space requirements of the requests, $W_{i,1}, W_{i,2}, \ldots, W_{i,n}$, are independent and identically distributed (i.i.d.) nonnegative integer-valued random variables. The i.i.d. assumption is needed for mathematical tractability, and it is quite common among operations research papers with stochastic model applications. (Moreover, we perform a statistical test on some real data collected from a major airline in Thailand, to test the null hypothesis that the space requirements are i.i.d. The p-value is 0.4661 for the shipments on 19 June 2008 and 0.7373 on 24 April 2008. We conclude that there is insufficient evidence to reject the null hypothesis at 5% significant level. A similar conclusion can be made for many other dates.) Our assumption that the space requirements are integer-valued is not restrictive, because the units can be chosen arbitrarily. As mentioned earlier, most airlines manage their capacity in terms on weight. In practice, the unit of space requirement may be that of a weight measuring device. Furthermore, the freight charge is usually based on the nearest whole pound or kilogram.

2 Optimal Solution

We will derive a closed-form expression for $E[U_i(x)]$, the expected actual allotment usage of forwarder i that receives allotment $x \in \mathbb{N}$. It follows from the assumptions in Section 1 and the recursive equation (2) that the stochastic process $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ is a discrete Markov chain with the state space $\{0, 1, \ldots, x\} := \mathcal{X}(x)$. Its initial distribution is $P(X_{i,0}(x) = 0) = 1$. Denote the (one-step) transition probability as $p_i(b, c|x) = P(X_{i,k}(x) = c|X_{i,k-1}(x) = b)$ for each $c, b \in \mathcal{X}(x)$.

Let h_i (resp., H_i) denote the probability mass (resp., cumulative distribution) function of a generic random variable $W_{i,1}$. We use a bar atop a distribution function to denote its complement; e.g., $\bar{H}_i(t) = 1 - H_i(t)$.

Lemma 1. *The transition probability of the Markov chain* $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ *is given* *as follows:*

$$p_i(b, c|x) = \begin{cases} h_i(c-b) & \textit{if } b < c \leq x \\ \bar{H}_i(x-b) + h_i(0) & \textit{if } c = b \\ 0 & \textit{if } 0 \leq c < b \end{cases} \tag{3}$$

for each $b < x$*, and*

$$p_i(x, c|x) = \begin{cases} 1 & \textit{if } c = x \\ 0 & \textit{if } c \neq x \end{cases} \tag{4}$$

Proof. Suppose that the cumulative usage prior to the k-th arrival of the booking request is b. Then, the unused portion of the allotment is $(x-b)$. Let c denote the total usage prior to the $(k+1)$-st arrival of the booking request. Note that the sequence of total usages are nondecreasing: $X_{i,k-1}(x) \leq X_{i,k}(x)$ with probability one. Therefore, $p_i(b, c|x) = 0$ if $c < b$, which is the third case in (3). For the first case, the total usage strictly increases from b to c, if and only if the k-th booking request with the space requirement $(c-b)$ is accepted; i.e., $c - b \leq x - b$, or equivalently $c \leq x$. The event that the arriving request has space requirement $(c-b)$ occurs with probability $h_i(c-b)$. For the second case, the total usage remains the same, if and only if the k-th booking request is rejected, or its space requirement is zero. The first occurs with probability $\bar{H}_i(x-b)$, whereas the latter occurs with probability $h_i(0)$. Finally, in equation (4), the cumulative usage prior to the k-th arrival is equal to the allotment. The forwarder has no allotment left, and all of future requests are rejected; thus, the cumulative usage must remain the same. □

Let $\mathbf{P}_i(x)$ be a (one-step) transition matrix, whose element in the b-th row and c-th column is $p_i(b, c|x)$ for each $b, c \in \mathcal{X}(x)$. From the Chapman-Kolmogorov equation, the k-step transition matrix, denoted by $\mathbf{P}_i^{(k)}(x)$, can be obtained by multiplying the one-step transition matrix with itself k times: $\mathbf{P}_i^{(k)}(x) = (\mathbf{P}_i(x))^k$ for each $k \in \mathbb{N}$. The distribution of the Markov chain $\{X_{i,k}(x) : k \in \mathbb{Z}_+\}$ is completely specified by the one-step transition probability $\mathbf{P}_i(x)$ and the initial distribution $P(X_{i,0}(x) = 0) = 1$.

Proposition 1. *The expected actual usage, if forwarder i receives allotment $a \in \mathbb{R}_+$, is given by*

$$E[U_i(a)] = \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=1}^{x} t p_i^{(n)}(0, t|x)$$

where $x = \lfloor a \rfloor$, and $E[U_i(0)] = 0$.

Proof. Since N_i and $W_{i,k}$ are nonnegative integer-valued random variables, the expected actual usage $E[U_i(a)] = E[U_i(x)]$, where $x = \lfloor a \rfloor$. Recall that $U_i(x) = X_{i,N_i}(x)$. We calculate the expected total usage by conditioning on the number of booking requests:

$$\begin{aligned}
E[X_{i,N_i}(x)] &= \sum_{n=0}^{\infty} P(N_i = n) E[X_{i,N_i}(x) | N_i = n] \\
&= 0P(N_i = 0) + \sum_{n=1}^{\infty} P(N_i = n) E[X_{i,n}(x)] && (5) \\
&= \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=0}^{x} tP\big(X_{i,n}(x) = t | X_{i,0}(x) = 0\big) && (6) \\
&= \sum_{n=1}^{\infty} P(N_i = n) \sum_{t=0}^{x} t p_i^{(n)}(0, t|x) && (7)
\end{aligned}$$

In equation (5), the total usage is zero, if the forwarder makes no booking request; i.e., $E[X_{i,N_i}(x) | N_i = 0] = 0$. In equation (6), we calculate $E[X_{i,n}(x)]$ by conditioning on the initial state $X_{i,0}(x)$. [Note that the initial distribution is $P(X_{i,0}(x) = 0) = 1$.] Equation (7) follows from the definition of the n-step transition probability $p_i^{(n)}(0, \cdot|x)$ given that the initial state is 0. Recall that $p_i^{(n)}(b, c|x)$ is the element in the b-th row and c-th column of the n-step transition matrix $\mathbf{P}_i^{(n)}(x) = (\mathbf{P}_i(x))^n$. The one-step transition matrix $\mathbf{P}_i(x)$ is given in Lemma 1. We can ignore the first term in the second summation in (7), since it equals zero. This completes the proof. $\square$

Note that the expected actual allotment usage $E[U_i(x)]$ is nondecreasing but might not be concave on $\mathbb{Z}_+$. Non-concavity partly results from the assumption that the shipment of size $W_{i,k}$ is accepted on an all-or-none basis. For instance, suppose that the sequence of space requirements is $(1, 3, 9, 5, 2, 4)$. Then, the actual allotment usages, if the allotment are $(x : x = 4, 5, \ldots, 11)$ are $(u(x) : x = 4, 5, \ldots, 11) = (4, 4, 6, 6, 6, 9, 9, 11)$. The differences between two consecutive usages are $(u(x) - u(x-1) : x = 5, 6, \ldots, 11) = (0, 2, 0, 0, 3, 0, 2)$; clearly, the actual usage is not concave on $\mathbb{Z}_+$. [Recall that a function $f : \mathbb{Z}_+ \to \mathbb{R}$ is concave on $\mathbb{Z}_+$, if $f(a) - f(a-1)$ is nonincreasing in $a \in \mathbb{N}$.]

For short-hand notation, denote $\pi_i(a) = p_i E[U_i(a)]$ for each $a \in \mathbb{R}_+$. The carrier's problem can be expressed equivalently as

$$\zeta = \max \left\{ \sum_{i=1}^{m} \pi_i(a_i) : \sum_{i=1}^{m} a_i \leq \kappa, \quad a_i \geq 0 \text{ for each } i \in \mathcal{M} \right\} \tag{8}$$

Note that (8) is a special case of the nonlinear resource allocation problem. For each $i \in \mathcal{M}$, $\pi_i(a)$ is neither concave nor differentiable on $\mathbb{R}_+$. Problem (8) can be solved via dynamic programming. The number of stages equals m, the number of the forwarders. Define the state at the beginning of stage i as the amount of cargo capacity to be allocated to forwarders $i, i+1, \ldots, m$. At the beginning of stage i, the carrier observes the state y and determines the allotment for forwarder i, denoted by a_i. Let $u_i(y)$ be the value function at stage i; i.e., $u_i(y)$ is the maximum expected total contribution that can be earned from forwarders i through m, if the current state is y. The value function can be computed recursively via the Bellman optimality equation

$$u_i(y) = \max_{a_i = 0, 1, \ldots, y} \{\pi_i(a_i) + u_{i+1}(y - a_i)\}, \qquad i \in \mathcal{M} \tag{9}$$

and the boundary condition is $u_{m+1}(y) = 0$ for all y. Let $a_i^*(y) = \text{argmax}_{a=0,1,\ldots,y}\{\pi_i(a) + u_{i+1}(y-a)\}$.

Proposition 2. *The optimal objective function in (8) is $\zeta = u_1(\kappa)$. The optimal allot-*