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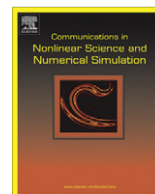
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Short communication

## Comment on “Symmetry breaking of systems of linear second-order ordinary differential equations with constant coefficients”

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### ABSTRACT

The present paper corrects the way of using Jordan canonical forms for studying the symmetry structures of systems of linear second-order ordinary differential equations with constant coefficients applied in [1]. The approach is demonstrated for a system consisting of two equations.

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## 1. Introduction

Recent works by Wafo Soh [1] have focused on the study of systems of second-order ordinary differential equations with constant coefficients. The studies deal with symmetries of systems of linear second-order ordinary differential equations with two and three equations are considered. The goal of the paper is to study the symmetry structure of a system of  $n = 2$  linear second-order ordinary differential equations with constant coefficients. Since a change of the dependent and independent variables does not change the structure of the admitted Lie group, the author at first simplify the system, and then calculate the admitted Lie group of the simplified system using the standard procedure.

The paper [1] concentrated attention on the system of  $n$  equations of the form

$$\ddot{\mathbf{x}} = M\mathbf{x}, \quad (1.1)$$

where the overdot denotes differentiation with respect to  $t$ ,  $\mathbf{x}$  is an  $n$ -dimensional vector with complex entries, and  $M$  is an  $n \times n$  matrix with complex entries.

A first simplification of system (1.1) is achieved by using the Jordan normal form  $J$  of the matrix  $M$

$$M = P^{-1}JP.$$

The change  $\mathbf{u} = P\mathbf{x}$  reduces system (1.1) to the system

$$\ddot{\mathbf{u}} = J\mathbf{u}. \quad (1.2)$$

Next, a simplification of system (1.2) is made for coefficients corresponding to diagonal blocks of the Jordan matrix  $J$ : the author applied scaling. This step is crucial in the paper. In fact, in case of a diagonal block of the Jordan matrix  $J$ , scaling of the dependent variables does not change the coefficients of this part of the system. Hence, one can conclude that scaling is

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applied to the independent variable. Because the independent variable is real valued, one can only make a real valued scaling. This allows one to reduce only one component of a diagonal coefficient: either the real or imaginary part of the eigenvalue of the Jordan matrix  $J$ . Whereas in [1] for any eigenvalue (including a complex eigenvalue) the corresponding coefficients are reduced to the real number 1. This means that the author [1] considered in the corresponding cases only real eigenvalues. Therefore, the results of [1] are not complete. For a complete study one also needs to study complex eigenvalues corresponding to diagonal Jordan blocks.

It is also worth to notice that from the paper it is unclear how the author calculated an admitted Lie algebra. In the standard procedure, the dependent and independent variables are real-valued, whereas the results of [1] are obtained for  $n$  complex-valued dependent variables. Does this mean that the authors considered  $2n$  real-valued equations for calculating the admitted Lie algebra?

The goal of the present paper is to correct the approach applied in [1]. The approach is illustrated by using a complete study of symmetry structures of systems of two real-valued linear second-order ordinary differential equations with constant coefficients. In application of this approach to a system with more than two equations one needs to take into account that if a real-valued matrix  $M$  has a complex eigenvalue, then the conjugate number is also an eigenvalue. Only systems of two second-order equations are considered here.

## 2. Equivalence Lie group

The first step in group classification is the step of obtaining an equivalence Lie group. The equivalence Lie group allows one to change the coefficients of the original system of equations. For completeness, this group is presented here.

Let us consider the system of equations

$$\ddot{x} = m_{11}x + m_{12}y,$$

$$\ddot{y} = m_{21}x + m_{22}y,$$

where  $m_{ij}$  are real-valued constants. The functions  $x(t)$  and  $y(t)$  are also real-valued. In matrix form, these equations are written as

$$\ddot{\mathbf{x}} = M\mathbf{x},$$

where

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad M = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}.$$

The equivalence Lie group of this system is defined by the generators:

$$X_1^e = \partial_t, \quad X_2^e = x\partial_x + m_{12}\partial_{m_{12}} - m_{21}\partial_{m_{21}}, \quad X_3^e = y\partial_y - m_{12}\partial_{m_{12}} + m_{21}\partial_{m_{21}},$$

$$X_4^e = t\partial_t - 2(m_{22}\partial_{m_{22}} + m_{21}\partial_{m_{21}} + m_{11}\partial_{m_{11}} + m_{12}\partial_{m_{12}}),$$

$$X_5^e = y\partial_x + m_{21}\partial_{m_{11}} + (m_{22} - m_{11})\partial_{m_{12}} - m_{21}\partial_{m_{22}},$$

$$X_6^e = x\partial_y - m_{12}\partial_{m_{11}} - (m_{22} - m_{11})\partial_{m_{21}} + m_{12}\partial_{m_{22}}.$$

The transformations corresponding to the generators  $X_2^e$ ,  $X_3^e$  and  $X_4^e$  define scaling of the coefficients if one scales the variables  $x$ ,  $y$  and  $t$ , respectively. The transformations corresponding to the generators  $X_5^e$  and  $X_6^e$  define the change of the coefficients if one takes linear combinations of the dependent variables  $x$  and  $y$ . Since the transformations corresponding to these generators are used for simplifying the canonical form, let us present them here:

$$X_2^e : \tilde{x} = xe^a, \quad \tilde{m}_{12} = m_{12}e^a, \quad \tilde{m}_{21} = m_{21}e^{-a};$$

$$X_3^e : \tilde{y} = ye^a, \quad \tilde{m}_{12} = m_{12}e^{-a}, \quad \tilde{m}_{21} = m_{21}e^a;$$

$$X_4^e : \tilde{t} = te^a, \quad \tilde{m}_{11} = m_{11}e^{-2a}, \quad \tilde{m}_{12} = m_{12}e^{-2a}, \quad \tilde{m}_{21} = m_{21}e^{-2a}, \quad \tilde{m}_{22} = m_{22}e^{-2a};$$

$$X_5^e : \tilde{x} = x + ay, \quad \tilde{m}_{11} = m_{11} + am_{21}, \quad \tilde{m}_{12} = m_{12} + a(m_{22} - m_{11}) - a^2m_{21}, \quad \tilde{m}_{22} = m_{22} - am_{21};$$

$$X_6^e : \tilde{y} = y + ax, \quad \tilde{m}_{11} = m_{11} - am_{12}, \quad \tilde{m}_{21} = m_{21} - a(m_{22} - m_{11}) - a^2m_{12}, \quad \tilde{m}_{22} = m_{22} + am_{12}.$$

Here only changeable variables are presented. There are also four involutions corresponding to the discrete transformations

$$E_1 : \tilde{x} = -x;$$

$$E_2 : \tilde{y} = -y;$$

$$E_3 : \tilde{t} = -t;$$

$$E_4 : \tilde{x} = y, \quad \tilde{y} = x.$$

### 3. Canonical forms

For a real-valued  $2 \times 2$  matrix  $M$ , the Jordan matrix is one of the following three types,

$$J_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad J_2 = \begin{pmatrix} a & c \\ -c & a \end{pmatrix}, \quad J_3 = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix},$$

where  $a, b$  and  $c > 0$  are real numbers. The matrix  $P$  has also real-valued entries in these cases. The simplification of system (1.2) through scaling of the dependent variables and independent variables depends on the form of the Jordan matrix.

#### 3.1. Case $J = J_1$

Since  $a$  and  $b$  are real-valued, it is well-known that for  $a = b$  the corresponding system of Eq. (1.2) with  $J = J_1$  is reduced to the free particle system. The admitted Lie algebra in this case is also well-known and consists of the generators:

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = \partial_y, \quad X_4 = t\partial_x, \quad X_5 = t\partial_y, \quad X_6 = x\partial_x, \quad X_7 = y\partial_y, \quad X_8 = t\partial_t, \quad X_9 = x\partial_t, \quad X_{10} = y\partial_t, \\ X_{11} = y\partial_x, \quad X_{12} = x\partial_y, \quad X_{13} = x(y\partial_y + x\partial_x + t\partial_t), \quad X_{14} = y(y\partial_y + x\partial_x + t\partial_t), \quad X_{15} = t(y\partial_y + x\partial_x + t\partial_t).$$

If  $a \neq b$ , then using the equivalence transformations, one part of the set of admitted generators is

$$X_1 = \partial_t, \quad X_2 = x\partial_x, \quad X_3 = y\partial_y.$$

The remaining generators are defined by the formulae

$$X = \varphi(t)\partial_x, \quad Y = \psi(t)\partial_y,$$

where

$$\varphi'' = a\varphi, \quad \psi'' = b\psi.$$

The general solution of these equations depends on the signs of the coefficients.

Notice also that using the involution  $E_4$  and the scaling of the independent variable corresponding to  $X_4^c$ , the coefficient  $a$  can be reduced to  $\pm 1$ , where the sign coincides with the sign of  $a$ . Thus one obtains the cases:

- (a.1)  $a = 1, b = 0 : X_4 = e^t\partial_x, X_5 = e^{-t}\partial_x, X_6 = \partial_y, X_7 = t\partial_y;$
- (a.2)  $a = 1, b = k^2 > 0 : X_4 = e^t\partial_x, X_5 = e^{-t}\partial_x, X_6 = e^{kt}\partial_y, X_7 = e^{-kt}\partial_y;$
- (a.3)  $a = 1, b = -k^2 < 0 : X_4 = e^t\partial_x, X_5 = e^{-t}\partial_x, X_6 = \sin(kt)\partial_y, X_7 = \cos(kt)\partial_y;$
- (a.4)  $a = -1, b = 0 : X_4 = \sin(t)\partial_x, X_5 = \cos(t)\partial_x, X_6 = \partial_y, X_7 = t\partial_y;$
- (a.5)  $a = -1, b = -k^2 < 0 : X_4 = \sin(t)\partial_x, X_5 = \cos(t)\partial_x, X_6 = \sin(kt)\partial_y, X_7 = \cos(kt)\partial_y;$

Notice that the last two cases are missing in [1]. Considering the commutators tables, one can show that the structure of these Lie algebras also differs from the structure of the Lie algebras presented in [1].

However, in term of the dimension of the symmetry Lie algebra no new dimension arises.

#### 3.2. Case $J = J_2$

Using the scaling corresponding to  $X_4^c$ , and the involutions  $E_1, E_2, E_4$  (if necessary), system (1.2) with  $J = J_2$  is reduced into the system

$$\ddot{x} = ax + y, \quad \ddot{y} = -x + ay.$$

Calculations give the admitted Lie algebra corresponding to the generators

$$X_1 = y\partial_x - x\partial_y, \quad X_2 = x\partial_x + y\partial_y, \quad X_3 = \partial_t, \quad X_4 = e^{tq_1}(\cos(tq_2)\partial_x - \sin(tq_2)\partial_y), \\ X_5 = e^{tq_1}(\sin(tq_2)\partial_x + \cos(tq_2)\partial_y), \quad X_6 = e^{-tq_1}(\cos(tq_2)\partial_x + \sin(tq_2)\partial_y), \quad X_7 = e^{-tq_1}(\sin(tq_2)\partial_x - \cos(tq_2)\partial_y),$$

where

$$q_1 = \sqrt{\frac{\sqrt{1+a^2}+a}{2}}, \quad q_2 = \sqrt{\frac{\sqrt{1+a^2}-a}{2}}.$$

This case is also missing in [1].

### 3.3. Case $J = J_3$

In this case system (1.2) is

$$\ddot{x} = ax + y, \quad \ddot{y} = ay. \quad (3.3)$$

One part of the set of admitted generators is

$$X_1 = \partial_t, \quad X_2 = y\partial_x, \quad X_3 = x\partial_x + y\partial_y.$$

The remaining generators are defined by the formula

$$X = c(t\partial_t - 2(y + ax)\partial_y) + \varphi\partial_x + (\varphi' - a\varphi)\partial_y,$$

where the constant  $k$  and the function  $\varphi = \varphi(t)$  satisfy the equations

$$ca = 0, \quad \varphi^{(4)} - 2a\varphi'' + a^2\varphi = 0.$$

Thus, one has three cases:

(c.1) if  $a = 0$ , then  $\varphi = p_4t^3 + p_3t^2 + p_2t + p_1$ , and the additional generators are

$$X_4 = \partial_x, \quad X_5 = t\partial_t - 2y\partial_y, \quad X_6 = t^3\partial_x + 6t\partial_y, \quad X_7 = t^2\partial_x + 2\partial_y, \quad X_8 = t\partial_x;$$

(c.2) if  $a = -k^2 < 0$ , then  $c = 0$  and  $\varphi = (p_1t + p_2)\sin(kt) + (p_3t + p_4)\cos(kt)$ , and the additional generators are

$$X_4 = \sin(kt)\partial_x, \quad X_5 = \cos(kt)\partial_x, \quad X_6 = 2k\cos(kt)\partial_y + t\sin(kt)\partial_x, \quad X_7 = t\cos(kt)\partial_x - 2k\sin(kt)\partial_y.$$

(c.3) if  $a = k^2 > 0$ , then  $c = 0$  and  $\varphi = (p_1t + p_2)e^{kt} + (p_3t + p_4)e^{-kt}$ , and the additional generators are

$$X_4 = e^{kt}\partial_x, \quad X_5 = e^{-kt}\partial_x, \quad X_6 = e^{kt}(t\partial_x + 2k\partial_y), \quad X_7 = e^{-kt}(t\partial_x - 2k\partial_y).$$

Here  $p_i$ , ( $i = 1, 2, 3, 4$ ) are constant.

In [1], the last case is presented incorrectly. One can check that the generator  $\partial_x$  is not admitted by system (3.3) in case (c.3).

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# On first integrals of second-order ordinary differential equations

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**Abstract** Here we discuss first integrals of a particular representation associated with second-order ordinary differential equations. The linearization problem is a particular case of the equivalence problem together with a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation. The relationship between the integral form, the associated equations, equivalence transformations, and some examples are considered as part of the discussion illustrating some important aspects and properties.

**Keywords** Equivalence transformations · Linearization · ODEs ·  $\lambda$ -Symmetry

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To Professor Peter Leach on his seventieth birthday.

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## 1 Introduction

Many methods of solving differential equations use a change of variables that transform a given differential equation into another equation with known properties. Since the class of linear equations is considered to be the simplest class of equations, it is attractive to transform a given differential equation into a linear equation. This problem, which is called a linearization problem, is a particular case of the equivalence problem. The equivalence problem can be formulated as follows. Let a set of invertible transformations be given. One can introduce the equivalence property according to these transformations: two differential equations are equivalent if there is a transformation of the given set that transforms one equation into another. The equivalence problem involves a number of related problems such as defining a class of transformations, finding invariants of these transformations, obtaining the equivalence criteria, and constructing the transformation.

### 1.1 Introduction to the problem

We give a short review of results related to an equivalence problem for a second-order ordinary differential equation (ODE). Furthermore, we distinguish two types of transformations used in the equivalence problem for second-order ODEs, namely point transformations and generalized Sundman transformations. Lie [1] also noted that all second-order ODEs can be transformed into each other by means of contact transformations and that this is not so for third-order equations. Thus this set of transformations cannot be applied to a classification of second-order ODEs.

Among the target equations, two classes of equations can be mentioned. One set of this class was obtained by Lie [2]. Lie's group classification of ODEs shows that the second-order equations can possess one, two, three, or eight infinitesimal symmetries. The equations with eight symmetries can be linearized by a change of variables. Lie showed that the latter equations are at most cubic in the first derivative and gave a convenient invariant description of all linearizable equations. A similar description of the equations with three symmetries was provided in [3,4]. Another set of target classes corresponds to the Painlevé equations. Analysis of the classes of equations corresponding to the first and second Painlevé equations was performed in [5,6].

For the linearization problem one studies those classes of equations that are equivalent to linear equations. The first linearization problem for ODEs was solved by Lie [1]. He found the general form of all ODEs of second order that can be reduced to a linear equation by changing the independent and dependent variables. He showed that any linearizable second-order equation should be at most cubic in the first-order derivative and provided a linearization test in terms of its coefficients. The linearization criterion is written through relative invariants of the equivalence group. Tresse [7] treated the equivalence problem for second-order ODEs in terms of relative invariants of the equivalence group of point transformations. In [8] an infinitesimal technique for obtaining relative invariants was applied to the linearization problem.

A different approach to tackling the equivalence problem of second-order ODEs was developed by Cartan [9]. The idea of his approach was to associate with every differential equation a uniquely defined geometric structure of a certain form. The Cartan approach was further applied by Chern [10] to third-order differential equations. Since none of the conditions given in [10] is an implicit expression that could be used as a test for determining the type of the studied equation, in a series of articles [11–15] the linearization problem was also considered. Linearization with respect to point transformations is studied in [11], with respect to contact transformations in [12–16]. The linearization problem was also investigated with respect to the generalized Sundman transformations [17–19].

The linearization problem via point transformations

$$\tau = \varphi(t, x), \quad u = \psi(t, x)$$

for a second-order equation  $\ddot{x} = F(t, x, \dot{x})$  is attractive because of the simplicity of the general solution of a linear equation: a linearizable second-order ODE is equivalent to the free particle equation  $u'' = 0$ . Thus, if one found the linearizing transformation, then the general solution of the original equation could be found easily. Note that for a linearizable equation  $\ddot{x} = F(t, x, \dot{x})$  the expression

$$u' = \frac{\dot{x}\psi_x + \psi_t}{\dot{x}\varphi_x + \varphi_t}$$

is a first integral of the equation. Here subscripts mean derivatives, for example,  $\varphi_t = \partial\varphi/\partial t$ ,  $\varphi_x = \partial\varphi/\partial x$  and so on. This motivated the authors of [20–22] to study equations possessing a first integral of the form

$$I = \frac{\dot{x}\tilde{A}(t, x) + \tilde{C}(t, x)}{\dot{x}\tilde{B}(t, x) + \tilde{Q}(t, x)}. \tag{1}$$

Notice that a second-order equation equivalent to the free particle equation via the generalized Sundman transformation also possesses a first integral of the form (1).

The authors of [20–22] came to the form of first integral (1) from the study of  $\lambda$ -symmetries for second-order equations that play a fundamental role in the study of  $\lambda$ -symmetries. Although the equation may lack Lie point symmetries, there always exists a  $\lambda$ -symmetry associated to a first integral  $I = I(t, x, \dot{x})$ . Such a  $\lambda$ -symmetry can be defined in canonical form by the vector field  $\mathbf{v} = \partial_x$  and the function  $\lambda = -I_x/I_{\dot{x}}$ . When  $I$  is of the form

$$I = C(t, x) + \frac{1}{A(t, x)\dot{x} + B(t, x)}, \quad (A \neq 0), \tag{2}$$

such a function  $\lambda$  is given by

$$\lambda(t, x, \dot{x}) = \gamma(t, x)\dot{x}^2 + \alpha(t, x)\dot{x} + \beta(t, x), \tag{3}$$

where

$$\gamma = AC_x = -a_3, \tag{4a}$$

$$\alpha = 2BC_x - A_x/A = -a_2 - AC_t, \tag{4b}$$

$$\beta = (C_x B^2 - B_x)/A = -a_1 + A_t/A - 2BC_t. \tag{4c}$$

In this way the study of ODEs that admit first integrals of the form (2) can be seen as a problem of classification of ODEs that admit  $\mathbf{v} = \partial_x$  as a  $\lambda$ -symmetry for some function  $\lambda$  of the form (3).

The case where  $C_x = 0$ ,

$$I = C(t) + \frac{1}{\dot{x}A(t, x) + C(t, x)},$$

was studied in [23]. It must be mentioned here that the case where  $\tilde{B} = 0$  in (1) was thoroughly examined in [22].

We denote by  $\mathcal{B}$  the class of equations corresponding to the particular case where  $\gamma = 0$  in (3). The equations in  $\mathcal{B}$  are ODEs of the form

$$\ddot{x} + a_2(t, x)\dot{x}^2 + a_1(t, x)\dot{x} + a_0(t, x) = 0 \tag{5}$$

that admit first integrals of the form (2) with  $C_x = 0$ .

A significant subclass of ODEs in  $\mathcal{B}$ , denoted by  $\mathcal{A}$ , is constituted by the equations that admit first integrals of the form  $A(t, x)\dot{x} + B(t, x)$  [that is,  $C = 0$  in (2)]. By (4b), the equations in  $\mathcal{A}$  are the equations of the form (5) that admit  $\mathbf{v} = \partial_x$  as  $\lambda$ -symmetry for some function  $\lambda = -a_2\dot{x} + \beta$ . According to the results in [22], the coefficients of the equations in  $\mathcal{A}$  must satisfy either  $S_1 = S_2 = 0$ , where

$$S_1(t, x) = a_{1x} - 2a_{2t}, \quad S_2(t, x) = (a_0a_2 + a_{0x})_x + (a_{2t} - a_{1x})_t + (a_{2t} - a_{1x})a_1, \tag{6}$$

or, if  $S_1 \neq 0$ ,  $S_3 = S_4 = 0$ , where

$$S_3(t, x) = \left(\frac{S_2}{S_1}\right)_x - (a_{2t} - a_{1x}), \quad S_4(t, x) = \left(\frac{S_2}{S_1}\right)_t + \left(\frac{S_2}{S_1}\right)^2 + a_1\left(\frac{S_2}{S_1}\right) + a_0a_2 + a_{0x}. \tag{7}$$

The equations in  $\mathcal{A}$  such that  $S_1 = S_2 = 0$  constitute the subclass  $\mathcal{A}_1$ , and they admit two functionally independent first integrals of the form  $A(t, x)\dot{x} + B(t, x)$ .

Several properties of the linearization through local and nonlocal transformations of the equations in  $\mathcal{B}$  are derived in [23, 24]. All the equations in  $\mathcal{A}_1$  pass the Lie test of linearization (i.e., their coefficients satisfy  $L_1 = L_2 = 0$ ). In

contrast, none of the equations in  $\mathcal{A}_2$  can be linearized through a local transformation; actually, there exist equations in  $\mathcal{A}_2$  that lack Lie point symmetries [see, for example, Eqs. (2.6) and (4.12) in [22]].

Although there exists Eq. (5) whose coefficients satisfy  $L_1 = L_2 = 0$ , which are not in  $\mathcal{A}_1$  (see Example 9 in [24]), they must all belong to  $\mathcal{B}$ . It is important to remark that there are equations in  $\mathcal{B}$  not linearizable through local transformations, apart from the subclass  $\mathcal{A}_2$  (as the family appearing in Example 2.1 in [25]). To linearize such types of equations, one must consider nonlocal transformations of the form

$$X = F(t, x), \quad dT = (G_1(t, x)\dot{x} + G_2(t, x)) dt. \quad (8)$$

The equations in  $\mathcal{B}$  can be characterized as the unique ODEs (5) that can be linearized through some nonlocal transformation of the form (8). When  $G_1(t, x) = 0$  in (8), the equation must belong to  $\mathcal{A}_2$  and vice versa. In other words, the equations in  $\mathcal{A}_2$  are the unique ODEs (5) that can be transformed into the linear equation  $X_{TT} = 0$  by means of some nonlocal transformation of the form

$$X = F(t, x), \quad dT = G(t, x)dt. \quad (9)$$

These transformations are known in the literature as *generalized Sundman transformations* (see [17, 18, 26–30] and references therein). Constructive methods to determine nonlocal linearizing transformations can be derived from the algorithms that calculate the first integrals [23, 24]. In particular, local changes of variables that linearize the equations in  $\mathcal{A}_1$  can be determined by just dealing with first-order ODEs. We remark that such linearizing point transformations usually appear in the literature as solutions of an involutive system of second-order partial differential equations [31, 32].

## 1.2 Invariants of a class of second-order equations

We recall some known properties of a second-order equation:

$$\ddot{x} + a_3(t, x)\dot{x}^3 + 3a_2(t, x)\dot{x}^2 + 3a_1(t, x)\dot{x} + a_0(t, x) = 0. \quad (10)$$

This form of equation is conserved with respect to any change of the independent and dependent variables:

$$\tau = \varphi(t, x), \quad u = \psi(t, x). \quad (11)$$

In fact, derivatives are changed by the formulae

$$\begin{aligned} u' &= g(t, x, \dot{x}) = \frac{D_t \psi}{D_t \varphi} = \frac{\psi_t + \dot{x} \psi_x}{\varphi_t + \dot{x} \varphi_x}, \\ u'' &= P(t, x, \dot{x}, \ddot{x}) = \frac{D_t g}{D_t \varphi} = \frac{g_t + \dot{x} g_x + \ddot{x} g_{\dot{x}}}{\varphi_t + \dot{x} \varphi_x} \\ &= (\varphi_t + \dot{x} \varphi_x)^{-3} \left( \ddot{x} (\varphi_t \psi_x - \varphi_x \psi_t) + \dot{x}^3 (\varphi_x \psi_{xx} - \varphi_{xx} \psi_x) \right. \\ &\quad \left. + \dot{x}^2 (\varphi_t \psi_{xx} - \varphi_{xx} \psi_t + 2 (\varphi_x \psi_{tx} - \varphi_{tx} \psi_x)) \right. \\ &\quad \left. + \dot{x} (\varphi_x \psi_{tt} - \varphi_{tt} \psi_x + 2 (\varphi_t \psi_{tx} - \varphi_{tx} \psi_t)) + \varphi_t \psi_{tt} - \varphi_{tt} \psi_t \right). \end{aligned} \quad (12)$$

Here  $D_t$  is the operator of the total derivative with respect to  $t$ , and

$$\Delta = \varphi_t \psi_x - \varphi_x \psi_t \neq 0.$$

Since the Jacobian of the change of variables  $\Delta \neq 0$ , the equation

$$u'' + b_3(\tau, u)u'^3 + 3b_2(\tau, u)u'^2 + 3b_1(\tau, u)u' + b_0(\tau, u) = 0 \quad (13)$$

becomes (10), where

$$\begin{aligned} a_1 &= \Delta^{-1} (\varphi_x \psi_{xx} - \varphi_{xx} \psi_x + \varphi_x^3 b_0 + 3\varphi_x^2 \psi_x b_1 + 3\varphi_x \psi_x^2 b_2 + \psi_x^3 b_3), \\ a_2 &= \Delta^{-1} (3^{-1} (\varphi_t \psi_{xx} - \varphi_{xx} \psi_t + 2 (\varphi_x \psi_{tx} - \varphi_{tx} \psi_x)) + \varphi_t \varphi_x^2 b_0 \\ &\quad + \varphi_x (2\varphi_t \psi_x + \varphi_x \psi_t) b_1 + (\varphi_t \psi_x^2 + 2\varphi_x \psi_t \psi_x) b_2 + \psi_t \psi_x^2 b_3), \\ a_3 &= \Delta^{-1} (3^{-1} (\varphi_x \psi_{tt} - \varphi_{tt} \psi_x + 2 (\varphi_t \psi_{tx} - \varphi_{tx} \psi_t)) + \varphi_t^2 \varphi_x b_0 \\ &\quad + (\varphi_t^2 \psi_x + 2\varphi_t \varphi_x \psi_t) b_1 + (2\varphi_t \psi_t \psi_x + \varphi_x \psi_t^2) b_2 + \psi_t^2 \psi_x b_3), \\ a_0 &= \Delta^{-1} (\varphi_t \psi_{tt} - \varphi_{tt} \psi_t + \varphi_t^3 b_0 + 3\varphi_t^2 \psi_t b_1 + 3\varphi_t \psi_t^2 b_2 + \psi_t^3 b_3). \end{aligned} \quad (14)$$

Two quantities play a major role in the study of Eqs. (13):

$$L_1 = -\frac{\partial \Pi_{11}}{\partial u} + \frac{\partial \Pi_{12}}{\partial \tau} - b_0 \Pi_{22} - b_2 \Pi_{11} + 2b_1 \Pi_{12},$$

$$L_2 = -\frac{\partial \Pi_{12}}{\partial u} + \frac{\partial \Pi_{22}}{\partial \tau} - b_3 \Pi_{11} - b_1 \Pi_{22} + 2b_2 \Pi_{12},$$

where

$$\Pi_{11} = 2(b_1^2 - b_2 b_0) + b_{1\tau} - b_{0u}, \quad \Pi_{22} = 2(b_2^2 - 3b_1 b_3) + b_{3\tau} - b_{2u},$$

$$\Pi_{12} = b_2 b_1 - b_3 b_0 + b_{2\tau} - b_{1u}.$$

Under point transformation (11) these components are transformed as follows [33]:

$$\tilde{L}_1 = \Delta(L_1 \varphi_t + L_2 \psi_t), \quad \tilde{L}_2 = \Delta(L_1 \varphi_x + L_2 \psi_x). \tag{15}$$

Here the tilde means that a value corresponds to system (10): the coefficients  $b_i$  are exchanged with  $a_i$ , the variables  $\tau$  and  $u$  are exchanged with  $t$  and  $x$ , respectively.

Lie [1] showed that any equation with  $L_1 = 0$  and  $L_2 = 0$  is equivalent to the equation  $u'' = 0$ . Liouville [33] also found other relative invariants, for example,

$$v_5 = L_2(L_1 L_{2\tau} - L_2 L_{1\tau}) + L_1(L_2 L_{1u} - L_1 L_{2u}) - b_3 L_1^3 + 3b_2 L_1^2 L_2 - 3b_1 L_1 L_2^2 + b_0 L_2^3$$

and

$$w_1 = L_1^{-4} \left( -L_1^3 (\Pi_{12} L_1 - \Pi_{11} L_2) + R_1 \left( L_1^2 \right)_\tau - L_1^2 R_{1\tau} + L_1 R_1 (b_1 L_1 - b_0 L_2) \right),$$

where

$$R_1 = L_1 L_{2\tau} - L_2 L_{1\tau} + b_2 L_1^2 - 2b_1 L_1 L_2 + b_0 L_2^2.$$

Notice that for the Painlevé equations  $L_1 \neq 0$  and  $L_2 = 0$ ,  $v_5 = 0$  and  $w_1 = 0$ .

*Remark 1.1* Without loss of generality one can assume that  $L_1 \neq 0$  and  $L_2 = 0$ ; otherwise a change of the dependent and independent variables such that the functions  $\varphi(t, x)$  and  $\psi(t, x)$  satisfy the equation

$$\varphi_y L_1 + \psi_y L_2 = 0$$

leads to this case. For the sake of simplicity we study equations with  $L_1 \neq 0$  and  $L_2 = 0$ .

### 1.3 General difficulties of the equivalence problem

Despite the fact that the criteria for linearizability can be simply checked, there are certain difficulties associated with finding the linearizing transformation. Let us consider a second-order ODE

$$y'' + b(x, y)y'^2 + c(x, y)y' + d(x, y) = 0, \tag{16}$$

where the coefficients satisfy the conditions

$$c_y = 2b_x, \quad d_{yy} - b_{xx} - b_x c + b_y d + d_y b = 0. \tag{17}$$

The transformation

$$t = \varphi(x), \quad u = \psi(x, y) \tag{18}$$

mapping Eq. (16) into the equation  $u'' = 0$  is found from the compatible conditions

$$\psi_{yy} = \psi_y b, \quad 2\psi_{xy} = \varphi_x^{-1} \psi_y \varphi_{xx} + c \psi_y, \quad \psi_{xx} = \varphi_x^{-1} \psi_x \varphi_{xx} + \psi_y d \tag{19}$$

and

$$\frac{2\varphi'\varphi''' - 3\varphi''^2}{\varphi'^2} = H, \quad (20)$$

where  $H = 4(d_y + bd) - (2c_x + c^2)$ . Notice that by virtue of the second equation of (17), the function  $H = H(x)$ . To solve systems (19) and (20), one must first solve Eq. (20). The change  $\varphi' = g^{-2}$  reduces Eq. (20) to the equation

$$g'' + \frac{1}{4}Hg = 0. \quad (21)$$

It is well known that the Riccati substitution

$$g' = gv$$

reduces Eq. (21) to the Riccati equation

$$v' + v^2 + \frac{1}{4}H = 0.$$

Thus, to solve Eq. (20), one must be able to solve the Riccati equation, which is not solvable in the general case.

The example presented above shows that the solution of the linearization problem is only theoretical: in many applications it becomes impossible to find the linearizing transformation. A similar problem is also encountered in finding the intermediate integral.

## 2 Existence of first integral

The existence of the first integral of the form

$$I = A(t, x) + \frac{1}{B(t, x)\dot{x} + Q(t, x)}, \quad (B \neq 0), \quad (22)$$

of a second-order equation requires that the necessary form of the equation be (10), where the coefficients are related by the equations

$$\begin{aligned} a_0 &= (Q_t - A_t Q^2)/B, & a_1 &= (B_t + Q_x - 2A_t B Q - A_x Q^2)/(3B), \\ a_2 &= (B_x - A_t B^2 - 2A_x B Q)/(3B), & a_3 &= -A_x B. \end{aligned} \quad (23)$$

The sufficient conditions for the existence of an intermediate integral of the form (22) are obtained if one considers (23) as equations for the functions  $A(t, x)$ ,  $B(t, x)$ , and  $Q(t, x)$  with given coefficients  $a_i(t, x)$ , ( $i = 0, 1, 2, 3$ ).

System (23) gives

$$A_t = B^{-1}G, \quad A_x = -B^{-1}a_3, \quad Q_t = a_0B + B^{-1}GQ^2, \quad B_t = -Q_x + 2GQ + 3a_1B - a_3B^{-1}Q^2, \quad (24)$$

where

$$G = B^{-1}(B_x - 3a_2B + 2a_3Q).$$

The function  $G(t, x)$  is introduced in order to simplify the calculations.

The equations  $(A_x)_t = (A_t)_x$  and  $(B_x)_t = (B_t)_x$  give

$$\begin{aligned} G_x &= -B^{-1}Q_x a_3 - a_{3t} + 3a_1 a_3 + 3a_2 G - B^{-2}a_3^2 Q^2 + G^2, \\ Q_{xx} &= B^{-2}(Q_x B(3a_2 B - 4a_3 Q + 3BG) - G_t B^3 + B^3(3a_{1x} - 3a_{2t} + 2a_0 a_3) \\ &\quad + (6a_2 a_3 - a_{3x})BQ^2 - 4a_3^2 Q^3 + 4a_3 B G Q^2). \end{aligned} \quad (25)$$

The equation  $(Q_{xx})_t = (Q_t)_{xx}$  becomes

$$\begin{aligned} G_{tt} &= B^{-4}(4G_t Q_x B^3 - 3G_t B^4 a_1 + 4G_t B^2 a_3 Q^2 - 2Q_x^2 B^2 G - 2G a_3^2 Q^4 \\ &\quad + 3Q_x B^3(2a_{2t} - a_{1x} - a_0 a_3) - 4Q_x B G a_3 Q^2 + B^4 G^2 a_0 + B^4 G(a_{0x} + 3a_0 a_2) \\ &\quad + B^4(a_{0t} a_3 + a_{1tx} + 3a_{1x} a_1 - 2a_{2tt} - 6a_{2t} a_1 + a_{3t} a_0 + 3a_0 a_1 a_3 - \lambda_1) \\ &\quad - 3B^2 a_3 Q^2(a_{1x} - 2a_{2t} + a_0 a_3)). \end{aligned} \quad (26)$$

The equation  $(G_{tt})_x - (G_x)_{tt} = 0$  leads to  $S = 0$ , where

$$S \equiv 12G_t Q_x B^3 G - 6G_t^2 B^4 - 6Q_x^2 B^2 G^2 + 12G_t B^2 G a_3 Q^2 - 12Q_x B G^2 a_3 Q^2 + 12(a_{1x} - 2a_{2t} + a_0 a_3) B^2 (G_t B^2 - Q_x B G - G a_3 Q^2) - 6G^2 a_3^2 Q^4 + 3B^4 G \lambda_1 - B^4 (\lambda_{1x} - 3a_2 \lambda_1 + 6(a_{1x} - 2a_{2t} + a_0 a_3)^2).$$

Furthermore, the equations

$$S_x - 6(G + a_2) S = 0, \quad B^2 S_t - 6(Q_x B - B^2 a_1 + a_3 Q^2) S = 0$$

are

$$Ba_3 Q_x = 3B^2 G \mu_1 - 5B^2 G^2 + B^2 (3a_1 a_3 - \mu_2) - a_3^2 Q^2, \tag{27a}$$

$$G_t = (15B^2 \lambda_1)^{-1} (6Q_x B \lambda_1 (5G - \mu_1) - 3B^2 G (\lambda_{1t} + 6a_1 \lambda_1) + 6a_3 \lambda_1 Q^2 (5G - \mu_1) + B^2 (\lambda_1 \mu_{1t} + 12a_{1x} \lambda_1 - 24a_{2t} \lambda_1 + \lambda_{1t} \mu_1 + 12a_0 a_3 \lambda_1 + 6a_1 \lambda_1 \mu_1)), \tag{27b}$$

where all coefficients  $\mu_i$ , ( $i = 1, 2, \dots, 7$ ) are presented in the [Appendix](#).

For further analysis one needs to consider two cases: (a)  $a_3 \neq 0$  and (b)<sup>1</sup>  $a_3 = 0$ . It is also worth noting that because of the relative invariant  $v_5$ , the property for  $a_3$  which is not equal to zero, is an invariant property of the point transformations conserving  $L_2 = 0$ .

### 2.1 Case $a_3 \neq 0$

Let  $a_3 \neq 0$ ; then Eq. (27b) gives

$$Q_x = (Ba_3)^{-1} (3B^2 G \mu_1 - 5B^2 G^2 + B^2 (3a_1 a_3 - \mu_2) - a_3^2 Q^2).$$

Thus, all first-order derivatives of the unknown functions  $A(t, x)$ ,  $B(t, x)$ ,  $Q(t, x)$ , and  $G(t, x)$  are found:

$$\begin{aligned} A_t &= B^{-1} G, & A_x &= -B^{-1} a_3, \\ Q_t &= Ba_0 + B^{-1} G Q^2, & Q_x &= (-5B^2 G^2 + 3B^2 G \mu_1 + B^2 (3a_1 a_3 - \mu_2) - a_3^2 Q^2) / (Ba_3), \\ B_t &= (5BG^2 - 3BG \mu_1 + B \mu_2 + 2Ga_3 Q) / a_3, & B_x &= BG + 3Ba_2 - 2a_3 Q, \\ G_t &= (-10G^3 + 8G^2 \mu_1 - G \mu_3 + \lambda_1 \mu_4) / a_3, & G_x &= 6G^2 + 3G (a_2 - \mu_1) - a_{3t} + \mu_2. \end{aligned} \tag{28}$$

The overdetermined system (28) is compatible if the conditions

$$\begin{aligned} (A_t)_x - (A_x)_t &= 0, & (B_t)_x - (B_x)_t &= 0, \\ (Q_x)_t - (Q_t)_x &= 0, & (G_t)_x - (G_x)_t &= 0 \end{aligned} \tag{29}$$

are satisfied. Notice also that by virtue of Eqs. (28), (24) are satisfied. Hence, it is not necessary to substitute the first-order derivatives into the intermediate Eqs. (25) and (26).

The conditions in (29) reduce to equations

$$H \equiv 12G^3 a_3 - G^2 \mu_5 - G \mu_6 - \mu_7 = 0, \tag{30a}$$

$$75G^4 - 80\mu_1 G^3 + 5q_2 G^2 - q_1 G - q_0 = 0, \tag{30b}$$

where the coefficients  $q_i$  ( $i = 0, 1, 2$ ) are presented in the [Appendix](#). Let us also add to this set of equations the following ones:

$$H_x = 0, \quad H_t = 0. \tag{31}$$

Equation (30a) is a polynomial equation of third degree with respect to  $G$ . If we exclude from Eqs. (30b) and (31) the value

$$G^3 = (G^2 \mu_5 + G \mu_6 + \mu_7) / (12a_3),$$

then Eqs. (30b) and (31) become

<sup>1</sup> This case has been studied in [23].

$$5\alpha_1 G^2 + \beta_1 G + \gamma_1 = 0, \quad \alpha_2 G^2 + \beta_2 G + \gamma_2 = 0, \quad 25\alpha_3 G^2 + \beta_3 G + 25\gamma_3 = 0, \quad (32)$$

where all coefficients  $\alpha_i$ ,  $\beta_i$ , and  $\gamma_i$ , ( $i = 1, 2, 3$ ) are presented in the [Appendix](#).

In solving Eq. (30a) with respect to  $G$ , one must also satisfy the conditions  $G_t = (G)_t$  and  $G_x = (G)_x$ . Satisfying these conditions is equivalent to satisfying Eq. (31). Thus, further study simply entails an algebraic study of Eqs. (30a) and (32). This study depends on the coefficients  $\alpha_i$ ,  $\beta_i$ , ( $i = 1, 2, 3$ ).

For example, assume that  $\alpha_1 \neq 0$ . From the first equation of (32) one finds  $G^2$ . Substituting  $G^2$  into (30a) and the remaining equations of (32), one obtains linear equations with respect to  $G$ . One needs to study resolving these linear equations with respect to  $G$ . This depends on the coefficients of these equations.

### 3 Case $G = 0$

Let us consider the case  $G = 0$  without restrictions for  $\lambda_2$ . Then

$$\begin{aligned} A_t &= 0, \quad A_x = -a_3/B, \quad Q_t = a_0 B, \\ B_t &= (-Q_x B + 3a_1 B^2 - a_3 Q^2)/B, \quad B_x = 3a_2 B - 2a_3 Q. \end{aligned} \quad (33)$$

The equations  $(A_x)_t - (A_t)_x = 0$  and  $(B_x)_t - (B_t)_x = 0$  give

$$Q_x a_3 B = -a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2, \quad (34a)$$

$$\begin{aligned} Q_{xx} B^2 + Q_x B (2a_3 Q - 3a_2 B) + B^3 (3a_{2t} - 3a_{1x} - 2a_0 a_3) \\ + B Q^2 (a_{3x} - 6a_2 a_3) + 2B^2 Q (3a_1 a_3 - a_{3t}) + 2a_3^2 Q^3 = 0. \end{aligned} \quad (34b)$$

#### 3.1 Case $a_3 \neq 0$

If  $a_3 \neq 0$ , then Eq. (34a) defines

$$Q_x = \left( -a_{3t} B^2 + 3a_1 a_3 B^2 - a_3^2 Q^2 \right) / (a_3 B). \quad (35)$$

This reduces Eq. (34b) and the equation  $(Q_x)_t - (Q_t)_x = 0$  to

$$\begin{aligned} (a_{3tx} - 3a_{2t} a_3 + 2a_0 a_3^2) a_3 - a_{3t} a_{3x} &= 0, \\ a_{3tt} - 3a_{3t} a_1 + (a_{0x} - 3a_{1t} + 3a_0 a_2) a_3 &= 0. \end{aligned} \quad (36)$$

Thus, if Eq. (10) satisfies condition (36), then the overdetermined system of equations consisting of Eqs. (33) and (35) is involutive.

For example, for  $a_3 = 1$  condition (36) can be solved as follows:

$$a_0 = 3a_{2t}/2, \quad a_1 = a_{2x}/2 + 3a_2^2/4 + \varphi,$$

where  $\varphi(x)$  is an arbitrary function. This means that all equations of the form

$$\ddot{x} + \dot{x}^3 + 3a_2 \dot{x}^2 + \left( a_{2x}/2 + 3a_2^2/4 + \varphi \right) \dot{x} + 3a_{2t}/2 = 0 \quad (37)$$

with arbitrary functions  $a_2(t, x)$  and  $\varphi(x)$  have the intermediate integral

$$I = A + \frac{1}{B(\dot{x} + \frac{3}{2}a_2) + H},$$

where the functions  $A(x)$ ,  $B(x)$ , and  $H(x)$  are solutions of the equations

$$A' = -1/B, \quad B' = -H, \quad H' = 3B\varphi - H^2/B.$$

Notice that for  $a_{2t} = 0$  Eq. (36) can be reduced to a first-order ODE by the standard change  $\dot{x} = y(x)$ , whereas for  $a_{2t} \neq 0$  this technique is not applicable.

### 3.2 Case $a_3 = 0$

In this case, Eq. (34a) is satisfied and Eq. (34b) becomes

$$Q_{xx} = 3Q_x a_2 - 3B(a_{2t} - a_{1x}). \tag{38}$$

The equation  $(Q_{xx})_t - (Q_t)_{xx} = 0$  gives

$$3Q_x \eta = B(\eta_t + 3a_1 \eta - \lambda_1), \tag{39}$$

where  $\eta = a_{1x} - 2a_{2t}$ . Hence, for  $\eta = 0$  one has that  $\lambda_1 = 0$ , and there are no other additional equations for the functions  $A(t, x)$ ,  $B(t, x)$ , and  $Q(t, x)$ . This means that the system of equations consisting of Eqs. (33) and (38) is involutive. If  $\eta \neq 0$ , then one can find  $Q_x$ . The equations  $(Q_x)_t - (Q_t)_x = 0$  and  $(Q_{xx})_x - (Q_x)_{xx} = 0$  give the conditions

$$\begin{aligned} 3\eta\eta_{tt} &= 4\eta_t^2 - 3\eta_t\eta a_1 + 15\eta_t\lambda_1 + 9\eta^2(a_{0x} - a_{1t} + 3a_0a_2 - 2a_1^2) - 9\eta(\lambda_{1t} + a_1\lambda_1) + 9\lambda_1^2, \\ \eta\eta_{tx} &= \eta_t\eta_x + 3\eta_x\lambda_1 - 2\eta^3 + 3\eta^2a_{2t} - 3\eta\lambda_{1x}. \end{aligned} \tag{40}$$

Thus, if Eq. (10) satisfies condition (40), then the overdetermined system of equations consisting of Eqs. (33) and (39) is involutive.

## 4 Examples

In this section we consider examples of first integrals of the form

$$I = \frac{A(t, x)\dot{x} + B(x, t)}{\dot{x} + Q(x, t)}. \tag{41}$$

*Example 4.1* The most general equation associated with the first integral  $I$  in (41) is given by

$$\begin{aligned} \ddot{x} + \frac{A_x}{\Delta}\dot{x}^3 + \frac{1}{\Delta}(A_t + B_x + A_x Q - A Q_x)\dot{x}^2 \\ + \frac{1}{\Delta}(B_t + A_t Q - A Q_t + B_x Q - B Q_x)\dot{x} + \frac{1}{\Delta}(Q B_t - B Q_t) = 0, \end{aligned} \tag{42}$$

where  $\Delta = A Q - B$ .

*Proof* Rearranging (41) we obtain

$$I(\dot{x} + Q(x, t)) = A(t, x)\dot{x} + B(x, t),$$

which immediately yields

$$I(\ddot{x} + Q_x \dot{x} + Q_t) = A\ddot{x} + A_x \dot{x}^2 + A_t \dot{x} + B_x \dot{x} + B_t,$$

$$(A\dot{x} + B)(\ddot{x} + Q_x \dot{x} + Q_t) = (\dot{x} + Q(x, t))(A\ddot{x} + A_x \dot{x}^2 + A_t \dot{x} + B_x \dot{x} + B_t). \quad \square$$

This equation is closely related to the (unparameterized) geodesic equations of some connection  $\Gamma$  on  $U \in \mathbf{R}^2$

$$\ddot{x}^c + \Gamma_{ab}^c \dot{x}^a \dot{x}^b = v \dot{x}^c$$

for  $x^a(t) = (x(t), y(t))$ . Eliminating the parameter  $t$  yields the second-order ODE for  $y$  as a function of  $x$

$$\frac{d^2 y}{dx^2} = a(x, y) \left(\frac{dy}{dx}\right)^3 + b(x, y) \left(\frac{dy}{dx}\right)^2 + c(x, y) \left(\frac{dy}{dx}\right) + d(x, y) = 0, \tag{43}$$

where

$$a(x, y) = -\Gamma_{11}^2, \quad b(x, y) = \Gamma_{11}^1 - 2\Gamma_{12}^2, \quad c(x, y) = 2\Gamma_{12}^1 - \Gamma_{22}^2, \quad d(x, y) = \Gamma_{22}^1.$$

<sup>2</sup> Notice that for Eq. (10), which is not linearizable, one can assume without loss of generality that  $\lambda_1 \neq 0$ .

In other words, any second-order ODEs with cubic nonlinearity in the first derivatives of the form (42) gives rise to some projective structures.

*Example 4.2* A quasimonomial  $q$  over  $\mathbb{K}$  is defined as

$$q = \mathbf{x}^c = \prod_{i=1}^n x_i^{c_i}, \quad c_i \in \mathbb{K}.$$

A quasimonomial function is a finite sum of quasimonomials  $f : \mathbb{C} \rightarrow \Sigma$ , where  $\Sigma = \mathbb{C} \cup \{\infty\}$ , defined as

$$\mathbf{x} \rightarrow \sum a_i \prod_{j=1}^n x_j^{c_{ij}}.$$

We assume  $A = x^\alpha t^\beta$ ,  $B = x^\gamma t^\delta$  and  $Q = 1$  in (42) to obtain the second-order equation

$$\ddot{x} + \frac{\alpha}{x(1-x^{\gamma-\alpha})} \dot{x}^3 + \frac{\alpha + \beta \frac{x}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})} \dot{x}^2 + \frac{\beta \frac{x}{t} + \beta \frac{x^{\gamma-\alpha+1}}{t} + \gamma x^{\gamma-\alpha}}{x(1-x^{\gamma-\alpha})} \dot{x} + \frac{\beta \frac{x^{\gamma-\alpha+1}}{t}}{x(1-x^{\gamma-\alpha})} = 0, \quad (44)$$

which admits the first integral

$$I = \frac{x^\alpha t^\beta (\dot{x} + x^{\gamma-\alpha})}{\dot{x} + 1}.$$

#### 4.1 Claim

Setting  $\alpha = -1$ ,  $\beta = 1 = \delta$ ,  $\gamma = 0$  we obtain the first integral of

$$\ddot{x} + \frac{1}{x(x-1)} \dot{x}^3 + \left( \frac{1}{x(x-1)} + \frac{1}{t(1-x)} \right) \dot{x}^2 + \frac{1}{t} \left( \frac{1+x}{1-x} \right) \dot{x} + \frac{x}{t(1-x)} = 0 \quad (45)$$

as

$$I = \left( \frac{t}{x} \right) \frac{\dot{x} + x}{\dot{x} + 1}.$$

*Example 4.3* The first integral of the second-order equation

$$\ddot{x} + \frac{t}{x^2(x-1)} \dot{x}^3 + \left( \frac{2}{1-x} - \frac{(1+x)t}{(1-x)x^2} + \frac{1}{x} \right) \dot{x}^2 + \left( \frac{1}{1-x} + \frac{1-t}{(1-x)x} \right) \dot{x} + \frac{1}{1-x} = 0 \quad (46)$$

is

$$I = e^{t/x} \frac{\dot{x} + x}{\dot{x} + 1}.$$

*Example 4.4* Let us set  $A = Q^{-1} = e^{\alpha(x)t}$  and  $B = b$  (constant) in (41). Then we obtain the equation

$$(1-b)\ddot{x} + e^{\alpha(x)t} (\alpha'(x)t\dot{x} + \alpha(x))\dot{x}^2 + 2(\alpha'(x)t\dot{x} + \alpha(x))\dot{x} + e^{-\alpha(x)t} (\alpha'(x)t\dot{x} + b\alpha(x)) = 0. \quad (47)$$

The corresponding first integral is

$$I = \frac{e^{\alpha(x)t} \dot{x} + b}{\dot{x} + e^{-\alpha(x)t}}.$$

### 4.2 Time-independent case

Consider  $A_t = B_t = Q_t = 0$ . Thus Eq. (41) becomes

$$\ddot{x} + \frac{A_x}{\Delta} \dot{x}^3 + \frac{1}{\Delta} (B_x + A_x Q - A Q_x) \dot{x}^2 + \frac{1}{\Delta} (A_t Q - A Q_t + B_x Q - B Q_x) \dot{x} = 0, \tag{48}$$

which can be expressed as

$$\dot{x} = y, \quad \dot{y} = -\frac{1}{\Delta} (A_x y^3 + (B_x + A_x Q - A Q_x) y^2 + (B_x Q - B Q_x) y). \tag{49}$$

This yields the flow equation

$$\frac{dy}{dx} = -\frac{1}{\Delta} (A_x y^2 + (B_x + A_x Q - A Q_x) y + (B_x Q - B Q_x)).$$

Assume  $A = 1$ ; thus,  $\Delta = Q - B$ . The flow becomes

$$\frac{dy}{dx} - \frac{\Delta'(x)}{\Delta(x)} y = \frac{B^2}{\Delta} \frac{d}{dx} \left( \frac{Q}{B} \right).$$

This immediately yields

$$y = -\frac{1}{Q/B - 1} + C_1.$$

Hence we obtain

$$t = \int \frac{dx}{C_1 Q - (C_1 + 1)B(x)}.$$

### 4.3 Reduction

Let  $A_x = 0$  and set

$$\frac{1}{\Delta} (A_t + B_x - A Q_x) = b(x, t) = \frac{1}{2} \phi_x, \quad \frac{1}{\Delta} (B_t + A_t Q - A Q_t + B_x Q - B Q_x) = c(x, t) = \phi_t. \tag{50}$$

A large number of second-order ODEs in the Painlevé–Gambier classification system belong to the following class of equations:

$$\ddot{x} + \frac{1}{2} \phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x) = 0.$$

This equation yields the Lagrangian description via Jacobi’s last multiplier. If we write this equation in the form

$$\ddot{x} = \mathcal{F}(t, x, \dot{x}) = -\left[ \frac{1}{2} \phi_x \dot{x}^2 + \phi_t \dot{x} + B(t, x) \right],$$

then the Jacobi last multiplier  $M$  is given by the solution of

$$\frac{d}{dt} \ln M = -\frac{\partial \mathcal{F}}{\partial \dot{x}}.$$

In the present case we have

$$M = \frac{\partial^2 L}{\partial \dot{x}^2} = e^{\phi(t,x)}.$$

We then obtain the Lagrangian as

$$L(t, x, \dot{x}) = e^{\phi(t,x)} \frac{\dot{x}^2}{2} + f_1(t, x) \dot{x} + f_2(t, x).$$

#### 4.4 Conditions for Lagrangians

Let us express  $\phi$  in terms  $A$ ,  $B$ ,  $Q$  and find the conditions for the Lagrangian. Defining

$$\phi_x = \frac{2}{\Delta} (A_t + B_x - A Q_x),$$

$$\phi_t = \frac{1}{\Delta} (B_t + A_t Q - A Q_t + B_x Q - B Q_x)$$

immediately yields

$$\phi_{xt} = \frac{2}{\Delta} (A''(t) + B_{xt} - A'(t) Q_x - A Q_{xt}) - \frac{2}{\Delta^2} (A'(t) Q + A Q_t - B_t) (A'(t) + B_x - A Q_x),$$

$$\phi_{tx} = \frac{1}{\Delta} (B_{tx} + A' Q_x - A Q_{tx} + B_{xx} Q - B Q_{xx}) - \frac{1}{\Delta^2} (A Q_x - B_x) (B_t + A' Q - A Q_t + B_x Q - B Q_x).$$

#### 4.5 Claim

A second-order nonlinear equation of the form

$$\ddot{x} + b(x, t)\dot{x}^2 + c(x, t)\dot{x} + d(x, t) = 0$$

admits a Lagrangian provided

$$2A'' + B_{xt} - 3A' Q_x - A Q_{xt} - B_{xx} Q + B Q_{xx} (A Q - B) \\ = A' Q (2A' + 3B_x - 3A Q_x) + (A Q_t - B_t) (2A' + B_x - A Q_x) - (A Q_x - B_x) (B_x Q - B Q_x),$$

where  $b(x, t)$  and  $c(x, t)$  are defined as (8) and  $d(x, t) = \frac{1}{\Delta} (Q B_t - B Q_t)$ .

**Outline of proof.** It follows analogously to the argument in Sect. 4.3 and makes use of the compatibility condition  $\phi_{xt} = \phi_{tx}$ .  $\square$

*Example 4.5* Set  $A = 1$ ,  $\Delta = A Q - B = x^\alpha t^\beta$ , and assume  $Q = x^\gamma$  in (51); the equation becomes

$$\ddot{x} - \frac{\alpha}{x} \dot{x}^2 + \left( (\gamma - \alpha) x^{\gamma-1} - \frac{\beta}{t} \right) \dot{x} - \frac{\beta x^\gamma}{t} = 0 \quad (51)$$

whose first integral is

$$I = \frac{\dot{x} + x^\gamma - x^\alpha t^\beta}{\dot{x} + x^\gamma}.$$

Let us find the conditions  $\gamma$  and  $\alpha$  for which Eq. (51) gives a Lagrangian description.

#### 4.6 Claim

For  $\gamma = \alpha$  or  $\gamma = 1$  Eq. (51) yields a Lagrangian description.

*Proof* Equate  $\frac{1}{2}\phi_x = -\frac{\alpha}{x}$  and  $\phi_t = (\gamma - \alpha)x^{\gamma-1} - \frac{\beta}{t}$ . This immediately yields  $\phi_{xt} = 0$  and  $\phi_{tx} = (\gamma - \alpha)(\gamma - 1)x^{\gamma-2}$ . Thus from the compatibility condition we obtain our criteria.  $\square$

## 5 Conclusion

Any second-order ODE that possesses a first integral of the form (1) must be cubic with respect to the first-order derivative (10). This paper gives complete criteria of the existence of a first integral of the form (1) for a second-order ODE (10), which is reduced to an equation with  $L_2 = 0$ . Despite the fact that any second-order ODE (10) can be reduced to an equation with  $L_2 = 0$ , the complete solution of the problem requires that sufficient conditions be given using coefficients of the original equation (not reduced). This is still an open problem.

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**Appendix**

The following coefficients  $\mu_i$  appear in Eqs. (27a) and (27b) in Sect. 2:

$$\begin{aligned} \mu_1 &= (\lambda_{1x} - 3a_2\lambda_1) / \lambda_1, \\ \mu_2 &= (\mu_{1x} + 3a_{3t} - 3a_2\mu_1 + \mu_1^2) / 3, \\ \mu_3 &= (\lambda_{1t}a_3 - 24a_1a_3\lambda_1 + 6\lambda_1\mu_1^2 + 10\lambda_1\mu_2) / (5\lambda_1), \\ \mu_4 &= (\mu_{1t}a_3 + 12a_3(a_{1x} - 2a_{2t} + a_0a_3 + a_1\mu_1) - 6\mu_1^3 - 4\mu_1\mu_2 + 5\mu_1\mu_3) / (15\lambda_1), \\ \mu_5 &= a_{3x} - 6a_2a_3 + 10a_3\mu_1, \\ \mu_6 &= (a_{3t}a_3 - 6a_1a_3^2 + 18a_3\mu_1^2 + a_3\mu_2 - a_3\mu_3 - 3\mu_1\mu_5) / 5, \\ \mu_7 &= (3a_{2t}a_3^2 - \mu_{2x}a_3 - 2a_0a_3^3 - 18a_1a_3^2\mu_1 + 6a_2a_3\mu_2 + a_3\lambda_1\mu_4 + 54a_3\mu_1^3 \\ &\quad - 4a_3\mu_1\mu_2 - 3a_3\mu_1\mu_3 - 9\mu_1^2\mu_5 - 15\mu_1\mu_6 + \mu_2\mu_5) / 5. \end{aligned}$$

In addition, the coefficients  $\alpha_i, \beta_i, \gamma_i$  appear in Sect. 2.1 and Eq. (32):

$$\begin{aligned} \alpha_1 &= 720a_3^2\mu_1^2 - 432a_1a_3^3 + 144a_3^2\mu_2 - 144a_3^2\mu_3 - 80a_3\mu_1\mu_5 - 300a_3\mu_6 - 5\mu_5^2, \\ \beta_1 &= 1728a_{1x}a_3^3 - 3456a_{2t}a_3^3 + 1776a_0a_3^4 + 3024a_1a_3^3\mu_1 - 1680a_3^2\lambda_1\mu_4 - 3456a_3^2\mu_1^3 \\ &\quad - 1008a_3^2\mu_1\mu_2 + 1008a_3^2\mu_1\mu_3 + 432a_3\mu_1^2\mu_5 + 1040a_3\mu_1\mu_6 - 300a_3\mu_7 - 25\mu_5\mu_6, \\ \gamma_1 &= 48a_{0x}a_3^4 - 144a_{1t}a_3^4 + 48\mu_{2t}a_3^3 + 144a_0a_2a_3^4 - 432a_1a_3^3\mu_2 - 144a_3^2\lambda_1\mu_1\mu_4 + 96a_3^2\mu_2^2 \\ &\quad + 864a_3^2\mu_1^2\mu_2 - 48a_3^2\mu_2\mu_3 - 144a_3\mu_1\mu_2\mu_5 + 320a_3\mu_1\mu_7 - 240a_3\mu_2\mu_6 - 25\mu_5\mu_7, \\ \alpha_2 &= -72\mu_{5t}a_3^3 + 432a_1a_3^3\mu_5 + 2592a_3^3\lambda_1\mu_4 - 1296a_3^2\mu_1^2\mu_5 + 1152a_3^2\mu_1\mu_6 - 72a_3^2\mu_2\mu_5 \\ &\quad - 2160a_3^2\mu_7 + 264a_3\mu_1\mu_5^2 + 180a_3\mu_5\mu_6 - 5\mu_5^3, \\ \beta_2 &= -72\mu_{6t}a_3^3 + 432a_1a_3^3\mu_6 - 144a_3^2\lambda_1\mu_4\mu_5 - 1296a_3^2\mu_1^2\mu_6 + 1728a_3^2\mu_1\mu_7 \\ &\quad - 72a_3^2\mu_2\mu_6 - 72a_3^2\mu_3\mu_6 + 264a_3\mu_1\mu_5\mu_6 - 60a_3\mu_5\mu_7 + 240a_3\mu_6^2 - 5\mu_5^2\mu_6, \\ \gamma_2 &= -72\mu_{7t}a_3^3 + 432a_1a_3^3\mu_7 - 72a_3^2\lambda_1\mu_4\mu_6 - 1296a_3^2\mu_1^2\mu_7 - 72a_3^2\mu_2\mu_7 - 144a_3^2\mu_3\mu_7 \\ &\quad + 264a_3\mu_1\mu_5\mu_7 + 240a_3\mu_6\mu_7 - 5\mu_5^2\mu_7, \\ \alpha_3 &= -2\mu_{5x}a_3 - 432a_1a_3^3 + 18a_2a_3\mu_5 + 1296a_3^2\mu_1^2 + 144a_3^2\mu_2 - 72a_3^2\mu_3 \\ &\quad - 242a_3\mu_1\mu_5 - 336a_3\mu_6 + 3\mu_5^2, \\ \beta_3 &= -1212a_{1x}a_3^3 + 2424a_{2t}a_3^3 - 10\mu_{5t}a_3^3 + 30\mu_{5x}a_3\mu_1 - 1244a_0a_3^4 + 4644a_1a_3^3\mu_1 \\ &\quad + 120a_1a_3^2\mu_5 - 270a_2a_3\mu_1\mu_5 + 1540a_3^2\lambda_1\mu_4 - 18336a_3^2\mu_1^3 - 1548a_3^2\mu_1\mu_2 + 468a_3^2\mu_1\mu_3 \\ &\quad + 3312a_3\mu_1^2\mu_5 + 5090a_3\mu_1\mu_6 - 30a_3\mu_2\mu_5 + 20a_3\mu_3\mu_5 + 850a_3\mu_7 + 75\mu_5\mu_6, \\ \gamma_3 &= -2\mu_{7x}a_3 + 12a_1a_3^2\mu_6 + 30a_2a_3\mu_7 - 36a_3\mu_1^2\mu_6 - 38a_3\mu_1\mu_7 - 4a_3\mu_2\mu_6 \\ &\quad + 2a_3\mu_3\mu_6 + 6\mu_1\mu_5\mu_6 + 3\mu_5\mu_7 + 10\mu_6^2. \end{aligned}$$

The  $q_i$  that follow appear in Eq. (30b):

$$\begin{aligned} q_0 &= a_{0x}a_3^2 - 3a_{1t}a_3^2 - a_{3t}\mu_2 + \mu_{2t}a_3 + 3a_0a_2a_3^2 - 3a_1a_3\mu_2 - 3\lambda_1\mu_1\mu_4 + \mu_2^2, \\ q_1 &= 36a_{1x}a_3 - 72a_{2t}a_3 + 3a_{3t}\mu_1 + 37a_0a_3^2 + 45a_1a_3\mu_1 - 35\lambda_1\mu_4 - 18\mu_1(\mu_1^2 + \mu_2 - \mu_3), \\ q_2 &= a_{3t} + 3a_1a_3 + 3\mu_1^2 - 2\mu_2 + 2\mu_3. \end{aligned}$$

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## Group analysis of evolutionary integro-differential equations describing nonlinear waves: the general model

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# Group analysis of evolutionary integro-differential equations describing nonlinear waves: the general model

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## Abstract

The paper deals with an evolutionary integro-differential equation describing nonlinear waves. A particular choice of the kernel in the integral leads to well-known equations such as the Khokhlov–Zabolotskaya equation, the Kadomtsev–Petviashvili equation and others. Since the solutions of these equations describe many physical phenomena, the analysis of the general model studied in this paper is important. One of the methods for obtaining solutions of differential equations is provided by the Lie group analysis. However, this method is not applicable to integro-differential equations. Therefore, we discuss new approaches developed in modern group analysis and apply them to the general model considered in this paper. Reduced equations and exact solutions are also presented.

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## 1. Introduction

One of the most general evolution equations used in nonlinear wave physics is as follows [1, 2]:

$$\begin{aligned} (u_x - uu_t - w_{tt})_t &= u_{yy} + u_{zz}, \\ w &= \int_0^\infty K(s)u(t-s) ds. \end{aligned} \tag{1}$$

Here the variable  $t$  is the time, and  $x, y, z$  are the spatial Cartesian coordinates. The coordinate  $x$  is distinguished as a ‘longitudinal’ one. It coincides with a preferred orientation of the wave propagation. The other coordinates  $y, z$  are identified as ‘transversal’ ones. They are commonly introduced in the cross-section of a wave beam.

Special cases of equation (1) are well known. In particular, if the kernel is identically zero,  $K(s) \equiv 0$ , the general equation (1) is reduced to the Khokhlov–Zabolotskaya (KZ) equation [3, 4], describing wave beams in nonlinear media:

$$(u_x - uu_t)_t = u_{yy} + u_{zz}. \tag{2}$$

If the kernel is the delta-function,  $K = 2\delta(s)$ , model (1) leads to the equation

$$(u_x - uu_t - u_{tt})_t = u_{yy} + u_{zz} \tag{3}$$

for nonlinear beams in a dissipative medium [5, 6]. Equation (3) is known as the Khokhlov–Zabolotskaya–Kuznetsov (KZK) equation. It is widely used in underwater acoustics for the engineering design of parametric radiating and receiving arrays [6].

If the kernel is proportional to the derivative of the delta-function,  $K = 2\delta'(s)$ , the integro-differential equation (1) becomes the Kadomtsev–Petviashvili (KP) equation

$$(u_x - uu_t - u_{ttt})_t = u_{yy} + u_{zz} \tag{4}$$

for nonlinear beams in a dispersive medium [7, 8]. The similar equation

$$(u_x - uu_t - u_{tttt})_t = u_{yy} + u_{zz} \tag{5}$$

for a scattering medium [9] follows from (1) when  $K = 2\delta''(s)$ .

There exist other models that specify or generalize equation (1), e.g. by including (1) in a coupled systems of nonlinear equations [10, 11].

If the wave field  $u = u(t, x)$  is a function of a single spatial (longitudinal) coordinate  $x$  and does not depend on the transverse coordinates  $y, z$ , equation (1) is reduced to well-known equations for plane waves [12]. In particular, the Riemann–Hopf equation follows from (2), the Burgers equation follows from (3) and the Korteweg–de Vries equation follows from (4). The one-dimensional equation with fourth-order derivative for the scattering medium suggested and solved in [9] follows from equation (5). 1D equations can be obtained by eliminating the  $y, z$  derivatives of 3D equations and the subsequent integration over  $dt$ , provided that the wave field vanishes at  $t \rightarrow \pm\infty$ .

A choice of the kernel as a linear combination of the delta-function and its derivatives of different orders gives a possibility of deriving from (1) various well-known differential equations of the physics of nonlinear waves. Symmetries of such equations either have already been studied (many results obtained until 1995 are collected in [13–15]) or can be studied by the standard Lie group methods [16–18]. However, to the best of our knowledge, particular versions of the general equation (1) with non-degenerate kernels which maintain the integro-differential feature of the model have not been studied yet.

The exponential kernel  $K = \exp(-s)$  is of particular applied interest. Equation (1) with such kernel describes wave beams in relaxing media. In this case, the integro-differential equation is also reduced to a differential equation [1]. To derive such an equation, it is sufficient to note that the integral term  $w$  in (1) and the variable  $u$  for the exponential kernel are related by the following equation:

$$w_t + w = u. \tag{6}$$

Reduction of (1) to a differential equation is also possible for some more complicated kernels. For example, if  $K = \exp(-s)\cos(\omega_0 s)$ , then the kernel describes the internal

dynamics of the medium with resonant inclusions. In this case, the differential relation between  $w$  and  $u$  in equation (1) has the form

$$(w_t + w - u)_t + (w_t + w - u) + \omega_0^2 w = 0. \tag{7}$$

A special class is formed by ‘model’ kernels which are non-zero on the finite segment, for example, within  $s \in (0, 1]$ . The simplest case is

$$K = \begin{cases} 1 & \text{if } s \leq 1, \\ 0 & \text{if } s > 1. \end{cases}$$

For this kernel the integro-differential equation (1) is reduced to the difference-differential equation

$$\begin{aligned} (u_x - uu_t - \Delta u_t)_t &= u_{yy} + u_{zz}, \\ \Delta u &\equiv u(t) - u(t - 1). \end{aligned} \tag{8}$$

Note that, using the finite shift operator, one rewrites the integral term of equation (1) in the form

$$w(t) = \hat{L}u(t), \quad \hat{L}(\partial_t) = \int_0^\infty K(s) \exp(-s \partial_t) ds, \tag{9}$$

where  $\partial_t$  is the partial derivative with respect to time. The second operator  $\hat{L}(\partial_t)$  is the Laplace transform of the function  $K(s)$  defining the kernel of equation (1).

For example, if a kernel has the form of the Bessel function of zero order, then one has

$$K = J_0(s), \quad \hat{L}(\partial_t) = (1 + \partial_t^2)^{-1/2}.$$

Using the tables of the Laplace transform and physical restrictions of the kernel forms, one can single out all cases when equation (1) can be reduced to a differential equation of finite order. In the general case, decomposing the exponential function of the integrand (9) into power series, one verifies that the resulting differential equation will contain derivatives of an arbitrary order.

The discussion of the properties of a kernel is continued below by considering the formulations of main physical problems.

## 2. Physical statement and main physical parameters

For definiteness, a concrete physical object is considered which is most simple and, at the same time, can be adequately described by models like (1). Namely, we will deal with high-intensity acoustic waves. The general equation (1), as well as majority of the particular models (2)–(7), has been written at first for nonlinear acoustic waves.

Note that equation (1) is written in certain dimensionless variables in order to reduce all coefficients of the equation into unity. To discuss a physical meaning of mathematical models, we rewrite equation (1) using initial physical notations:

$$\frac{\partial}{\partial \tau} \left[ \frac{\partial p}{\partial x} - \frac{\varepsilon}{c^3 \rho} p \frac{\partial p}{\partial \tau} - W \right] = \frac{c}{2} \Delta_\perp p, \tag{10}$$

$$\begin{aligned} W &= \frac{m}{2c} \frac{\partial}{\partial \tau} \int_{-\infty}^\tau K \left( \frac{\tau - \tau'}{t_0} \right) \frac{\partial p}{\partial \tau'} d\tau' \\ &= \frac{m}{2c} \frac{\partial^2}{\partial \tau^2} \int_0^\infty K \left( \frac{\xi}{t_0} \right) p(\vec{r}, \tau - \xi) d\xi. \end{aligned} \tag{11}$$

Here  $x$  is the coordinate along the direction of wave propagation,  $\Delta_{\perp}$  is the Laplace operator written in the coordinates  $y, z$  on the orthogonal plane,  $\tau = t - x/c$  is the time in the moving system of coordinates propagating with the sound velocity  $c$ ,  $\varepsilon$  is the parameter of nonlinearity and  $\rho$  is the density of a medium. The acoustic pressure  $p$  is chosen as the wave field variable. The constant  $m$  characterizes the ‘force’ of time delay processes and  $t_0$  is the typical ‘memorizing time’ of a medium.

Let us note an important point before passing to further discussion. The natural question arises: why is the coordinate  $x$  instead of time  $t$  used as a ‘slow’ (evolutionary) variable in equations (1) and (10)? The answer is that the difference between  $x$  and  $t$  depends only on the way of description which depends on the statement of problem and ease of analysis of results. In the case of non-wave problems (e.g. description of turbulence), the problem is posed as follows. At the initial moment  $t = 0$ , a distribution of the velocity field in space  $u(t = 0, x) = u_0(x)$  is given, and the solution  $u(t, x)$  is sought with growing time  $t > 0$ . In the corresponding experiment, sensors measuring the velocity field are placed in various locations, and the measurement is made by all sensors at the same time  $t_1$ . These results determine the spatial structure of the field  $u(t_1, x)$ . Then similar measurements performed at  $t_2$  give the field profile  $u(t_2, x)$ . Repeating the measurements we trace the field evolution with respect to time.

When propagating waves are of interest, the experiment is performed in a different way. The only sensor placed at the position  $x_1$  measures the variation of the signal with respect to time:  $u(x_1, t)$ . Then the sensor is moved to another position  $x_2 > x_1$  and the signal  $u(x_2, t)$  is measured. By moving the sensor of the vibration velocity (or the acoustic pressure) farther and farther from the source of wave, we trace the evolution of the form of the wave profile as the wave propagates. In real experiments a wave gets distorted at distances of the order of thousand wavelengths, whereas, for a good reconstruction of the wave profile within its each length  $\lambda$ , one has to place no fewer than ten sensors. In this case, the method of the ‘slow time’ is very inconvenient. Moreover, this method is completely inappropriate in those cases when the wave profile contains shock fronts whose extent is very small, e.g.  $10^{-4}\lambda$ .

But in various acoustic problems, e.g. those dealing with standing waves in a resonator, it is convenient to utilize the ‘slow time’ instead of the ‘slow coordinate’. It is clear that the resonator has a limited length and, by measuring the field at the lowest modes, it is quite realistic to place several sensors along the length of the resonator and perform simultaneous measurement with them at various moments of time.

Let us return to the physical model (10), (11). The integration within the limits  $-\infty < \tau' < \tau$  in the first integral (11) means that the wave behavior at a given moment  $\tau$  is determined by the values of the field variable at the preceding moments from  $\tau$  to the infinitely distant past. Consequently, the kernel  $K(\tau)$  describing the ‘memory’ of a medium must be nonzero only at positive values of its argument and tend to zero for  $\tau \rightarrow +\infty$ . This decrease can be non-monotone and can look like oscillatory damping (see the example leading to formula (7)).

In order to understand how the concrete form of the kernel is related to the measured characteristics of the medium, we shall consider the simplest model of a plane wave moving in a medium without nonlinearity. In other words, let us consider the equation

$$\frac{\partial p}{\partial x} - \frac{m}{2c} \frac{\partial^2}{\partial \tau^2} \int_0^{\infty} K\left(\frac{\xi}{t_0}\right) p(x, \tau - \xi) d\xi = 0. \tag{12}$$

Let us establish a relation of a kernel with the dispersion law. A solution is sought in the form

$$p = \exp(-i\omega t + ik x), \quad k = k' + ik''. \tag{13}$$

Here  $k$  is the wave number, and  $k'$  and  $k''$  are its real and imaginary parts, respectively. Substituting (13) into (12), we find

$$\begin{aligned} k' &= -\frac{m\omega^2 t_0}{2c} \int_0^\infty K(s) \sin(\omega t_0 s) ds, \\ k'' &= \frac{m\omega^2 t_0}{2c} \int_0^\infty K(s) \cos(\omega t_0 s) ds. \end{aligned} \tag{14}$$

The first formula in (14) gives a frequency-dependent addition to the velocity of the wave propagation:  $c(\omega) = c(1 - ck'(\omega)/\omega)$ . The second one defines the absorption coefficient or the law of spatial decrease of the wave amplitude:  $p_0 \exp(-k''x)$ .

Evidently, integrals (14) must be convergent for physically feasible kernels. The concrete form of a kernel can be reconstructed on the base of the corresponding physical model, or on the base of experimental measurements.

A relaxing medium provides an important model known as the Mandelstam–Leontovich model (see [1, 19]). The kernel for this model has the exponential form (see the example leading to formula (6)). In this case,

$$k' = -\frac{m\omega}{2c} \frac{(\omega t_0)^2}{1 + (\omega t_0)^2}, \quad k'' = \frac{m\omega}{2c} \frac{\omega t_0}{1 + (\omega t_0)^2}. \tag{15}$$

The frequency dependences (15) of the dispersion  $k'$  and the absorption  $k''$  were confirmed repeatedly in experiments. One could proceed in an opposite way. First, establish the dependences (15) as an empirical generalization of measured data, and then reconstruct the kernel by means of a standard procedure. This procedure exploits the causality principle according to which two functions  $k'$  and  $k''$  cannot be arbitrary but should be connected by the relations of Kramers–Kronig’s type [12].

The method of kernel reconstruction has been utilized for deriving mathematical models used in the medical applications of ultrasound [20]. It is known that, within the most interesting frequency range, the absorption of the ultrasound in soft tissues behaves like  $k'' \sim \omega^{2-\nu}$ ,  $0 < \nu < 1$ . It is easy to reconstruct the kernel  $K(s) = s^{\nu-1}$  and verify that the corresponding absorption coefficient

$$k'' = \frac{m}{2ct_0} \Gamma(\nu) \cos\left(\frac{\pi}{2}\nu\right) (\omega t_0)^{2-\nu} \tag{16}$$

has the correct frequency dependence. Note that the considered power kernel has a singularity at  $s = 0$  and is not integrable in semi-infinite limits. However, the convolution of this kernel with the oscillating function describing a wave provides the convergence of the integral for  $k''$ . This example demonstrates a wide variety of situations which can be met in applications.

In the conclusion of this section we demonstrate how one has to change variables in equations (10) and (11) to reduce these to the simplest normalized form (1). One has to set

$$\tau \rightarrow t_0 t, \quad p \rightarrow p_0 u, \quad x \rightarrow x_0 x, \quad y \rightarrow y_0 y, \quad z \rightarrow z_0 z, \tag{17}$$

where the constant  $t_0$  (the ‘memory’ time) is defined by the structure of kernel and the other constants are

$$p_0 = \frac{m}{2\varepsilon} c^2 \rho, \quad x_0 = \frac{2}{m} c t_0, \quad y_0 = z_0 = \frac{c t_0}{\sqrt{m}}. \tag{18}$$

### 3. Admitted Lie group

As for differential equations, an admitted Lie group of integro-differential equation (1) is defined by the determining equations. These equations are integro-differential equations for the coordinates of the infinitesimal generator

$$X = \xi^t \partial_t + \xi^x \partial_x + \xi^y \partial_y + \xi^z \partial_z + \zeta^u \partial_u + \zeta^w \partial_w,$$

where the coordinates  $\xi^t, \xi^x, \xi^y, \xi^z, \zeta^u$  and  $\zeta^w$  are functions depending on the variables  $(t, x, y, z, u, w)$ . The system (1) comprises a partial differential equation and an integro-differential equation. The determining equation related to the partial differential equation is obtained by the standard procedure:

$$\zeta^{u_{tx}} - u \zeta^{u_{tt}} - 2u_t \zeta^{u_t} - \zeta^{w_{ttt}} = \zeta^{u_{yy}} + \zeta^{u_{zz}}, \tag{19}$$

where the coefficients  $\zeta^{u_{tx}}, \zeta^{u_{tt}}, \zeta^{u_t}, \zeta^{w_{ttt}}, \zeta^{u_{yy}}$  and  $\zeta^{u_{zz}}$  are the coefficients of the prolonged generator  $X$ :

$$\bar{X} = X + \zeta^{u_{tx}} \partial_{u_{tx}} + \zeta^{u_{tt}} \partial_{u_{tt}} + \zeta^{u_t} \partial_{u_t} + \zeta^{w_{ttt}} \partial_{w_{ttt}} + \zeta^{u_{yy}} \partial_{u_{yy}} + \zeta^{u_{zz}} \partial_{u_{zz}}.$$

The general theory of constructing determining equations for integro-differential equations can be found in [21]. Formerly the determining equation related to the integro-differential equation is obtained applying the following strategy. First, one has to construct the canonical Lie–Bäcklund operator equivalent to the generator  $X$ :

$$\tilde{X} = (\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z) \partial_u + (\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z) \partial_w.$$

Then the Lie–Bäcklund operator has to be prolonged up to the maximum order of derivatives of the equation. Finally, the determining equation is obtained by applying the prolonged Lie–Bäcklund operator to the equation, where the actions of the derivatives are considered in terms of the Frechet derivatives:

$$\psi^w(t, x, y, z) = \int_0^\infty K(s) \psi^u(t - s, x, y, z) ds. \tag{20}$$

Here

$$\begin{aligned} \psi^u(h_1) &= \zeta^u(h_2) - \xi^t(h_2)u_t(h_1) - \xi^x(h_2)u_x(h_1) - \xi^y(h_2)u_y(h_1) - \xi^z(h_2)u_z(h_1), \\ \psi^w(h_1) &= \zeta^w(h_2) - \xi^t(h_2)w_t(h_1) - \xi^x(h_2)w_x(h_1) - \xi^y(h_2)w_y(h_1) - \xi^z(h_2)w_z(h_1), \end{aligned}$$

where for the sake of simplicity of the presentation we denoted

$$h_1 = (t, x, y, z), \quad h_2 = (t, x, y, z, u(t, x, y, z), w(t, x, y, z)).$$

The determining equations (19) and (20) have to be satisfied for any solution of equations (1). Note that the determining equation (20) is still an integral equation.

Since it is difficult to find the general solution of the determining equations (19) and (20), the following simplification is considered. One can assume that the determining equation (19) is valid for any functions  $u(t, x, y, z)$  and  $w(t, x, y, z)$  only satisfying the first equation of (1). This allows us to use the standard procedure for solving determining equations developed for partial differential equations. After solving the determining equation (19), one can use the found solution for solving the integral determining equation (20). It has to be noted that this way of solving the determining equations (19) and (20) can give a particular solution. In this paper, this method is used.

The described method of solving the determining equations (19) and (20) will be illustrated on the one-dimensional case of equations (1). For the other cases, final results will be presented in the following sections.

3.1. One-dimensional case

In the one-dimensional case, equations (1) are

$$\frac{\partial}{\partial t}(u_x - uu_t - w_{tt}) = 0, \quad w(t, x) = \int_0^\infty K(s)u(t - s, x) ds. \quad (21)$$

The admitted generator is sought in the form

$$X = \xi^t(t, x, u, w)\partial_t + \xi^x(t, x, u, w)\partial_x + \zeta^u(t, x, u, w)\partial_u + \zeta^w(t, x, u, w)\partial_w.$$

Applying the group analysis method to the first equation, one finds that

$$\begin{aligned} \xi^t &= t(\xi'/2 + k_1)t + g, & \zeta^u &= u(k_1 - \xi'/2) - t\xi''/2 - g', \\ \zeta^w &= -(6\xi'w + t^3\xi''')/12 + 3wk_1 + t^2\mu + t\eta + \zeta, \end{aligned} \quad (22)$$

where  $k_1$  is constant,  $\xi = \xi(x)$ ,  $\zeta = \zeta(x)$ ,  $\eta = \eta(x)$ ,  $\mu = \mu(x)$  and  $g = g(x)$  are arbitrary functions.

**Remark.** There is another representation of the first equation of (21). This equation is obtained by integrating with respect to  $t$  and setting the arbitrary function of the integration to zero:

$$u_x - uu_t - w_{tt} = 0, \quad w(t, x) = \int_0^\infty K(s)u(t - s, x) ds. \quad (23)$$

In this case, the first step in finding admitted Lie group leads to (22) with the particular case of the function  $\mu = -\xi''/2$ .

The determining equation for the second equation of (21)

$$w(t, x) = \int_0^\infty K(s)u(t - s, x) ds$$

is the equation

$$(\tilde{\zeta}^w - \tilde{\xi}^t w_t - \tilde{\xi}^x w_x)(t, x) = \int_0^\infty K(s)(\tilde{\zeta}^u - \tilde{\xi}^t u_t - \tilde{\xi}^x u_x)(t - s, x) ds, \quad (24)$$

where

$$\begin{aligned} \tilde{\xi}^t(t, x) &= \xi^t(t, x, u(t, x), w(t, x)), & \tilde{\xi}^x(t, x) &= \xi^x(t, x, u(t, x), w(t, x)), \\ \tilde{\zeta}^w(t, x) &= \zeta^w(t, x, u(t, x), w(t, x)), & \tilde{\zeta}^u(t, x) &= \zeta^u(t, x, u(t, x), w(t, x)). \end{aligned}$$

Substituting the coefficients (22) into (24), let us satisfy equation (24). Note that

$$(\tilde{\xi}^x w_x)(t, x) = \int_0^\infty K(s)(\tilde{\xi}^x u_x)(t - s, x) ds,$$

and

$$\begin{aligned} \tilde{\xi}^t(t - s) &= \tilde{\xi}^t(t) - s(k_1 + \frac{1}{2}\xi'), \\ \tilde{\zeta}^u(t - s) &= (k_1 - \frac{1}{2}\xi')u(t - s) - \frac{1}{2}(t - s)\xi'' - g'. \end{aligned}$$

Here and further, the argument  $x$  is omitted, and further tilde is also omitted. The determining equation (24) becomes

$$\begin{aligned} -\zeta^w(t) + \int_0^\infty K(s) \left( \zeta^u(t - s) - s \left( k_1 + \frac{1}{2}\xi' \right) u_t(t - s) \right) ds \\ = (6\xi'w + t^3\xi''')/12 - 3wk_1 - t^2\mu - t\eta - \zeta + \left( k_1 - \frac{1}{2}\xi' \right) w - g' \int_0^\infty K(s) ds \end{aligned}$$

$$\begin{aligned}
 & -\frac{1}{2}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) + \left( \frac{1}{2}\xi' + k_1 \right) \int_0^\infty sK(s)u_t(t-s) ds \\
 & = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\
 & -\frac{1}{2}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) + \left( \frac{1}{2}\xi' + k_1 \right) \int_0^\infty sK(s)u_t(t-s) ds.
 \end{aligned}$$

Let us calculate

$$\begin{aligned}
 \int_0^\infty sK(s)u_t(t-s) ds & = - \int_0^\infty sK(s) du(t-s) \\
 & = -sK(s)u(t-s)|_0^\infty + \int_0^\infty (K(s) + sK'(s))u(t-s) ds \\
 & = w + \int_0^\infty sK'(s)u(t-s) ds.
 \end{aligned}$$

Here it is assumed that

$$sK(s)u(t-s)|_0^\infty = 0.$$

The determining equation becomes

$$\begin{aligned}
 & t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\
 & -\frac{1}{2}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) + \left( \frac{1}{2}\xi' + k_1 \right) \int_0^\infty sK(s)u_t(t-s) ds \\
 & = t^3\xi'''/12 - 2wk_1 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\
 & -\frac{1}{2}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) \\
 & + \left( \frac{1}{2}\xi' + k_1 \right) \left( w + \int_0^\infty sK'(s)u(t-s) ds \right) \\
 & = t^3\xi'''/12 + w \left( \frac{1}{2}\xi' - k_1 \right) - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\
 & -\frac{1}{2}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) \\
 & + \left( \frac{1}{2}\xi' + k_1 \right) \int_0^\infty sK'(s)u(t-s) ds \\
 & = t^3\xi'''/12 - t^2\mu - t\eta - \zeta - g' \int_0^\infty K(s) ds \\
 & -\frac{1}{3}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) \\
 & + \int_0^\infty \left( \left( \frac{1}{2}\xi' + k_1 \right) sK'(s) + \left( \frac{1}{2}\xi' - k_1 \right) K(s) \right) u(t-s) ds = 0.
 \end{aligned}$$

Since  $u = 0, w = 0$  is a solution of equations (21), the determining equation has to be satisfied on this solution. Thus, one obtains

$$-t^3\xi'''/12 + t^2\mu + t\eta + \zeta + g' \int_0^\infty K(s) ds + \frac{1}{3}\xi'' \left( t \int_0^\infty K(s) ds - \int_0^\infty sK(s) ds \right) = 0, \tag{25}$$

and hence,

$$\int_0^\infty \left( \left( \frac{1}{2}\xi' + k_1 \right) s K'(s) + \left( \frac{1}{2}\xi' - k_1 \right) K(s) \right) u(t-s) ds = 0.$$

Since  $u(t, x)$  is an arbitrary, the last equation gives

$$\left( \frac{1}{2}\xi' + k_1 \right) s K'(s) + \left( \frac{1}{2}\xi' - k_1 \right) K(s) = 0,$$

which means that  $\xi'$  is constant, for example,  $\xi = 2kx + k_0$ . Thus,

$$(k + k_1)s K'(s) + (k - k_1)K(s) = 0,$$

or  $K(s) = K_0 s^\alpha$ , and

$$(k + k_1)\alpha + (k - k_1) = 0.$$

Equation (25) becomes

$$t^2\mu + t\eta + \zeta + g' \int_0^\infty K(s) ds = 0.$$

This equation implies

$$\mu = 0, \quad \eta = 0, \quad \zeta = -g' \int_0^\infty K(s) ds.$$

To prevent problems with the convergence of the integral  $\int_0^\infty K(s) ds$ , one can assume that

$$g' = 0.$$

**Remark.** One can call the transformations corresponding to the generators

$$X = g\partial_t - g'\partial_u$$

formally admitted. These generators can be used for constructing invariant solutions.

Thus, one obtains that the Lie group corresponding to the generators

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_t, \\ X_3 &= k^t t \partial_t + k^x x \partial_x + k^u u \partial_u + k^w w \partial_w \end{aligned}$$

is admitted by equations (21). Here

$$\begin{aligned} k^t &= k + k_1, & k^x &= 2k, & k^u &= k_1 - k, & k^w &= 3k_1 - k, \\ (k + k_1)\alpha + (k - k_1) &= 0. \end{aligned} \tag{26}$$

Note also that

$$k^u = \alpha k^t, \quad k^t \neq 0.$$

Since the integro-differential equations are nonlocal, not every admitted Lie group has the property to transform a solution of integro-differential equations into a solution. However, for the transformations corresponding to the generators  $X_1$  and  $X_2$ , it is trivial to check that these transformations possess this property. Let us also check that the scaling group corresponding to the generator  $X_3$  maps any solution of equations (21) into a solution of the same equations.

The transformation corresponding to the generator  $X_3$  is

$$t' = t e^{ak^t}, \quad x' = x e^{ak^x}, \quad u' = u e^{ak^u},$$

which maps a function  $u(t, x)$  into the function

$$u'(t', x') = e^{ak^u} u(t' e^{-ak^t}, x' e^{-ak^x}).$$

Let us consider the transformation of the integral

$$\begin{aligned}
 \int_0^\infty K(s')u'(t' - s', x') ds' &= e^{ak^t} \int_0^\infty K(s')u((t' - s')e^{-ak^t}, x'e^{-ak^x}) ds' \\
 &= e^{ak^t} \int_0^\infty K(s')u(t'e^{-ak^t} - s'e^{-ak^t}) ds' \\
 &= e^{\alpha(k^u+k^t)} \int_0^\infty K(s'e^{-ak^t} e^{ak^t})u(t - s'e^{-ak^t}) d(s'e^{-ak^t}) \\
 &= e^{\alpha(k^u+k^t)} \int_0^\infty K(s e^{ak^t})u(t - s) ds \\
 &= e^{\alpha(k^u+k^t)} \int_0^\infty K_0(s e^{ak^t})^\alpha u(t - s) ds \\
 &= e^{\alpha(k^u+(\alpha+1)k^t)} \int_0^\infty K_0 s^\alpha u(t - s) ds \\
 &= e^{\alpha(k^u+(\alpha+1)k^t)} w(t, x).
 \end{aligned}$$

Thus, for checking one needs to only check that

$$-k^w + k^u + (\alpha + 1)k^t = 0.$$

Indeed

$$\begin{aligned}
 -k^w + k^u + (\alpha + 1)k^t &= -3k_1 + k + k_1 - k + (\alpha + 1)(k + k_1) \\
 &= k_1(-3 + 1 + \alpha + 1) + k(1 - 1 + \alpha + 1) \\
 &= k_1(\alpha - 1) + k(\alpha + 1) = 0.
 \end{aligned}$$

Here condition (26) was used. One also needs to check the other conditions:

$$k^u - k^t - k^x = 2k^u - 2k^t = k^w - 3k^t.$$

Indeed,

$$\begin{aligned}
 k^u - k^t - k^x &= k_1 - k - (k + k_1) - 2k = -4k, \\
 2k^u - 2k^t &= 2(k_1 - k) - 2(k + k_1) = -4k, \\
 k^w - 3k^t &= 3k_1 - k - 3(k + k_1) = -4k.
 \end{aligned}$$

### 3.2. Classification of subalgebras

The commutator table of the Lie algebra  $L_3 = \{X_1, X_2, X_3\}$  is

	$X_1$	$X_2$	$X_3$
$X_1$	0	0	$k^x X_1$
$X_2$	0	0	$k^t X_2$
$X_3$	$-k^x X_1$	$-k^t X_2$	0.

The set of automorphisms is defined by the commutator table:

$$\begin{aligned}
 A_1 : \quad x'_1 &= x_1 + a_1 k^x x_3, \\
 A_2 : \quad x'_2 &= x_2 + a_2 k^t x_3, \\
 A_3 : \quad x'_1 &= x_1 e^{a_3 k^x}, \quad x'_2 = x_2 e^{a_3 k^t},
 \end{aligned}$$

where only changeable coordinates of the automorphisms are presented.

If  $\alpha = 1$ , then  $k^x = 0$ , and the operator  $X_1$  composes a center of the Lie algebra. Thus, the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, X_2 + \gamma X_1, X_1\},$$

where  $\gamma = \pm 1$ .

If  $\alpha \neq 1$ , then  $k^x \neq 0$ , and the one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3, X_2 + \lambda X_1, X_1\}.$$

For  $\alpha \neq 0$ , using the automorphism  $A_3$ , the subalgebra  $X_2 + \lambda X_1$  can also be reduced to  $X_2 \pm X_1$ .

### 3.3. Invariant solutions

The optimal system of subalgebras of the Lie algebra  $L_3$  defines the complete set of representations of solutions invariant with respect to  $L_3$ .

#### 3.3.1. Case $\alpha = 1$

The subalgebra  $\{X_3 + \lambda X_1\}$ . The generator  $\{X_3 + \lambda X_1\}$  is

$$t\partial_t + u\partial_u + \lambda\partial_x.$$

It is convenient to separate two cases (a)  $\lambda = 0$  and (b)  $\lambda \neq 0$ . The invariants are

$$\begin{aligned} \lambda = 0: & \quad u/t, x; \\ \lambda \neq 0: & \quad u/t, t e^{x\lambda_0} \quad (\lambda_0 = -1/\lambda). \end{aligned}$$

Invariant solutions have the representations

$$\begin{aligned} \lambda = 0: & \quad u = t\varphi(x); \\ \lambda \neq 0: & \quad u = t\varphi(y) \quad (y = t e^{x\lambda_0}). \end{aligned}$$

Note that for  $\lambda = 0$ , the integral

$$w = \int_0^\infty K_0 s(t-s)\varphi(x) ds = K_0\varphi(x) \int_0^\infty s(t-s) ds$$

is divergent. Hence, one has only to consider the case  $\lambda \neq 0$ . In this case,

$$\begin{aligned} w(t, x) &= K_0 \int_0^\infty s(t-s)\varphi((t-s)e^{x\lambda_0}) ds = K_0 e^{-3x\lambda_0} \int_y^{-\infty} (y-z)z\varphi(z) d(y-z) \\ &= K_0 e^{-3x\lambda_0} \int_0^\infty s(y-s)\varphi(y-s) ds = e^{-3x\lambda_0} W(y). \end{aligned}$$

$$w_t = e^{-2x\lambda_0} W', \quad w_{tt} = e^{-x\lambda_0} W'',$$

where we used the relations

$$s = t - z e^{-x\lambda_0} = e^{-x\lambda_0} (t e^{x\lambda_0} - z) = e^{-x\lambda_0} (y - z).$$

Equation (23) becomes

$$y((\lambda\varphi + 1)\varphi' - \varphi^2) = K_0 \frac{d^2}{dy^2} \left( \int_0^\infty s(y-s)\varphi(y-s) ds \right).$$

The subalgebra  $\{X_2 + \gamma X_1\}$ . The generator  $\{X_2 + \gamma X_1\}$  is  $\partial_t + \gamma \partial_x$ . The invariants are

$$u, y = x - \gamma t.$$

An invariant solution has the representation

$$u = \varphi(y).$$

Hence, one has

$$\begin{aligned} w(t, x) &= K_0 \int_0^\infty s \varphi(x - \gamma(t - s)) \, ds = K_0 \int_y^{-\infty} (y - z) \varphi(z) \, d(y - z) \\ &= K_0 \int_0^\infty s \varphi(y - s) \, ds = W(y). \\ w_t &= -\gamma W', \quad w_{tt} = W'', \end{aligned}$$

where we used the relations

$$s = t + \gamma(z - x) = -\gamma(x - \gamma t) + \gamma z = -\gamma(y - z).$$

Equation (23) becomes

$$(1 + \gamma \varphi) \varphi' = K_0 \frac{d^2}{dy^2} \left( \int_0^\infty s(y - s) \varphi(y - s) \, ds \right).$$

The subalgebra  $\{X_1\}$ . The generator  $\{X_1\}$  is  $\partial_x$ . The invariants are

$$u, t.$$

An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t, x) = K_0 \int_0^\infty s \varphi(t - s) \, ds.$$

Equation (23) becomes

$$-\varphi \varphi' = K_0 \frac{d^2}{dt^2} \left( \int_0^\infty s \varphi(t - s) \, ds \right).$$

### 3.4. Case $\alpha \neq 1$ .

The subalgebra  $\{X_3\}$ . The generator  $\{X_3\}$  is

$$X_3 = k^t t \partial_t + k^x x \partial_x + k^u u \partial_u \quad (k^x \neq 0),$$

where the constants  $k^t$ ,  $k^x$  and  $k^u$  are defined by (26). Substituting the representation of the invariant solution

$$u = x^{k^u/(2k)} \varphi(y), \quad t = y x^{k^t/(2k)}$$

into the second equation of (23), one has

$$\begin{aligned} w &= \int_0^\infty K_0 s^\alpha u(t - s) \, ds = K_0 x^{k^u/(2k)} \int_0^\infty s^\alpha \varphi((t - s) x^{-k^t/(2k)}) \, ds \\ &= -K_0 x^{((1+\alpha)k^t + k^u)/(2k)} \int_{t x^{-k^t/(2k)}}^{-\infty} (y - z)^\alpha \varphi(z) \, dz \\ &= K_0 x^{((1+\alpha)k^t + k^u)/(2k)} \int_{-\infty}^y (y - z)^\alpha \varphi(z) \, dz \\ &= K_0 x^{((1+\alpha)k^t + k^u)/(2k)} W(y) = K_0 x^{(k^t + 2k^u)/(2k)} W(y), \end{aligned}$$

where we used the relations

$$s = t - zx^{k^t/(2k)} = x^{k^t/(2k)}(tx^{-k^t/(2k)} - z) \\ = x^{k^t/(2k)}(y - z), \quad (1 + \alpha)k^t + k^u = k^t + 2k^u,$$

and

$$W(y) = \int_{-\infty}^y (y - z)^\alpha \varphi(z) dz.$$

The derivatives are changed as follows:

$$\frac{\partial \varphi}{\partial t} = \frac{y}{t} \varphi', \quad \frac{\partial \varphi}{\partial x} = -\frac{k^t}{2k} \frac{y}{x} \varphi', \\ \frac{\partial}{\partial t} \left( \frac{\partial W}{\partial t} \right) = \frac{\partial}{\partial t} \left( \frac{y}{t} W' \right) = \frac{1}{t^2} (-yW' + y(yW')') = \frac{y^2}{t^2} W''.$$

Thus, the first equation of (23) becomes

$$u_x - uu_t - v_{tt} = u_x - uu_t - \frac{\partial^2}{\partial t^2} \int_0^\infty K_0 s^\alpha u(t - s) ds \\ = \frac{k^u}{2k} \frac{x^{k^u/(2k)}}{x} \varphi - \frac{k^t}{2k} \frac{x^{k^u/(2k)} y}{x} \varphi' - x^{k^u/(2k)} \varphi x^{k^u/(2k)} \frac{y}{t} \varphi' - K_0 \frac{x^{(k^t+2k^u)/(2k)} y^2}{t^2} W'' \\ = \frac{k^u}{2k} \frac{x^{k^u/(2k)}}{x} \varphi - \frac{k^t}{2k} \frac{x^{k^u/(2k)} y}{x} \varphi' - x^{k^u/(2k)} \varphi x^{k^u/(2k)} \frac{y}{t} \varphi' - K_0 y x^{(2k^u - k^t)/(2k)} W'' \\ = \frac{x^{k^u/(2k)}}{x} \left( \frac{k^u}{2k} \varphi - \frac{k^t}{2k} y \varphi' - \varphi x^{(k^x+k^u-k^t)/(2k)} \varphi' - K_0 x^{(k^u-k^t+k^x)/(2k)} W'' \right) \\ = \frac{x^{k^u/(2k)}}{x} \left( \frac{k^u}{2k} \varphi - \frac{k^t}{2k} y \varphi' - \varphi \varphi' - K_0 W'' \right) = 0.$$

Here, the following relation was used:

$$k^u + k^x - k^t = k_1 - k + 2k - (k + k_1) = 0.$$

Thus, the reduced equation is

$$\alpha \varphi + ((\alpha - 1)\varphi - y)\varphi' = (1 - \alpha)K_0 \int_0^\infty s^\alpha \varphi''(y - s) ds,$$

where

$$W''(y) = \frac{d^2}{dy^2} \left( \int_{-\infty}^y (y - z)^\alpha \varphi(z) dz \right) = \frac{d^2}{dy^2} \left( \int_0^\infty s^\alpha \varphi(y - s) ds \right) \\ = \int_0^\infty s^\alpha \varphi''(y - s) ds.$$

The subalgebra  $\{X_2 + \lambda X_1\}$ . The generator  $\{X_2 + \lambda X_1\}$  is  $\partial_t + \lambda \partial_x$ . The invariants are

$$u, y = x - \lambda t.$$

An invariant solution has the representation of a traveling wave type

$$u = \varphi(y).$$

As in the previous case, one needs to study only the case  $\lambda \neq 0$ . If  $\lambda > 0$ , then one has

$$w(t, x) = K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t - s)) ds = \lambda^{-\alpha-1} K_0 \int_y^\infty (z - y)^\alpha \varphi(z) d(z - y) \\ = \lambda^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y + s) ds = W(y).$$

$$w_t = -\lambda W', \quad w_{tt} = \lambda^2 W'',$$

where

$$s = t + \frac{z - x}{\lambda} = -\frac{x - \lambda t}{\lambda} + \frac{z}{\lambda} = \frac{z - y}{\lambda}.$$

The reduced equation is

$$(1 + \lambda\varphi)\varphi' = \lambda^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y + s) ds. \tag{27}$$

If  $\lambda < 0$ , then

$$\begin{aligned} w(t, x) &= K_0 \int_0^\infty s^\alpha \varphi(x - \lambda(t - s)) ds = K_0 (-\lambda)^{-\alpha-1} \int_y^{-\infty} (y - z)^\alpha \varphi(z) d(y - z) \\ &= (-\lambda)^{-\alpha-1} K_0 \int_0^\infty s^\alpha \varphi(y + s) ds = W(y). \end{aligned}$$

Equation (23) becomes

$$(1 + \lambda\varphi)\varphi' = (-\lambda)^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y + s) ds. \tag{28}$$

Combining equations (27) and (28), one has

$$(1 + \lambda\varphi)\varphi' = |\lambda|^{1-\alpha} K_0 \int_0^\infty s^\alpha \varphi''(y + s) ds. \tag{29}$$

*The subalgebra*  $\{X_1\}$ . The generator  $\{X_1\}$  is  $\partial_x$ . The invariants are  $u, t$ . An invariant solution has the representation

$$u = \varphi(t).$$

Hence, one has

$$w(t, x) = K_0 \int_0^\infty s^\alpha \varphi(t - s) ds.$$

Equation (23) becomes

$$-\varphi\varphi' = K_0 \left( \int_0^\infty s^\alpha \varphi''(t - s) ds \right).$$

#### 4. Two-dimensional equation

The studied equations are

$$\frac{\partial}{\partial t}(u_x - upu_t - w_{tt}) = u_{yy}, \quad w(t, x, y) = \int_0^\infty K(s)u(t - s, x, y) ds.$$

The first step gives the generator

$$\begin{aligned} X &= \xi^t(t, x, y, p, w)\partial_t + \xi^x(t, x, y, p, w)\partial_x + \xi^y(t, x, y, p, w)\partial_y \\ &\quad + \zeta^u(t, x, y, p, w)\partial_p + \zeta^v(t, x, y, p, w)\partial_w, \end{aligned}$$

where the coefficients are

$$\begin{aligned} \xi^t &= (2t\xi' + \xi''y^2 + 3h'y)/6 + tk_1 + g, & \xi^x &= \xi, & \xi^y &= (4y\xi' + 3yk_1)/6 + h, \\ \zeta^u &= ((6k_1 - 4\xi') - \xi'''y^2 - 2t\xi'' - 6g' - 3yh'')/6, \\ \zeta^v &= (3k_1 - \xi')w + t^2\mu + t\eta + \zeta, \end{aligned} \tag{30}$$

Here  $k_1$  is constant,  $\xi = \xi(x)$ ,  $h = h(x)$ ,  $g = g(x)$ ,  $\zeta = \zeta(t, x, y)$ ,  $\eta = \eta(t, x, y)$  and  $\mu = \mu(t, x, y)$  are arbitrary functions.

The determining equation for the second equation

$$w(t, x, y) = \int_0^\infty K(s)u(t - s, x, y) ds$$

is the equation

$$\begin{aligned} &(\zeta^v - \xi^t w_t - \xi^x w_x - \xi^y w_y)(t, x, y) \\ &= \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y)(t - s, x, y) ds. \end{aligned}$$

Substituting the coefficients (30) into the last equation, one obtains

$$\mu = 0, \quad \eta = 0, \quad \zeta = 0, \quad g' = 0, \quad h'' = 0,$$

$$\left(\frac{1}{3}\xi' + k_1\right) sK'(s) + \left(\frac{2}{3}\xi' - k_1\right) K(s) = 0,$$

which means that  $\xi'$  is constant, for example,  $\xi = 3kx + k_0$ . Thus,

$$(k + k_1)sK'(s) + (2k - k_1)K(s) = 0,$$

or

$$K(s) = K_0 s^\alpha,$$

and

$$(k + k_1)\alpha + (2k - k_1) = 0.$$

Therefore, the admitted Lie algebra is defined by the generators

$$\begin{aligned} X_1 &= \partial_x, & X_2 &= \partial_t, & X_3 &= y\partial_t + 2x\partial_y, & X_4 &= \partial_y, \\ X_5 &= 2t(k + k_1)\partial_t + 6kx\partial_x + (4k + k_1)y\partial_y + 2(k_1 - 2k)u\partial_u + 6v(k_1 - k)\partial_w. \end{aligned}$$

### 5. Three-dimensional case

The studied equations are

$$\frac{\partial}{\partial t}(u_x - uu_t - w_{tt}) = u_{yy} + u_{zz}, \quad w(t, x, y, z) = \int_0^\infty K(s)u(t - s, x, y, z) ds.$$

The first step gives the generator

$$\begin{aligned} X &= \xi^t(t, x, y, u, z, w)\partial_t + \xi^x(t, x, y, u, z, w)\partial_x + \xi^y(t, x, y, u, z, w)\partial_y \\ &+ \zeta^u(t, x, y, u, z, w)\partial_u + \zeta^w(t, x, y, z, u, w)\partial_w \end{aligned}$$

with the coefficients

$$\xi^t = t(\xi' + 2k_1) + 3\xi''(z^2 + y^2)/4 + (h'y + f'z)/2 + g,$$

$$\xi^x = 5\xi, \quad \xi^y = 3\xi'y + zk_2 + yk_1 + h, \quad \xi^z = 3\xi'z + zk_1 - yk_2 + f,$$

$$\zeta^u = 2(k_1 - 2\xi')u - 3\xi'''(z^2 + y^2)/4 - \xi''t - g' - (h''y + f''z)/2,$$

$$\zeta^w = w(6k_1 - 7\xi') + \xi'''t^3/3 + t^2\mu + t\eta + \zeta,$$

where  $k_1, k_2$  are constant, and  $\xi = \xi(x), h = h(x), g = g(x), f = f(x), \zeta = \zeta(x, y, z), \eta = \eta(x, y, z)$  and  $\mu = \mu(x, y, z)$  are arbitrary functions. It is obvious that the generator corresponding to the rotation in the planes  $y$  and  $z$  is admitted. This generator is defined by  $k_2$ :

$$X = z\partial_y - y\partial_z.$$

Excluding this generator, one obtains

$$\xi^t = t(\xi' + 2k_1) + 3\xi''(z^2 + y^2)/4 + (h'y + f'z)/2 + g,$$

$$\xi^x = 5\xi, \quad \xi^y = y(3\xi' + k_1) + h, \quad \xi^z = z(3\xi' + k_1) + f,$$

$$\zeta^u = 2(k_1 - 2\xi')u - 3\xi'''(z^2 + y^2)/4 - \xi''t - g' - (h''y + f''z)/2,$$

$$\zeta^w = w(6k_1 - 7\xi') + \xi'''t^3/3 + t^2\mu + t\eta + \zeta.$$

The determining equation for the second equation

$$w(t, x, y, z) = \int_0^\infty K(s)u(t - s, x, y, z) ds$$

is the equation

$$\begin{aligned} &(\zeta^w - \xi^t w_t - \xi^x w_x - \xi^y w_y - \xi^z w_z)(t, x, y, z) \\ &= \int_0^\infty K(s)(\zeta^u - \xi^t u_t - \xi^x u_x - \xi^y u_y - \xi^z u_z)(t - s, x, y, z) ds. \end{aligned}$$

Substituting the coefficients into the last equation, one obtains

$$g = k_3, \quad h = 2k_4x + k_5, \quad f = 2k_6x + k_7$$

$$\xi^t = t(k + 2k_1) + k_4y + k_6z + k_3, \quad \xi^x = 5(kx + k_0), \quad \xi^y = y(3k + k_1) + 2k_4x + k_5,$$

$$\xi^z = z(3k + k_1) + 2k_6x + k_7, \quad \zeta^u = 2(k_1 - 2k)u, \quad \zeta^w = w(6k_1 - 7k).$$

$$(k + 2k_1)sK'(s) + 2(2k - k_1)K(s) = 0,$$

which means that

$$K(s) = K_0s^\alpha,$$

and

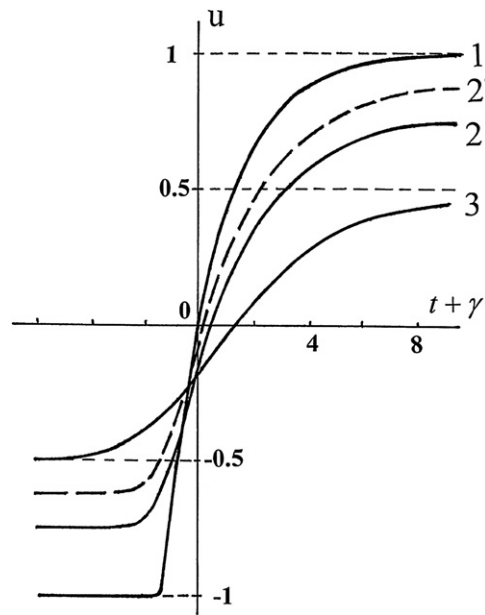
$$k(\alpha + 4) + 2k_1(\alpha - 1) = 0.$$

The admitted generators are

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = z\partial_y - y\partial_z, \quad X_4 = y\partial_t + 2x\partial_y,$$

$$X_5 = z\partial_t + 2x\partial_z, \quad X_6 = \partial_y, \quad X_7 = \partial_z,$$

$$\begin{aligned} X_8 = &t(k + 2k_1)\partial_t + 5kx\partial_x + (3k + k_1)y\partial_y + (3k + k_1)z\partial_z \\ &+ 2(k_1 - 2k)u\partial_u + w(6k_1 - 7k)\partial_w. \end{aligned}$$



**Figure 1.** The solution describing the shape of front in a relaxing medium. Curves 1, 2 and 3 are constructed for  $\alpha = 0$  and  $\Delta = 1, 0.75$  and  $0.5$ , correspondingly. The dashed curve 2' is constructed for the values  $\Delta = 0.75$  and  $\alpha = 0.15$ .

### 6. Exponential kernel

There exists one known exact solution to the 1D equation for an exponential kernel [1, 19]:

$$u_x - uu_t - w_{tt} = 0,$$

$$w = \int_0^\infty \exp(-s)u(t - s) ds.$$

One can seek for it in the form of a traveling wave:

$$u = u(t + \alpha x).$$

The solution has the form

$$t + \gamma = \frac{1}{\Delta} \ln \frac{|u - \alpha + \Delta|^{1-\Delta}}{|u - \alpha - \Delta|^{1+\Delta}}.$$

Here  $\alpha$ ,  $\Delta$  and  $\gamma$  are the constants. This solution has evident physical meaning for the parameters  $\Delta > |\alpha|$ ,  $0 < \Delta < 1$ . This is shown in figure 1 and the solution describes the shape of a single shock front in a relaxing medium.

It is interesting to derive this solution using computed symmetries.

Since for the exponential kernel  $K(s) = e^{-s}$  one has relation (6):

$$w_t = u - w,$$

equations

$$u_{xt} - uu_{tt} - u_t^2 = w_{ttt} \tag{31}$$

and

$$u_x - uu_t = w_{tt} \tag{32}$$

can be reduced to the partial differential equations, respectively,

$$u_{xtt} - (1 + u)u_{ttt} - 3u_t u_{tt} + u_{xt} - uu_{tt} - u_t^2 = 0, \tag{33}$$

and

$$u_{xt} - (1 + u)u_{tt} - u_t^2 + u_x - uu_t = 0. \tag{34}$$

Admitted Lie groups of these equations are as follows. The admitted Lie group of equation (33) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = g(x)\partial_t - g'(x)\partial_u,$$

where the function  $g = g(x)$  is an arbitrary function. The admitted Lie group of equation (34) is defined by the generators

$$X_1 = \partial_x, \quad X_2 = \partial_t, \quad X_3 = x\partial_t - \partial_u.$$

The commutator table of the algebra  $\{X_1, X_2, X_3\}$  is

	$X_1$	$X_2$	$X_3$
$X_1$	0	0	$X_2$
$X_2$	0	0	0
$X_3$	$-X_2$	0	0.

The generator  $X_2$  is a center of the Lie algebra. The set of automorphisms is defined by the commutator table:

$$\begin{aligned} A_1: \quad x'_2 &= x_2 + a_1 x_3, \\ A_3: \quad x'_2 &= x_2 - a_3 x_1. \end{aligned}$$

The one-dimensional optimal system of subalgebras consists of the subalgebras

$$\{X_3 + \lambda X_1, X_2, X_1\}.$$

The representations of invariant solutions are

$$\begin{aligned} X_3 + \lambda X_1: \quad u &= -\frac{x}{\lambda} + \varphi(x^2 - 2\lambda t) \quad (\lambda \neq 0), \\ X_3: \quad u &= -\frac{t}{x} + \varphi(x), \\ X_2: \quad u &= \varphi(x), \\ X_1: \quad u &= \varphi(t), \end{aligned}$$

and the reduced equations are

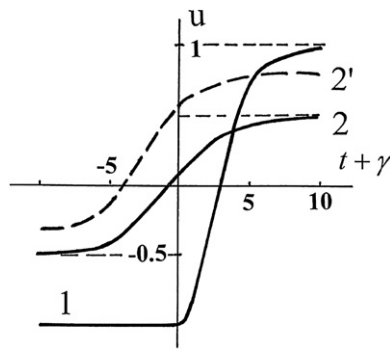
$$\begin{aligned} X_3 + \lambda X_1: \quad (4\lambda^3(\varphi' + \varphi\varphi') - \lambda^2\varphi^2)' + 1 &= 0, \\ X_3: \quad \varphi' + \frac{1}{x}\varphi &= 0, \\ X_2: \quad \varphi' &= 0, \\ X_1: \quad (\varphi' + \varphi\varphi' + \varphi^2/2)' &= 0. \end{aligned}$$

The solution

$$u = u(t + \alpha x)$$

is invariant with respect to the operator  $X_1 - \alpha X_2 = \partial_x - \alpha\partial_t$ . The subalgebra corresponding to this operator is equivalent to the subalgebra with the generator  $X_1$ . The reduced equation of an invariant solution corresponding to the generator  $X_1$  is

$$\varphi' = \frac{k - \varphi^2}{2(1 + \varphi)},$$



**Figure 2.** Solution describing the shape of wave front in the medium with the constant ‘memory’ inside the time interval  $[0, 1]$ . Curves 1 and 2 are constructed for  $\alpha = 0$  and the values of  $\beta = 1$  and 0.5, correspondingly. The dashed curve 2’ is constructed for the values  $\beta = 0.5$  and  $\alpha = 0.25$ .

where  $k$  is an arbitrary constant of integration. The general solution of this equation depends on the constant  $k$ :

$$\begin{aligned}
 k = \alpha^2 > 0: \quad t + c_0 &= \frac{1}{\alpha} \ln \left( \frac{|\varphi + \alpha|^{1-\alpha}}{|\varphi - \alpha|^{1+\alpha}} \right), \\
 k = -\alpha^2 < 0: \quad t + c_0 &= -\ln(\varphi^2 + \alpha^2) - \frac{2}{\alpha} \arctan \left( \frac{\varphi}{\alpha} \right), \\
 k = 0: \quad t + c_0 &= -2 \ln(\varphi) + \frac{2}{\varphi}.
 \end{aligned}$$

### 7. Delay equation

It is desirable to derive an exact solution for any kernel which does not permit the reduction of an integral equation to the differential one. Such an example exists. That is the model kernel which is nonzero on the finite segment, say  $s \in [0, 1]$ . The simplest case is  $K = 1, s \leq 1; K = 0, s > 1$ . For this kernel the integral equation is reduced to the difference-differential equation

$$u_x - uu_t - \Delta u_t = 0, \quad \Delta u \equiv u(t) - u(t - 1).$$

Its solution can be sought in the form of a traveling wave:  $u = u(t + \alpha x)$ . The reduced equation can be integrated one time. The solution is  $u(t - 1) = \frac{1}{2}[u^2(t) + 2(1 - \alpha)u(t) - \beta^2]$ . Here  $\alpha$  and  $\beta$  are constant. This formula defines a mapping  $u(t) \rightarrow u(t - 1)$  which offers easy possibility of constructing curves representing the profiles of the wave. These profiles are shown in figure 2. They display the image of a shock front in the medium with constant ‘memory’ within the segment  $[0, 1]$ .

The equation

$$u_x(t, x) = (u(t, x) + 1)u_t(t, x) - u_t(t - 1, x)$$

is a delay differential equation. The algorithm for applying the group analysis method to delay differential equations is given in [21–23]. Calculations show that the admitted Lie group is defined by the generators

$$X_1 = \partial_t, \quad X_2 = \partial_x, \quad X_3 = x\partial_t - \partial_u.$$

The representations of invariant solutions are given in the previous case. The reduced equations are

$$\begin{aligned} X_3 + \lambda X_1: & 2\lambda^2(\varphi'(z)(\varphi(z) + 1) + \varphi'(z + 2\lambda)) = 1, \\ X_3: & \varphi' + \frac{1}{x}\varphi = 0, \\ X_2: & \varphi' = 0, \\ X_1: & \varphi'(t)(\varphi(t) + 1) + \varphi'(t - 1) = 0. \end{aligned}$$

The reduced equations can be integrated:

$$\begin{aligned} X_3 + \lambda X_1: & \lambda^2(\varphi^2(z) + 2\varphi(z) + \varphi^2(z + 2\lambda)) = z + c_0, \\ X_3: & \varphi = c_0/x, \\ X_2: & \varphi = c_0, \\ X_1: & \varphi^2(t) + 2\varphi(t) + 2\varphi(t - 1) = c_0. \end{aligned}$$

## 8. Conclusion

The nonlinear *integro-differential* evolution equation (1) considered in this paper is not an exotic model. It encapsulates numerous mathematical models formulated by *differential* evolution equations and differs from them significantly not only in its form, but mostly due to its physical content meaning. Namely, any dispersion (frequency-dependent phase velocity) must be strongly connected with frequency-dependent absorption. Such connection follows from the causality principle. For example, waves having infinite velocities of propagation which are allowed by the differential equations of Burgers and Korteweg–de Vries type must disappear on their way, since otherwise a cause appears at a certain point later than its effect. The causality principle is given in physical models by integral Kramers–Kronig relations. Consequently, a consistent model must contain integral terms, in other words, be represented in an *integro-differential* form. Although this conclusion is well known, mathematical models described by purely differential evolution equations have been widely accepted in the nonlinear wave physics due to their simplicity compared to the *integro-differential* models. It seems that the consistent *integro-differential* nonlinear models will meet more applications in future.

This paper provides a first step in the application of the Lie group analysis to equation (1). The approach used in this paper is described in [21]. The analysis of the determining equation for the *integro-differential* equation allows us, in particular, to single out a class of kernels used for deriving mathematical models in the medical applications of ultrasound [20].

Note that for particular kernels, the *integro-differential* equation (1) becomes a partial differential equation or a delay partial differential equation. In these cases the complete group classification of equation (1) can be obtained. In the case of partial differential equations, the classical group analysis is used. For delay partial differential equations, the analysis developed in [22, 23] and described in [21] is applied. A complete study of particular cases is given in the paper. This provides a new result in the application of the group analysis method to partial and delay partial differential equations.

Along with admitted Lie groups, the representations of exact solutions and reduced equations are constructed in the paper. A complete solution and a physical interpretation of some of them are presented.

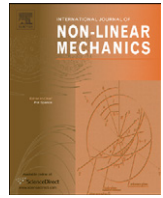
We hope that more results will be obtained in future by applying the above approach for solving the concrete models of physical significance as well as for new mathematical developments. In particular, it is interesting to make the preliminary group classification of *exceptional kernels* by applying the method of an *a priori* use of symmetries [24] to the *integro-differential* equations of the form (1).

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# On group classification of the spatially homogeneous and isotropic Boltzmann equation with sources

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## ABSTRACT

In Nonenmacher (1984) [1] an admitted Lie group of transformations for the spatially homogeneous and isotropic Boltzmann equation with sources was studied. In fact, the author is Nonenmacher (1984) [1] considered the equation for a generating function of the power moments of the Boltzmann equation solution. However, this equation is still a non-local partial differential equation, and this property was not taken into account there. In the present paper the admitted Lie group of this equation is studied, using our original method developed for group analysis of equations with non-local operators (Grigoriev and Meleshko, 1986; Meleshko, 2005; Grigoriev et al., 2010 [2–4]). The Lie groups obtained are compared with Nonenmacher (1984) [1]. The deficiency of Nonenmacher (1984) [1] is corrected.

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## 1. Introduction

The Boltzmann kinetic equation is the basis of the classical kinetic theory of rarefied gases. Considerable interest in the study of the Boltzmann equation was always the search for exact (invariant) solutions directly associated with the fundamental properties of the equation. After the studies of the class of the local Maxwellians [5–7] new classes of invariant solutions were constructed in the 1960s in [8–10]. A decade later the BKW-solution was almost simultaneously derived in [11,12]. Contrary to the Maxwellians, the Boltzmann collision integral does not vanish for this solution. The discovery of the BKW-solution stimulated a great splash of studies of exact solutions of various kinetic equations. However, the progress at that time was really limited to obtaining BKW-type solutions for different simplified models of the Boltzmann equation.<sup>1</sup>

The Boltzmann equation is an integro-differential equation. Whereas the classical group analysis method has been developed for studying partial differential equations, the main obstacle for applying group analysis to integro-differential equations comes from presence of non-local integral operators. The direct group analysis for equations with non-local operators was worked out and successfully used in [2–4]. In particular, a complete group classification of the spatially homogeneous and isotropic Boltzmann equation without sources was obtained in [2,14].

One of the alternative approaches for studying solutions of the Boltzmann equation, by transition to an equation for a moment generating function, was first considered in [12]. The BKW-solution was obtained there. In [1], such an approach was applied to the spatially homogeneous and isotropic Boltzmann equation with sources. The author of [1] used the group analysis method for studying solutions of the equation for the generating function. However, it was not taken into account that this equation is still a non-local one. In the present paper we use our method [2,14] to amend the results of [1]. A group classification of the equation for a moment generating function with respect to a source function is obtained.

## 2. General equations

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form [15]

$$\varphi_t(y,t) + \varphi(y,t)\varphi(0,t) = \int_0^1 \varphi(ys,t)\varphi(y(1-s),t) ds + \hat{q}(y,t), \quad (1)$$

where the Fourier transform

$$\tilde{\varphi}(k,t) = \frac{4\pi}{k} \int_0^\infty v \sin(kv) f(v,t) dv,$$

of the distribution function  $f(v,t)$  is defined by the formula

$$\tilde{\varphi}(k,t) = \varphi(k^2/2,t).$$

Similarly, the transform of the source function  $q(v,t)$  is

$$\tilde{\tilde{q}}(k,t) = \frac{4\pi}{k} \int_0^\infty v \sin(kv) q(v,t) dv,$$

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<sup>1</sup> See the review [13].

and

$$\tilde{q}(k, t) = \hat{q}(k^2/2, t).$$

The inverse Fourier transform of  $\tilde{\varphi}(k, t)$  gives the distribution function

$$f(v, t) = \frac{4\pi}{v} \int_0^\infty k \sin(kv) \tilde{\varphi}(k, t) dk.$$

Normalized moments of the distribution function are introduced by the formulae

$$M_n(t) = \frac{4\pi}{(2n+1)!!} \int_0^\infty f(v, t) v^{2n+2} dv, \quad (n = 0, 1, 2, \dots). \quad (2)$$

Following [16], one can obtain a system of equations for the moments (2) from (1). It is sufficient to substitute the expansions in power series

$$\varphi(y, t) = \sum_{n=0}^\infty (-1)^n M_n(t) \frac{y^n}{n!}, \quad \hat{q}(y, t) = \sum_{n=0}^\infty (-1)^n q_n(t) \frac{y^n}{n!},$$

into (1), where

$$q_n(t) = \frac{1}{(2n+1)!!} 4\pi \int_0^\infty f(v, t) v^{2n+2} dv, \quad (n = 0, 1, 2, \dots),$$

are the normalized moments of the source function. As a result, one derives the moment system considered in [1]

$$\frac{dM_n(t)}{dt} + M_n(t)M_0(t) = \frac{1}{n+1} \sum_{k=0}^n M_k(t)M_{n-k}(t) + q_n(t). \quad (3)$$

For  $q(v, t) = 0$  this system was derived in [12] in a very complicated way.

Let us define moment generation functions for the distribution function  $f(v, t)$  and for the source function  $q(v, t)$ :

$$G(\omega, t) = \sum_{n=0}^\infty \omega^n M_n(t), \quad S(\omega, t) = \sum_{n=0}^\infty \omega^n q_n(t).$$

Multiplying Eq. (3) by  $\omega^n$ , and summing over all  $n$ , one obtains for  $G(\omega, t)$  the equation

$$\frac{\partial^2(\omega G)}{\partial t \partial \omega} + M_0(t) \frac{\partial(\omega G)}{\partial \omega} = G^2 + \frac{\partial(\omega S)}{\partial \omega}. \quad (4)$$

Here the obvious relations are used

$$\sum_{n=0}^\infty (n+1)\omega^n M_n(t) = \frac{\partial(\omega G)}{\partial \omega}, \quad \sum_{n=0}^\infty (n+1)\omega^n q_n(t) = \frac{\partial(\omega S)}{\partial \omega},$$

$$\sum_{n=0}^\infty \omega^n \sum_{k=0}^n M_k(t)M_{n-k}(t) = G^2.$$

In contrast to the case of homogeneous relaxation with  $q(v, t) = 0$ , the gas density  $M_0(t) \equiv \varphi(0, t)$  is not constant. From Eq. (3) for  $n=0$  one can obtain

$$M_0(t) = \int_0^t q_0(t') dt' + M_0(0).$$

Notice also that

$$M_0(t) = G(t, 0). \quad (5)$$

This is the reason why Eq. (4) has a non-local term. This fact was not taken into account in [1] in the process of finding an admitted Lie group. The lack of this condition can lead to uncorrected admitted Lie groups. In the present paper this omission is corrected.

### 3. Admitted Lie algebra of the generating function equation

Eq. (4) is conveniently rewritten in the form

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \quad (6)$$

where  $u(0) = u(t, 0)$ . Here  $\omega = x$ ,  $G = u$  and  $(\omega S)_\omega = g$ .

As mentioned, because of the presence of the term  $u(0)$ , Eq. (6) differs from a partial differential equation. Hence, the classical group analysis method cannot be applied to this equation. One needs to use the method developed for equations with non-local terms [2–4]. In this section the latter method is applied for finding an admitted Lie group of Eq. (6).

The admitted generator is sought in the form

$$X = \tau(t, x, u)\partial_t + \zeta(t, x, u)\partial_x + \zeta(t, x, u)\partial_u.$$

According to the algorithm [2–4], the determining equation for Eq. (6) is

$$x\psi_{tx} + \psi_t + u(0)(x\psi_x + \psi) - 2\psi u + \psi(0)(xu)_x = 0, \quad (7)$$

where

$$\psi(t, x) = \zeta(t, x, u(t, x)) - u_t(t, x)\tau(t, x, u(t, x)) - u_x(t, x)\zeta(t, x, u(t, x)), \\ \psi(0) = \psi(t, 0).$$

After substituting the derivatives  $u_{tx}$ ,  $u_{txx}$  and  $u_{ttx}$  found from Eq. (6) and its derivatives with respect to  $x$  and  $t$  into (7), one obtains the determining equation

$$\begin{aligned} &\zeta_{tx}x^2 + \zeta_t x + \zeta_{ux}gx + \zeta_{uu}u^2x + gx^2 + u^2\zeta - 2ux\zeta + ux\zeta(0) \\ &\quad - x(g_t\tau + g_x\zeta + g(\tau_t + \zeta_x)) \\ &\quad - \tau_t u^2x - \zeta_x u^2x + u(0)(\zeta_x x^2 - \zeta_{ux}ux - u\zeta + x\zeta + x\zeta_x u + x\tau_t u) \\ &\quad - xu_x(0)(u_x x + u)\zeta(0) \\ &\quad - \tau_x u_{tt}x^2 - x^2 u_x u_{tt} \tau_u - u_t u_{xx} \zeta_{ux} x^2 - u_{xx} \zeta_t x^2 \\ &\quad + u_t u_x x(\zeta_{uu}x - \tau_{tu}x + \tau_u u(0)x - \zeta_{xii}x + \zeta_{uu}) \\ &\quad + u_t(\zeta_x x + \zeta_{xu}x^2 - \zeta - \tau_{tx}x^2 - 2\tau_u x(g + u^2)) \\ &\quad + u(0)x(2\tau_u u - x\tau_x) \\ &\quad + u^2 x(\tau_u - x\tau_{xii}) + xu_t(0)(\tau - \tau(0))(u_x x + u) + u_x^2 x^2(\zeta_{uu}(0) - \zeta_{uu}) \\ &\quad - u_t^2 u_x \tau_{uu} x^2 \\ &\quad - u_t u_x^2 \zeta_{uu} x^2 + xu_x(x(\tau_t u(0) + \zeta(0) + \zeta_{tu}) - \zeta_{tx}x - \zeta_t \\ &\quad - 2\zeta_{ug} - 2\zeta_{uu}u^2 + 2\zeta_{uu}u(0)) = 0. \end{aligned} \quad (8)$$

Here

$$\tau(0) = \tau(t, 0, u(t, 0)), \quad \zeta(0) = \zeta(t, 0, u(t, 0)), \quad \zeta(0) = \zeta(t, 0, u(t, 0)),$$

$$u_t(0) = u_t(t, 0), \quad u_x(0) = u_x(t, 0).$$

Differentiating the determining Eq. (8) with respect to  $u_{tt}$ ,  $u_{xx}$ , and then with respect to  $u_t$  and  $u_x$ , one gets

$$\tau_u = 0, \quad \tau_x = 0, \quad \zeta_u = 0, \quad \zeta_t = 0.$$

Hence,

$$\tau = \tau(t), \quad \zeta = \zeta(x),$$

and

$$\tau(0) = \tau.$$

Differentiating the determining equation with respect to  $u_t$ , and then  $u_x$ , one finds

$$\zeta_{uu} = 0,$$

or

$$\zeta(t, x, u) = u\zeta_1(t, x) + \zeta_0(t, x).$$

The coefficient with  $u_x u_x(0)$  in the determining Eq. (8) gives  $\zeta(0) = 0$ . Continuing splitting the determining Eq. (8) with respect to  $u_t$ , and then  $u_x$ , one finds

$$\zeta_1(t, x) = -x^{-1}\zeta(x) + \zeta_{10}(t).$$

Hence,

$$\zeta(0) = \zeta(t, 0) = u(0)(\zeta_{10}(t) - \zeta'(0)) + \zeta_0(t, 0).$$

The coefficient with  $u_x u(0)$  leads to the condition

$$\zeta_{10} = -\tau_t + \zeta'(0).$$

Differentiating the determining equation with respect to  $u$  twice, one has

$$\xi_x = 2 \frac{\xi}{x} - \zeta'(0).$$

The general solution of this equation is

$$\xi = x(c_1 x + c_0).$$

Equating the coefficient with  $u_x$  to zero, one derives

$$\tau_{tt}(t) = \zeta_0(t, 0).$$

The coefficient with  $u(0)$  in the determining Eq. (8) gives

$$x \zeta_{0x} + \zeta_0 = 0.$$

This equation only has one solution which is non-singular at  $x=0$

$$\zeta_0(t, x) = 0.$$

Hence,  $\zeta_0(t, 0) = 0$ , and

$$\tau = c_2 t + c_3.$$

The remaining part of the determining Eq. (8) becomes

$$g_t(c_2 t + c_3) + x g_x(c_1 x + c_0) = -2g(c_1 x + c_2). \tag{9}$$

Thus, the admitted generator has the form

$$X = c_0 X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3,$$

where

$$X_0 = x \partial_x, \quad X_1 = x(x \partial_x - u \partial_u), \quad X_2 = t \partial_t - u \partial_u, \quad X_3 = \partial_t. \tag{10}$$

The values of the constants  $c_0, c_1, c_2, c_3$  and relations between them depend on the function  $g(t, x)$ .

The trivial case of the function

$$g = 0,$$

satisfies Eq. (9), and corresponds to the case of the spatially homogeneous and isotropic Boltzmann equation without a source term. In this case, using its Fourier image (1) with  $\hat{q}(y, t) = 0$ , the complete group classification of the Boltzmann equation was carried out in [2,14]. The four-dimensional Lie algebra  $L^4 = \{Y_1, Y_2, Y_3, Y_4\}$  spanned by the generators

$$Y_0 = y \partial_y, \quad Y_1 = y \varphi \partial_\varphi, \quad Y_2 = t \partial_t - \varphi \partial_\varphi, \quad Y_3 = \partial_t, \tag{11}$$

defines the complete admitted Lie group  $G^4$  of (1). There are direct relations between the generators (10) and (11).

Indeed, since the functions  $\varphi(y, t)$  and  $u(x, t)$  are related through the moments  $M_n(t)$  ( $n = 0, 1, 2, \dots$ ), it is sufficient to check that the transformations of moments defined through these functions coincide.

Let us consider the transformations corresponding to the generators  $Y_0$  and  $X_0$

$$Y_0 = y \partial_y : \bar{t} = t, \quad \bar{y} = ye^a, \quad \bar{\varphi} = \varphi,$$

$$X_0 = x \partial_x : \bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u.$$

The transformed functions are  $\bar{\varphi}(\bar{y}, \bar{t}) = \varphi(\bar{y}e^{-a}, \bar{t})$  and  $\bar{u}(\bar{x}, \bar{t}) = u(\bar{x}e^{-a}, \bar{t})$ . The transformations of moments are, respectively,

$$\begin{aligned} \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n \frac{\partial^n \varphi(\bar{y}e^{-a}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n e^{-na} \frac{\partial^n \varphi}{\partial y^n} (0, \bar{t}) = e^{-na} M_n(\bar{t}), \end{aligned}$$

$$\bar{M}_n(\bar{t}) = n! \frac{\partial^n u(\bar{x}e^{-a}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = e^{-na} n! \frac{\partial^n u}{\partial x^n} (0, \bar{t}) = e^{-na} M_n(\bar{t}).$$

Hence, one can see that the transformations of moments defined through the functions  $\varphi(y, t)$  and  $u(x, t)$  coincide.

The Lie groups of transformations corresponding to the generators  $Y_1$  and  $X_1$  are

$$Y_1 = y \varphi \partial_\varphi : \bar{t} = t, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{y^a},$$

$$X_1 = x(x \partial_x - u \partial_u) : \bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u.$$

These transformations map the functions  $\varphi(y, t)$  and  $u(x, t)$  to  $\bar{\varphi}(\bar{y}, \bar{t}) = e^{y^a} \varphi(\bar{y}, \bar{t})$  and  $\bar{u}(\bar{x}, \bar{t}) = (1/(1+a\bar{x}))u(\bar{t}, \bar{x}/(1+a\bar{x}))$ . The transformations of moments are, respectively,

$$\begin{aligned} \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n \frac{\partial^n (e^{y^a} \varphi(\bar{y}, \bar{t}))}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n \left( \left( \frac{\partial}{\partial y} + a \right)^n \varphi \right) (0, \bar{t}), \end{aligned}$$

$$\bar{M}_n(\bar{t}) = n! \frac{\partial^n \bar{u}(\bar{x}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = n! \frac{\partial^n}{\partial \bar{x}^n} \left( \frac{1}{1+a\bar{x}} u(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}) \right) \Big|_{\bar{x}=0}.$$

Using computer symbolic calculations on Reduce [17] one can check that these transformations of moments also coincide.

The Lie groups of transformations corresponding to the generators  $Y_2$  and  $X_2$  are

$$Y_2 = t \partial_t - \varphi \partial_\varphi : \bar{t} = te^a, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{-a},$$

$$X_2 = t \partial_t - u \partial_u : \bar{t} = te^a, \quad \bar{x} = x, \quad \bar{u} = ue^{-a}.$$

These transformations map the functions  $\varphi(y, t)$  and  $u(x, t)$  to  $\bar{\varphi}(\bar{y}, \bar{t}) = e^{-a} \varphi(\bar{y}, \bar{t}e^{-a})$  and  $\bar{u}(\bar{x}, \bar{t}) = e^{-a} u(\bar{x}, \bar{t}e^{-a})$ . The transformations of moments are, respectively,

$$\begin{aligned} \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n e^{-a} \frac{\partial^n \varphi(\bar{y}, \bar{t}e^{-a})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n e^{-a} \frac{\partial^n \varphi}{\partial y^n} (0, \bar{t}e^{-a}) = M_n(\bar{t}e^{-a}) e^{-a}, \end{aligned}$$

$$\begin{aligned} \bar{M}_n(\bar{t}) &= n! \frac{\partial^n \bar{u}(\bar{x}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = n! e^{-a} \frac{\partial^n u(\bar{x}, \bar{t}e^{-a})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} \\ &= n! e^{-a} \frac{\partial^n u}{\partial x^n} (0, \bar{t}e^{-a}) = M_n(\bar{t}e^{-a}) e^{-a}. \end{aligned}$$

The case where the transformations of moments corresponding to the generators  $Y_3 = \partial_t$  and  $X_3 = \partial_t$  coincide is trivial. These direct relations between the Lie algebras confirm correctness of our calculations.

#### 4. Comparison with algebra [1]

Using the variables of the present paper, let us formulate the results of [1]. The admitted generator obtained in [1] has the form

$$Z_g = \tau(t)(\partial_t - M_0(t)u \partial_u) + \alpha u \partial_u + (\gamma - \delta)x(x \partial_x - u \partial_u) - \gamma x \partial_x, \tag{12}$$

where

$$m_0(t) = \int_0^t M_0(t') dt', \quad \tau(t) = \left( \beta - \alpha \int_0^t e^{-m_0(t')} dt' \right) e^{m_0(t)},$$

$\alpha, \beta, \gamma$  and  $\delta$  are constant. The function  $g(t, x)$  has to satisfy the equation

$$\tau(t) \frac{\partial g}{\partial t} + x(x(\gamma - \delta) - \gamma) \frac{\partial g}{\partial x} = -2x(\gamma - \delta) + M_0(t)\tau(t) - \alpha g. \tag{13}$$

Since  $M_0(t)$  is unknown, comparison of our results is only possible for  $g = 0$ . Moreover, in contrast to Eq. (9), the source function  $g(t, x)$  in (13) as a solution of Eq. (13) depends on the function  $M_0(t)$ , whereas

the function  $M_0(t)$  also depends on the source function. This makes Eq. (13) non-local and very complicated.

Comparing the operator  $Z_g$  for  $g=0$  with (10), one obtains that the part related with the constants  $\gamma$  and  $\delta$  coincides with the result of the present paper, whereas the part related with the constants  $\alpha$  and  $\beta$  is completely different. Indeed, in this case Eq. (13) is satisfied identically,  $M_0(t) = M_0(0)$ , and for

$$M_0(0) \neq 0 : m_0(t) = tM_0(0), \quad \tau(t) = \beta e^{tM_0(0)} + \frac{\alpha}{M_0(0)}(1 - e^{tM_0(0)}),$$

$$M_0(0) = 0 : m_0(t) = 0, \quad \tau(t) = \beta - \alpha t.$$

The admitted generator (13) becomes

$$M_0(0) \neq 0 : Z_0 = \left( \beta - \frac{\alpha}{M_0(0)} \right) e^{tM_0(0)} (\partial_t - M_0(0)u\partial_u) + \frac{\alpha}{M_0(0)} \partial_t + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x,$$

$$M_0(0) = 0 : Z_0 = \beta\partial_t - \alpha(t\partial_t - u\partial_u) + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x.$$

One can see that our results only coincide with [1] for  $M_0(0) = 0$ . The case  $M_0(0) = 0$  corresponds to a gas with zero density which is not realistic. For  $M_0(0) \neq 0$ , the coefficient with the exponent  $e^{tM_0(0)}$  plays a crucial role. This coefficient only vanishes for

$$\alpha = M_0(0)\beta. \tag{14}$$

In this case the admitted Lie algebra found in [1] is a proper subalgebra of the Lie algebra defined by the generators (10). Thus, all invariant solutions with  $(\alpha, \beta, \gamma, \delta) = (M_0(0)\beta, \beta, \gamma, \delta)$  considered in [1] are particular cases of invariant solutions obtained in [2,4]. In particular, the well-known BKW-solution is an invariant solution with respect to the generator  $Y_{BKW} = c(Y_1 - Y_0) + Y_3$ . In the Lie algebra (10) this solution is related with the generator  $X_{BKW} = c(X_1 - X_0) + X_3$ . Other classes of invariant solutions studied in [1] correspond to (14) with the particular choice  $\beta = 0$ .

**5. On equivalence transformations of the generating function equation**

For the group classification one needs to know equivalence transformations. Let us find some of them using the generators (10) and considering their transformations of the left hand side of Eq. (6)

$$Lu = xu_{tx} + u_t - u^2 + u(0)(xu_x + u).$$

The transformations corresponding to the generator  $X_0 = x\partial_x$  map a function  $u(t,x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = u(\bar{t}, \bar{x}e^{-a}),$$

where  $a$  is the group parameter. Hence,

$$\begin{aligned} \bar{L}\bar{u} &= \bar{x} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{t}}(\bar{t}, \bar{x}) + \frac{\partial \bar{u}}{\partial \bar{t}}(\bar{t}, \bar{x}) - \bar{u}^2(\bar{t}, \bar{x}) + \bar{u}(\bar{t}, 0) \left( \bar{x} \frac{\partial \bar{u}}{\partial \bar{x}}(\bar{t}, \bar{x}) + \bar{u}(\bar{t}, \bar{x}) \right) \\ &= x \frac{\partial^2 u}{\partial x \partial t}(t, x) + \frac{\partial u}{\partial t}(t, x) - u^2(t, x) + u(t, 0) \left( x \frac{\partial u}{\partial x}(t, x) + u(t, x) \right). \end{aligned}$$

One can check that the Lie group of transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = g,$$

is an equivalence Lie group of Eq. (6).

Similarly, one derives that the transformations corresponding to the generator  $X_3 = \partial_t$  define the equivalence Lie group

$$\bar{t} = t + a, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{g} = g.$$

The transformations corresponding to the generator  $X_2 = t\partial_t - u\partial_u$  map a function  $u(t,x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = e^{-a} u(\bar{t}e^{-a}, \bar{x}).$$

Hence,

$$\begin{aligned} \bar{L}\bar{u} &= \bar{x} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{t}}(\bar{t}, \bar{x}) + \frac{\partial \bar{u}}{\partial \bar{t}}(\bar{t}, \bar{x}) - \bar{u}^2(\bar{t}, \bar{x}) + \bar{u}(\bar{t}, 0) \left( \bar{x} \frac{\partial \bar{u}}{\partial \bar{x}}(\bar{t}, \bar{x}) + \bar{u}(\bar{t}, \bar{x}) \right) \\ &= e^{-2a} \left( x \frac{\partial^2 u}{\partial x \partial t}(t, x) + \frac{\partial u}{\partial t}(t, x) - u^2(t, x) \right. \\ &\quad \left. + u(t, 0) \left( x \frac{\partial u}{\partial x}(t, x) + u(t, x) \right) \right). \end{aligned}$$

One can conclude that the transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = ge^{-2a},$$

compose an equivalence Lie group of Eq. (6).

The transformations corresponding to the generator  $X_1 = x(x\partial_x - u\partial_u)$  map a function  $u(t,x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = \frac{1}{1 + a\bar{x}} u \left( \bar{t}, \frac{\bar{x}}{1 + a\bar{x}} \right).$$

Hence,

$$\begin{aligned} \bar{L}\bar{u} &= \bar{x} \frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{t}}(\bar{t}, \bar{x}) + \frac{\partial \bar{u}}{\partial \bar{t}}(\bar{t}, \bar{x}) - \bar{u}^2(\bar{t}, \bar{x}) \\ &\quad + \bar{u}(\bar{t}, 0) \left( \bar{x} \frac{\partial \bar{u}}{\partial \bar{x}}(\bar{t}, \bar{x}) + \bar{u}(\bar{t}, \bar{x}) \right) \\ &= (1 - ax)^2 \left( x \frac{\partial^2 u}{\partial x \partial t}(t, x) + \frac{\partial u}{\partial t}(t, x) - u^2(t, x) \right. \\ &\quad \left. + u(t, 0) \left( x \frac{\partial u}{\partial x}(t, x) + u(t, x) \right) \right), \end{aligned}$$

and the transformations

$$\bar{t} = t, \quad \bar{x} = \frac{x}{1 - ax}, \quad \bar{u} = (1 - ax)u, \quad \bar{g} = (1 - ax)^2g,$$

compose an equivalence Lie group of transformations.

Thus, it has been shown that the Lie group corresponding to the generators

$$X_0^e = x\partial_x, \quad X_1^e = x(x\partial_x - u\partial_u - 2g\partial_g), \quad X_2^e = t\partial_t - u\partial_u - 2g\partial_g, \quad X_3^e = \partial_t,$$

is an equivalence Lie group of Eq. (6).

**6. Group classification**

Group classification of Eq. (6) is performed with respect to the equivalence transformations considered above.

Eq. (9) can be rewritten in the form

$$c_0 h_0 + c_1 h_1 + c_2 h_2 + c_3 h_3 = 0, \tag{15}$$

where

$$h_0 = xg_x, \quad h_1 = x(xg_x + 2g), \quad h_2 = tg_t + 2g, \quad h_3 = g_t. \tag{16}$$

For the classification of all possibilities, it is convenient to consider the functions  $h_i$ , ( $i=0,1,2,3$ ) as coordinates of the four-dimensional vector

$$v = (h_0, h_1, h_2, h_3).$$

The analysis of the relations between the constants  $c_0, c_1, c_2$  and  $c_3$  follows the algorithm developed for the gas dynamics equations [18]: one analyzes the vector space  $Span(V)$ , where the set  $V$  consists of the vectors  $v$  with  $t$  and  $x$  are changed. This algorithm allows one to study all possible admitted Lie algebras of Eq. (6) without omission.

First of all, if the function  $g(t,x)$  is such that changing  $t$  and  $x$ , one can find four linear independent vectors  $v$ , then Eq. (15) can only be satisfied in the case

$$c_0 = 0, \quad c_1 = 0, \quad c_2 = 0, \quad c_3 = 0.$$

In this case Eq. (6) does not admit a Lie group. Hence, for the existence of an admitted Lie group one needs to study  $\dim(\text{Span}(V)) \leq 3$ .

Another trivial case is  $\dim(\text{Span}(V)) = 0$ . One can derive that  $\dim(\text{Span}(V)) = 0$  if and only if  $g = 0$ . Indeed, if  $g = 0$ , then  $v = 0$ . Conversely, if  $\dim(\text{Span}(V)) = 0$ , then the vector  $v = (h_0, h_1, h_2, h_3)$  is constant with  $t$  and  $x$  are changed. This leads to the function  $g = 0$ .

Further study deals with  $1 \leq \dim(\text{Span}(V)) \leq 3$ .

6.1. Case  $\dim(\text{Span}(V)) = 1$

In this case there is a constant vector

$$(\alpha, \beta, \gamma, \theta) \neq 0,$$

such that

$$h_0 = \alpha p, \quad h_1 = \beta p, \quad h_2 = \gamma p, \quad h_3 = \theta p, \tag{17}$$

where  $p(t, x) \neq 0$  is some function which is not constant. Substituting (16) into (17), one obtains that

$$g = p(\gamma - t\theta)/2,$$

and

$$\beta - (\alpha + \gamma)x + t\theta = 0, \quad p_t(\gamma - t\theta) = 3p\theta, \quad xp_x(\gamma - t\theta) = 2p\alpha.$$

Hence,

$$\beta = 0, \quad \theta = 0, \quad \gamma = -\alpha \neq 0, \quad p_t = 0, \quad xp_x = -2p.$$

In this case one derives that  $g(t, x) = qx^{-2}$ , where  $q \neq 0$  is constant. Eq. (9) becomes

$$c_2 = c_0.$$

Thus, for the function  $g = qx^{-2}$  the admitted generators are

$$X_0 + X_2, \quad X_1, \quad X_3.$$

For constructing all invariant solutions related with the Lie algebra  $\{X_0 + X_2, X_1, X_3\}$  one needs to have an optimal system of subalgebras. Using the commutator table

	$X_0 + X_2$	$X_1$	$X_3$
$X_0 + X_2$	0	$X_1$	$-X_3$
$X_1$	$-X_1$	0	0
$X_3$	$X_3$	0	0

one obtains the inner automorphisms [18]

$$A_y : \tilde{x}_1 = x_1 e^{a_1}, \quad \tilde{x}_3 = x_3 e^{-a_1},$$

$$X_1 : \tilde{x}_1 = x_1 - a_2 y,$$

$$X_3 : \tilde{x}_3 = x_3 + a_3 y,$$

where only the transformed coordinates are presented, and  $y$  is used for the coordinate corresponding to the generator  $X_0 + X_2$ . Hence, an optimal system of subalgebras consists of the subalgebras

$$\{X_0 + X_2\}, \{X_1 + X_3\}, \{X_1\}, \{X_3\}.$$

Here the involution

$$E : \tilde{x}_1 = -x_1,$$

corresponding to the change  $x \rightarrow -x$  was also applied.

The representations of the invariant solutions are

$$X_0 + X_2 : u = \frac{1}{t} \varphi(z), \quad z = \frac{x}{t},$$

$$X_1 + X_3 : u = (xt + 1)\varphi(z), \quad z = \frac{x}{xt + 1},$$

$$X_1 : u = \frac{1}{x} \varphi(z), \quad z = t,$$

$$X_3 : u = \varphi(z), \quad z = x.$$

Substituting these representations into Eq. (6), one obtains the reduced equations

$$X_0 + X_2 : z^4 \varphi'' - z^3 (\varphi(0) - 3)\varphi' + z^2 \varphi(\varphi - \varphi(0) + 1) = -q,$$

$$X_1 + X_3 : z^5 \varphi'' + z^3 (4z - \varphi(0))\varphi' + z^2 \varphi(2z + \varphi - \varphi(0)) = -q,$$

$$X_3 : z^3 \varphi(0)\varphi' - z^2 \varphi(\varphi - \varphi(0)) = q.$$

Notice that since  $u(0)$  is not defined for the representation of an invariant solution corresponding to the subalgebra  $\{X_1\}$ , there is no reduced equation for this subalgebra.

6.2. Case  $\dim(\text{Span}(V)) = 2$

In this case there are two constant vectors  $(\alpha_i, \beta_i, \gamma_i, \theta_i)$  ( $i = 1, 2$ ) and functions  $p_i(t, x)$  such that  $p_1 p_2 \neq 0$ ,

$$\text{rank} \begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \theta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \theta_2 \end{pmatrix} = 2, \tag{18}$$

and

$$v = p_1(\alpha_1, \beta_1, \gamma_1, \theta_1) + p_2(\alpha_2, \beta_2, \gamma_2, \theta_2).$$

The last equation leads to the conditions

$$g = \frac{1}{2}(p_1(\gamma_1 - t\theta_1) + p_2(\gamma_2 - t\theta_2)),$$

$$p_1(\beta_1 - (\alpha_1 + \gamma_1)x + \theta_1 tx) + p_2(\beta_2 - (\alpha_2 + \gamma_2)x + \theta_2 tx) = 0,$$

$$p_{1t}(t\theta_1 - \gamma_1) + p_{2t}(t\theta_2 - \gamma_2) + 3p_1\theta_1 + 3p_2\theta_2 = 0,$$

$$p_{1x}x(t\theta_1 - \gamma_1) + p_{2x}x(t\theta_2 - \gamma_2) + 2p_1\alpha_1 + 2p_2\alpha_2 = 0. \tag{19}$$

Notice that  $\beta_1 - (\alpha_1 + \gamma_1)x + \theta_1 tx \neq 0$ , because otherwise condition (18) is violated. Hence,

$$p_1 = -p_2 \frac{\beta_2 - (\alpha_2 + \gamma_2)x + \theta_2 tx}{\beta_1 - (\alpha_1 + \gamma_1)x + \theta_1 tx},$$

and the remaining equations in (19) compose an overdetermined system of equations for the function  $p_2(t, x)$ . Since the analysis of compatibility of this system is cumbersome (although is not complicated), we omit it here. After finding the function  $g(t, x)$ , and substituting this function into Eq. (9), one finds relations between the constants  $c_0, c_1, c_2$  and  $c_3$  as it demonstrated for the  $\dim(\text{Span}(V)) = 1$ .

6.3. Case  $\dim(\text{Span}(V)) = 3$

Analysis of this case differs from the previous case using a single constant vector

$$(\alpha, \beta, \gamma, \theta) \neq 0,$$

which is orthogonal to the set  $V$

$$h_0\alpha + h_1\beta + h_2\gamma + h_3\theta = 0. \tag{20}$$

Substituting (16) into (20), one obtains a first-order partial differential equation which has the characteristic system of equations

$$\frac{dt}{\beta t + \gamma} = \frac{dx}{x(\alpha x + \theta)} = \frac{dg}{-2g(\alpha x + \beta)}.$$

As in the previous case, a solution of Eq. (20) depends on the values of  $\alpha, \beta, \gamma$  and  $\theta$ , and after finding the function  $g(t, x)$ , and substituting it into Eq. (9), one finds relations between the constants  $c_0, c_1, c_2$  and  $c_3$ .

## 7. Conclusion

The present paper deals with the group classification of Eq. (6) which, by the presence of the term  $u(0)$ , differs from a partial differential equation. The method developed for equations with non-local terms [2–4] was applied in the present paper. The result obtained matches with [2,14], and is essentially different in comparison with [1]. It is also worth to notice that after obtaining the function  $g(t,x)$ , one has to coordinate it with the source term of the original Boltzmann equation. Only then the invariant solutions of the spatially homogeneous and isotropic Boltzmann equation with isotropic scattering model can be considered.

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# On the Equation for the Power Moment Generating Function of the Boltzmann Equation. Group Classification with respect to a Source Function

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Lie symmetries of the spatially homogeneous and isotropic Boltzmann equation with sources were first studied by Nonnenmacher (1984). In fact, he considered the associated equations for the generating function of the power moments of the unknown distribution function. However, it was not taken into account that this equation is still a nonlocal partial differential equation. In the present paper their Lie symmetries are studied using the original approach developed by Grigoriev and Meleshko (1986) for group analysis of equations with nonlocal operators, which allows us to correct Nonnenmacher’s results. The group classification with respect to sources is carried out for the equations under consideration using the algebraic method.

## 1 Introduction

The Boltzmann kinetic equation is the basis of the classical kinetic theory of rarefied gases. Considerable interest in the study of the Boltzmann equation was always the search for exact (invariant) solutions directly associated with the fundamental properties of the equation. After the studies of the class of the local Maxwellians [3, 4, 11] new classes of invariant solutions were constructed in the 1960s in [13–15]. A decade later the BKW-solution was almost simultaneously derived in [1] and in [10]. Contrary to the Maxwellians, the Boltzmann collision integral does not vanish for this solution. The discovery of the BKW-solution stimulated a great splash of studies of exact solutions of various kinetic equations. However, the progress at that time was really limited to the construction of BKW-type solutions for different simplified models of the Boltzmann equation<sup>1</sup>.

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<sup>1</sup>See [5] for the review.

The Boltzmann equation is an integro-differential equation. Whereas the classical group analysis method has been developed for studying partial differential equations, the main obstacle for applying group analysis to integro-differential equations comes from presence of nonlocal integral operators. The direct group analysis for equations with nonlocal operators was worked out and successfully used in [6, 7, 12]. In particular, a complete group classification of the spatially homogeneous and isotropic Boltzmann equation without sources was obtained in [7, 8].

One of the alternative approaches for studying solutions of the Boltzmann equation, by transition to an equation for a moment generating function, was first considered in [10]. The BKW-solution was obtained there. In [16], such an approach was applied to the spatially homogeneous and isotropic Boltzmann equation with sources. The author of [16] used the group analysis method for studying solutions of the equation for the generating function. However, it was not taken into account that this equation is still a nonlocal one. In the present paper we use our method [7, 8] to amend the results of [16]. A group classification of the equation for a moment generating function with respect to a source function is obtained.

## 2 General equations

The Fourier image of the spatially homogeneous and isotropic Boltzmann equation with sources has the form [1]

$$\varphi_t(y, t) + \varphi(y, t)\varphi(0, t) = \int_0^1 \varphi(ys, t)\varphi(y(1-s), t) ds + \hat{q}(y, t). \quad (1)$$

Here the function  $\varphi(y, t)$  is related with the Fourier transform  $\tilde{\varphi}(k, t)$  of the isotropic distribution function  $f(v, t)$  by the formulae

$$\varphi(k^2/2, t) = \tilde{\varphi}(k, t) = \frac{4\pi}{k} \int_0^\infty v \sin(kv) f(v, t) dv.$$

The function  $\hat{q}(y, t)$  is defined by the Fourier transform of the source function  $q(v, t)$  in a similar way:

$$\hat{q}(k^2/2, t) = \tilde{\hat{q}}(k, t) = \frac{4\pi}{k} \int_0^\infty v \sin(kv) q(v, t) dv.$$

The inverse Fourier transform of  $\tilde{\varphi}(k, t)$  gives the distribution function

$$f(v, t) = \frac{4\pi}{v} \int_0^\infty k \sin(kv) \tilde{\varphi}(k, t) dk.$$

Normalized moments of the distribution function are introduced by the formulae

$$M_n(t) = \frac{4\pi}{(2n+1)!!} \int_0^\infty f(v, t) v^{2n+2} dv, \quad n = 0, 1, 2, \dots \quad (2)$$

Following [2], one can obtain a system of equations for the moments (2) from (1). It is sufficient to substitute the expansions in power series

$$\varphi(y, t) = \sum_{n=0}^{\infty} (-1)^n M_n(t) \frac{y^n}{n!}, \quad \hat{q}(y, t) = \sum_{n=0}^{\infty} (-1)^n q_n(t) \frac{y^n}{n!},$$

into (1), where

$$q_n(t) = \frac{1}{(2n+1)!!} 4\pi \int_0^{\infty} q(v, t) v^{2n+2} dv, \quad n = 0, 1, 2, \dots,$$

are the normalized moments of the source function. As a result, one derives the moment system considered in [16]:

$$\frac{dM_n(t)}{dt} + M_n(t)M_0(t) = \frac{1}{n+1} \sum_{k=0}^n M_k(t)M_{n-k}(t) + q_n(t). \quad (3)$$

For  $q(v, t) = 0$  this system was derived in [10] in a very complicated way.

Let us define moment generation functions for the distribution function  $f(v, t)$  and for the source function  $q(v, t)$ :

$$G(\omega, t) = \sum_{n=0}^{\infty} \omega^n M_n(t), \quad S(\omega, t) = \sum_{n=0}^{\infty} \omega^n q_n(t).$$

Multiplying equations (3) by  $\omega^n$ , and summing over all  $n$ , one obtains for  $G(\omega, t)$  the equation

$$\frac{\partial^2(\omega G)}{\partial t \partial \omega} + M_0(t) \frac{\partial(\omega G)}{\partial \omega} = G^2 + \frac{\partial(\omega S)}{\partial \omega}. \quad (4)$$

Here the obvious relations are used

$$\sum_{n=0}^{\infty} (n+1) \omega^n M_n(t) = \frac{\partial(\omega G)}{\partial \omega}, \quad \sum_{n=0}^{\infty} (n+1) \omega^n q_n(t) = \frac{\partial(\omega S)}{\partial \omega},$$

$$\sum_{n=0}^{\infty} \omega^n \sum_{k=0}^n M_k(t) M_{n-k}(t) = G^2.$$

In contrast to the case of homogeneous relaxation with  $q(v, t) = 0$ , the gas density  $M_0(t) \equiv \varphi(0, t)$  is not constant. From equation (3) for  $n = 0$  one can obtain

$$M_0(t) = \int_0^t q_0(t') dt' + M_0(0).$$

Notice also that

$$M_0(t) = G(t, 0). \quad (5)$$

This is the reason why equation (4) has a nonlocal term. This fact was not taken into account in [16] in the process of finding an admitted Lie group. The lack of this condition can lead to incorrect admitted Lie groups. In the present paper this omission is corrected.

### 3 Admitted Lie algebra of the equation for the generating function

Equation (4) is conveniently rewritten in the form

$$(xu_t)_x - u^2 + u(0)(xu)_x = g, \quad (6)$$

where  $u(0) = u(t, 0)$ . Here  $\omega = x$ ,  $G = u$  and  $(\omega S)_\omega = g$ .

As mentioned, because of the presence of the term  $u(0)$ , equation (6) is not a partial differential equation. Therefore, the classical group analysis method cannot be applied to this equation. A method that can be used for such equations with nonlocal terms was developed in [6, 7, 12]. In this section the latter method is applied for finding an admitted Lie group of equation (6).

Admitted generators are sought in the form

$$X = \tau(t, x, u)\partial_t + \xi(t, x, u)\partial_x + \zeta(t, x, u)\partial_u.$$

According to the algorithm [6, 7, 12], the determining equation for equation (6) is

$$xu_{tx} + \psi_t + u(0)(x\psi_x + \psi) - 2\psi u + \psi(0)(xu)_x = 0, \quad (7)$$

where

$$\begin{aligned} \psi(t, x) &= \zeta(t, x, u(t, x)) - u_t(t, x)\tau(t, x, u(t, x)) - u_x(t, x)\xi(t, x, u(t, x)), \\ \psi(0) &= \psi(t, 0). \end{aligned}$$

After substituting the derivatives  $u_{tx}$ ,  $u_{txx}$  and  $u_{ttx}$  found from equation (6) and its derivatives with respect to  $x$  and  $t$  into (7), one obtains the determining equation

$$\begin{aligned} &\zeta_{tx}x^2 + \zeta_tx + \zeta_u gx + \zeta_u u^2 x + g\xi + u^2\xi - 2ux\zeta + ux\zeta(0) \\ &- x(g_t\tau + g_x\xi + g(\tau_t + \xi_x)) - \tau_t u^2 x - \xi_x u^2 x - xu_x(0)(u_x x + u)\xi(0) \\ &+ u(0)(\zeta_x x^2 - \zeta_u ux - u\xi + x\zeta + x\xi_x u + x\tau_t u) - \tau_x u_{tt} x^2 - x^2 u_x u_{tt} \tau_u \\ &- u_t u_{xx} \xi_u x^2 - u_{xx} \xi_t x^2 + u_t u_{xx} (\zeta_{uu} x - \tau_{tu} x + \tau_u u(0)x - \xi_{xu} x + \xi_u) \\ &+ u_t (\xi_x x + \zeta_{xu} x^2 - \xi - \tau_{tx} x^2 - 2\tau_u x(g + u^2) + u(0)x(2\tau_u u - x\tau_x)) \\ &+ u_t^2 x(\tau_u - x\tau_{xu}) + xu_t(0)(\tau - \tau(0))(u_x x + u) + u_x^2 x^2 (\xi_u u(0) - \xi_{tu}) \\ &+ xu_x (x(\tau_t u(0) + \zeta(0) + \zeta_{tu}) - \xi_{tx} x - \xi_t - 2\xi_u g - 2\xi_u u^2 + 2\xi_u uu(0)) \\ &- u_t^2 u_x \tau_{uu} x^2 - u_t u_x^2 \xi_{uu} x^2 = 0. \end{aligned} \quad (8)$$

Here

$$\begin{aligned} \tau(0) &= \tau(t, 0, u(t, 0)), & \xi(0) &= \xi(t, 0, u(t, 0)), & \zeta(0) &= \zeta(t, 0, u(t, 0)), \\ u_t(0) &= u_t(t, 0), & u_x(0) &= u_x(t, 0). \end{aligned}$$

Differentiating the determining equation (8) with respect to  $u_{tt}$ ,  $u_{xx}$ , and then with respect to  $u_t$  and  $u_x$ , one gets  $\tau_u = 0$ ,  $\tau_x = 0$ ,  $\xi_u = 0$ ,  $\xi_t = 0$ . Therefore,

$$\tau = \tau(t), \quad \xi = \xi(x),$$

and hence  $\tau(0) = \tau$ . Differentiating the determining equation with respect to  $u_t$ , and then  $u_x$ , one finds  $\zeta_{uu} = 0$ , i.e.,

$$\zeta(t, x, u) = u\zeta_1(t, x) + \zeta_0(t, x).$$

The coefficient with  $u_x u_x(0)$  in the determining equation (8) gives  $\xi(0) = 0$ . Continuing splitting the determining equation (8) with respect to  $u_t$  and then with respect to  $u_x$ , one finds

$$\zeta_1(t, x) = -x^{-1}\xi(x) + \zeta_{10}(t).$$

Hence  $\zeta(0) = \zeta(t, 0) = u(0)(\zeta_{10}(t) - \xi'(0)) + \zeta_0(t, 0)$ . The coefficient with  $u_x u(0)$  leads to the condition

$$\zeta_{10} = -\tau_t + \xi'(0).$$

Differentiating the determining equation with respect to  $u$  twice, one has

$$\xi_x = 2\frac{\xi}{x} - \xi'(0).$$

The general solution of this equation is

$$\xi = x(c_1 x + c_0).$$

Equating the coefficient with  $u_x$  to zero, one derives  $\tau_{tt}(t) = \zeta_0(t, 0)$ . The coefficient with  $u(0)$  in the determining equation (8) gives  $x\zeta_{0x} + \zeta_0 = 0$ . This equation has a unique solution which is nonsingular at  $x = 0$ ,

$$\zeta_0(t, x) = 0.$$

Therefore,  $\zeta_0(t, 0) = 0$  and

$$\tau = c_2 t + c_3.$$

The remaining part of the determining equation (8) becomes

$$g_t(c_2 t + c_3) + x g_x(c_1 x + c_0) = -2g(c_1 x + c_2). \quad (9)$$

Thus, each admitted generator has the form

$$X = c_0 X_0 + c_1 X_1 + c_2 X_2 + c_3 X_3,$$

where

$$X_0 = x\partial_x, \quad X_1 = x(x\partial_x - u\partial_u), \quad X_2 = t\partial_t - u\partial_u, \quad X_3 = \partial_t. \quad (10)$$

The values of the constants  $c_0$ ,  $c_1$ ,  $c_2$  and  $c_3$  and relations between them depend on the function  $g(t, x)$ .

The trivial case of the function

$$g = 0$$

satisfies equation (9), and corresponds to the case of the spatially homogeneous and isotropic Boltzmann equation without a source term. In this case, the complete group classification of the Boltzmann equation was carried out in [7, 8] using its Fourier image (1) with  $\hat{q}(y, t) = 0$ . The four-dimensional Lie algebra  $L^4 = \{Y_1, Y_2, Y_3, Y_4\}$  spanned by the generators

$$Y_0 = y\partial_y, \quad Y_1 = y\varphi\partial_\varphi, \quad Y_2 = t\partial_t - \varphi\partial_\varphi, \quad Y_3 = \partial_t \quad (11)$$

defines the complete admitted Lie group  $G^4$  of (1). There are direct relations between the generators (10) and (11).

Indeed, since the functions  $\varphi(y, t)$  and  $u(x, t)$  are related through the moments  $M_n(t)$ ,  $n = 0, 1, 2, \dots$ , it is sufficient to check that the transformations of moments defined through these functions coincide.

Consider the transformations corresponding to the generators  $Y_0$  and  $X_0$ ,

$$\begin{aligned} Y_0 = y\partial_y: \quad & \bar{t} = t, \quad \bar{y} = ye^a, \quad \bar{\varphi} = \varphi; \\ X_0 = x\partial_x: \quad & \bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u. \end{aligned}$$

The transformed functions are

$$\bar{\varphi}(\bar{y}, \bar{t}) = \varphi(\bar{y}e^{-a}, \bar{t}), \quad \bar{u}(\bar{x}, \bar{t}) = u(\bar{x}e^{-a}, \bar{t}).$$

The transformations of moments are, respectively:

$$\begin{aligned} \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n \frac{\partial^n \varphi(\bar{y}e^{-a}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n e^{-na} \frac{\partial^n \varphi}{\partial y^n}(0, \bar{t}) = e^{-na} M_n(\bar{t}); \\ \bar{M}_n(\bar{t}) &= n! \frac{\partial^n u(\bar{x}e^{-a}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = e^{-na} n! \frac{\partial^n u}{\partial x^n}(0, \bar{t}) = e^{-na} M_n(\bar{t}). \end{aligned}$$

Hence, one can see that the transformations of moments defined through the functions  $\varphi(y, t)$  and  $u(x, t)$  coincide.

The transformations corresponding to the generators  $Y_1$  and  $X_1$ ,

$$\begin{aligned} Y_1 = y\varphi\partial_\varphi: \quad & \bar{t} = t, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{ya}; \\ X_1 = x(x\partial_x - u\partial_u): \quad & \bar{t} = t, \quad \bar{x} = \frac{x}{1-ax}, \quad \bar{u} = (1-ax)u, \end{aligned}$$

act on the functions  $\varphi(y, t)$  and  $u(x, t)$  and their moments in the following way:

$$\begin{aligned}\bar{\varphi}(\bar{y}, \bar{t}) &= e^{\bar{y}a} \varphi(\bar{y}, \bar{t}), \quad \bar{u}(\bar{x}, \bar{t}) = \frac{1}{1+a\bar{x}} u\left(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}\right), \\ \bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n \frac{\partial^n (e^{\bar{y}a} \varphi(\bar{y}, \bar{t}))}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n \left( \left( \frac{\partial}{\partial y} + a \right)^n \varphi \right) (0, \bar{t}); \\ \bar{M}_n(\bar{t}) &= n! \frac{\partial^n \bar{u}(\bar{x}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = n! \frac{\partial^n}{\partial \bar{x}^n} \left( \frac{1}{1+a\bar{x}} u\left(\bar{t}, \frac{\bar{x}}{1+a\bar{x}}\right) \right) \Big|_{\bar{x}=0},\end{aligned}$$

respectively. Using computer symbolic calculations with REDUCE [9] one can check that these transformations of moments also coincide.

The vector fields  $Y_2$  and  $X_2$  generate the following transformations:

$$\begin{aligned}Y_2 = t\partial_t - \varphi\partial_\varphi: \quad & \bar{t} = te^a, \quad \bar{y} = y, \quad \bar{\varphi} = \varphi e^{-a}; \\ X_2 = t\partial_t - u\partial_u: \quad & \bar{t} = te^a, \quad \bar{x} = x, \quad \bar{u} = ue^{-a},\end{aligned}$$

which map the functions  $\varphi(y, t)$  and  $u(x, t)$  to the functions  $\bar{\varphi}(\bar{y}, \bar{t}) = e^{-a} \varphi(\bar{y}, \bar{t}e^{-a})$  and  $\bar{u}(\bar{x}, \bar{t}) = e^{-a} u(\bar{x}, \bar{t}e^{-a})$ , respectively. The transformations of moments are

$$\begin{aligned}\bar{M}_n(\bar{t}) &= (-1)^n \frac{\partial^n \bar{\varphi}(\bar{y}, \bar{t})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} = (-1)^n e^{-a} \frac{\partial^n \varphi(\bar{y}, \bar{t}e^{-a})}{\partial \bar{y}^n} \Big|_{\bar{y}=0} \\ &= (-1)^n e^{-a} \frac{\partial^n \varphi}{\partial y^n} (0, \bar{t}e^{-a}) = M_n(\bar{t}e^{-a}) e^{-a}; \\ \bar{M}_n(\bar{t}) &= n! \frac{\partial^n \bar{u}(\bar{x}, \bar{t})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} = n! e^{-a} \frac{\partial^n u(\bar{x}, \bar{t}e^{-a})}{\partial \bar{x}^n} \Big|_{\bar{x}=0} \\ &= n! e^{-a} \frac{\partial^n u}{\partial x^n} (0, \bar{t}e^{-a}) = M_n(\bar{t}e^{-a}) e^{-a}.\end{aligned}$$

The case where the transformations of moments corresponding to the generators  $Y_3 = \partial_t$  and  $X_3 = \partial_t$  coincide is trivial. These direct relations between the Lie algebras confirm correctness of our calculations.

## 4 Comparison with results of [16]

Let us formulate the results of [16] using the variables of the present paper. The admitted generator obtained in [16] has the form

$$Z_g = \tau(t) (\partial_t - M_0(t)u\partial_u) + \alpha u\partial_u + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x, \quad (12)$$

where

$$m_0(t) = \int_0^t M_0(t') dt', \quad \tau(t) = \left( \beta - \alpha \int_0^t e^{-m_0(t')} dt' \right) e^{m_0(t)},$$

$\alpha, \beta, \gamma$  and  $\delta$  are constants. The function  $g(t, x)$  has to satisfy the equation

$$\tau(t) \frac{\partial g}{\partial t} + x(x(\gamma - \delta) - \gamma) \frac{\partial g}{\partial x} = -2(x(\gamma - \delta) + M_0(t)\tau(t) - \alpha)g. \quad (13)$$

Since  $M_0(t)$  is unknown, comparison of our results is only possible for  $g = 0$ . Moreover, in contrast to equation (9), the source function  $g(t, x)$  in (13) as a solution of equation (13) depends on the function  $M_0(t)$ , whereas the function  $M_0(t)$  also depends on the source function. This makes equation (13) nonlocal and very complicated.

Comparing the operator  $Z_g$  for  $g = 0$  with (10), one obtains that the part related with the constants  $\gamma$  and  $\delta$  coincides with the result of the present paper, whereas the part related with the constants  $\alpha$  and  $\beta$  is completely different. Indeed, in this case equation (13) is satisfied identically,  $M_0(t) = M_0(0)$ , and for

$$M_0(0) \neq 0: \quad m_0(t) = tM_0(0), \quad \tau(t) = \beta e^{tM_0(0)} + \frac{\alpha}{M_0(0)} \left(1 - e^{tM_0(0)}\right);$$

$$M_0(0) = 0: \quad m_0(t) = 0, \quad \tau(t) = \beta - \alpha t.$$

The admitted generator (13) becomes

$$M_0(0) \neq 0: \quad Z_0 = \left(\beta - \frac{\alpha}{M_0(0)}\right) e^{tM_0(0)} (\partial_t - M_0(0)u\partial_u) + \frac{\alpha}{M_0(0)} \partial_t \\ + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x;$$

$$M_0(0) = 0: \quad Z_0 = \beta\partial_t - \alpha(t\partial_t - u\partial_u) + (\gamma - \delta)x(x\partial_x - u\partial_u) - \gamma x\partial_x.$$

One can see that the above results coincide with [16] only for  $M_0(0) = 0$ . The case  $M_0(0) = 0$  corresponds to a gas with zero density which is not realistic. For  $M_0(0) \neq 0$ , the coefficient with the exponent  $e^{tM_0(0)}$  plays a crucial role. This coefficient only vanishes for

$$\alpha = M_0(0)\beta. \quad (14)$$

In this case the admitted Lie algebra found in [16] is a proper subalgebra of the Lie algebra defined by the generators (10). Thus, all invariant solutions with  $(\alpha, \beta, \gamma, \delta) = (M_0(0)\beta, \beta, \gamma, \delta)$  considered in [16] are particular cases of invariant solutions obtained in [6, 7]. In particular, the well-known BKW-solution is an invariant solution with respect to the generator  $Y_{\text{BKW}} = c(Y_1 - Y_0) + Y_3$ . In the Lie algebra (10), this solution is related with the generator  $X_{\text{BKW}} = c(X_1 - X_0) + X_3$ . Other classes of invariant solutions studied in [16] correspond to (14) with the particular choice of  $\beta = 0$ .

## 5 On equivalence transformations of the equation for the generating function

For the group classification, one needs to know equivalence transformations. Let us find some of them using the generators (10) and considering their transforma-

tions of the left hand side of equation (6)

$$Lu = xu_{tx} + u_t - u^2 + u(0)(xu_x + u).$$

The transformations corresponding to the generator  $X_0 = x\partial_x$  map a function  $u(t, x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = u(\bar{t}, \bar{x}e^{-a}),$$

where  $a$  is the group parameter. Hence  $\bar{L}\bar{u} = Lu$ . One can check that the Lie group of transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = g$$

is an equivalence Lie group of equation (6).

Similarly, one derives that the transformations corresponding to the generator  $X_3 = \partial_t$  define the equivalence Lie group:

$$\bar{t} = t + a, \quad \bar{x} = x, \quad \bar{u} = u, \quad \bar{g} = g.$$

The transformations corresponding to the generator  $X_2 = t\partial_t - u\partial_u$  map a function  $u(t, x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = e^{-a}u(\bar{t}e^{-a}, \bar{x}).$$

Hence  $\bar{L}\bar{u} = e^{-2a}Lu$ . One can conclude that the transformations

$$\bar{t} = t, \quad \bar{x} = xe^a, \quad \bar{u} = u, \quad \bar{g} = ge^{-2a}$$

compose an equivalence Lie group of equation (6).

The transformations corresponding to the generator  $X_1 = x(x\partial_x - u\partial_u)$  map a function  $u(t, x)$  into the function

$$\bar{u}(\bar{t}, \bar{x}) = \frac{1}{1 + a\bar{x}}u\left(\bar{t}, \frac{\bar{x}}{1 + a\bar{x}}\right).$$

Hence  $\bar{L}\bar{u} = (1 - ax)^2Lu$  and the transformations

$$\bar{t} = t, \quad \bar{x} = \frac{x}{1 - ax}, \quad \bar{u} = (1 - ax)u, \quad \bar{g} = (1 - ax)^2g$$

compose an equivalence Lie group of transformations.

Thus, it has been shown that the Lie group corresponding to the generators

$$X_0^e = x\partial_x, \quad X_1^e = x(x\partial_x - u\partial_u - 2g\partial_g), \quad X_2^e = t\partial_t - u\partial_u - 2g\partial_g, \quad X_3^e = \partial_t$$

is an equivalence Lie group of equation (6).

There are also two involutions corresponding to the changes

$$E_1: \bar{x} = -x; \quad E_2: \bar{t} = -t, \quad \bar{u} = -u.$$

## 6 Group classification

Group classification of equation (6) is carried out up to the equivalence transformations considered above.

Equation (9) can be rewritten in the form

$$c_0 h_0 + c_1 h_1 + c_2 h_2 + c_3 h_3 = 0, \quad (15)$$

where

$$h_0 = xg_x, \quad h_1 = x(xg_x + 2g), \quad h_2 = tg_t + 2g, \quad h_3 = g_t. \quad (16)$$

One of the methods for analyzing relations between the constants  $c_0$ ,  $c_1$ ,  $c_2$  and  $c_3$  is employing the algorithm developed for the gas dynamics equations [17]: one analyzes the vector space  $\text{Span}(V)$ , where the set  $V$  consists of the vectors

$$v = (h_0, h_1, h_2, h_3)$$

with  $t$  and  $x$  are varied. This algorithm allows one to study all possible admitted Lie algebras of equation (6) without omission. Unfortunately, it is difficult to implement.

In [20]<sup>2</sup> an algebraic algorithm for group classification was applied, which essentially reduces this study to a simpler problem. Here we follow this algorithm<sup>3</sup>. Observe here that because of the nonlinearity of the equivalence transformations corresponding to the generator  $X_1$ , it is difficult to select out equivalent cases with respect to these transformations, whereas the algebraic algorithm does not have such complication.

First we study the Lie algebra  $L_4$  composed by the generators  $X_0$ ,  $X_1$ ,  $X_2$  and  $X_3$ . The commutator table is

	$X_0$	$X_1$	$X_2$	$X_3$
$X_0$	0	$X_1$	0	0
$X_1$	$-X_1$	0	0	0
$X_2$	0	0	0	$-X_3$
$X_3$	0	0	$X_3$	0

The inner automorphisms are

$$\begin{aligned} A_0: \quad \hat{x}_1 &= x_1 e^a, \\ A_1: \quad \hat{x}_1 &= x_1 + ax_0, \\ A_2: \quad \hat{x}_3 &= x_3 e^a, \\ A_3: \quad \hat{x}_3 &= x_3 + ax_2, \end{aligned}$$

where only the changed coordinates are presented.

<sup>2</sup>See also references therein.

<sup>3</sup>The authors thank the anonymous referee for pointing to the possibility of applying to the analysis of equation (15) the algorithm considered in [20].

Second, one can notice that the results of using the equivalence transformations corresponding to the generators  $X_0^e, X_1^e, X_2^e, X_3^e$  are similar to changing coordinates of a generator  $X$  with regarding to the basis change. These changes are similar to the inner automorphisms. Indeed, the coefficients of the generator  $X$  are changed according to the relation [17]:

$$X = (X\bar{t})\partial_{\bar{t}} + (X\bar{x})\partial_{\bar{x}} + (X\bar{u})\partial_{\bar{u}}.$$

Any generator  $X$  can be expressed as a linear combination of the basis generators:

$$\hat{x}_0\hat{X}_0 + \hat{x}_1\hat{X}_1 + \hat{x}_2\hat{X}_2 + \hat{x}_3\hat{X}_3 = x_0X_0 + x_1X_1 + x_2X_2 + x_3X_3, \quad (17)$$

where

$$\hat{X}_0 = \bar{x}\partial_{\bar{x}}, \quad \hat{X}_1 = \bar{x}(\bar{x}\partial_{\bar{x}} - \bar{u}\partial_{\bar{u}}), \quad \hat{X}_2 = \bar{t}\partial_{\bar{t}} - \bar{u}\partial_{\bar{u}}, \quad \hat{X}_3 = \partial_{\bar{t}}.$$

Using the invariance of a generator with respect to a change of the variables, the basis generators  $X_i$  ( $i = 0, 1, 2, 3$ ) and  $\hat{X}_j$  ( $j = 0, 1, 2, 3$ ) in corresponding equivalence transformations are related as follows

$$\begin{aligned} X_0^e: & \quad X_0 = \hat{X}_0, \quad X_1 = e^{-a}\hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3; \\ X_1^e: & \quad X_0 = \hat{X}_0 + a\hat{X}_1, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = \hat{X}_3; \\ X_2^e: & \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2, \quad X_3 = e^a\hat{X}_3; \\ X_3^e: & \quad X_0 = \hat{X}_0, \quad X_1 = \hat{X}_1, \quad X_2 = \hat{X}_2 - a\hat{X}_3, \quad X_3 = \hat{X}_3. \end{aligned}$$

Substituting these relations into the identity (17), one obtains that the coordinates of the generator  $X$  in the basis  $X_0, X_1, X_2, X_3$  and in the basis  $\hat{X}_0, \hat{X}_1, \hat{X}_2, \hat{X}_3$  are related similar to the changes defined by the inner automorphisms.

This observation allows us to use an optimal system of subalgebras of the Lie algebra  $L_4$  for studying equation (15). Construction of such optimal system is not difficult. Moreover, it may be simplified if one notices that  $L_4 = F_1 \oplus F_2$ , where  $F_1 = \{X_0, X_1\}$  and  $F_2 = \{X_2, X_3\}$  are ideals of the Lie algebra  $L_4$ . This decomposition gives possibility to apply a two-step algorithm [18, 19]. The result of construction of an optimal system of subalgebras is presented in Table 1.

Notice also that in constructing the optimal system of subalgebras we also used transformations corresponding to the involutions  $E_1$  and  $E_2$ :

$$E_1: \hat{x}_1 = -x_1; \quad E_2: \hat{x}_3 = -x_3.$$

To obtain functions  $g(t, x)$  using the optimal system of subalgebras one needs to substitute the constants  $c_i$  corresponding to the basis generators of a subalgebra into equation (15), and solve the system of equations thus obtained. The result of group classification is presented in Table 2, where  $\alpha, \beta \neq 1, \gamma \neq -2$  and  $k$  are constant, and the function  $\Phi$  is an arbitrary function of its argument.

**Table 1.** Optimal system of subalgebras

	Basis		Basis
1	$X_0, X_1, X_2, X_3$	11	$X_2 + \alpha X_0, X_1$
2	$X_2 + \alpha X_0, X_1, X_3$	12	$X_0 + X_3, X_1$
3	$X_0, X_1, X_3$	13	$X_3, X_1$
4	$X_0, X_1, X_2$	14	$X_0, X_1$
5	$X_0, X_2, X_3$	15	$X_2 + \alpha X_0$
6	$X_2, X_3$	16	$X_2 + X_1$
7	$X_2 - X_0, X_1 + X_3$	17	$X_3 + X_0$
8	$\alpha X_2 - 2X_0, X_3$	18	$X_3 + X_1$
9	$X_1 + X_2, X_3$	19	$X_0$
10	$X_0, X_2$	20	$X_1$
		21	$X_3$

**Table 2.** Group classification

	$g(t, x)$	Generators		$g(t, x)$	Generators
1	0	$X_0, X_1, X_2, X_3$	9	$kx^{-2}$	$X_1, X_3$
2	$kx^{-2}$	$X_2 + X_0, X_3, X_1$	10	$t^{-2}\Phi(xt^{-\alpha})$	$X_2 + \alpha X_0$
3	$kx^2(xt + 1)^{-4}$	$X_2 - X_0, X_1 + X_3$	11	$x^{-2}e^{2x^{-1}}\Phi(te^{x^{-1}})$	$X_2 + X_1$
4	$kx^\gamma$	$\gamma X_2 - 2X_0, X_3$	12	$\Phi(xe^{-t})$	$X_3 + X_0$
5	$kx^{-2}e^{2x^{-1}}$	$X_2 + X_1, X_3$	13	$x^{-2}\Phi(t + x^{-1})$	$X_3 + X_1$
6	$kt^{-2}$	$X_0, X_2$	14	$\Phi(t)$	$X_0$
7	$kx^{-2}t^{2(\beta-1)}$	$X_2 + \beta X_0, X_1$	15	$x^{-2}\Phi(t)$	$X_1$
8	$kx^{-2}e^{2t}$	$X_3 + X_0, X_1$	16	$\Phi(x)$	$X_3$

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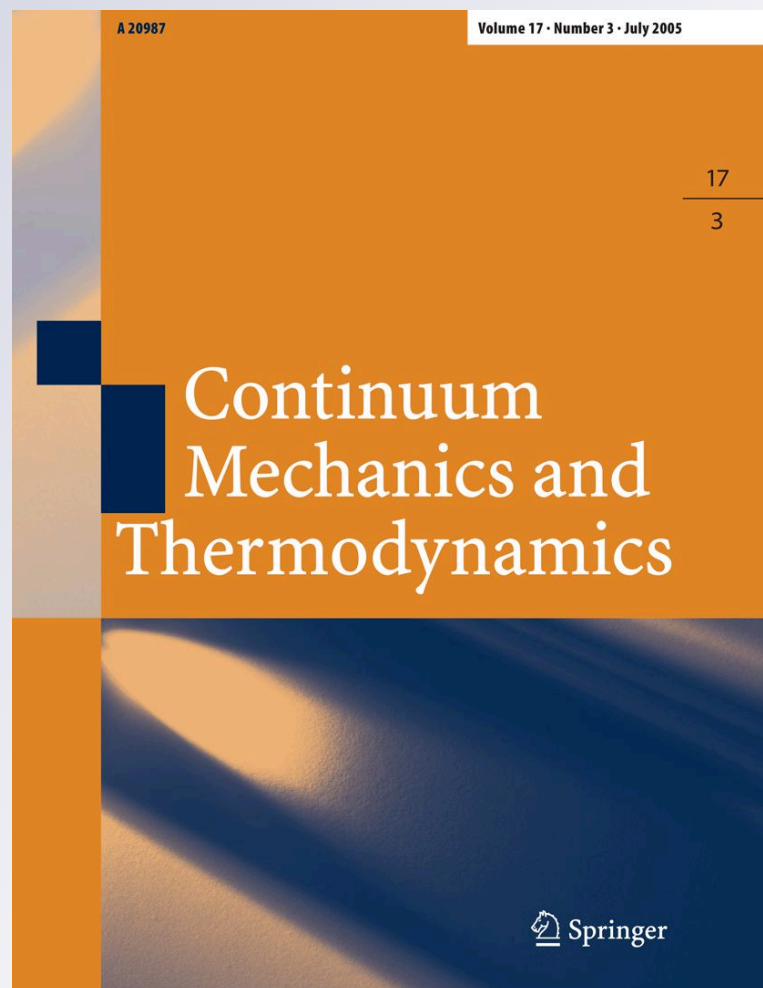
# *Group classification of one-dimensional nonisentropic equations of fluids with internal inertia*

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# Group classification of one-dimensional nonisentropic equations of fluids with internal inertia

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**Abstract** A systematic application of the group analysis method for modeling fluids with internal inertia is presented. The equations studied include models such as the nonlinear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles (Iordanski Zhurnal Prikladnoj Mekhaniki i Tekhnitheskoj Fiziki **3**, 102–111, 1960; Kogarko Dokl. AS USSR **137**, 1331–1333, 1961; Wijngaarden J. Fluid Mech. **33**, 465–474, 1968), and the dispersive shallow water model (Green and Naghdi J. Fluid Mech. **78**, 237–246, 1976; Salmon 1988). These models are obtained for special types of the potential function  $W(\rho, \dot{\rho}, S)$  (Gavrilyuk and Teshukov Continuum Mech. Thermodyn. **13**, 365–382, 2001). The main feature of the present paper is the study of the potential functions with  $W_S \neq 0$ . The group classification separates these models into 73 different classes.

**Keywords** Group classification · Equivalence Lie group · Admitted Lie group · Fluids with internal inertia

**Mathematics Subject Classification (2000)** 58D19 · 58Z05

## 1 Introduction

Symmetry is a fundamental topic in many areas of physics and mathematics [1–4]. Whereas group-theoretical methods play a prominent role in modern theoretical physics, a systematic use of them in constructing models of continuum mechanics has not been widely applied yet [5]. The present paper tries to help fill this niche.

This manuscript is focused on group classification of a class of dispersive models [6, see also references therein]

$$\begin{aligned} \dot{\rho} + \rho \operatorname{div}(u) &= 0, \quad \rho \dot{u} + \nabla p = 0, \quad \dot{S} = 0, \\ p &= \rho \frac{\delta W}{\delta \rho} - W = \rho \left( \frac{\partial W}{\partial \rho} - \frac{\partial}{\partial t} \left( \frac{\partial W}{\partial \dot{\rho}} \right) - \operatorname{div} \left( \frac{\partial W}{\partial \dot{\rho}} u \right) \right) - W, \end{aligned} \quad (1)$$

where  $t$  is time,  $\nabla$  is the gradient operator with respect to space variables,  $\rho$  is the fluid density,  $u$  is the velocity field,  $W(\rho, \dot{\rho}, S)$  is a given potential, "dot" denotes the material time derivative:  $\dot{f} = \frac{df}{dt} = f_t + u \nabla f$  and

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$\frac{\delta W}{\delta \rho}$  denotes the variational derivative of  $W$  with respect to  $\rho$  at a fixed value of  $u$ . These models include the nonlinear one-velocity model of a bubbly fluid (with incompressible liquid phase) at small volume concentration of gas bubbles [7–9], and the dispersive shallow water model [10, 11]. Equation (1) were obtained in [6] using the Lagrangian of the form

$$L = \frac{1}{2}|u|^2 - W(\rho, \dot{\rho}, S).$$

This is an example of a medium behavior dependent not only on thermodynamical variables but also on their derivatives with respect to space and time. In this particular case, the potential function depends on the total derivative of the density which reflects the dependence of the medium on its inertia. Another example of models where the medium behavior depends on the derivatives is constructed in [12] by assuming that the Lagrangian is of the form:

$$L = \frac{1}{2}|u|^2 - \varepsilon(\rho, |\nabla\rho|, S).$$

One of the methods for studying properties of differential equations is group analysis [1, 2, 13]. This method is a basic method for constructing exact solutions of partial differential equations. A wide range of applications of group analysis to partial differential equations are collected in [14–16]. Group analysis, besides facilitating the construction of exact solutions, provides a regular procedure for mathematical modeling by classifying differential equations with respect to arbitrary elements. This feature of group analysis is the fundamental basis for mathematical modeling in the present paper.

An application of group analysis employs several steps. The first step is a group classification with respect to arbitrary elements. An algorithm of the group classification is applied in case where a system of differential equations has arbitrary elements in the form of undefined parameters and functions. This algorithm is necessary since a specialization of the arbitrary elements can lead to an extension of admitted Lie groups. Group classification selects the functions  $W(\rho, \dot{\rho}, S)$  such that the fluid dynamics equations (1) possess additional symmetry properties extending the kernel of admitted Lie groups. Algorithms of finding equivalence and admitted Lie groups are particular parts of the algorithm of the group classification.

A complete group classification of Eq. (1), where  $W = W(\rho, \dot{\rho})$  is performed in [17] (one-dimensional case) and [18] (three-dimensional case). Invariant solutions of some particular cases which are separated out by the group classification are considered in [17–19]. Group classification of the class of models describing the behavior of a dispersive continuum with  $\varepsilon = \varepsilon(\rho, |\nabla\rho|)$  was studied in [20]. It is also worth to notice that the classical gas dynamics model corresponds to  $W = W(\rho, S)$  (or  $\varepsilon = \varepsilon(\rho, S)$ ). A complete group classification of the gas dynamics equations was presented in [1]. Later, an exhausted program of studying the models appeared in the group classification of the gas dynamics equations was announced in [5]. Some results of this program were summarized in [21].

The present paper is focused on the group classification of the one-dimensional equations of fluids (1), where the function  $W = W(\rho, \dot{\rho}, S)$  satisfies the conditions  $W_{S\dot{\rho}\dot{\rho}} = 0$  and  $W_S \neq 0$ .

The paper is organized as follows. The following section studies the equivalence Lie group of transformations. The equivalence transformations are applied for simplifying the function  $W(\rho, \dot{\rho}, S)$  in the process of the classification. In Sect. 3, the defining equations of the admitted Lie group are presented. Analysis of these equations separates Eq. (1) into equivalent classes. It should be noted that these classes are defined by the function  $W(\rho, \dot{\rho}, S)$ . For convenience of the reader, this analysis is split into two parts. A complete study of one particular case is given in Sect. 4. Analysis of the other cases is similar but cumbersome. A complete study of the other cases is provided in Appendix. The result of the group classification of Eq. (1) where  $W_{S\dot{\rho}\dot{\rho}} = 0$  and  $W_S \neq 0$  is summarized in Table 1. The admitted Lie algebras are also presented in this table.

## 2 Equivalence Lie group

For finding an equivalence Lie group, the algorithm described in [22, 23] is applied. This algorithm differs from the classical one [1] by assuming dependence of all coefficients from all variables including the arbitrary elements. Since the function  $W$  depends on the derivatives of the dependent variables and in order to simplify the process of finding an equivalence Lie group, new dependent variables are introduced:

$$u_3 = \dot{\rho}, \quad u_4 = S.$$

Here,  $u_1 = \rho$ ,  $u_2 = u$ ,  $u_3 = \dot{\rho}$  and  $u_4 = S$ ,  $x_1 = x$ ,  $x_2 = t$ . An infinitesimal operator  $X^e$  of the equivalence Lie group is sought in the form [23]

$$X^e = \xi^i \partial_{x_i} + \zeta^{u_j} \partial_{u_j} + \zeta^W \partial_W,$$

where all the coefficients  $\xi^i$ ,  $\zeta^{u_j}$ , ( $i = 1, 2$ ;  $j = 1, 2, 3, 4$ ) and  $\zeta^W$  are functions of the variables  $x, t, \rho, u, \dot{\rho}, S, W$ . Hereafter, a sum over repeated indices is implied. The coefficients of the prolonged operator are obtained by using the prolongation formulae:

$$\begin{aligned} \zeta^{u_{\beta,i}} &= D_i^e \zeta^{u_{\beta}} - u_{\beta,1} D_i^e \xi^x - u_{\beta,2} D_i^e \xi^t, \quad (i = 1, 2), \\ D_1^e &= \partial_x + u_{\beta,1} \partial_{u_{\beta}} + (\rho_x W_{\alpha,1} + \dot{\rho}_x W_{\alpha,2} + S_x W_{\alpha,3}) \partial_{W_{\alpha}}, \\ D_2^e &= \partial_t + u_{\beta,2} \partial_{u_{\beta}} + (\rho_t W_{\alpha,1} + \dot{\rho}_t W_{\alpha,2} + S_t W_{\alpha,3}) \partial_{W_{\alpha}}, \end{aligned}$$

where  $\alpha = (\alpha_1, \alpha_2, \alpha_3)$  and  $\beta = (\beta_1, \beta_2)$  are multi-indexes ( $\alpha_i \geq 0$ ), ( $\beta_i \geq 0$ )

$$\begin{aligned} (\alpha_1, \alpha_2, \alpha_3), 1 &= (\alpha_1 + 1, \alpha_2, \alpha_3), \quad (\alpha_1, \alpha_2, \alpha_3), 2 = (\alpha_1, \alpha_2 + 1, \alpha_3), \quad (\alpha_1, \alpha_2, \alpha_3), 3 = (\alpha_1, \alpha_2, \alpha_3 + 1) \\ (\beta_1, \beta_2), 1 &= (\beta_1 + 1, \beta_2), \quad (\beta_1, \beta_2), 2 = (\beta_1, \beta_2 + 1) \\ u_{(\beta_1, \beta_2)} &= \frac{\partial^{\beta_1 + \beta_2} u}{\partial x^{\beta_1} \partial t^{\beta_2}}, \quad W_{(\alpha_1, \alpha_2, \alpha_3)} = \frac{\partial^{\alpha_1 + \alpha_2 + \alpha_3} W}{\partial \rho^{\alpha_1} \partial \dot{\rho}^{\alpha_2} \partial S^{\alpha_3}}. \end{aligned}$$

The conditions that  $W$  does not depend on  $t, x$  and  $u$  give

$$\zeta_{x_i}^{u_1} = 0, \quad \zeta_u^{u_1} = 0, \quad \zeta_{x_i}^{u_3} = 0, \quad \zeta_u^{u_3} = 0, \quad \zeta_{x_i}^{u_4} = 0, \quad \zeta_u^{u_4} = 0, \quad \zeta_{x_i}^W = 0, \quad \zeta_{u_j}^W = 0, \quad (i = 1, 2).$$

Using these relations, the prolongation formulae for the coefficients  $\zeta^{W_{\alpha}}$  become:

$$\begin{aligned} \zeta^{W_{\alpha,i}} &= \tilde{D}_i^e \zeta^{W_{\alpha}} - W_{\alpha,1} \tilde{D}_i^e \zeta^{u_1} - W_{\alpha,2} \tilde{D}_i^e \zeta^{u_3} - W_{\alpha,3} \tilde{D}_i^e \zeta^{u_4}, \quad (i = 1, 2), \\ \tilde{D}_1^e &= \partial_{\rho} + W_{\alpha,1} \partial_{W_{\alpha}}, \quad \tilde{D}_2^e = \partial_{\dot{\rho}} + W_{\alpha,2} \partial_{W_{\alpha}}, \quad \tilde{D}_3^e = \partial_S + W_{\alpha,3} \partial_{W_{\alpha}}. \end{aligned}$$

For constructing the determining equations and for their solution, the symbolic computer Reduce [24] program was applied. Calculations give the following basis of generators of the equivalence Lie group

$$\begin{aligned} X_1^e &= \partial_x, \quad X_2^e = \partial_t, \quad X_3^e = t \partial_x + \partial_u, \quad X_4^e = t \partial_t + x \partial_x, \\ X_5^e &= t \partial_t + 2\rho \partial_{\rho} - u \partial_u, \quad X_6^e = \partial_W, \quad X_7^e = -u \partial_u + \rho \partial_{\rho} - W \partial_W + t \partial_t, \\ X_8^e &= \rho \varphi(S) \partial_W, \quad X_9^e = \dot{\rho} g(\rho, S) \partial_W, \quad X_{10}^e = h(S) \partial_S, \end{aligned}$$

where the functions  $g(\rho, S)$ ,  $\varphi(S)$  and  $h(S)$  are arbitrary. Here, only the essential part of the operators  $X_i^e$ , ( $i = 5, 6, \dots, 10$ ) is written.

Since the equivalence transformations corresponding to the operators  $X_5^e, X_6^e, X_7^e, X_8^e, X_9^e$  and  $X_{10}^e$  are applied for simplifying the function  $W$  in the process of the group classification, let us present these transformations. Because the function  $W$  depends on  $\rho, \dot{\rho}$  and  $S$  only, the transformations of these variables are presented:

$$\begin{aligned} X_5^e : \rho' &= \rho e^{2a}, \quad \dot{\rho}' = \dot{\rho}, \quad S' = S & W' &= W; \\ X_6^e : \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad S' = S & W' &= W e^{-2a}; \\ X_7^e : \rho' &= \rho e^a, \quad \dot{\rho}' = \dot{\rho}, \quad S' = S & W' &= W + a; \\ X_8^e : \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad S' = S & W' &= \rho \varphi(S) a + W; \\ X_9^e : \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad S' = S & W' &= \dot{\rho} g(\rho, S) a + W \\ X_{10}^e : \rho' &= \rho, \quad \dot{\rho}' = \dot{\rho}, \quad S' = q(S, a) & W' &= W; \end{aligned}$$

Here,  $a$  is the group parameter.

### 3 Defining equations of the admitted Lie group

An admitted generator  $X$  is sought in the form

$$X = \xi^x \partial_x + \xi^t \partial_t + \zeta^\rho \partial_\rho + \zeta^u \partial_u + \zeta^S \partial_S,$$

where the coefficients  $\xi^x, \xi^t, \zeta^\rho, \zeta^u, \zeta^S$  are functions of  $(x, t, \rho, u, S)$ . Calculations showed that

$$\begin{aligned} \xi^x &= (k_2 t + k_3)x + k_4 t + k_5, & \xi^t &= k_2 t^2 + (k_1 + 2k_3)t + k_7. \\ \zeta^\rho &= \rho(k_8 - k_2 t), & \zeta^u &= k_2(-ut + x) - u(k_1 + k_3) + k_4, \\ \zeta^S &= \zeta^S(S), \end{aligned}$$

$$k_2(3W_{\dot{\rho}\dot{\rho}\dot{\rho}} + W_{\dot{\rho}\dot{\rho}\rho} + 3W_{\dot{\rho}\dot{\rho}}) = 0, \tag{2}$$

$$\begin{aligned} -3W_{\dot{\rho}\dot{\rho}S}\dot{\rho}\zeta^S + 3W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho}^2(k_1 + 2k_3 - k_8) - 3W_{\dot{\rho}\dot{\rho}\rho}\dot{\rho}k_8 \\ + 3W_{\dot{\rho}\dot{\rho}\dot{\rho}}(2k_3 - k_8) - \rho k_2(3W_{\dot{\rho}\dot{\rho}} + W_{\dot{\rho}\dot{\rho}\rho}) = 0, \end{aligned} \tag{3}$$

$$k_2(W_{\rho\rho\rho}\dot{\rho}\rho - W_{\rho\rho\dot{\rho}}\dot{\rho} + 3W_{\rho\rho\dot{\rho}\dot{\rho}}\dot{\rho}^2 - W_{\rho\rho\rho}\rho + W_{\rho\rho}) = 0, \tag{4}$$

$$\begin{aligned} \zeta^S \rho(W_{\rho\rho S} - W_{\rho\rho S}\dot{\rho}) - W_{\rho\rho\rho}\dot{\rho}\rho^2 k_8 - W_{\rho\rho\dot{\rho}}\dot{\rho}(2k_1 + 2k_3 + k_8) \\ + W_{\rho\rho\dot{\rho}\dot{\rho}}\dot{\rho}^2 \rho(k_1 + 2k_3 - k_8) + W_{\rho\rho\rho}\rho^2 k_8 + W_{\rho\rho\rho}(2k_1 + 2k_3 + k_8) \\ + k_2 \dot{\rho}(W_{\rho\rho\dot{\rho}\dot{\rho}}\rho^2 + 5W_{\dot{\rho}\dot{\rho}\rho}\rho + 3W_{\dot{\rho}\dot{\rho}}) = 0, \end{aligned} \tag{5}$$

$$k_2(W_{\rho\rho\dot{\rho}S}\dot{\rho}\rho^2 - 3W_{\rho\dot{\rho}S}\dot{\rho}\rho + 3W_{\rho\dot{\rho}\dot{\rho}S}\dot{\rho}^2\rho - 3W_{\dot{\rho}\dot{\rho}S}\dot{\rho}^2 + 3W_{\dot{\rho}S}\dot{\rho} - W_{\rho\rho S}\rho^2 + 3W_{\rho S}\rho - 3W_S) = 0, \tag{6}$$

$$\begin{aligned} -W_{\rho\rho\dot{\rho}S}\dot{\rho}\rho^2 k_8 - 2W_{\rho\dot{\rho}S}\dot{\rho}\rho k_1 - 2W_{\rho\dot{\rho}\dot{\rho}S}\dot{\rho}\rho k_3 + W_{\rho\dot{\rho}S}\dot{\rho}\rho k_8 + W_{\rho\dot{\rho}\dot{\rho}S}\dot{\rho}^2 \rho(2k_3 + k_1 - k_8) \\ + W_{\dot{\rho}\dot{\rho}S}\dot{\rho}^2(k_8 - k_1 - 2k_3) + W_{\dot{\rho}S}\dot{\rho}(2k_1 + 2k_3 - k_8) + W_{\rho\rho S}\rho^2 k_8 + W_{\rho S}\rho(2k_1 + 2k_3 - k_8) \\ + W_S(k_8 - 2k_1 - 2k_3) + \dot{\rho}\rho k_2(W_{\rho\dot{\rho}\dot{\rho}S}\rho + 3W_{\dot{\rho}\dot{\rho}S}) + \zeta_S^S(-W_{\rho\dot{\rho}S}\dot{\rho}\rho + W_{\dot{\rho}S}\dot{\rho} + W_{\rho S}\rho - W_S) \\ + \zeta^S(-W_{\rho\dot{\rho}SS}\dot{\rho}\rho + W_{\dot{\rho}SS}\dot{\rho} + W_{\rho SS}\rho - W_{SS}) = 0, \end{aligned} \tag{7}$$

where  $k_i$ , ( $i = 1, 2, \dots, 8$ ) are constant. The determining Eqs. (2)–(7) define the kernel of admitted Lie algebras and its extensions. The kernel of admitted Lie algebras is determined for all functions  $W(\rho, \dot{\rho}, S)$  and it consists of the generators

$$Y_4 = \partial_x, \quad Y_5 = \partial_t, \quad Y_6 = t\partial_x + \partial_u.$$

Extensions of the kernel depend on the value of the function  $W(\rho, \dot{\rho}, S)$ . They can only be operators of the form

$$k_1 X_1 + k_2 X_2 + k_3 X_3 + k_8 X_8 + \zeta^S \partial_S,$$

where

$$\begin{aligned} X_1 &= t\partial_t - u\partial_u - \dot{\rho}\partial_{\dot{\rho}}, \\ X_2 &= t^2\partial_t + tx\partial_x + (x - ut)\partial_u - t\rho\partial_\rho - (\rho + 3t\dot{\rho})\partial_{\dot{\rho}}, \\ X_3 &= 2t\partial_t + x\partial_x - u\partial_u - 3\dot{\rho}\partial_{\dot{\rho}} - \rho\partial_\rho, \\ X_8 &= \rho\partial_\rho + \dot{\rho}\partial_{\dot{\rho}}. \end{aligned}$$

Since the function  $W(\rho, \dot{\rho}, S)$  depends on  $\dot{\rho}$ , the term with  $\partial_{\dot{\rho}}$  is also presented in the generators.

Relations between the constants  $k_1, k_2, k_3, k_8$  and  $\zeta^S(S)$  depend on the function  $W(\rho, \dot{\rho}, S)$ .

### 4 Case $k_2 \neq 0$

If  $k_2 \neq 0$ , then Eq. (2) gives

$$3W_{\dot{\rho}\dot{\rho}\dot{\rho}} + W_{\dot{\rho}\dot{\rho}\rho} + 3W_{\dot{\rho}\dot{\rho}} = 0.$$

The general solution of this equation is

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^3 g(z, S) + \varphi_0(\rho, S), \tag{8}$$

where  $z = \dot{\rho}\rho^{-3}$ . Substituting (8) into (4), one obtains

$$\rho\varphi_{0\rho\rho\rho} - \varphi_{0\rho\rho} = 0.$$

The general solution of this equation is

$$\varphi_0 = \rho^3\mu(S) + \rho I(S) + J(S), \tag{9}$$

where without loss of the generality by virtue of the equivalence transformation corresponding to the operator  $X_8^e$ , it can be assumed that  $I(S) = 0$ . Equation (6) gives that  $J' = 0$ . By virtue of the equivalence transformation corresponding to  $X_7^e$ , it can also be assumed that  $J = 0$ . Substituting the obtained  $W$  into (3) and splitting it with respect to  $\rho$ , one obtains  $g_{zzz} = 0$  or  $g = \varphi_2(S)z^2$ , where  $\varphi_2 \neq 0$ . It should be noted that the linear part of the function  $\varphi_2$  is also omitted because of the equivalence transformations corresponding to the generator  $X_9^e$ . The remaining part of Eq. (3) becomes

$$\varphi_2'\zeta^S - 2\varphi_2(k_3 + k_8) = 0. \tag{10}$$

If  $\varphi_2' = 0$  or  $\varphi_2 = q \neq 0$ , then  $k_3 = -k_8$  and Eq. (5) becomes

$$\mu'\zeta^S + 2k_1\mu = 0. \tag{11}$$

For  $\mu' = 0$ , the function  $W$  does not depend on  $S$ . Since this case has been studied in [17], it is excluded from further study in the present paper. Thus, one has to assume that  $\mu' \neq 0$ . From (11), one gets  $\zeta^S = -2k_1\mu/\mu'$ . Changing the entropy  $\tilde{S} = \mu(S)$ , one has

$$W(\rho, \dot{\rho}, \tilde{S}) = q\frac{\dot{\rho}^2}{\rho^3} + \rho^3\tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_2, \quad X_3 - X_8.$$

In the final Table 1, this is model  $M_1$ , where the tilde sign is omitted.

If  $\varphi_2' \neq 0$ , then from (3) and (10), one obtains

$$\zeta^S = 2\frac{\varphi_2'}{\varphi_2}(k_3 + k_8),$$

$$\mu'\varphi_2(k_3 + k_8) + \varphi_2'\mu(k_1 + k_3 + k_8) = 0. \tag{12}$$

If  $\mu \neq 0$  then, the last equation defines

$$k_1 = -(k_3 + k_8) \left( 1 + \frac{\mu'\varphi_2}{\mu\varphi_2'} \right). \tag{13}$$

Differentiation (13) with respect to  $S$  gives

$$(k_3 + k_8) \left( \frac{\mu'\varphi_2}{\mu\varphi_2'} \right)' = 0. \tag{14}$$

If  $\left( \frac{\mu'\varphi_2}{\mu\varphi_2'} \right)' = 0$  or  $\mu = q_1\varphi_2^k$ , then the general solution of Eqs. (2)–(7) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3}\tilde{S} + q_1\rho^3\tilde{S}^k,$$

where  $\tilde{S} = \varphi_2(S)$ . The extension of the kernel is given by the generators

$$X_2, \quad X_3 - X_8, \quad X_8 - (k + 1)X_1 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this is model  $M_2$ .

If  $(\frac{\mu'\varphi_2}{\mu\varphi_2}') \neq 0$ , then the general solution of Eqs. (2)–(7) is

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3} \tilde{S} + \rho^3 \mu(\tilde{S}), \quad (\mu \neq q_1 \tilde{S}^k),$$

and the extension of the kernel is given by the generators

$$X_2, X_3 - X_8.$$

In the final Table 1, this is model M<sub>3</sub>.

If  $\mu = 0$ , then

$$W(\rho, \dot{\rho}, \tilde{S}) = \frac{\dot{\rho}^2}{\rho^3} \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1, X_2, X_3 - X_8, X_8 + 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this is model M<sub>4</sub>.

*Remark* The last two cases do not satisfy the restriction  $W_{\dot{\rho}\dot{\rho}S} \neq 0$  announced in the title. For the case where  $k_2 \neq 0$ , it is not necessary to separate the study into the cases  $W_{\dot{\rho}\dot{\rho}S} \neq 0$  and  $W_{\dot{\rho}\dot{\rho}S} = 0$ . Whereas for the analysis of the case where  $k_2 = 0$ , one needs to make this separation.

### 5 Results of the group classification

The result of the group classification of Eq. (1) is summarized in Table 1. The linear part with respect to  $\dot{\rho}$  of the function  $W(\rho, \dot{\rho}, S)$  is omitted. The equivalence transformation corresponding to the operator  $X_{10}^e$  is also used. This transformation allows one to simplify the dependence on entropy of the function  $W(\rho, \dot{\rho}, S)$ .

The first column in Table 1 presents the number of the extension, forms of the function  $W(\rho, \dot{\rho}, S)$  are presented in the second column, extensions of the kernel of admitted Lie algebras are given in the third column, restrictions for functions and constants are in the fourth column.

**Table 1** Group classification

	$W(\rho, \dot{\rho}, S)$	Extensions	Remarks
M <sub>1</sub>	$q_0\rho^{-3}\dot{\rho}^2 + \rho^3S$	$X_1 - 2S\partial_S, X_2, X_3 - X_8$	
M <sub>2</sub>	$\rho^{-3}\dot{\rho}^2S + q_1\rho^3S^k$	$X_2, X_3 - X_8, X_8 - (k + 1)X_1 + 2S\partial_S$	
M <sub>3</sub>	$\rho^{-3}\dot{\rho}^2S + \rho^3\mu(S)$	$X_2, X_3 - X_8$	$\mu' \neq q_1S^k$
M <sub>4</sub>	$\rho^{-3}\dot{\rho}^2S$	$X_1, X_2, X_3 - X_8, X_8 + 2S\partial_S$	
M <sub>5</sub>	$\phi(\rho, \dot{\rho}) + S$	$\partial_S$	
M <sub>6</sub>	$\varphi(\rho)\dot{\rho}^2 + S$	$\partial_S, X_1 - 2S\partial_S$	
M <sub>7</sub>	$\varphi(\rho)\dot{\rho}^2 + \eta(\rho) + S$	$\partial_S$	$\eta'' \neq 0$
M <sub>8</sub>	$\varphi(\rho)\dot{\rho}^2 + \eta(\rho)S$	$X_1 - 2S\partial_S$	$\eta'' \neq 0$
M <sub>9</sub>	$\varphi(\rho)\dot{\rho} \ln  \dot{\rho}  + S$	$X_3 - 2S\partial_S, \partial_S$	
M <sub>10</sub>	$\varphi(\rho)\dot{\rho} \ln  \dot{\rho}  + \eta(\rho) + S$	$\partial_S$	$\eta'' \neq 0$
M <sub>11</sub>	$\varphi(\rho)\dot{\rho} \ln  \dot{\rho}  + \eta(\rho)S$	$X_3 - 2S\partial_S$	$\eta'' \neq 0$
M <sub>12</sub>	$(q_1\rho + q_0) \ln  \dot{\rho}  + \eta(\rho) + S$	$X_3 - X_1, \partial_S$	$q_0^2 + q_1^2 \neq 0$
M <sub>13</sub>	$\varphi(\rho) \ln  \dot{\rho}  + \eta(\rho) + S$	$\partial_S$	$\varphi'' \neq 0$
M <sub>14</sub>	$(q_1\rho + q_0) \ln  \dot{\rho}  + h(\rho, S)$	$X_3 - X_1$	$(q_0^2 + q_1^2)h_{\rho\rho S} \neq 0$
M <sub>15</sub>	$\varphi(\rho)(\ln  \dot{\rho}  + S) + \eta(\rho) + q_2S$	$X_3 - X_1 + \partial_S$	$\varphi'' \neq 0$
M <sub>16</sub>	$\varphi(\rho)\dot{\rho}^{k+2} + S$	$kX_3 - 2(k + 1)X_1 + 2(k + 2)S\partial_S, \partial_S$	$k(k + 1)(k + 2) \neq 0$
M <sub>17</sub>	$\varphi(\rho)\dot{\rho}^{k+2} + \eta(\rho) + S$	$\partial_S$	$\eta'' \neq 0$
M <sub>18</sub>	$\varphi(\rho)\dot{\rho}^{k+2} + \eta(\rho)S$	$kX_3 - 2(k + 1)X_1 + 2(k + 2)S\partial_S$	$\eta'' \neq 0$
M <sub>19</sub>	$\rho^\lambda g(\dot{\rho}\rho^k) + \eta(\rho) + S$	$\partial_S$	
M <sub>20</sub>	$g(\dot{\rho}\rho^k) + q_1\rho^2 + S$	$-2kX_1 + (2k + 1)X_3 + 2X_8, \partial_S$	
M <sub>21</sub>	$\rho g(\dot{\rho}\rho^k) + q_1\rho \ln \rho + S$	$(k + 1)(X_3 - X_1) + X_8 + S\partial_S, \partial_S$	

**Table 1** continued

$W(\rho, \dot{\rho}, S)$	Extensions	Remarks
$M_{22} \rho^\lambda g(\dot{\rho}\rho^k) + S$	$-2(k + \lambda)X_1 + (2k + \lambda + 1)X_3 + 2X_8 + 2\lambda S\partial_S, \partial_S$	$\lambda(\lambda - 1) \neq 0$
$M_{23} g(\dot{\rho}\rho^k) + S \ln \rho$	$-2kX_1 + (2k + 1)X_3 + 2X_8$	
$M_{24} \rho g(\dot{\rho}\rho^k) + S\rho \ln \rho$	$(k + 1)(X_3 - X_1) + X_8$	
$M_{25} \rho^\lambda (g(\dot{\rho}\rho^k) + S)$	$-2(k + \lambda)X_1 + (2k + \lambda + 1)X_3 + 2X_8$	$\lambda(\lambda - 1) \neq 0$
$M_{26} g(\dot{\rho}\rho^k) + Q(\rho S) + q_1 \ln S$	$-2kX_1 + (2k + 1)X_3 + 2X_8 - 2S\partial_S$	
$M_{27} \rho^\lambda g(\dot{\rho}\rho^k) + Q(\rho S)$	$-2(k + \lambda)X_1 + (2k + \lambda + 1)X_3 + 2X_8 - 2S\partial_S$	$\lambda \neq 0$
$M_{28} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S$	$X_3 - 2S\partial_S, X_8 - \lambda X_1 + (2\lambda + 1)S\partial_S, \partial_S$	
$M_{29} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + \eta(\rho) + S$	$2X_8 - 2\lambda X_1 + (2\lambda - \nu - 1)X_3 + 2(\nu + 2)S\partial_S, \partial_S$	$\eta'' = q_1 \rho^\nu \neq 0$
$M_{30} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + \eta(\rho) + S$	$\partial_S$	$\eta'' \neq q_1 \rho^\nu, \eta'' \neq 0$
$M_{31} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + \eta(\rho)S$	$X_3 - 2S\partial_S$	$\eta'' \neq 0$
$M_{32} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho \ln \rho$	$X_3 - 2S\partial_S, -\lambda X_1 + X_8 + 2\lambda S\partial_S$	
$M_{33} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S(\rho \ln \rho + q_1)$	$X_3 - 2S\partial_S$	$q_1 \neq 0$
$M_{34} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho \ln \rho + q_2 \ln S$	$2(X_8 - \lambda X_1 + \lambda X_3) + X_3 - 2S\partial_S$	$q_2 \neq 0$
$M_{35} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho \ln \rho + q_2 S^\alpha$	$2(1 - \alpha)(X_8 - \lambda X_1 + \lambda X_3) + X_3 - 2S\partial_S$	$q_2 \alpha(\alpha - 1) \neq 0$
$M_{36} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S(\ln \rho + q_1)$	$X_3 - 2S\partial_S$	
$M_{37} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S(\ln \rho + q_1) + \alpha S \ln S$	$-2\lambda \alpha X_1 + (2\alpha \lambda + 1)X_3 + 2\alpha X_8 + 2(\alpha - 1)S\partial_S$	$\alpha \neq 0$
$M_{38} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho^{\nu+2}$	$X_3 - 2S\partial_S, -\lambda X_1 + X_8 + (2\lambda - \nu - 1)S\partial_S$	$(\nu + 2)(\nu + 1) \neq 0$
$M_{39} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S(\rho^{\nu+2} + q_1)$	$X_3 - 2S\partial_S$	$q_1(\nu + 2)(\nu + 1) \neq 0$
$M_{40} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho^{\nu+2} + q_1 \ln S$	$-2\lambda X_1 + (2\lambda + 1)X_3 + 2X_8 - 2(\nu + 2)S\partial_S$	$q_1(\nu + 2)(\nu + 1) \neq 0$
$M_{41} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + S\rho^{\nu+2} + q_2 S^\alpha$	$2\lambda(\alpha - 1)X_1 + (\alpha(\nu + 1 - 2\lambda) + 2\lambda + 1)X_3 + 2(1 - \alpha)X_8 - 2(\nu + 2)S\partial_S$	$(\nu + 2)(\nu + 1) \neq 0, q_2 \alpha(\alpha - 1) \neq 0$
$M_{42} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + g(\rho S) + q_2 \ln S$	$-2\lambda X_1 + (2\lambda + 1)X_3 + 2X_8 - 2S\partial_S$	
$M_{43} q_0 \dot{\rho} \rho^\lambda \ln  \dot{\rho}  + \rho^\nu g(\rho S) + q_2 S^{-\nu}$	$-2\lambda X_1 + (2\lambda - \nu + 1)X_3 + 2X_8 - 2S\partial_S$	$\nu \neq 0$
$M_{44} \rho^\lambda (q_1 + q_0 \ln(\rho^\nu  \dot{\rho} )) + S$	$-2(\lambda + \nu)X_1 + (\lambda + 2\nu + 1)X_3 + 2X_8 + 2\lambda S\partial_S, \partial_S$	
$M_{45} q_0 \rho^\lambda \ln  \dot{\rho}  + \eta(\rho) + S$	$\partial_S$	$\eta \neq \rho^\lambda (q_1 + q_2 \ln \rho)$
$M_{46} q_0 \rho^\lambda \ln  \dot{\rho}  + \eta(\rho) + S$	$X_3 - X_1, \partial_S$	$\eta'' \neq q_1 \rho^{(\lambda-2)}, \lambda(\lambda - 1) = 0$
$M_{47} q_0 \rho^\lambda \ln( \dot{\rho}  \rho^\nu) + S$	$X_3 - X_1, (1 - \lambda)X_1 + 2X_8 + 2\lambda S\partial_S, \partial_S$	$\lambda(\lambda - 1) = 0$
$M_{48} q_0 \ln  \dot{\rho}  + S \ln \rho + f(S)$	$X_3 - X_1$	
$M_{49} \rho(q_0 \ln  \dot{\rho}  + S \ln \rho) + f(S)$	$X_3 - X_1$	$f' \neq 0$
$M_{50} \rho(q_0 \ln  \dot{\rho}  + S \ln \rho)$	$X_3 - X_1, X_8$	
$M_{51} q_0 \ln  \dot{\rho}  + \phi(\rho S) + q_1 \ln S$	$X_3 - X_1, X_1 + 2X_8 - 2S\partial_S$	$(z\phi(z))'' \neq 0$
$M_{52} \rho(q_0 \ln  \dot{\rho}  + \phi(\rho S)) + q_1 S^{-1}$	$X_3 - X_1, X_8 - S\partial_S$	$(z\phi(z))'' \neq 0$
$M_{53} \rho^\lambda (q_0 \ln  \dot{\rho}  + \phi(\rho S)) + f(S)$	$X_3 - X_1$	$\lambda(\lambda - 1) = 0, (Sf' + \lambda f)' \neq 0, (z\phi(z))'' \neq 0$
$M_{54} q_0 \rho^\lambda (\ln  \dot{\rho}  + S) + \eta(\rho) + q_1 S$	$X_3 - X_1 + \partial_S$	$\lambda(\lambda - 1) \neq 0, \eta'' \neq \rho^{\lambda-2}(\nu \ln \rho + q_2)$
$M_{55} q_0 \rho^\lambda (\ln( \dot{\rho}  \rho^\nu) + S) + q_1 \rho^\lambda$	$X_3 - X_1 + \partial_S, 2(\lambda - 1)X_1 + 2X_8 - (\lambda + 2\nu + 1)\partial_S$	$\lambda(\lambda - 1) \neq 0$
$M_{56} q_0 \rho^\lambda (\ln( \dot{\rho}  \rho^\nu) + S) + q_1 \rho^\lambda + q_2 S$	$X_3 - X_1 + \partial_S$	$q_2 \lambda(\lambda - 1) \neq 0$
$M_{57} q_0 \rho^\lambda (\ln( \dot{\rho}  \rho^\nu) + S) + q_1 \rho^\lambda + q_2 e^{\kappa S}$	$-2(\kappa(\lambda + \nu) + \lambda)X_1 + 2\kappa X_8 + 2\lambda \partial_S + (2\lambda + \kappa(\lambda + 2\nu + 1))X_3$	$q_2 \lambda(\lambda - 1) \neq 0$
$M_{58} q_0 \rho^\lambda (\ln( \dot{\rho}  \rho^\nu) + g(\rho S)) + q_2 S^{-\lambda}$	$-2(\lambda + \nu)X_1 + (\lambda + 2\nu + 1)X_3 + 2X_8 - 2S\partial_S$	$\lambda(\lambda - 1) \neq 0, (z^{\lambda+1} g'(z))'' \neq 0$
$M_{59} q_0 \rho^\lambda \dot{\rho}^{k+2} + \eta(\rho) + S$	$-2(2k + \lambda + 2 + (k + 1)\nu)X_1 + (k\nu + 3k + 2\lambda + 2)X_3 + 2(k + 2)(X_8 + (\nu + 2)S\partial_S), \partial_S$	$(k + 1)(k + 2) \neq 0, \eta'' = q_1 \rho^\nu \neq 0$
$M_{60} q_0 \rho^\lambda \dot{\rho}^{k+2} + \eta(\rho) + S$	$\partial_S$	$(k + 1)(k + 2) \neq 0, \eta'' \neq q_1 \rho^\nu, \eta'' \neq 0$
$M_{61} q_0 \rho^\lambda \dot{\rho}^{k+2} + S$	$2(k + 1)X_1 - kX_3 - 2(k + 2)S\partial_S, (k + \lambda + 1)X_3 + 2(k + 1)X_8 - 2\lambda S\partial_S, \partial_S$	$(k + 1)(k + 2) \neq 0$
$M_{62} q_0 \rho^\lambda \dot{\rho}^{k+2} + g(\rho S) + q_2 \ln S$	$-2\lambda X_1 + (k + 2\lambda + 2)X_3 + 2(k + 2)(X_8 - S\partial_S)$	$(k + 1)(k + 2) \neq 0, (zg'(z))'' \neq 0$
$M_{63} q_0 \rho^\lambda \dot{\rho}^{k+2} + \rho^\nu g(\rho S) + q_2 S^{-\nu}$	$-2(\nu(k + 1) + \lambda)X_1 + (k(\nu + 1) + 2\lambda + 2)X_3 + 2(k + 2)(X_8 - S\partial_S)$	$\nu(k + 1)(k + 2) \neq 0, (z^{\nu+1} g'(z))'' \neq 0$
$M_{64} q_0 \rho^\lambda \dot{\rho}^{k+2} + S\eta(\rho)$	$2(k + 1)X_1 - kX_3 - 2(k + 2)S\partial_S$	$(k + 1)(k + 2) \neq 0, \eta'' \neq q_1 \rho^\nu, \eta'' \neq 0$

**Table 1** continued

	$W(\rho, \dot{\rho}, S)$	Extensions	Remarks
$M_{65}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S(\ln(\rho S^\beta) + q_2)$	$2(k+1-\beta\lambda)X_1 + (\beta(k+2\lambda+2)-k)X_3 + 2(k+2)(\beta X_8 - S\partial_S)$	$(k+1)(k+2) \neq 0$
$M_{66}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + \rho \ln(\rho)S$	$2(k+1)X_1 - kX_3 - 2(k+2)S\partial_S, 2\lambda(X_3 - X_1) + (k+2)(X_3 + 2X_8 - 2S\partial_S)$	$(k+1)(k+2) \neq 0$
$M_{67}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S(\rho \ln(\rho) + q_2)$	$2(k+1)X_1 - kX_3 - 2(k+2)S\partial_S$	$q_2 \neq 0, (k+1)(k+2) \neq 0$
$M_{68}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S\rho \ln \rho + q_2 \ln S$	$2\lambda(X_3 - X_1) + (k+2)(X_3 + 2X_8 - 2S\partial_S)$	$q_2 \neq 0, (k+1)(k+2) \neq 0$
$M_{69}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S(\rho \ln \rho + q_2 S^\beta)$	$2(\beta(k+\lambda+1)+k+1)(X_3 - X_1) - (k+2)X_3 + 2(k+2)(\beta X_8 + S\partial_S)$	$\beta q_2 \neq 0, (k+1)(k+2) \neq 0$
$M_{70}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S\rho^\nu$	$2(k+1)X_1 - kX_3 - 2(k+2)S\partial_S, (k+\lambda+1)X_3 + 2(k+1)X_8 - 2(\lambda+\nu(k+1))S\partial_S$	$\nu \neq 0, (k+1)(k+2) \neq 0$
$M_{71}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S(\rho^\nu + q_2)$	$2(k+1)X_1 - kX_3 - 2(k+2)S\partial_S$	$q_2 \neq 0, (k+1)(k+2) \neq 0$
$M_{72}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S\rho^\nu + q_2 \ln S$	$2\lambda(X_3 - X_1) + (k+2)(X_3 + 2X_8 - 2\nu S\partial_S)$	$q_2 \neq 0, (k+1)(k+2) \neq 0$
$M_{73}$	$q_0 \rho^\lambda \dot{\rho}^{k+2} + S(\rho^\nu + q_2 S^\beta)$	$2\beta(\lambda+\nu(k+1)) + \nu(k+1)(X_3 - X_1) - (k+2)(\beta\nu - \beta + \nu)X_3 + 2(k+2)(\beta X_8 + \nu S\partial_S)$	$q_2 \beta \neq 0, (k+1)(k+2) \neq 0$

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**Appendix: Case  $k_2 = 0$**

For further study, the knowledge of  $\zeta^S(S)$  plays a key role. For example, for  $k_2 = 0$ , Eq. (3) becomes

$$W_{\dot{\rho}\dot{\rho}S}\zeta^S = W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho}k_1 + 2k_3(W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + W_{\dot{\rho}\dot{\rho}}) - k_8(W_{\dot{\rho}\dot{\rho}\dot{\rho}}\dot{\rho} + W_{\dot{\rho}\dot{\rho}\rho} + W_{\dot{\rho}\dot{\rho}}). \tag{15}$$

In the present paper, we study the case where

$$W_{\dot{\rho}\dot{\rho}S} = 0.$$

By virtue of the equivalence transformation corresponding to the generator  $X_9^e$ , the general solution of the equation  $W_{\dot{\rho}\dot{\rho}S} = 0$  is

$$W(\rho, \dot{\rho}, \tilde{S}) = \phi(\rho, \dot{\rho}) + h(\rho, S),$$

where  $\phi_{\dot{\rho}}h_S \neq 0$ . Since for  $\phi_{\dot{\rho}\dot{\rho}} = 0$ , Eq. (1) are equivalent to the gas dynamics equations, it is assumed that  $\phi_{\dot{\rho}\dot{\rho}} \neq 0$ . Equation (15) reduces to

$$k_1a + k_3b - k_8c = 0, \tag{16}$$

where

$$a = \dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}}, \quad b = 2(\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \phi_{\dot{\rho}\dot{\rho}}), \quad c = -(\dot{\rho}\phi_{\dot{\rho}\dot{\rho}\dot{\rho}} + \rho\phi_{\dot{\rho}\dot{\rho}\rho} + \phi_{\dot{\rho}\dot{\rho}}).$$

In the further study, the following strategy is used. Notice that Eq. (5) is linear with respect to  $\zeta^S$  with the coefficient  $h_{\rho\rho S}$ , i.e.,

$$h_{\rho\rho S}\zeta^S = A$$

with some function  $A = A(\rho, \dot{\rho}, S)$  which is independent of  $\zeta^S$ . If  $h_{\rho\rho S} = 0$ , then due to equivalence transformations one can also assume that

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where  $\mu' \neq 0$ . In this case, Eq. (7) leads to

$$\zeta^S = (-2k_1\mu - 2k_3\mu + k_8\mu + c_0)/\mu',$$

where  $c_0$  is an arbitrary constant. The admitted generator takes the form

$$X = k_1(X_1 - 2\tilde{S}\partial_{\tilde{\zeta}}) + k_3(X_3 - 2\tilde{S}\partial_{\tilde{\zeta}}) + k_8(X_8 + \tilde{S}\partial_{\tilde{\zeta}}) + c_0\partial_{\tilde{\zeta}}, \quad (17)$$

where  $\tilde{S} = \mu(S)$ . Remaining equations are (3) and (5). The relations between constants  $k_1, k_3$  and  $k_8$  depend on the functions  $\eta(\rho)$  and  $\phi(\rho, \dot{\rho})$ . If  $h_{\rho\rho S} \neq 0$ , then the function  $\zeta^S$  is defined by Eq. (5). In this case, one needs to satisfy the system of Eqs. (3), (7) and the condition that  $\zeta^S = \zeta^S(S)$ .

The analysis of the relations between the constants  $k_1, k_3$  and  $k_8$ , follows to the algorithm developed for the gas dynamics equations [1]: the vector space  $Span(V)$ , where the set  $V$  consists of the vectors  $(a, b, c)$  with  $\rho, \dot{\rho}$  and  $S$  are changed, is analyzed. This algorithm allows one to study all possible subalgebras without omission.

$$\dim(Span(V)) = 3$$

If the function  $W(\rho, \dot{\rho}, S)$  is such that  $\dim(Span(V)) = 3$ , then Eq. (16) is only satisfied for

$$k_1 = 0, \quad k_3 = 0, \quad k_8 = 0.$$

In this case, Eqs. (5) and (7) become

$$\zeta^S h_{\rho\rho S} = 0, \quad \zeta_S^S(\rho h_{\rho S} - h_S) + \zeta^S(\rho h_{\rho SS} - h_{SS}) = 0.$$

Since for  $\zeta^S = 0$ , there are no extensions of the kernel of admitted Lie algebras, one has to consider  $\zeta^S \neq 0$ . The general solution of the first equation, after using the equivalence transformation corresponding to the generator  $X_8^c$ , is

$$h = \mu(S),$$

where  $\mu' \neq 0$ . The general solution of the second equation is  $\zeta^S = c/\mu'$ . Hence,

$$W(\rho, \dot{\rho}, \tilde{S}) = \phi(\rho, \dot{\rho}) + \tilde{S},$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{\zeta}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model  $M_5$ .

$$\dim(Span(V)) = 2$$

There exists a constant vector  $(\alpha, \beta, \gamma) \neq 0$ , which is orthogonal to the set  $V$ :

$$\alpha a + \beta b + \gamma c = 0. \quad (18)$$

This means that the function  $\phi(\rho, \dot{\rho})$  satisfies the equation

$$(\alpha + 2\beta + \gamma)\dot{\rho}\phi_{\dot{\rho}\dot{\rho}} + \gamma\rho\phi_{\dot{\rho}\rho} = -(2\beta + \gamma)\phi_{\dot{\rho}}. \quad (19)$$

The characteristic system of this equation is

$$\frac{d\dot{\rho}}{(\alpha + 2\beta + \gamma)\dot{\rho}} = \frac{d\rho}{\gamma\rho} = \frac{d\phi_{\dot{\rho}}}{-(2\beta + \gamma)\phi_{\dot{\rho}}}. \quad (20)$$

Case  $\gamma = 0$

Because  $\phi_{\dot{\rho}\dot{\rho}} \neq 0$  and  $(\alpha, \beta, \gamma) \neq 0$ , one has that  $\alpha + 2\beta \neq 0$ . The general solution of Eq. (19) is

$$\phi_{\dot{\rho}\dot{\rho}} = \tilde{\varphi}(\rho)\dot{\rho}^k, \quad (21)$$

where  $\tilde{\varphi}(\rho) \neq 0$  is an arbitrary function and  $k = 2\beta/(\alpha + 2\beta)$ . Since  $\dim(\text{Span}(V)) = 0$  for  $(\rho\tilde{\varphi}'/\tilde{\varphi})' = 0$ , one has to assume that  $(\rho\tilde{\varphi}'/\tilde{\varphi})' \neq 0$ .

Substitution of (21) into (16) gives

$$k_8\tilde{\varphi}'\rho - \tilde{\varphi}(k(k_1 + 2k_3 - k_8) + 2k_3 - k_8) = 0. \quad (22)$$

The case  $k_8 \neq 0$  leads to  $(\rho\tilde{\varphi}'/\tilde{\varphi})' = 0$ . Hence,  $k_8 = 0$  and Eq. (22) becomes

$$k(k_1 + 2k_3) + 2k_3 = 0. \quad (23)$$

Let  $k = 0$ . Due to Eq. (22), one gets  $k_3 = 0$ . Integrating (21), one finds  $\phi = \varphi(\rho)\dot{\rho}^2$ . Equation (5) becomes

$$h_{\rho\rho S}\zeta^S + 2h_{\rho\rho}k_1 = 0. \quad (24)$$

Assume that  $h_{\rho\rho} = 0$ , this means that after using the equivalence transformation corresponding to the generator  $X_8^\xi$ , one has that  $h = \mu(S)$ , where  $\mu' \neq 0$ . Equation (7) after integration gives

$$\zeta^S = -2k_1\mu/\mu' + c_0/\mu',$$

where  $c_0$  is a constant of the integration. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \tilde{S}.$$

and the extension of the kernel is given by the generators

$$\partial_{\tilde{S}}, \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model  $M_6$ .

Assume that  $h_{\rho\rho} \neq 0$ . For the existence of an extension of the kernel, Eq. (24) implies that  $h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S)$ , where  $\mu\eta'' \neq 0$ . In this case, Eq. (5) becomes

$$\mu'\zeta^S + 2k_1\mu = 0.$$

If  $\mu' = 0$ , then  $\mu'_2 \neq 0$ ,  $k_1 = 0$  and Eq. (7) gives  $\zeta^S = c_0/\mu'_2$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu_2(S)$ . In the final Table 1, this is model  $M_7$ .

If  $\mu' \neq 0$ , then  $\zeta^S = -2k_1\mu/\mu'$ , and Eq. (7) gives

$$(\mu'_2/\mu')' = 0.$$

Hence, without loss of generality one can assume that  $\mu_2 = 0$ . Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)\dot{\rho}^2 + \eta(\rho)\tilde{S}.$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model  $M_8$ .

*Remark* In the cases where  $\mu' \neq 0$ , one can assume that  $\mu_2(S) = f(\mu(S))$ . This simplifies calculations.

Let  $k \neq 0$ . Equation (23) gives

$$k_1 = -2k_3 \frac{1+k}{k}.$$

The function  $\phi(\rho, \dot{\rho})$  is obtained by integrating Eq. (21). The integration depends on the value of  $k$ .

Let  $k = -1$ , then

$$\phi = \varphi(\rho) \dot{\rho} \ln |\dot{\rho}|. \quad (25)$$

Substituting (25) into (5), one obtains

$$\zeta^S h_{\rho\rho S} + 2k_3 h_{\rho\rho} = 0. \quad (26)$$

If  $h_{\rho\rho} = 0$ , then  $h = \mu(S)$  with  $\mu' \neq 0$ , and Eq. (7) leads to

$$\zeta^S = -2k_3 \mu / \mu' + c_0 / \mu'.$$

Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho) \dot{\rho} \ln |\dot{\rho}| + \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S} \partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model M<sub>9</sub>.

If  $h_{\rho\rho} \neq 0$ , then

$$h(\rho, S) = \mu(S) \eta(\rho) + \mu_2(S), \quad (\mu \eta'' \neq 0).$$

Equation (5) becomes  $\mu' \zeta^S + 2k_3 \mu = 0$ .

If  $\mu' = 0$ , then  $\mu'_2 \neq 0$ ,  $k_3 = 0$  and Eq. (7) gives  $\zeta^S = c_0 / \mu'_2$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho) \dot{\rho} \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is defined by the generator

$$\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu_2(S)$ . In the final Table 1, this is model M<sub>10</sub>.

If  $\mu' \neq 0$ , then

$$\zeta^S = -2k_3 \mu(S) / \mu'.$$

Similar to the case  $k = 0$ , Eq. (7) gives  $\mu_2 = 0$ . Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho) \dot{\rho} \ln |\dot{\rho}| + \eta(\rho) \tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$X_3 - 2\tilde{S} \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model M<sub>11</sub>.

Let  $k = -2$ , then

$$\phi = \varphi(\rho) \ln |\dot{\rho}|. \quad (27)$$

Equation (5) becomes

$$\zeta^S h_{\rho\rho S} - k_3 \varphi'' = 0 \quad (28)$$

Assuming that  $h_{\rho\rho S} = 0$ , one has

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where  $\mu' \neq 0$ . Equation (7) leads to

$$\zeta^S = c_0/\mu'. \tag{29}$$

Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho) \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and (a) for  $\varphi'' = 0$ , one has two admitted generators

$$X_3 - X_1, \partial_{\tilde{S}},$$

(b) for  $\varphi'' \neq 0$ , there is the only admitted generator

$$\partial_{\tilde{S}}.$$

Here,  $\tilde{S} = \mu(S)$ . In the final Table, 1 case (a) is model M<sub>12</sub> and case (b) is model M<sub>13</sub>.

Assuming that  $h_{\rho\rho S} \neq 0$ , one has

$$\zeta^S = k_3 \frac{\varphi''}{h_{\rho\rho S}}.$$

It should be noted that here  $k_3 \neq 0$ , otherwise there is no an extension of the kernel of admitted Lie algebras. Hence,

$$\left( \frac{\varphi''}{h_{\rho\rho S}} \right)_\rho = 0. \tag{30}$$

If  $\varphi'' = 0$ , then Eq. (7) is also satisfied. Therefore, there is the only extension

$$X_3 - X_1,$$

and

$$W(\rho, \dot{\rho}, \tilde{S}) = (q_1\rho + q_0) \ln |\dot{\rho}| + h(\rho, S),$$

where  $(q_0^2 + q_1^2)h_{\rho\rho S} \neq 0$ . In the final Table 1, this is model M<sub>14</sub>.

If  $\varphi'' \neq 0$ , then Eqs. (30) and (7) give

$$h(\rho, S) = \varphi(\rho)\mu(S) + \eta(\rho) + q_2\mu(S),$$

where  $\mu' \neq 0$ . Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \varphi(\rho)(\ln |\dot{\rho}| + \tilde{S}) + \eta(\rho) + q_2\tilde{S}, \quad (\varphi'' \neq 0),$$

and the extension of the kernel is

$$X_3 - X_1 + \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model M<sub>15</sub>.

Let  $k(k+1)(k+2) \neq 0$  in (21), then

$$\phi = \varphi(\rho)\dot{\rho}^{(k+2)} \tag{31}$$

Substituting (31) into (5), one obtains

$$\zeta^S h_{S\rho\rho} k - 2k_3(k+2)h_{\rho\rho} = 0. \tag{32}$$

If  $h_{\rho\rho} = 0$ , then one can consider that  $h = \mu(S)$ , where  $\mu' \neq 0$ . Equation (7) is

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu' + c_0/\mu'.$$

In this case,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2} \varphi(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model  $M_{16}$ .

If  $h_{\rho\rho} \neq 0$ , then for an existence of an extension of the kernel, Eq. (32) requires that

$$h(\rho, S) = \eta(\rho)\mu(S) + \mu_2(S),$$

where  $\mu\eta'' \neq 0$ . Equation (32) becomes

$$\zeta^S \mu' k - 2k_3(k+2)\mu = 0.$$

If  $\mu' = 0$ , then  $\mu'_2 \neq 0$ ,  $k_3 = 0$  and Eq. (7) gives  $\zeta^S = c_0/\mu'_2$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2} \varphi(\rho) + \eta(\rho) + \tilde{S}, \quad (\eta'' \neq 0).$$

and the extension of the kernel is given by the generator

$$\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu_2(S)$ . In the final Table 1, this is model  $M_{17}$ .

If  $\mu' \neq 0$ , then

$$\zeta^S = 2k_3 \frac{(k+2)}{k} \mu/\mu',$$

Similar to the case  $k = 0$ , Eq. (7) gives  $\mu_2 = 0$ . Therefore,

$$W(\rho, \dot{\rho}, \tilde{S}) = \dot{\rho}^{k+2} \varphi(\rho) + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the generator

$$kX_3 - 2(k+1)X_1 + 2(k+2)\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model  $M_{18}$ .

Case  $\gamma \neq 0$ .

In this case, the general solution of (20) is

$$\phi = \rho^\lambda g(z), \quad (g'' \neq 0), \tag{33}$$

where  $z = \dot{\rho}\rho^k$ ,  $k = -(\alpha + 2\beta)/\gamma - 1$ ,  $\lambda = 2(\beta + \alpha)/\gamma + 1$ . Substituting  $\phi$  into (15), one obtains

$$zg'''k_0 + g''\tilde{k}_0 = 0, \tag{34}$$

where  $k_0 = k_1 + 2k_3 - k_8(k+1)$  and  $\tilde{k}_0 = 2k_3 - k_8(2k + \lambda + 1)$ . If  $k_0 \neq 0$ , then  $\dim(\text{Span}(V)) \leq 1$ , hence,  $k_0 = 0$  and  $\tilde{k}_0 = 0$ , which mean that

$$k_1 = -k_8(k + \lambda), \quad k_3 = k_8(2k + \lambda + 1)/2.$$

Equation (5) becomes

$$\zeta^S h_{S\rho\rho} + k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho}) = 0. \quad (35)$$

Assume that  $h_{\rho\rho S} = 0$  or

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where  $\mu' \neq 0$ . Equation (5) and (7) become, respectively,

$$k_8(\rho\eta''' - (\lambda - 2)\eta'') = 0, \quad \zeta^S = k_8\lambda\mu/\mu' + c_0/\mu'.$$

If  $\rho\eta''' - (\lambda - 2)\eta'' \neq 0$ , then  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \eta(\rho) + \tilde{S},$$

and there is the only extension of the kernel of admitted Lie algebras corresponding to the generator

$$\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this is model M<sub>19</sub>.

If  $\rho\eta''' - (\lambda - 2)\eta'' = 0$  or

$$\eta = \begin{cases} q_1\rho^2, & \lambda = 0, \\ q_1\rho \ln(\rho), & \lambda = 1, \\ q_1\rho^\lambda, & \lambda(\lambda - 1) \neq 0. \end{cases}$$

Then,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho}\rho^k) + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras corresponding to the generators is

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8 + \lambda\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, these models correspond to M<sub>20</sub>–M<sub>22</sub>.

Assume that  $h_{S\rho\rho} \neq 0$  in (35), then

$$\zeta^S = -k_8(\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho})/h_{S\rho\rho}.$$

Since  $\zeta^S = \zeta^S(S)$ , one has

$$\frac{-\rho h_{\rho\rho\rho} + (\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} = H(S), \quad (36)$$

and  $\zeta^S = k_8 H(S)$ .

If  $H = 0$ , then the general solution of (36) is

$$h_{\rho\rho}(\rho, S) = \mu(S)\rho^{\lambda-2}. \quad (37)$$

Hence,

$$h(\rho, S) = \mu(S)\eta(\rho) + \mu_2(S),$$

where  $\mu'(S) \neq 0$  and

$$\eta = \begin{cases} \ln \rho, & \lambda = 0, \\ \rho \ln(\rho), & \lambda = 1, \\ \rho^\lambda, & \lambda(\lambda - 1) \neq 0. \end{cases}$$

Equation (7) gives

$$k_8 (\lambda \mu'_2 + \mu' (\rho^2 \eta'' - \lambda (\rho \eta' - \eta))) = 0.$$

This equation leads to: (a) if  $\lambda = 0$ , then  $k_8 = 0$ , (b) if  $\lambda \neq 0$ , then  $\mu'_2 k_8 = 0$ . Hence, an extension of the kernel of admitted Lie algebras occurs for  $\lambda \neq 0$ . In this case,  $\mu'_2 = 0$ , which allows one to assume that  $\mu_2 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho} \rho^k) + \tilde{S} \eta(\rho),$$

and the extension is given by the generator

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8.$$

In the final Table 1, these models correspond to  $M_{23}$ – $M_{25}$ .

If  $H \neq 0$ , then Eq. (36) leads to

$$h = \rho^\lambda Q + \mu_2,$$

where  $\mu = \mu(S)$ ,  $\mu_2 = f(\mu(S))$ ,  $Q = Q(z)$ ,  $z = \rho\mu$  and  $\mu' \neq 0$ . Here,  $H(S) = \mu/\mu' \neq 0$ . Substitution of

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda g(\dot{\rho} \rho^k) + \rho^{-\lambda} Q(\rho\mu(S)) + f(\mu(S))$$

into (7) gives

$$\mu f'' + (\lambda + 1)f' = 0.$$

Hence,

$$f' = c\mu^{-(\lambda+1)}.$$

Integration of this equation depends on  $\lambda$ :

$$\mu_2 = \begin{cases} q_1 \ln \mu, & \lambda = 0, \\ q_1 \mu^{-\lambda}, & \lambda \neq 0. \end{cases}$$

Thus,

$$\begin{aligned} \lambda = 0 : W(\rho, \dot{\rho}, \tilde{S}) &= g(\dot{\rho} \rho^k) + Q(\rho \tilde{S}) + q_1 \ln \tilde{S}, \\ \lambda \neq 0 : W(\rho, \dot{\rho}, S) &= \rho^\lambda (g(\dot{\rho} \rho^k) + Q(\rho \tilde{S})). \end{aligned}$$

The extension of the kernel is given by the generator

$$-(k + \lambda)X_1 + \frac{(2k + \lambda + 1)}{2}X_3 + X_8 - \tilde{S} \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, these models correspond to  $M_{26}$ – $M_{27}$ .

$$\dim(\text{Span}(V)) = 1$$

Let  $\dim(\text{Span}(V)) = 1$ . There exists a constant vector  $(\alpha, \beta, \gamma) \neq 0$  such that

$$(a, b, c) = (\alpha, \beta, \gamma)B$$

with some function  $B(\rho, \dot{\rho}, S) \neq 0$ . Since  $\phi_{\dot{\rho}\dot{\rho}} \neq 0$ , one has  $\beta - 2\alpha \neq 0$ , and

$$\rho \phi_{\rho\dot{\rho}\dot{\rho}} = \lambda \phi_{\dot{\rho}\dot{\rho}}, \quad \dot{\rho} \phi_{\dot{\rho}\dot{\rho}\dot{\rho}} = k \phi_{\dot{\rho}\dot{\rho}},$$

where

$$\lambda = \frac{3\alpha - \beta - \gamma}{\beta - 2\alpha}, \quad k = \frac{\alpha}{\beta - 2\alpha}.$$

These relations give

$$\phi_{\dot{\rho}\dot{\rho}} = c_1 \rho^\lambda \dot{\rho}^k, \tag{38}$$

where  $c_1 \neq 0$  is constant. Equation (3) becomes

$$k_1 k + 2k_3(k + 1) - k_8(k + \lambda + 1) = 0. \tag{39}$$

Integration of (38) depends on the value of  $k$ . It should be noted that  $k^2 + \lambda^2 \neq 0$ , otherwise  $\dim(\text{Span}(V)) = 0$ .

Case  $k = -1$ .

Integrating (38), one obtains

$$\phi = q_0 \rho^\lambda \dot{\rho} \ln |\dot{\rho}|. \quad (40)$$

Equation (39) gives

$$k_1 = -\lambda k_8,$$

and Eq. (5) becomes

$$h_{S\rho\rho} \zeta^S = -2k_3 h_{\rho\rho} - k_8 (\rho h_{\rho\rho\rho} - h_{\rho\rho} (2\lambda - 1)). \quad (41)$$

Assuming that  $h_{S\rho\rho} = 0$  or

$$h(\rho, S) = \eta(\rho) + \mu(S), \quad (\mu' \neq 0),$$

Equation (41) is reduced to the equation

$$\rho \eta''' k_8 - (k_8 (2\lambda - 1) - 2k_3) \eta'' = 0. \quad (42)$$

The general solution of Eq. (7) is

$$\zeta^S = (k_8 (2\lambda + 1) - 2k_3) \frac{\mu}{\mu'} + \frac{c_0}{\mu'},$$

where  $c_0$  is an arbitrary constant.

If  $\eta'' = 0$ , then without loss of the generality one can assume that  $\eta = 0$ . Equation (42) is satisfied. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_8 - \lambda X_1 + (2\lambda + 1)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this model corresponds to  $M_{28}$ .

If  $\eta'' \neq 0$ , then

$$k_3 = k_8 \left( \lambda - \frac{1}{2} - \frac{\rho \eta'''}{2\eta''} \right). \quad (43)$$

Because  $k_3$  is constant, one has

$$k_8 \left( \frac{\rho \eta'''}{\eta''} \right)' = 0.$$

Assume that

$$\left( \frac{\rho \eta'''}{\eta''} \right)' = 0$$

or  $\eta'' = q_1 \rho^\nu$ , where  $\nu$  is constant. Substituting  $\eta''$  into (43), one gets

$$k_3 = k_8 \left( \lambda - \frac{\nu + 1}{2} \right).$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1 \rho^\nu, \quad q_1 \neq 0),$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$2X_8 - 2\lambda X_1 + (2\lambda - \nu - 1)X_3 + 2(\nu + 2)\tilde{S}\partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$  and  $q_1 \neq 0$ . In the final Table 1, this model corresponds to  $M_{29}$ .

If

$$\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0,$$

then  $k_8 = 0$ ,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^\lambda \ln|\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{30}$ .

Assuming that  $h_{S\rho\rho} \neq 0$ , Eq. (41) gives

$$\zeta^S = -2k_3 \frac{h_{\rho\rho}}{h_{S\rho\rho}} - k_8 \frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}. \quad (44)$$

Differentiating equation (44) with respect to  $\rho$ , one obtains

$$2k_3 \left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho + k_8 \left(\frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}}\right)_\rho = 0. \quad (45)$$

If  $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$ , then  $h = \eta(\rho)\mu(S) + f(\mu(S))$ , and Eq. (45) becomes

$$k_8 \left(\frac{\rho\eta'''}{\eta''}\right)' = 0. \quad (46)$$

Here,  $\mu'\eta'' \neq 0$ .

If  $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$ , then  $k_8 = 0$ . Equation (7) gives

$$f(\mu) = c_0\mu. \quad (47)$$

Changing the function  $\eta$  such that  $\eta + c_0 \rightarrow \eta$ , one obtains

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\dot{\rho}\rho^\lambda \ln|\dot{\rho}| + \eta(\rho)\tilde{S}, \quad (\eta'' \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this model corresponds to  $M_{31}$ .

If  $\left(\frac{\rho\eta'''}{\eta''}\right)' = 0$ , then  $\eta'' = \rho^\nu$ , where  $\nu$  is constant. Further study depends on  $\nu$ .

If  $\nu = -1$ , then

$$\eta = \rho \ln \rho. \quad (48)$$

Substitution of (48) into (7) gives

$$2(k_3 - \lambda k_8)(f'\mu - f) = (c_1 - k_8 f), \quad (49)$$

where  $c_1$  is a constant of the integration.

Assume that  $f'\mu - f = 0$ , then  $f = q_1\mu$ , and Eq. (49) becomes

$$q_1 k_8 \mu = c_1.$$

Because  $\mu' \neq 0$ , one obtains  $q_1 k_8 = 0$  and  $c_1 = 0$ . If  $q_1 = 0$ , then

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S} \rho \ln \rho,$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad -\lambda X_1 + X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this model corresponds to  $M_{32}$ . If  $q_1 \neq 0$ , then  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S}(\rho \ln \rho + q_1), \quad (q_1 \neq 0),$$

and the extension is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{33}$ .

If  $f'\mu - f \neq 0$ , then

$$k_3 = \lambda k_8 + \frac{c_1 - k_8 f}{2(f'\mu - f)}. \tag{50}$$

Differentiating the last equation with respect to  $\mu$ , one gets

$$\left( \frac{c_1 - k_8 f}{f'\mu - f} \right)' = 0$$

or

$$c_0(f'\mu - f) = c_1 - k_8 f,$$

where  $c_0$  is constant. It should be noted that if  $c_0 = 0$ , then an extension of the kernel only occurs for  $k_8 \neq 0$ . This means that  $f = \text{const}$  which is without loss of generality can be assumed  $f = 0$ , and then  $f'\mu - f = 0$ . Hence, one has to assume that  $c_0 \neq 0$ . This implies that

$$f'\mu - \alpha f = q_3,$$

where  $k_8 = c_0(1 - \alpha)$  and  $c_1 = c_0 q_3$ . It should be noted that by virtue of the equivalence transformation corresponding to the generator  $X_7^e$ , one can assume that  $\alpha q_3 = 0$ . We also note that for  $\alpha = 1$ , one obtains  $k_8 = q_3 = 0$ , which prohibits an extension of the kernel. Hence,  $\alpha \neq 1$ . The extension of the kernel of admitted Lie algebras is given by the only generator

$$2(1 - \alpha)(X_8 - \lambda X_1 + \lambda X_3) + X_3 - 2\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S} \rho \ln \rho + f(\tilde{S}),$$

$\tilde{S} = \mu(S)$  and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^\alpha, & \alpha(\alpha - 1) \neq 0. \end{cases}$$

In the final Table 1, these models correspond to  $M_{34}$  and  $M_{35}$ .

If  $\nu = -2$ , then  $h = \mu(S) \ln \rho + f(\mu(S))$ . Integrating Eq. (7), one has

$$(2k_3 - (2\lambda + 1)k_8)(\mu f' - f) - \mu k_8 = c_1, \tag{51}$$

where  $c_1$  is a constant of the integration. If  $f'\mu - f = 0$  or  $f = q_1\mu$ , then  $k_8 = 0$ , and  $c_1 = 0$ , and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S}(\ln \rho + q_1). \quad (52)$$

The extension of the kernel in this case is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{36}$ . If  $f'\mu - f \neq 0$ , then

$$2k_3 = (2\lambda + 1)k_8 + \frac{c_1 + \mu k_8}{\mu f' - f},$$

and, hence,

$$\left( \frac{c_1 + k_8 \mu}{\mu f' - f} \right)' = 0$$

or

$$c_0(\mu f' - f) = c_1 + k_8 \mu,$$

where  $c_0$  is constant. It should be noted that if  $c_0 = 0$ , then  $k_8 = 0$ , and there is not an extension of the kernel of admitted Lie algebras. Hence,  $c_0 \neq 0$ , and

$$f'\mu - f = q_3 + \alpha\mu,$$

where  $k_8 = c_0\alpha$  and  $c_1 = c_0q_3$ . The general solution of the last equation is

$$f = \alpha\mu \ln(\mu) + q_1\mu - q_3.$$

Thus, the extension of the kernel of admitted Lie algebras is given by the generator

$$-2\lambda\alpha X_1 + (2\alpha\lambda + 1)X_3 + 2\alpha X_8 + 2(\alpha - 1)\tilde{S}\partial_{\tilde{S}},$$

where

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S}(\ln \rho + q_1) + \alpha \tilde{S} \ln(\tilde{S}),$$

and  $\tilde{S} = \mu(S)$ . Notice also that the previous case (52) is included in the present case by setting  $\alpha = 0$ . In the final Table 1, this model corresponds to  $M_{37}$ .

Let  $(v + 1)(v + 2) \neq 0$ , then  $h = \rho^{v+2}\mu(S) + f(\mu(S))$ , and Eq. (7) gives

$$(2k_3 - (2\lambda - v - 1)k_8)(\mu f' - f) + (v + 2)fk_8 = c_1. \quad (53)$$

If  $f'\mu - f = 0$  or  $\mu_2 = q_1\mu$ , then

$$(v + 2)\mu q_1 k_8 = c_1.$$

Because  $(v + 2)\mu' \neq 0$ , one obtains that  $q_1 k_8 = 0$  and  $c_1 = 0$ . If  $q_1 = 0$ , then

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S}\rho^{v+2},$$

and the extension of the kernel is given by the generators

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}, \quad -\lambda X_1 + X_8 + (2\lambda - v - 1)\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this model corresponds to  $M_{38}$ .

If  $q_1 \neq 0$ , then  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S}(\rho^{v+2} + q_1), \quad (q_1 \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{39}$ .

If  $\mu f' - f \neq 0$ , then

$$2k_3 = (2\lambda - \nu - 1)k_8 + \frac{c_1 - (\nu + 2)fk_8}{\mu f' - f},$$

and, hence,

$$\frac{c_1 - (\nu + 2)fk_8}{\mu f' - f} = c_0,$$

where  $c_0$  is constant. It should be noted that if  $c_0 = 0$ , then an extension of the kernel only occurs for  $f = const$ , whereas by virtue of the equivalence transformation corresponding to the generator  $X_7^e$  one can assume that  $f = 0$ , and then  $f'\mu - f = 0$ . Hence,  $c_0 \neq 0$ , and

$$f'\mu - \alpha f = q_2,$$

where

$$c_1 = c_0 q_2, \quad k_8 = c_0 \frac{1 - \alpha}{\nu + 2}.$$

Here, as in the previous case, one has to require that  $\alpha \neq 1$ . Hence,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \dot{\rho} \rho^\lambda \ln |\dot{\rho}| + \tilde{S} \rho^{\nu+2} + f(\tilde{S}),$$

and the admitted generator is

$$2\lambda(\alpha - 1)X_1 + (\alpha(\nu + 1 - 2\lambda) + 2\lambda + 1)X_3 + 2(1 - \alpha)X_8 - 2(\nu + 2)\tilde{S}\partial_{\tilde{S}},$$

where

$$f = \begin{cases} q_2 \ln(\tilde{S}), & \alpha = 0; \\ q_2 \tilde{S}^\alpha, & \alpha(\alpha - 1) \neq 0. \end{cases}$$

In the final Table 1, these models correspond to  $M_{40}$  and  $M_{41}$ .

Returning to (45), if  $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho \neq 0$ , then Eq. (45) gives

$$2k_3 = -k_8 \left( \frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}} \right)_\rho / \left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho. \quad (54)$$

Thus,

$$\left( \frac{\rho h_{\rho\rho\rho} - h_{\rho\rho}(2\lambda - 1)}{h_{S\rho\rho}} \right)_\rho / \left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho = const$$

or

$$\rho h_{\rho\rho\rho} - H h_{S\rho\rho} = k_0 h_{\rho\rho}$$

where  $k_0$  is constant and  $H = H(S)$  is some function. It should be noted that for  $H = 0$ , one has the contradiction  $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$ . Hence,  $H(S) \neq 0$ . The general solution of the last equation (up to an equivalence transformation) is

$$h(\rho, S) = \rho^\nu g(\rho\mu(S)) + f(\mu(S)), \quad (55)$$

where  $\mu' \neq 0$ . Equation (7) becomes

$$\mu f'' + (\nu + 1)f' = 0. \quad (56)$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho} \ln |\dot{\rho}| + \rho^\nu g(\rho \tilde{S}) + f(\tilde{S}),$$

and the extension is given by the only generator

$$-2\lambda X_1 + (2\lambda - \nu + 1)X_3 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

Here,  $\tilde{S} = \mu(S)$ , and

$$f(\tilde{S}) = \begin{cases} q_2 \ln(\tilde{S}), & \nu = 0; \\ q_2 \tilde{S}^{-\nu}, & \nu \neq 0. \end{cases}$$

In the final Table 1, these models correspond to  $M_{42}$  and  $M_{43}$ .

Case  $k = -2$ .

Integrating (38), one obtains

$$\phi = q_0 \rho^\lambda \ln |\dot{\rho}|, \quad (q_0 \neq 0). \quad (57)$$

Substituting (57) into (3), one gets

$$k_1 = -k_3 + k_8 \frac{1 - \lambda}{2}.$$

Equation (5) becomes

$$2h_{S\rho\rho}\zeta^S = 2q_0 k_3 \rho^{\lambda-2} \lambda(\lambda - 1) - k_8(2\rho h_{\rho\rho\rho} - 2(\lambda - 2)h_{\rho\rho} + q_0 \rho^{\lambda-2} \lambda(\lambda^2 - 1)). \quad (58)$$

Assuming that  $h_{S\rho\rho} = 0$  or

$$h(\rho, S) = \eta(\rho) + \mu(S), \quad (\mu' \neq 0),$$

Equations (7) and (5) are reduced to the equations, respectively,

$$\zeta^S = \frac{k_8 \lambda \mu + c_0}{\mu'}, \quad (59)$$

$$q_0 \lambda(\lambda - 1)(2k_3 - k_8(\lambda + 1)) = 2k_8 \rho^{2-\lambda} (\eta''' \rho - \eta''(\lambda - 2)), \quad (60)$$

where  $c_0$  is the constant of integration.

Let  $\lambda(\lambda - 1) \neq 0$ . Equation (60) gives

$$k_3 = k_8 \left( \frac{\lambda + 1}{2} + \frac{(\eta''' \rho - \eta''(\lambda - 2))}{q_0 \lambda(\lambda - 1) \rho^{\lambda-2}} \right),$$

Differentiating this equation with respect to  $\rho$ , one has

$$k_8 (\rho^{2-\lambda} (\eta''' \rho - \eta''(\lambda - 2)))_\rho = 0.$$

If  $(\rho^{2-\lambda} (\eta''' \rho - \eta''(\lambda - 2)))_\rho = 0$ , then  $\eta'' = \rho^{\lambda-2} (\tilde{q}_1 + q_0 \lambda(\lambda - 1) \nu \ln(\rho))$  or, by virtue of equivalence transformations,

$$\eta = \rho^\lambda (q_1 + q_0 \nu \ln(\rho)).$$

Here,  $\nu$  and  $q_1$  are constant. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda (q_1 + q_0 \ln(\rho^\nu |\dot{\rho}|)) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-2(\lambda + \nu)X_1 + (\lambda + 2\nu + 1)X_3 + 2X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}}, \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{44}$ . If  $(\rho^{2-\lambda}(\eta''' \rho - \eta''(\lambda - 2)))_\rho \neq 0$ , then  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{45}$ .

Let  $\lambda(\lambda - 1) = 0$ . Equation (60) becomes

$$k_8(\eta''' \rho - \eta''(\lambda - 2)) = 0.$$

If  $\eta'' \neq q_1 \rho^{(\lambda-2)}$ , then  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \ln |\dot{\rho}| + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the generators

$$-X_1 + X_3, \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{46}$ . If  $\eta'' = q_1 \rho^{(\lambda-2)}$  or

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \ln(|\dot{\rho}| \rho^\nu) + \tilde{S}, \quad (\lambda(\lambda - 1) = 0),$$

then the extension of the kernel is given by the generators

$$-X_1 + X_3, (1 - \lambda)X_1 + 2X_8 + 2\lambda\tilde{S}\partial_{\tilde{S}}, \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{47}$ .

Assuming that  $h_{S\rho\rho} \neq 0$  in Eq. (58), one obtains

$$2\zeta^S = q_0 \lambda(\lambda - 1)(2k_3 - k_8(\lambda + 1)) \frac{\rho^{\lambda-2}}{h_{S\rho\rho}} - 2k_8 \frac{\rho h_{\rho\rho\rho} - (\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}}. \quad (61)$$

Differentiating the last equation with respect to  $\rho$ , one gets

$$q_0 \lambda(\lambda - 1)(2k_3 - k_8(\lambda + 1)) \left( \frac{\rho^{\lambda-2}}{h_{S\rho\rho}} \right)_\rho = 2k_8 \left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho. \quad (62)$$

If  $\lambda(\lambda - 1) = 0$ , then Eq. (62) becomes

$$k_8 \left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho = 0.$$

Let

$$\left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} - \frac{(\lambda - 2)h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho = 0,$$

then

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\lambda - 2)h_{\rho\rho}, \quad (63)$$

where  $H = H(S)$  is a function of the integration. A solution of the last equation depends on the function  $H(S)$ .

Assuming that  $H = 0$ , one has  $\zeta^S = 0$ ,

$$h(\rho, S) = \mu(S)\rho^\lambda \ln \rho + f(\mu(S)),$$

where  $\mu' \neq 0$ . Equation (7) becomes

$$k_8(\lambda f' + q_0(\lambda - 1)\rho^\lambda) = 0. \quad (64)$$

If  $\lambda = 0$ , then Eq. (64) gives  $k_8 = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \ln |\dot{\rho}| + \tilde{S} \ln \rho + f(\tilde{S}),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

If  $\lambda = 1$ , then Eq. (64) becomes  $k_8 f' = 0$ . For  $f' \neq 0$ , one has  $k_8 = 0$ ,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho(q_0 \ln |\dot{\rho}| + \tilde{S} \ln \rho) + f(\tilde{S}), \quad (f' \neq 0),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1, these models correspond to  $M_{48}$  and  $M_{49}$ . For  $f' = 0$ , one has

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho(q_0 \ln |\dot{\rho}| + \tilde{S} \ln \rho),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \quad X_8.$$

In the final Table 1, this model corresponds to  $M_{50}$ .

Assuming that  $H \neq 0$  in (63), one obtains

$$h(\rho, S) = \rho^\lambda \phi(\rho\mu(S)) + f(\mu(S)), \quad (65)$$

where  $\mu' \neq 0$ . Substitution of  $h(\rho, S)$  into (7) gives

$$k_8(\mu f' + \lambda f)' = 0. \quad (66)$$

If  $(\mu f' + \lambda f)' = 0$  or

$$f = \begin{cases} q_1 \ln(\mu), & \lambda = 0, \\ q_1 \mu^{-1}, & \lambda = 1, \end{cases}$$

then

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda(q_0 \ln |\dot{\rho}| + \phi(\rho\tilde{S})) + f(\tilde{S}),$$

and the extension of the kernel is given by the generators

$$X_1 - X_3, \quad (1 - \lambda)X_1 + 2X_8 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, these models correspond to  $M_{51}$  and  $M_{52}$ . If  $(\mu f' + \lambda f)' \neq 0$ , then  $k_8 = 0$ ,

$$W(\rho, \dot{\rho}, \tilde{S}) = \rho^\lambda(q_0 \ln |\dot{\rho}| + \phi(\rho\tilde{S})) + f(\tilde{S}),$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3.$$

In the final Table 1, this model corresponds to  $M_{53}$ .

Returning to (62), let  $\lambda(\lambda - 1) \neq 0$ . Assume also that

$$\left( \frac{\rho^{\lambda-2}}{h_{S\rho\rho}} \right)_\rho = 0,$$

which means that

$$h(\rho, S) = q_0\mu(S)\rho^\lambda + \eta(\rho) + f(\mu(S)),$$

where  $\mu' \neq 0$ . Then Eq. (62) becomes

$$k_8 (\rho^{2-\lambda}(\rho\eta''' - (\lambda - 2)\eta''))_\rho = 0.$$

If  $(\rho^{2-\lambda}(\rho\eta''' - (\lambda - 2)\eta''))_\rho \neq 0$ , then  $k_8 = 0$ . Substituting into (7), one obtains

$$k_3 f'' = 0. \tag{67}$$

Since for  $k_3 = 0$ , there is no extension of the kernel, one has  $f'' = 0$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln|\dot{\rho}| + \tilde{S}) + \eta(\rho) + q_1\tilde{S},$$

and the extension of the kernel is given by the only generator

$$X_1 - X_3 - \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{54}$ .

If  $(\rho^{2-\lambda}(\rho\eta''' - (\lambda - 2)\eta''))_\rho = 0$ , then  $\eta'' = \rho^{\lambda-2}(\tilde{\nu} \ln \rho + \tilde{q}_1)$ , where  $\tilde{\nu}$  and  $\tilde{q}_1$  are constant. Using equivalence transformations, one finds that  $\eta = \rho^\lambda(q_0\nu \ln \rho + q_1)$ , where  $\tilde{\nu} = q_0\nu\lambda(\lambda - 1)$  and  $\tilde{q}_1 = q_1\lambda(\lambda - 1) + q_0\nu(2\lambda - 1)$ . In this case,

$$k_1 = - \left( k_3 - k_8 \frac{\lambda - 1}{2} \right), \quad \zeta^S = (2k_3 - k_8(\lambda + 2\nu + 1))/(2\mu'),$$

and Eq. (7) becomes

$$(2k_3 - k_8(\lambda + 2\nu + 1))f' - 2k_8\lambda f = \tilde{q}_2,$$

where  $\tilde{q}_2$  is constant. The last equation can be rewritten in the form

$$\alpha f' - lf = \tilde{q}_2,$$

where

$$k_8 = \frac{l}{2\lambda}, \quad k_3 = \frac{\alpha}{2} + \frac{l}{2\lambda} \frac{\lambda + 2\nu + 1}{2}.$$

Further analysis depends on the constants  $\alpha$  and  $l$ . It should be noted that for the existence of an extension of the kernel of admitted Lie algebras, one needs to require that  $\alpha^2 + l^2 \neq 0$ . Hence, for  $\alpha = 0$ , one has  $l \neq 0$ , which means that without loss of generality one can assume that  $f = 0$ . In the case  $f = 0$ , one obtains

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \tilde{S}) + q_1\rho^\lambda,$$

and the extension of the kernel is given by the generators

$$X_1 - X_3 - \partial_{\tilde{S}}, \quad 2(\lambda - 1)X_1 + 2X_8 - (\lambda + 2\nu + 1)\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{55}$ . For  $\alpha \neq 0$ , one has

$$f = \begin{cases} q_2\mu, & l = 0; \\ q_2e^{-\kappa\mu} & l \neq 0, \end{cases}$$

and

$$k_1 = -\frac{\kappa(\lambda + \nu) + \lambda}{2\lambda}, \quad k_8 = \frac{\kappa}{2\lambda}, \quad k_3 = \frac{1}{2} + \frac{\kappa}{4\lambda}(\lambda + 2\nu + 1),$$

where  $l = \kappa\alpha$  and  $q_2 \neq 0$  is constant. Thus, one obtains:

(a) for the function  $f(\tilde{S}) = q_2\tilde{S}$ :

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \tilde{S}) + q_1\rho^\lambda + q_2\tilde{S}, \quad (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-X_1 + X_3 + \partial_{\tilde{S}};$$

(b) for the function  $f(\tilde{S}) = q_2e^{\kappa\tilde{S}}$ :

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + \tilde{S}) + q_1\rho^\lambda + q_2e^{\kappa\tilde{S}}, \quad (q_2 \neq 0),$$

and the extension of the kernel is given by the only generator

$$-2(\kappa(\lambda + \nu) + \lambda)X_1 + 2\kappa X_8 + (2\lambda + \kappa(\lambda + 2\nu + 1))X_3 + 2\lambda\partial_{\tilde{S}}.$$

In the final Table 1 these models correspond to  $M_{56}$  and  $M_{57}$ .

Assume that  $\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho \neq 0$ , then from (62) one finds

$$k_3 = k_8 \left( \frac{\left(\frac{\rho h_{\rho\rho\rho} - (\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{q_0\lambda(\lambda-1)\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho} + \frac{\lambda+1}{2} \right). \quad (68)$$

Since for  $k_8 = 0$ , there is not an extension, then

$$\frac{\left(\frac{\rho h_{\rho\rho\rho} - (\lambda-2)h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho}{\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho} = \text{const}$$

or

$$\rho h_{\rho\rho\rho} + H(S)h_{S\rho\rho} = (\lambda-2)h_{\rho\rho} + \nu\rho^{\lambda-2}, \quad (69)$$

where  $\nu$  is constant and  $H(S)$  is some function. It should be noted that for  $H(S) = 0$ , one obtains

$$h_{\rho\rho} = (\nu \ln \rho + \mu(S))\rho^{\lambda-2},$$

which leads to the contradiction

$$\left(\frac{\rho^{\lambda-2}}{h_{S\rho\rho}}\right)_\rho = 0.$$

Hence, one has to assume that  $H(S) \neq 0$ , which gives

$$h_{\rho\rho}(\rho, S) = \rho^{\lambda-2}(\tilde{\nu} \ln \rho + \tilde{g}(\rho\mu(S)))$$

or

$$h(\rho, S) = \rho^\lambda(\nu \ln \rho + g(\rho\mu(S))) + f(\mu(S)),$$

where  $\mu' \neq 0$ . Equation (7) gives  $f = q_2\mu^{-\lambda}$ . Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0\rho^\lambda(\ln(|\dot{\rho}|\rho^\nu) + g(\rho\tilde{S})) + q_2\tilde{S}^{-\lambda},$$

and the extension of the kernel is given by the only generator

$$-2(\lambda + \nu)X_1 + 2X_8 + (\lambda + 2\nu + 1)X_3 - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{58}$ .

Case  $(k + 1)(k + 2) \neq 0$

Returning to integration of (38), if  $(k + 2)(k + 1) \neq 0$ , then one obtains

$$\phi = q_0 \rho^\lambda \dot{\rho}^{k+2} \tag{70}$$

Substituting (70) into (3), one has

$$k_3 = -k_1 \frac{k}{2(k+1)} + k_8 \frac{k+\lambda+1}{2(k+1)},$$

and Eq. (5) becomes

$$\zeta^S h_{S\rho\rho} + h_{\rho\rho} \left( k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) + k_8 \rho h_{\rho\rho\rho} = 0. \tag{71}$$

Assuming that  $h_{S\rho\rho} = 0$ , one finds

$$h(\rho, S) = \eta(\rho) + \mu(S),$$

where  $\mu' \neq 0$ . Equation (71) becomes

$$\eta'' \left( k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) + k_8 \rho \eta''' = 0. \tag{72}$$

Let  $\eta'' \neq 0$ , then

$$k_1 = -k_8 \frac{k+1}{k+2} \left( \frac{2k+\lambda+2}{k+1} + \frac{\rho \eta'''}{\eta''} \right).$$

Differentiating the last equation with respect to  $\rho$ , one gets

$$k_8 \left( \frac{\rho \eta'''}{\eta''} \right)' = 0. \tag{73}$$

If  $\frac{\rho \eta'''}{\eta''} = k_0 = const$ , then  $\eta'' = q_1 \rho^\nu$ , where  $\nu = k_0$ . This gives that

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1 \rho^\nu \neq 0),$$

and the extension of the kernel is given by the generators

$$-\frac{k+1}{k+2} \left( \frac{2k+\lambda+2}{k+1} + \nu \right) X_1 + \frac{k\nu+3k+2\lambda+2}{2(k+2)} X_3 + X_8 + (\nu+2) \tilde{S} \partial_{\tilde{S}}, \quad \partial_{\tilde{S}},$$

where  $\eta'' = q_1 \rho^\nu$ ,  $\tilde{S} = \mu(S)$  and  $q_1 \neq 0$ . In the final Table 1, this model corresponds to  $M_{59}$ ,  $(k^2 + \lambda^2 \neq 0)$ .

If  $\left( \frac{\rho \eta'''}{\eta''} \right)' \neq 0$ , then  $k_8 = 0$ ,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \eta(\rho) + \tilde{S},$$

and the extension of the kernel is given by the only generator

$$\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{60}$ ,  $(k^2 + \lambda^2 \neq 0)$ .

Considering (72), let  $\eta'' = 0$ . Without loss of the generality, one can assume that  $\eta = 0$ . Equation (7) gives

$$\zeta^S = -\frac{\mu}{\mu'} \left( k_1 \frac{k+2}{k+1} + k_8 \frac{\lambda}{k+1} \right) + \frac{c_0}{\mu'}.$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S},$$

and the extension of the kernel of admitted Lie algebras is defined by the generators

$$X_1 - \frac{k}{2(k+1)} X_3 - \frac{k+2}{k+1} \tilde{S} \partial_{\tilde{S}}, \quad \frac{k+\lambda+1}{2(k+1)} X_3 + X_8 - \frac{\lambda}{k+1} \tilde{S} \partial_{\tilde{S}}, \quad \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{61}$ , ( $k^2 + \lambda^2 \neq 0$ ). Returning to (71), assume that  $h_{S\rho\rho} \neq 0$ . Then

$$\zeta^S = -\frac{h_{\rho\rho}}{h_{S\rho\rho}} \left( k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) - k_8 \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}}.$$

Differentiating this equation with respect to  $\rho$ , one finds

$$\left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho \left( k_1 \frac{k+2}{k+1} + k_8 \frac{2k+\lambda+2}{k+1} \right) + k_8 \left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} \right)_\rho = 0. \quad (74)$$

If  $\left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho \neq 0$ , then

$$k_1 = -k_8 \frac{k+1}{k+2} \left( \frac{2k+\lambda+2}{k+1} + \frac{\left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} \right)_\rho}{\left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho} \right).$$

Extension of the kernel occurs only for

$$\frac{\left( \frac{\rho h_{\rho\rho\rho}}{h_{S\rho\rho}} \right)_\rho}{\left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho} = \text{const},$$

which means that

$$\rho h_{\rho\rho\rho} - H(S) h_{S\rho\rho} = \tilde{v} h_{\rho\rho},$$

where  $H(S)$  is some function and  $v$  is constant. It should be noted that for  $H(S) = 0$  one has

$$h_{\rho\rho}(\rho, S) = \rho^{\tilde{v}} \mu(S)$$

which leads to the contradiction  $\left( \frac{h_{\rho\rho}}{h_{S\rho\rho}} \right)_\rho = 0$ . Hence,  $H(S) \neq 0$ , and then

$$h_{\rho\rho}(\rho, S) = \rho^{\tilde{v}} \tilde{g}(\rho \mu(S)),$$

or

$$h(\rho, S) = \rho^v g(\rho \mu(S)) + f(\mu(S)),$$

where  $\mu' \neq 0$  and  $(z^{\nu+1} g'(z))'' \neq 0$ . Equation (7) leads to the condition

$$\mu f' + v f = \tilde{q}_2,$$

where  $\tilde{q}_2$  is constant. The general solution of the last equation depends on  $v$ :

$$f(\mu) = \begin{cases} q_2 \ln(\mu), & v = 0, \\ q_2 \mu^{-v}, & v \neq 0. \end{cases}$$

Thus, setting  $\tilde{S} = \mu(S)$ , one gets

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \rho^\nu g(\rho \tilde{S}) + f(\tilde{S}),$$

and the extension of the kernel of admitted Lie algebras is defined by the generator

$$-\frac{\nu(k+1)+\lambda}{k+2}X_1 + \frac{k(\nu+1)+2\lambda+2}{2(k+2)}X_3 + X_8 - \tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, these models correspond to  $M_{62}$  and  $M_{63}$ , ( $k^2 + \lambda^2 \neq 0$ ).

If  $\left(\frac{h_{\rho\rho}}{h_{S\rho\rho}}\right)_\rho = 0$ , then  $h(\rho, S) = \mu(S)(\eta(\rho) + f(\mu(S)))$ , where  $\eta''\mu' \neq 0$ . Equation (74) becomes

$$k_8 \left(\frac{\rho\eta'''}{\eta''}\right)' = 0.$$

If  $\left(\frac{\rho\eta'''}{\eta''}\right)' \neq 0$ , then  $k_8 = 0$ , Eq. (7) leads to the equation

$$\mu f'' + 2f' = 0.$$

A solution of the last equation is  $f(\mu) = c_1/\mu + c_0$ , where  $c_0$  and  $c_1$  are constant. Without loss of generality, one can assume that  $c_1 = c_0 = 0$ . Thus,

$$k_3 = -k_1 \frac{k}{2(k+1)}, \quad \zeta^S = -k_1 \frac{k+2}{k+1} \frac{\mu}{\mu'},$$

and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S}\eta(\rho).$$

The extension of the kernel consists of the generator

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{64}$ , ( $k^2 + \lambda^2 \neq 0$ ).

If  $\frac{\rho\eta'''}{\eta''} = k_0 = const$ , then  $\eta'' = \tilde{q}_1 \rho^{\nu-2}$ , where  $\nu = 2(k_0 + 1)$ . One can choose the function  $\eta(\rho)$  as follows

$$\eta = \begin{cases} \ln(\rho), & \nu = 0, \\ \rho \ln(\rho), & \nu = 1, \\ \rho^\nu, & \nu(\nu-1) \neq 0. \end{cases}$$

This reduces Eq. (7) to the equations

$$\begin{aligned} \nu = 0 : a\mu f' &= b + \tilde{q}_2 \mu^{-1}, \\ \nu = 1 : a\mu f' + bf &= \tilde{q}_2 \mu^{-1}, \\ \nu(\nu-1) \neq 0 : a\mu f' + \nu bf &= \tilde{q}_2 \mu^{-1}. \end{aligned} \tag{75}$$

where  $a = k_1(k+2) + k_8(\lambda + \nu(k+1))$ ,  $b = k_8(k+1)$  and  $\tilde{q}_2$  is constant. It should be noted that the condition  $a^2 + b^2 = 0$  leads to the relations  $k_1 = 0$  and  $k_8 = 0$ . These conditions do not allow an extension of the kernel of admitted Lie algebras. Hence, one has to assume that  $a^2 + b^2 \neq 0$ .

Let us consider the case  $\nu = 0$ , where  $\eta = \ln(\rho)$ . In this case  $a \neq 0$ , because otherwise  $b = 0$ . Using equivalence transformations, the general solution of Eq. (75) <sub>$\nu=0$</sub>  has the representation:

$$f = \beta \ln(\mu) + q_2,$$

where  $\beta$  and  $q_2$  are constant. Substituting the representation of the function  $f(\mu)$  into Eq. (75) <sub>$v=0$</sub> , one finds that  $\beta a = b$  and  $\tilde{q}_2 = 0$ . Therefore,

$$k_1 = a \frac{k+1-\lambda\beta}{(k+1)(k+2)}, \quad k_3 = a \frac{\beta(k+2\lambda+2)-k}{2(k+1)(k+2)}, \quad k_8 = a \frac{\beta}{k+1},$$

and

$$W = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S} \left( \ln \left( \rho \tilde{S}^\beta \right) + q_2 \right),$$

where  $\tilde{S} = \mu(S)$ . The extension of the kernel of admitted Lie algebras is defined by the only generator

$$\frac{k+1-\beta\lambda}{k+2} X_1 + \frac{\beta(k+2\lambda+2)-k}{2(k+2)} X_3 + \beta X_8 - \tilde{S} \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{65}$ , ( $k^2 + \lambda^2 \neq 0$ ). In other two cases,  $v = 1$  and  $v(v-1) \neq 0$  one has to solve the equation

$$a\mu f' + vbf = \tilde{q}_2 \mu^{-1}, \quad (v \neq 0). \tag{76}$$

By virtue of equivalence transformations, the function  $f$  is equivalent to the function  $\tilde{f} = f - r\mu^{-1}$ , where  $r$  is constant. The change  $f = \tilde{f} + r\mu^{-1}$  reduces Eq. (76) to the equation

$$a\mu \tilde{f}' + v b \tilde{f} = (\tilde{q}_2 + (a - vb)r)\mu^{-1}.$$

This means that for  $a - vb \neq 0$ , one can assume in (76) that  $\tilde{q}_2 = 0$ . Therefore, the analysis of solutions of Eq. (76) is reduced to the study of solutions of either the homogeneous equation

$$a\mu f' + vbf = 0, \tag{77}$$

or the nonhomogeneous equation

$$\mu f' + f = q_2 \mu^{-1}, \quad (q_2 \neq 0). \tag{78}$$

The function  $f = 0$  is the trivial solution of Eq. (77). In this case,  $k_1$  and  $k_3$  are arbitrary. Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S} \eta(\rho),$$

and the extension of the kernel consists of the generators

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}, \quad (k+\lambda+1)X_3 + 2(k+1)X_8 - 2(\lambda+v(k+1))\tilde{S}\partial_{\tilde{S}}.$$

Here,  $\tilde{S} = \mu(S)$ . In the final Table 1, these models correspond to  $M_{66}$  and  $M_{70}$ , ( $k^2 + \lambda^2 \neq 0$ ).

The only nontrivial solution of Eq. (77) has the representation

$$f(\mu) = q_2 \mu^\beta, \quad (q_2 \neq 0, \quad \beta \neq -1).$$

Substituting the representation into Eq. (77), it becomes

$$\beta(k_1(k+2) + k_8(\lambda + v(k+1))) + k_8 v(k+1) = 0. \tag{79}$$

If  $\beta = 0$ , then  $k_8 = 0$ , and

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S}(\eta(\rho) + q_2),$$

with the extension

$$2(k+1)X_1 - kX_3 - 2(k+2)\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, these models correspond to  $M_{67}$  and  $M_{71}$ , ( $k^2 + \lambda^2 \neq 0$ ). If  $\beta \neq 0$ , then Eq. (79) gives

$$k_1 = -k_8 \frac{\beta(\lambda + v(k+1)) + v(k+1)}{\beta(k+2)}.$$

Thus,

$$k_3 = k_8 \frac{\beta(kv + k + 2\lambda + 2) + kv}{2(k + 2)\beta}, \quad \zeta^S = k_8 \frac{v}{\beta} \frac{\mu}{\mu'},$$

and the potential function is

$$W(\rho, \dot{\rho}, S) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S}(\eta(\rho) + q_2 \tilde{S}^\beta), \quad (q_2 \beta(\beta + 1) \neq 0).$$

The extension of the kernel of admitted Lie algebras is defined by the only generator

$$2 \frac{\beta(\lambda + v(k + 1)) + v(k + 1)}{(k + 2)} (X_3 - X_1) - (\beta v - \beta + v) X_3 + 2\beta X_8 + 2v \tilde{S} \partial_{\tilde{S}}.$$

In the final Table 1, these models correspond to  $M_{69}$  and  $M_{73}$ ,  $(k^2 + \lambda^2 \neq 0)$ .

The representation of the general solution of Eq. (78) is  $f = q_2 \mu^{-1} \ln(\mu)$ . Substituting the representation into Eq. (76), it gives

$$\tilde{q}_2 = a q_2, \quad a - vb = 0.$$

Hence,

$$k_1 = -k_8 \frac{\lambda}{k + 2}.$$

Thus,

$$W(\rho, \dot{\rho}, \tilde{S}) = q_0 \rho^\lambda \dot{\rho}^{k+2} + \tilde{S} \eta(\rho) + q_2 \ln(\tilde{S}), \quad (q_2 \neq 0),$$

and the extension of the kernel is defined by the generator

$$2\lambda(X_3 - X_1) + (k + 2)(X_3 + 2X_8 - 2v \tilde{S} \partial_{\tilde{S}}).$$

In the final Table 1, these models correspond to  $M_{68}$  and  $M_{72}$ ,  $(k^2 + \lambda^2 \neq 0)$ .

$$\dim(\text{Span}(V)) = 0$$

In this case, the vector

$$(\dot{\rho} \phi_{\dot{\rho} \dot{\rho} \dot{\rho}}, 2(\dot{\rho} \phi_{\dot{\rho} \dot{\rho} \dot{\rho}} + \phi_{\dot{\rho} \dot{\rho}}), -(\dot{\rho} \phi_{\dot{\rho} \dot{\rho} \dot{\rho}} + \rho \phi_{\dot{\rho} \dot{\rho} \dot{\rho}} + \phi_{\dot{\rho} \dot{\rho}}))$$

is constant. This condition implies that

$$\phi = q_0 \dot{\rho}^2.$$

Substituting  $\phi$  into (3) and (5), one gets, respectively,

$$k_3 = \frac{1}{2} k_8,$$

$$\zeta^S h_{S\rho\rho} + 2k_1 h_{\rho\rho} + k_8 (\rho h_{\rho\rho\rho} + 2h_{\rho\rho}) = 0. \tag{80}$$

Assume that  $h_{S\rho\rho} \neq 0$ , then

$$\zeta^S = -2ak_1 - k_8 b, \tag{81}$$

where  $a = \frac{h_{\rho\rho}}{h_{S\rho\rho}}$ ,  $b = \frac{\rho h_{\rho\rho\rho} + 2h_{\rho\rho}}{h_{S\rho\rho}}$ . Differentiating (81) with respect to  $\rho$ , one obtains

$$2k_1 a_\rho + k_8 b_\rho = 0. \tag{82}$$

If  $a_\rho = 0$  then,  $h(\rho, S) = \eta(\rho)\mu(S) + f(\mu(S))$ , where  $\eta''\mu' \neq 0$ . Equation (82) becomes

$$k_8 \left( \frac{\rho\eta'''}{\eta''} \right)' = 0.$$

If  $\left( \frac{\rho\eta'''}{\eta''} \right)' \neq 0$ , then  $k_8 = 0$ , and Eq. (7) becomes

$$k_1 f'' = 0.$$

Since for  $k_1 = 0$  there is no extension of the kernel, without loss of generality one can assume that  $f = 0$ . Thus,

$$W = \dot{\rho}^2 q_0 + \eta(\rho)\tilde{S},$$

and the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

where  $\tilde{S} = \mu(S)$ . In the final Table 1, this model corresponds to  $M_{64}$ , ( $k = \lambda = 0$ ).

If  $\left( \frac{\rho\eta'''}{\eta''} \right)' = 0$  or  $\eta'' = \rho^{\nu-2}$ . Finding the function  $\eta(\rho)$  depends on the value of  $\nu$ .

Let  $\nu(\nu - 1) \neq 0$ , then  $\eta = \rho^\nu$  and Eq. (7) becomes

$$2k_1\mu f'' + \nu k_8 (\mu f'' + f') = 0. \tag{83}$$

If  $f'' = 0$ , then  $f = q_1\mu$  and Eq. (83) is reduced to the equation

$$k_8 q_1 = 0.$$

Hence, if  $q_1 \neq 0$ , then  $k_8 = 0$  and

$$W = \dot{\rho}^2 q_0 + (\rho^\nu + q_1)\tilde{S}, \quad (q_1 \neq 0),$$

the extension of the kernel is given by the generator

$$X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

In the final Table 1, this model corresponds to  $M_{71}$ , ( $k = \lambda = 0$ ).

If  $q_1 = 0$ , then  $k_8$  is arbitrary, and

$$W = \dot{\rho}^2 q_0 + \rho^\nu \tilde{S}.$$

The extension of the kernel is given by the generators

$$X_1 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_3 + 2X_8 - 2\nu\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{70}$ , ( $k = \lambda = 0$ ).

If  $f'' \neq 0$ , then Eq. (83) gives that

$$\mu f'' - \beta f' = 0, \quad (\mu \neq 0),$$

where  $\beta$  is constant and

$$k_1 = -\nu k_8 \frac{(\beta + 1)}{2\beta}. \tag{84}$$

Thus,

$$W = \dot{\rho}^2 q_0 + \rho^\nu \tilde{S} + f(\tilde{S}),$$

and the extension of the kernel is given by the generator

$$-\nu(\beta + 1)X_1 + \beta X_3 + 2\beta X_8 + 2\nu\tilde{S}\partial_{\tilde{S}}, \quad (\beta \neq 0).$$

Here

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1. \end{cases}$$

In the final Table 1, these models correspond to  $M_{72}$  and  $M_{73}$ , ( $k = \lambda = 0$ ).

For  $\nu = 1$ , one has  $\eta = \rho \ln(\rho)$ . Further analysis of this equation is similar to the previous case:

$$W = q_0 \dot{\rho}^2 + \tilde{S}(\rho \ln \rho + q_1), \quad (q_1 \neq 0): \quad X_1 - 2\tilde{S}\partial_{\tilde{S}},$$

$$W = q_0 \dot{\rho}^2 + \tilde{S}\rho \ln \rho: \quad X_1 - 2\tilde{S}\partial_{\tilde{S}}, \quad X_3 + 2X_8 - 2\lambda\tilde{S}\partial_{\tilde{S}},$$

$$W = q_0 \dot{\rho}^2 + \tilde{S}\rho \ln \rho + f(\tilde{S}): \quad -(k + 1)X_1 + kX_3 + 2kX_8 + 2\tilde{S}\partial_{\tilde{S}}, \quad (k \neq 0),$$

where

$$f = \begin{cases} q_1 \ln(\tilde{S}), & \beta = -1, \\ q_1 \tilde{S}^{\beta+1}, & \beta \neq -1, \end{cases}$$

and  $q_1 \neq 0$ . In the final Table 1, these models correspond to  $M_{67}$ ,  $M_{66}$ ,  $M_{68}$  and  $M_{69}$ , ( $k = \lambda = 0$ ), respectively.

Let  $\nu = 0$ , then  $\eta = \ln(\rho)$ , and Eq. (7) becomes

$$k_8 = 2k_1 \mu f''.$$

This equation gives

$$k_1(\mu f'')' = 0.$$

Since for  $k_1 = 0$  there is no extension, one has that  $\mu f''$  is constant or after using equivalence transformation, one finds

$$f = \mu(\beta \ln(\mu) + q_2).$$

Thus,

$$W = q_0 \dot{\rho}^2 + \tilde{S}(\ln(\rho\tilde{S}^\beta) + q_2): \quad X_1 + \beta(X_3 + 2X_8) - 2\tilde{S}\partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{65}$ , ( $k = \lambda = 0$ ).

If in Eq. (82)  $a_\rho \neq 0$ , then there exists a constant  $\nu$  and a function  $H(S)$  such that

$$b - \nu a + H(S) = 0$$

or

$$\rho h_{\rho\rho} + H(S)h_{S\rho} = (\nu - 2)h_{\rho\rho}.$$

Hence,

$$k_1 = \nu k_8 / 2.$$

It should be noted that if  $H = 0$  then  $a_\rho = 0$ , hence  $H \neq 0$ . In this case,

$$h = \rho^\nu g(\rho\mu(S)) + f(\mu(S)), \tag{85}$$

where  $\mu' \neq 0$ . Equation (7) becomes

$$\mu f'' + (\nu + 1)f' = 0. \tag{86}$$

Thus,

$$W = q_0 \dot{\rho}^2 + \rho^\nu g(\rho \tilde{S}) + f(\tilde{S}) : -\nu X_1 + X_3 + 2X_8 - 2\tilde{S} \partial_{\tilde{S}}.$$

In the final Table 1, these models correspond to  $M_{62}$  and  $M_{63}$ , ( $k = \lambda = 0$ ).

If  $h_{S\rho\rho} = 0$ , then

$$h = \eta(\rho) + \mu(S),$$

where  $\mu' \neq 0$ , and Eqs. (5) (or 80) and (7) become, respectively,

$$2k_1 \eta'' + k_8(\rho \eta''' + 2\eta'') = 0. \quad (87)$$

$$(\zeta^S \mu')' = -2k_1 \mu' \quad (88)$$

Equation (88) gives

$$\zeta^S = (-2k_1 \mu + c_0) / \mu' \quad (89)$$

Hence, if  $\eta'' = 0$ , then one can assume that  $\eta = 0$ . In this case,

$$W = q_0 \dot{\rho}^2 + \tilde{S},$$

and the extension of the kernel is given by the generators

$$X_1 - 2\tilde{S} \partial_{\tilde{S}}, \quad X_3 + 2X_8, \quad \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{61}$ , ( $k = \lambda = 0$ ).

If  $\eta'' \neq 0$ , then Eq. (87) leads to

$$k_1 = -k_8 \left( \frac{\rho \eta'''}{2\eta''} + 1 \right).$$

This gives that

$$k_8 \left( \frac{\rho \eta'''}{\eta''} \right)' = 0.$$

For  $\rho \eta''' = \nu \eta''$ , one has

$$2k_1 + k_8(\nu + 2) = 0.$$

In this case,

$$W = q_0 \dot{\rho}^2 + \eta(\rho) + \tilde{S}, \quad (\eta'' = q_1 \rho^\nu),$$

and the extension of the kernel is given by the generators

$$-(\nu + 2)X_1 + X_3 + 2X_8 + 2(\nu + 2)\tilde{S} \partial_{\tilde{S}}, \quad \partial_{\tilde{S}}.$$

In the final Table 1, this model corresponds to  $M_{59}$ , ( $k = \lambda = 0$ ).

For  $\left( \frac{\rho \eta'''}{\eta''} \right)' \neq 0$ , one has the only generator  $\partial_{\tilde{S}}$ . In the final Table 1, this model corresponds to  $M_{60}$ , ( $k = \lambda = 0$ ).

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