tions 1 through k can be determined. With the bounds in hand, we obtain the updated Lagrange multiplier ν_{k+1} using a subgradient optimization method. The process continues until the best lower and upper bounds are close enough. Steps in Table 2 are similar to those in a generic Lagrangian relaxation algorithm; cf. Fisher (1985).

4 Numerical Examples

We illustrate the performance of our heuristic solutions via numerical examples. We also compare them with a proportional allocation scheme, whose variants may be used in practice because of their simplicity. In this scheme, the allotment for forwarder i is $a_i = \frac{\mu_i}{\sum_{i=1}^m \mu_i}\kappa$ for each $i \in \mathcal{M}$, where μ_i is defined as in (13). Note that the allotment that forwarder i receives is proportional to its mean. The larger the mean demand, the larger the allotment.

Below, we describe the setup in the numerical experiments. Suppose that the carrier manages its air-cargo capacity in terms of weight. Assume that there are $m = 3$ forwarders, whose per-kilogram contributions are 1.2, 1.0, and 0.8 respectively. We consider two sets of experiments. In the first (resp., second) set, small (resp., large) problem instances are considered; the capacity is discretized so that one unit equals 300 (resp., 50) kilograms, and the per-unit contribution (p_1, p_2, p_3) is $(360, 200, 240)$ [resp., $(60, 50, 40)$]. The small problem instances are solved to optimality. The number of shipment requests from forwarder i is a Poisson random variable with mean $E[N_i] = \eta_i = 12 - 0.03p_i$. Note that the mean number of arrivals (demand) is linearly decreasing in the per-unit contribution (margin). Assume that the random requirements of all forwarders are i.i.d. negative binomial random variables with parameters r and p. Then, $E[W_{i,k}] = rq/p$ and $\text{var}(W_{i,k}) = rq/p^2$ where $q = 1 - p$.

In the continuous approximation heuristic, we model $\hat{D}_i$ using the gamma distribution with shape and scale parameters a_i and b_i, respectively. Then, $E[\hat{D}_i] = a_i b_i$ and $\text{var}(\hat{D}_i) =$

1. Initialization:

   (a) Determine initial multiplier $\nu_1 = \frac{1}{m}\sum_{i=1}^{m} p_i$.

   (b) Select a small tolerance $\epsilon > 0$.

   (c) Set the best upper bound $\mathcal{U}_0 = \infty$, the best lower bound $\mathcal{L}_0 = -\infty$, and a constant $\alpha_1 = 2$.

2. Iteration $k \geq 1$:

   (a) Obtain $\mathbf{a}^*(\nu_k)$ in Proposition 5, and $\mathbf{x}^*(\nu)$ as in (22).

   (b) Find an upper bound $\mathcal{U}_k^* = \sum_{i=1}^{m} c_i(a_i^*(\nu_k)|\nu_k)$, and the best upper bound $\mathcal{U}_k = \min\{\mathcal{U}_k^*, \mathcal{U}_{k-1}\}$.

   (c) Find a lower bound $\mathcal{L}_k^* = \sum_{i=1}^{m} \rho_i(x_i^*(\nu_k))$, and the best lower bound $\mathcal{L}_k = \max\{\mathcal{L}_k^*, \mathcal{L}_{k-1}\}$.

   (d) Compute a stepsize
$$t_k = \frac{\alpha_k(\mathcal{U}_k - \mathcal{L}_k)}{\left(\kappa - \sum_{i=1}^{m} a_i^*(\nu_k)\right)^2}$$

   (e) Update the Lagrange multiplier
$$\nu_{k+1} = \max\left\{0, \nu_k - t_k\left(\kappa - \sum_{i=1}^{m} a_i^*(\nu_k)\right)\right\}$$

   (f) Modify the constant α_k. If the best upper bound $\mathcal{U}_k$ fails to go down for some consecutive number of iterations (e.g., 4 consecutive iterations in the numerical example), then the value of α_k is halved; i.e., $\alpha_{k+1} = \alpha_k/2$. Otherwise, it remains unchanged; i.e., $\alpha_{k+1} = \alpha_k$.

   (g) Stop if $\mathcal{U}_k - \mathcal{L}_k < \epsilon$. Otherwise, perform the next iteration $k = k + 1$.

Table 2: Lagrangian relaxation heuristic procedure

%Δ	Proportional	Continuous	Lagrangian
Minimum	3.05	1.78	0.00
Maximum	13.19	14.07	12.71
Average	6.10	5.83	2.48

Table 3: Percent differences between the optimal expected contribution and the expected contribution if the heuristic solution is used

$a_i b_i^2$. It follows from (13) that $a_i = \eta_i rq/(1+rq)$ and $b_i = (1+rq)/p$.

In the Lagragian relaxation algorithm, we compute the distribution of D_i by conditioning on N_i. If $N_i = n \in \mathbb{N}$, then the n-fold convolution $\sum_{k=1}^{n} W_{i,k}$ follows the negative binomial distribution with parameters nr and p, since $W_{i,1}, W_{i,2}, \ldots$ are i.i.d. negative binomial random variables with parameters r and p.

Example 1 (Small Problem Instances). Let $(r,p) = (12, 0.79)$. Then, the mean of total capacity requirements of booking requests from all three forwarders is $E[\sum_{i=1}^{m} D_i] = 28.7$, or equivalently 8,610 kilograms. This is about two thirds of the cargo capacity of Airbus A330-300 based on the normal operating conditions and full passenger loads. Table 3 shows the maximum, minimum, and average, of the percent differences between the optimal expected contribution and the expected contribution if the heuristic solution is used, when the capacity is varied from 18 to 38. From Table 3, the average, minimum, and maximum from the Lagrangian relaxation algorithm are smaller than those from the other heuristics. The average and minimum from the continuous approximation algorithm are smaller than those from the proportional allocation scheme, whereas the maximum from the proportional allocation scheme is smaller than that from the continuous approximation algorithm. If the proportional allocation is replaced with the Lagrangian relaxation heuristic, then the expected incremental benefit is on average 3.62 percent.

Example 2 (Large Problem Instances). Let $(r,p) = (36, 0.79)$. Then, the mean of total capacity requirements of booking requests from all three forwarders is $E[\sum_{i=1}^{m} D_i] = 301$, or equivalently 15,050 kilograms. This is approximately the cargo capacity of Airbus A330-300. Figure 1 shows the expected total contributions from different schemes and