1 Case $a_3 \neq 0$

If $a_3 \neq 0$, then equation (5.27) defines

$$Q_x = (-a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2)/(a_3 B). \tag{5.29}$$

This reduces equation (5.28) and the equation $(Q_x)_t - (Q_t)_x = 0$:

$$\begin{array}{c} (a_{3tx} - 3a_{2t} a_3 + 2a_0 a_3^2) a_3 - a_{3t} a_{3x} = 0, \\ a_{3tt} - 3a_{3t} a_1 + (a_{0x} - 3a_{1t} + 3a_0 a_2) a_3 = 0. \end{array} \tag{5.30}$$

Thus, if equation (5.1) satisfies the conditions (5.30), then the overdetermined system of equations consisting of equations (5.26) and (5.29) is involutive.

For example, for $a_3 = 1$ the conditions (5.30) can be solved

$$a_0 = 3a_{2t}/2, \qquad a_1 = a_{2x}/2 + 3a_2^2/4 + \varphi,$$

where $\varphi(x)$ is an arbitrary function. This means that all equations of the form

$$\ddot{x} + \dot{x}^3 + 3a_2 \dot{x}^2 + (a_{2x}/2 + 3a_2^2/4 + \varphi)\dot{x} + 3a_{2t}/2 = 0 \tag{5.31}$$

with arbitrary functions $a_2(t, x)$ and $\varphi(x)$ have the intermediate integral

$$I = A + \frac{1}{B(\dot{x} + \frac{3}{2} a_2) + H},$$

where the functions $A(x)$, $B(x)$ and $H(x)$ are solutions of the equations

$$A' = -1/B, \quad B' = -H, \quad H' = 3B\varphi - H^2/B.$$

Notice that for $a_{2t} = 0$ equation (5.30) can be reduced to a first-order ordinary differential equation by the standard change $\dot{x} = y(x)$ whereas for $a_{2t} \neq 0$ this technique is not applicable.

[2]There is no restrictions for λ_2

5.4.2 Case $a_3 = 0$

In this case equation (5.27) is satisfied and equation (5.28) becomes

$$Q_{xx} = 3Q_x a_2 - 3B(a_{2t} - a_{1x}). \tag{5.32}$$

The equation $(Q_{xx})_t - (Q_t)_{xx} = 0$ gives

$$3Q_x\eta = B(\eta_t + 3a_1\eta - \lambda_1), \tag{5.33}$$

where $\eta = a_{1x} - 2a_{2t}$. Hence, for $\eta = 0$ one has that[3] $\lambda_1 = 0$, and there are no other additional equations for the functions $A(t,x)$, $B(t,x)$ and $Q(t,x)$. This means that the system of equations consisting of equations (5.26) and (5.32) is involutive. If $\eta \neq 0$, then one can find Q_x. The equations $(Q_x)_t - (Q_t)_x = 0$ and $(Q_{xx})_x - (Q_x)_x = 0$ give the conditions

$$\begin{aligned} 3\eta\eta_{tt} &= 4\eta_t^2 - 3\eta_t\eta a_1 + 15\eta_t\lambda_1 + 9\eta^2(a_{0x} - a_{1t} + 3a_0a_2 - 2a_1^2) \\ &\quad -9\eta(\lambda_{1t} + a_1\lambda_1) + 9\lambda_1^2, \\ \eta\eta_{tx} &= \eta_t\eta_x + 3\eta_x\lambda_1 - 2\eta^3 + 3\eta^2 a_{2t} - 3\eta\lambda_{1x}. \end{aligned} \tag{5.34}$$

Thus, if equation (5.1) satisfies the conditions (5.34), then the overdetermined system of equations consisting of equations (5.26) and (5.33) is involutive.

5.5 Examples

In this section we consider examples of first integrals of the form

$$I = \frac{A(t,x)\dot{x} + B(x,t)}{\dot{x} + Q(x,t)}. \tag{5.35}$$

Example 5.5.1. The equation associated with the first integral I in (5.35) is given by

$$\ddot{x} + \frac{A_x}{\Delta}\dot{x}^3 + \frac{1}{\Delta}(A_t + B_x + A_xQ - AQ_x)\dot{x}^2 + \frac{1}{\Delta}(B_t + A_tQ - AQ_t + B_xQ - BQ_x)\dot{x} + \frac{1}{\Delta}(QB_t - BQ_t) = 0, \tag{5.36}$$

where $\Delta = AQ - B$.

Proof. Re-arranging (5.35), we obtain

$$I(\dot{x} + Q(x,t)) = A(t,x)\dot{x} + B(x,t)$$

which immediately yields

$$I(\ddot{x} + Q_x\dot{x} + Q_t) = A\ddot{x} + A_x\dot{x}^2 + A_t\dot{x} + B_x\dot{x} + B_t$$

$$(A\dot{x} + B)(\ddot{x} + Q_x\dot{x} + Q_t) = (\dot{x} + Q(x,t))(A\ddot{x} + A_x\dot{x}^2 + A_t\dot{x} + B_x\dot{x} + B_t)$$

□

This equation is closely related to the (unparametrised) geodesic equations of some connection Γ on $U \in \mathbf{R}^2$

$$\ddot{x}^c + \Gamma^c_{ab}\dot{x}^a\dot{x}^b = v\dot{x}^c,$$

[3]Notice that for not linearizable equation (5.1) without loss of generality one can assume that $\lambda_1 \neq 0$.

for $x^a(t) = (x(t), y(t))$. Eliminating the parameter t yields the second-order ODE for y as a function of x

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = a(x,y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + b(x,y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + c(x,y)\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) + d(x,y) = 0, \tag{5.37}$$

where

$$a(x,y) = -\Gamma^2_{11}, \qquad b(x,y) = \Gamma^1_{11} - 2\Gamma^2_{12}, \qquad c(x,y) = 2\Gamma^1_{12} - \Gamma^2_{22}, \qquad d(x,y) = \Gamma^1_{22}.$$

In other words, any second order ODEs with cubic nonlinearity in the first derivatives of the form (5.36) gives rise to some projective structures.

Example 5.5.2. A quasimonomial q over $\mathbb{K}$ is defined as

$$q = \mathbf{x}^c = \prod_{i=1}^{n} x_i^{c_i} \qquad c_i \in \mathbb{K}.$$

A quasimonomial function is a finite sum of quasimonomials $f : \mathbb{C} \to \Sigma$, where $\Sigma = \mathbb{C} \cup \{\infty\}$, defined as

$$\mathbf{x} \longrightarrow \sum a_i \prod_{j=1}^{n} x_j^{c_{ij}}.$$

We assume $A = x^\alpha t^\beta$, $B = x^\gamma t^\delta$ and $Q = 1$ in (5.36) to obtain the second-order equation

$$\ddot{x} + \frac{\alpha}{x(1-x^{\gamma-\alpha})}\dot{x}^3 + \frac{\alpha + \beta\frac{x}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})}\dot{x}^2 + \frac{\beta\frac{x}{t} + \beta\frac{x^{\gamma-\alpha+1}}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})}\dot{x} + \frac{\beta\frac{x^{\gamma-\alpha+1}}{t}}{x(1-x^{\gamma-\alpha})} = 0, \tag{5.38}$$

which admits the first integral

$$I = \frac{x^\alpha t^\beta(\dot{x} + x^{\gamma-\alpha})}{\dot{x}+1}.$$

Claim 5.5.1. Setting $\alpha = -1, \ \beta = 1 = \delta, \gamma = 0$, we obtain the first integral of

$$\ddot{x} + \frac{1}{x(x-1)}\dot{x}^3 + \left(\frac{1}{x(x-1)} + \frac{1}{t(1-x)}\right)\dot{x}^2 + \frac{1}{t}\left(\frac{1+x}{1-x}\right)\dot{x} + \frac{x}{t(1-x)} = 0 \tag{5.39}$$

as

$$I = \left(\frac{t}{x}\right) \cdot \frac{\dot{x}+x}{\dot{x}+1}.$$

Example 5.5.3. The first integral of another second-order equation

$$\ddot{x} + \frac{t}{x^2(x-1)}\dot{x}^3 + \left(\frac{2}{1-x} - \frac{(1+x)t}{(1-x)x^2} + \frac{1}{x}\right)\dot{x}^2 + \left(\frac{1}{1-x} + \frac{1-t}{(1-x)x}\right)\dot{x} + \frac{1}{1-x} = 0 \tag{5.40}$$

is

$$I = e^{t/x}\frac{\dot{x}+x}{\dot{x}+1}.$$

Example 5.5.4. Let us set $A = Q^{-1} = e^{\alpha(x)t}$ and $B = b$ (constant) in (5.35). Then we obtain the equation

$$(1-b)\ddot{x} + e^{\alpha(x)t}(\alpha'(x)t\dot{x} + \alpha(x))\dot{x}^2 + 2(\alpha'(x)t\dot{x} + \alpha(x))\dot{x} + e^{-\alpha(x)t}(\alpha'(x)t\dot{x} + b\alpha(x)) = 0, \tag{5.41}$$

corresponding first integral is

$$I = \frac{e^{\alpha(x)t}\dot{x} + b}{\dot{x} + e^{-\alpha(x)t}}.$$

5.5.1 Time-independent case

Consider $A_t = B_t = Q_t = 0$. Thus the equation (5.35) becomes

$$\ddot{x} + \frac{A_x}{\Delta}\dot{x}^3 + \frac{1}{\Delta}(B_x + A_x Q - AQ_x)\dot{x}^2 + \frac{1}{\Delta}(A_t Q - AQ_t + B_x Q - BQ_x)\dot{x} = 0, \qquad (5.42)$$

which can be expressed as

$$\dot{x} = y, \qquad \dot{y} = -\frac{1}{\Delta}\big(A_x y^3 + (B_x + A_x Q - AQ_x)y^2 + (B_x Q - BQ_x)y\big). \qquad (5.43)$$

This yields the flow equation

$$\frac{dy}{dx} = -\frac{1}{\Delta}\big(A_x y^2 + (B_x + A_x Q - AQ_x)y + (B_x Q - BQ_x).\big)$$

Assume $A = 1$, thus $\Delta = Q - B$. The flow becomes

$$\frac{dy}{dx} - \frac{\Delta'(x)}{\Delta(x)}y = \frac{B^2}{\Delta}\frac{d}{dx}(\frac{Q}{B}).$$

This immediately yields

$$y = -\frac{1}{Q/B - 1} + C_1.$$

Thus we obtain

$$t = \int \frac{dx}{C_1 Q - (C_1 + 1)B(x)}.$$

5.5.2 Reduction

Let $A_x = 0$ and set

$$\frac{1}{\Delta}(A_t + B_x - AQ_x) = b(x,t) = \frac{1}{2}\phi_x \qquad \frac{1}{\Delta}(B_t + A_t Q - AQ_t + B_x Q - BQ_x) = c(x,t) = \phi_t. \qquad (5.44)$$

A large number of second-order ODEs in the Painlevé-Gambier classification system belong to the following class of equations, namely

$$\ddot{x} + \frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,x) = 0.$$

This equation yields the Lagragian description via Jacobi's last multiplier. Writing this equation in the form

$$\ddot{x} = \mathcal{F}(t, x, \dot{x}) = -[\frac{1}{2}\phi_x \dot{x}^2 + \phi_t \dot{x} + B(t,x)],$$

the Jacobi last multiplier M is given by the solution of

$$\frac{d}{dt}\ln M = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} = \exp \phi(t,x).$$

We then obtain the Lagrangian as

$$L(t,x,\dot{x}) = e^{\phi(t,x)}\frac{\dot{x}^2}{2} + f_1(t,x)\dot{x} + f_2(t,x).$$

Conditions for Lagrangians

Let us express ϕ in terms A, B, Q and find the conditions for Lagrangian. Defining

$$\phi_x = \frac{2}{\Delta}(A_t + B_x - AQ_x),$$

$$\phi_t = \frac{1}{\Delta}(B_t + A_tQ - AQ_t + B_xQ - BQ_x),$$

immediately yields

$$\phi_{xt} = \frac{2}{\Delta}(A''(t) + B_{xt} - A'(t)Q_x - AQ_{xt}) - \frac{2}{\Delta^2}(A'(t)Q + AQ_t - B_t)(A'(t) + B_x - AQ_x),$$

$$\phi_{tx} = \frac{1}{\Delta}(B_{tx}+A'Q_x-AQ_{tx}+B_{xx}Q-BQ_{xx}) - \frac{1}{\Delta^2}(AQ_x-B_x)(B_t+A'Q-AQ_t+B_xQ-BQ_x).$$

Claim 5.5.2. The second-order nonlinear equation of the form

$$\ddot{x} + b(x,t)\dot{x}^2 + c(x,t)\dot{x} + d(x,t) = 0$$

admits Lagrangian provided

$$\begin{aligned} 2A'' + B_{xt} - 3A'Q_x - AQ_{xt} - B_{xx}Q + BQ_{xx})(AQ - B) = \\ A'Q(2A' + 3B_x - 3AQ_x) + (AQ_t - B_t)(2A' + B_x - AQ_x) \\ -(AQ_x - B_x)(B_xQ - BQ_x), \end{aligned}$$

where $b(x,t), c(x,t)$ defined as (8) and $d(x,t) = \frac{1}{\Delta}(QB_t - BQ_t)$.

Outline of proof. It follows from the compatibility condition $\phi_{xt} = \phi_{tx}$.
□

Example 5.5.5. Set $A = 1$, $\Delta = AQ - B = x^\alpha t^\beta$ and assume $Q = x^\gamma$ in (5.45), the equation becomes

$$\ddot{x} - \frac{\alpha}{x}\dot{x}^2 + \left((\gamma - \alpha)x^{\gamma-1} - \frac{\beta}{t}\right)\dot{x} - \frac{\beta x^\gamma}{t} = 0 \tag{5.45}$$

whose first integral is

$$I = \frac{\dot{x} + x^\gamma - x^\alpha t^\beta}{\dot{x} + x^\gamma}.$$

Let us find the condition γ and α for which equation (5.45) gives Lagrangian description.

Claim 5.5.3. For $\gamma = \alpha$ or $\gamma = 1$ equation (5.45) yields a Lagrangian description.

Proof. Equate $\frac{1}{2}\phi_x = -\frac{\alpha}{x}$ and $\phi_t = (\gamma-\alpha)x^{\gamma-1} - \frac{\beta}{t}$. It immediately yields $\phi_{xt} = 0$ and $\phi_{tx} = (\gamma - \alpha)(\gamma - 1)x^{\gamma-2}$. Thus from the compatibility condition we obtain our criteria.
□

5.6 Conclusion

Any second-order ordinary differential equation which possesses a first integral of the form (1.11) has to be cubic with respect to the first-order derivative (5.1). The present research gives a complete criteria of the existence of a first integral of the form (1.11) for a second-order ordinary differential equation (5.1) which is reduced to the equation with $L_2 = 0$. Despite that any second-order ordinary differential equation (5.1) can be reduced to the equation with $L_2 = 0$, the complete solving of the problem requires the sufficient conditions be given using coefficients of the original equation (not reduced). This is still an open problem.

Appendix

Coefficients are

$$
\begin{aligned}
\mu_1 =& \ (\lambda_{1x} - 3a_2\lambda_1)/\lambda_1,\\
\mu_2 =& \ (\mu_{1x} + 3a_{3t} - 3a_2\mu_1 + \mu_1^2)/3,\\
\mu_3 =& \ (\lambda_{1t}a_3 - 24a_1a_3\lambda_1 + 6\lambda_1\mu_1^2 + 10\lambda_1\mu_2)/(5\lambda_1),\\
\mu_4 =& \ (\mu_{1t}a_3 + 12a_3(a_{1x} - 2a_{2t} + a_0a_3 + a_1\mu_1) - 6\mu_1^3 - 4\mu_1\mu_2 + 5\mu_1\mu_3)/(15\lambda_1),\\
\mu_5 =& \ a_{3x} - 6a_2a_3 + 10a_3\mu_1,\\
\mu_6 =& \ (a_{3t}a_3 - 6a_1a_3^2 + 18a_3\mu_1^2 + a_3\mu_2 - a_3\mu_3 - 3\mu_1\mu_5)/5,\\
\mu_7 =& \ (3a_{2t}a_3^2 - \mu_{2x}a_3 - 2a_0a_3^3 - 18a_1a_3^2\mu_1 + 6a_2a_3\mu_2 + a_3\lambda_1\mu_4 + 54a_3\mu_1^3\\
& -4a_3\mu_1\mu_2 - 3a_3\mu_1\mu_3 - 9\mu_1^2\mu_5 - 15\mu_1\mu_6 + \mu_2\mu_5)/5,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_1 =& \ 720a_3^2\mu_1^2 - 432a_1a_3^3 + 144a_3^2\mu_2 - 144a_3^2\mu_3 - 80a_3\mu_1\mu_5 - 300a_3\mu_6 - 5\mu_5^2,\\
\beta_1 =& \ 1728a_{1x}a_3^3 - 3456a_{2t}a_3^3 + 1776a_0a_3^4 + 3024a_1a_3^3\mu_1 - 1680a_3^2\lambda_1\mu_4 - 3456a_3^2\mu_1^3\\
& -1008a_3^2\mu_1\mu_2 + 1008a_3^2\mu_1\mu_3 + 432a_3\mu_1^2\mu_5 + 1040a_3\mu_1\mu_6 - 300a_3\mu_7 - 25\mu_5\mu_6,\\
\gamma_1 =& \ 48a_{0x}a_3^4 - 144a_{1t}a_3^4 + 48\mu_{2t}a_3^3 + 144a_0a_2a_3^4 - 432a_1a_3^3\mu_2 - 144a_3^2\lambda_1\mu_1\mu_4 + 96a_3^2\mu_2^2\\
& +864a_3^2\mu_1^2\mu_2 - 48a_3^2\mu_2\mu_3 - 144a_3\mu_1\mu_2\mu_5 + 320a_3\mu_1\mu_7 - 240a_3\mu_2\mu_6 - 25\mu_5\mu_7,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2 =& \ -72\mu_{5t}a_3^3 + 432a_1a_3^3\mu_5 + 2592a_3^3\lambda_1\mu_4 - 1296a_3^2\mu_1^2\mu_5 + 1152a_3^2\mu_1\mu_6 - 72a_3^2\mu_2\mu_5\\
& -2160a_3^2\mu_7 + 264a_3\mu_1\mu_5^2 + 180a_3\mu_5\mu_6 - 5\mu_5^3,\\
\beta_2 =& \ -72\mu_{6t}a_3^3 + 432a_1a_3^3\mu_6 - 144a_3^2\lambda_1\mu_4\mu_5 - 1296a_3^2\mu_1^2\mu_6 + 1728a_3^2\mu_1\mu_7\\
& -72a_3^2\mu_2\mu_6 - 72a_3^2\mu_3\mu_6 + 264a_3\mu_1\mu_5\mu_6 - 60a_3\mu_5\mu_7 + 240a_3\mu_6^2 - 5\mu_5^2\mu_6,\\
\gamma_2 =& \ -72\mu_{7t}a_3^3 + 432a_1a_3^3\mu_7 - 72a_3^2\lambda_1\mu_4\mu_6 - 1296a_3^2\mu_1^2\mu_7 - 72a_3^2\mu_2\mu_7 - 144a_3^2\mu_3\mu_7\\
& +264a_3\mu_1\mu_5\mu_7 + 240a_3\mu_6\mu_7 - 5\mu_5^2\mu_7,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_3 =& \ -2\mu_{5x}a_3 - 432a_1a_3^3 + 18a_2a_3\mu_5 + 1296a_3^2\mu_1^2 + 144a_3^2\mu_2 - 72a_3^2\mu_3\\
& -242a_3\mu_1\mu_5 - 336a_3\mu_6 + 3\mu_5^2,\\
\beta_3 =& \ -1212a_{1x}a_3^3 + 2424a_{2t}a_3^3 - 10\mu_{5t}a_3^2 + 30\mu_{5x}a_3\mu_1 - 1244a_0a_3^4 + 4644a_1a_3^3\mu_1\\
& +120a_1a_3^2\mu_5 - 270a_2a_3\mu_1\mu_5 + 1540a_3^2\lambda_1\mu_4 - 18336a_3^2\mu_1^3 - 1548a_3^2\mu_1\mu_2 + 468a_3^2\mu_1\mu_3\\
& +3312a_3\mu_1^2\mu_5 + 5090a_3\mu_1\mu_6 - 30a_3\mu_2\mu_5 + 20a_3\mu_3\mu_5 + 850a_3\mu_7 + 75\mu_5\mu_6,\\
\gamma_3 =& \ -2\mu_{7x}a_3 + 12a_1a_3^2\mu_6 + 30a_2a_3\mu_7 - 36a_3\mu_1^2\mu_6 - 38a_3\mu_1\mu_7 - 4a_3\mu_2\mu_6\\
& +2a_3\mu_3\mu_6 + 6\mu_1\mu_5\mu_6 + 3\mu_5\mu_7 + 10\mu_6^2,
\end{aligned}
$$

$$
\begin{aligned}
q_0 = &\ a_{0x}a_3^2 - 3a_{1t}a_3^2 - a_{3t}\mu_2 + \mu_{2t}a_3 + 3a_0a_2a_3^2 - 3a_1a_3\mu_2 - 3\lambda_1\mu_1\mu_4 + \mu_2^2, \\
q_1 = &\ 36a_{1x}a_3 - 72a_{2t}a_3 + 3a_{3t}\mu_1 + 37a_0a_3^2 + 45a_1a_3\mu_1 - 35\lambda_1\mu_4 - 18\mu_1(\mu_1^2 + \mu_2 - \mu_3), \\
q_2 = &\ a_{3t} + 3a_1a_3 + 3\mu_1^2 - 2\mu_2 + 2\mu_3.
\end{aligned}
$$

Chapter 6

Complete group classification of systems of linear second-order ordinary differential equations with constant coefficients

Abstract.

The present research corrects the way of using Jordan canonical forms for studying the symmetry structures of systems of linear second-order ordinary differential equations with constant coefficients applied in [81]. The approach is demonstrated for a system consisting of two equations.

6.1 Equivalence Lie group

The first step in group classification is the step of obtaining an equivalence Lie group. The equivalence Lie group allows one to change the coefficients of the original system of equations. For completeness, this group is presented here.

Let us consider the system of equations

$$\ddot{x} = m_{11}x + m_{12}y,$$
$$\ddot{y} = m_{21}x + m_{22}y,$$

where m_{ij} are real-valued constants. The functions $x(t)$ and $y(t)$ are also real-valued. In matrix form, these equations are written as

$$\ddot{\mathbf{x}} = M\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

The equivalence Lie group of this system is defined by the generators:

$$X_1^e = \partial_t, \quad X_2^e = x\partial_x + m_{12}\partial_{m_{12}} - m_{21}\partial_{m_{21}}, \quad X_3^e = y\partial_y - m_{12}\partial_{m_{12}} + m_{21}\partial_{m_{21}},$$
$$X_4^e = t\partial_t - 2(m_{22}\partial_{m_{22}} + m_{21}\partial_{m_{21}} + m_{11}\partial_{m_{11}} + m_{12}\partial_{m_{12}}),$$
$$X_5^e = y\partial_x + m_{21}\partial_{m_{11}} + (m_{22} - m_{11})\partial_{m_{12}} - m_{21}\partial_{m_{22}},$$
$$X_6^e = x\partial_y - m_{12}\partial_{m_{11}} - (m_{22} - m_{11})\partial_{m_{21}} + m_{12}\partial_{m_{22}}.$$

The transformations corresponding to the generators X_2^e , X_3^e and X_4^e define scaling of the coefficients if one scales the variables x, y and t, respectively. The transformations corresponding to the generators X_5^e and X_6^e define the change of the coefficients if one takes linear combinations of the dependent variables x and y. Since the transformations corresponding to these generators are used for simplifying the canonical form, let us present them here:

$$\begin{array}{ll}
X_2^e: & \widetilde{x} = xe^a,\ \widetilde{m}_{12} = m_{12}e^a,\ \widetilde{m}_{21} = m_{21}e^{-a};\\
X_3^e: & \widetilde{y} = ye^a,\ \widetilde{m}_{12} = m_{12}e^{-a},\ \widetilde{m}_{21} = m_{21}e^a;\\
X_4^e: & \widetilde{t} = te^a,\ \widetilde{m}_{11} = m_{11}e^{-2a},\ \widetilde{m}_{12} = m_{12}e^{-2a},\ \widetilde{m}_{21} = m_{21}e^{-2a},\ \widetilde{m}_{22} = m_{22}e^{-2a};\\
X_5^e: & \widetilde{x} = x + ay,\ \widetilde{m}_{11} = m_{11} + am_{21},\ \widetilde{m}_{12} = m_{12} + a(m_{22} - m_{11}) - a^2m_{21},\ \widetilde{m}_{22} = m_{22} - am_{21};\\
X_6^e: & \widetilde{y} = y + ax,\ \widetilde{m}_{11} = m_{11} - am_{12},\ \widetilde{m}_{21} = m_{21} - a(m_{22} - m_{11}) - a^2m_{12},\ \widetilde{m}_{22} = m_{22} + am_{12}.
\end{array}$$

Here only changeable variables are presented. There are also four involutions corresponding to the discrete transformations

$$\begin{array}{ll}
E_1: & \widetilde{x} = -x;\\
E_2: & \widetilde{y} = -y;\\
E_3: & \widetilde{t} = -t;\\
E_4: & \widetilde{x} = y,\ \widetilde{y} = x.
\end{array}$$

6.2 Canonical forms

For a real-valued 2×2 matrix M, the Jordan matrix is one of the following three types,

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix},\quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix},\quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},$$

where a, b and $c > 0$ are real numbers. The matrix P has also real-valued entries in these cases. The simplification of system (1.23) through scaling of the dependent variables and independent variables depends on the form of the Jordan matrix.

6.2.1 Case $J = J_1$

Since a and b are real-valued, it is well-known that for $a = b$ the corresponding system of equations (1.23) with $J = J_1$ is reduced to the free particle system. The admitted Lie algebra in this case is also well-known and consists of the generators:

$$\begin{gathered}
X_1 = \partial_t,\ \ X_2 = \partial_x,\ \ X_3 = \partial_y,\ \ X_4 = t\partial_x,\ \ X_5 = t\partial_y,\ \ X_6 = x\partial_x,\ \ X_7 = y\partial_y,\ \ X_8 = t\partial_t,\\
X_9 = x\partial_t,\ \ X_{10} = y\partial_t,\ \ X_{11} = y\partial_x,\ \ X_{12} = x\partial_y,\\
X_{13} = x(y\partial_y + x\partial_x + t\partial_t),\ \ X_{14} = y(y\partial_y + x\partial_x + t\partial_t),\ \ X_{15} = t(y\partial_y + x\partial_x + t\partial_t).
\end{gathered}$$

If $a \neq b$, then using the equivalence transformations corresponding one part of the set of admitted generators is

$$X_1 = \partial_t,\ \ X_2 = x\partial_x,\ \ X_3 = y\partial_y.$$

The remaining generators are defined by the formulae

$$X = \varphi(t)\partial_x,\ \ Y = \psi(t)\partial_y,$$

where

$$\varphi'' = a\varphi,\ \ \psi'' = b\psi.$$

The general solution of these equations depends on the signs of the coefficients. Notice also that using the involution E_4 and the scaling of the independent variable corresponding to X_4^e, the coefficient a can be reduced to ± 1, where the sign coincides with the sign of a. Thus one obtains the cases:

(a.1) $a = 1$, $b = 0$: $X_4 = e^t \partial_x$, $X_5 = e^{-t}\partial_x$, $X_6 = \partial_y$, $X_7 = t\partial_y$;

(a.2) $a = 1$, $b = k^2 > 0$: $X_4 = e^t \partial_x$, $X_5 = e^{-t}\partial_x$, $X_6 = e^{kt}\partial_y$, $X_7 = e^{-kt}\partial_y$;

(a.3) $a = 1$, $b = -k^2 < 0$: $X_4 = e^t \partial_x$, $X_5 = e^{-t}\partial_x$, $X_6 = \sin(kt)\partial_y$, $X_7 = \cos(kt)\partial_y$;

(a.4) $a = -1$, $b = 0$: $X_4 = \sin(t)\partial_x$, $X_5 = \cos(t)\partial_x$, $X_6 = \partial_y$, $X_7 = t\partial_y$;

(a.5) $a = -1$, $b = -k^2 < 0$: $X_4 = \sin(t)\partial_x$, $X_5 = \cos(t)\partial_x$, $X_6 = \sin(kt)\partial_y$, $X_7 = \cos(kt)\partial_y$;

Notice that the last two cases are missing in [81]. Considering the commutators tables, one can show that the structure of these Lie algebras also differs from the structure of the Lie algebras presented in [81]. However, in term of the dimension of the symmetry Lie algebra no new dimension arises.

6.2.2 Case $J = J_2$

Using the scaling corresponding to X_4^e, and the involutions E_1, E_2, E_4 (if necessary), system (1.23) with $J = J_2$ is reduced into the system

$$\ddot{x} = ax + y, \quad \ddot{y} = -x + ay.$$

Calculations give the admitted Lie algebra corresponding to the generators

$$\begin{gathered} X_1 = y\partial_x - x\partial_y, \quad X_2 = x\partial_x + y\partial_y, \quad X_3 = \partial_t, \\ X_4 = e^{tq_1}\left(\cos(tq_2)\partial_x - \sin(tq_2)\partial_y\right), \quad X_5 = e^{tq_1}\left(\sin(tq_2)\partial_x + \cos(tq_2)\partial_y\right), \\ X_6 = e^{-tq_1}\left(\cos(tq_2)\partial_x + \sin(tq_2)\partial_y\right), \quad X_7 = e^{-tq_1}\left(\sin(tq_2)\partial_x - \cos(tq_2)\partial_y\right), \end{gathered}$$

where

$$q_1 = \sqrt{\frac{\sqrt{1+a^2}+a}{2}}, \quad q_2 = \sqrt{\frac{\sqrt{1+a^2}-a}{2}}.$$

This case is also missing in [81].

6.2.3 Case $J = J_3$

In this case system (1.23) is

$$\ddot{x} = ax + y, \quad \ddot{y} = ay. \tag{6.1}$$

One part of the set of admitted generators is

$$X_1 = \partial_t, \quad X_2 = y\partial_x, \quad X_3 = x\partial_x + y\partial_y.$$

The remaining generators are defined by the formula

$$X = c(t\partial_t - 2(y + ax)\partial_y) + \varphi\partial_x + (\varphi'' - a\varphi)\partial_y,$$

where the constant k and the function $\varphi = \varphi(t)$ satisfy the equations

$$ca = 0, \quad \varphi^{(4)} - 2a\varphi'' + a^2\varphi = 0.$$

Thus, one has three cases:

(c.1) if $a = 0$, then $\varphi = p_4 t^3 + p_3 t^2 + p_2 t + p_1$, and the additional generators are

$$X_4 = \partial_x,\ X_5 = t\partial_t - 2y\partial_y,\ X_6 = t^3\partial_x + 6t\partial_y,\ X_7 = t^2\partial_x + 2\partial_y,\ X_8 = t\partial_x;$$

(c.2) if $a = -k^2 < 0$, then $c = 0$ and $\varphi = (p_1 t + p_2)\sin(kt) + (p_3 t + p_4)\cos(kt)$, and the additional generators are

$$X_4 = \sin(kt)\partial_x,\quad X_5 = \cos(kt)\partial_x,$$
$$X_6 = 2k\cos(kt)\partial_y + t\sin(kt)\partial_x,\quad X_7 = t\cos(kt)\partial_x - 2k\sin(kt)\partial_y.$$

(c.3) if $a = k^2 > 0$, then $c = 0$ and $\varphi = (p_1 t + p_2)e^{kt} + (p_3 t + p_4)e^{-kt}$, and the additional generators are

$$X_4 = e^{kt}\partial_x,\quad X_5 = e^{-kt}\partial_x,\ X_6 = e^{kt}(t\partial_x + 2k\partial_y),\quad X_7 = e^{-kt}(t\partial_x - 2k\partial_y).$$

Here p_i, $(i = 1, 2, 3, 4)$ are constant.

In [81], the last case is presented incorrectly. One can check that the generator ∂_x is not admitted by system (6.1) in case (c.3).

Chapter 7

Summary

7.1 Fluids with internal inertia

A systematic application of the group analysis method for modeling fluids with internal inertia is considered in the third part of the research. The equations studied include models such as the nonlinear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles, and the dispersive shallow water model. These models are obtained for special types of the potential function $W(\rho, \dot{\rho}, S)$. The main feature of the present research is the study of the potential functions with $W_{\dot{\rho}\dot{\rho}S} \neq 0$. The group classification separates these models into 73 different classes. The result is published in [93].

7.2 Applications of group analysis to integro-differential equations

7.2.1 The Rudenko equation

The research deals with an evolutionary integro-differential equation describing nonlinear waves. A particular choice of the kernel in the integral leads to well-known equations such as the Khokhlov-Zabolotskaya equation, the Kadomtsev-Petviashvili equation and others. Since the solutions of these equations describe many physical phenomena, the analysis of the general model studied in this paper is important. One of the methods for obtaining solutions of differential equations is provided by the Lie group analysis. However, this method is not applicable to integro-differential equations. Therefore, in the research we discuss new approaches developed in modern group analysis and apply them to the general model considered in this paper. Reduced equations and exact solutions are also presented. The result is published in [94].

7.2.2 The Boltzmann equation with sources

This part of the research considered in the project is started after visiting Suranaree University of Technology by Professor Yu.N.Grigoriev. We started with the following problem. In [48] the classical group analysis method was applied to the equation which was obtained from the spatially homogeneous and isotropic Boltzmann equation with sources. The derived equation is still a nonlocal partial differential equation. However, this property was not taken into account there. In the present paper this lack of [48] is corrected. The result is published in [95, 96].

7.3 Application of group analysis to ordinary differential equations

7.3.1 On first integrals of second-order ordinary differential equations

This part of the research considered in the project is devoted to the study of intermediate integrals of a second-order ordinary differential equation of the form

$$I = \frac{\dot{x}A(t,x) + C(t,x)}{\dot{x}B(t,x) + Q(t,x)}.$$

This research is started after visiting Suranaree University of Technology by Professor Sibusiso Moyo (DUT, South Africa). The sufficient conditions for existence of an intermediate integral of the form presented above are obtained. The result is published in [97].

7.3.2 Complete group classification of systems of two linear second-order ordinary differential equations with constant coefficients

Recent works by Wafo Soh [81] have focused on the study of systems of second-order ordinary differential equations with constant coefficients. The studies deal with symmetries of systems of linear second-order ordinary differential equations with two and three equations are considered. The research conducted in the project corrects the way of using Jordan canonical forms for studying the symmetry structures of systems of linear second-order ordinary differential equations with constant coefficients applied by Wafo Soh. The result is published in [98].

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