A VARIATION OF THE OPAQUE PROBLEM

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ABSTRACT

The opaque problem on a plane is a problem which seeks, among curves (or unions of curves) that intersect all the straight lines passing through a given shape (e.g., a square, a circle, a polygon, etc.), the curve (or union of curves) with the shortest length. In this study, we simplify this problem by fixing a point on the plane outside the given shape and considering only the straight lines that pass through this point. Our problem is then to find a "minimizer" inside the given shape that intersects all those straight lines that intersect the given shape. We prove that, for most of the convex shapes on the plane, solutions to this problem always fail to exist. However, the greatest lower bound of the length of the candidates is found to be simply $\int_{\theta_1}^{\theta_2} r \, d\theta$, where $r = r(\theta)$ is the distance, in polar form, from the fixed point to a certain part of the boundary of the region in question. We also extend our study to non-convex shapes, whose boundary is a simple closed curve. We prove that the solution exists if its boundary satisfies certain conditions.

KEY WORDS : OPAQUE PROBLEM/CONVEX SET/CURVE

69 pp.

การแปรผันของปัญหารูปทึบแสง
(A VARIATION OF THE OPAQUE PROBLEM)

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บทคัดย่อ

ปัญหารูปทึบแสง เป็นปัญหาที่ต้องการหาเส้นโค้ง หรือยูเนียนของเส้นโค้ง ซึ่งตัดกับเส้นตรงทุกเส้นที่มาตัดกับรูปที่กำลังพิจารณา (เช่น รูปสี่เหลี่ยมจัตุรัส, วงกลม, รูปหลายเหลี่ยม ฯลฯ) และมีความยาวสั้นที่สุดด้วย ในการศึกษาครั้งนี้ได้กำหนดเงื่อนไขบางอย่างเพิ่มเข้ามา นั่นคือ กำหนดจุดบนระนาบขึ้นมาหนึ่งจุดที่อยู่ภายนอกรูปที่กำหนด และพิจารณาเฉพาะเส้นตรงที่ผ่านจุดนี้เท่านั้น โดยปัญหาของการศึกษาครั้งนี้คือ การหาเส้นโค้ง หรือยูเนียนของเส้นโค้งที่สั้นที่สุด ที่อยู่ภายในรูปที่กำหนดให้ และตัดกับเส้นตรงเหล่านี้ทุกเส้นที่มาตัดกับรูปที่กำหนดให้

ในการศึกษานี้ได้พิสูจน์ว่า สำหรับรูปนูนเกือบทั้งหมดจะไม่สามารถหาคำตอบสำหรับปัญหานี้ได้ แต่สามารถหาขอบเขตล่างมากสุดของความยาวของเส้นที่สามารถเป็นคำตอบได้ ซึ่งมีค่าเท่ากับ $\int_{\theta_1}^{\theta_2} r\,d\theta$ โดยที่ $r = r(\theta)$ เป็น สมการในรูปเชิงขั้วของส่วนของขอบที่เฉพาะเจาะจงของรูปที่กำหนดให้ และได้ขยายการศึกษาไปยัง รูปไม่นูนที่มีขอบเป็นเส้นโค้งปิดเชิงเดียว โดยพิสูจน์ว่า ด้วยคุณสมบัติบางอย่างของขอบของรูปสามารถทำให้หาคำตอบสำหรับปัญหานี้ได้

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CHAPTER I

INTRODUCTION

The opaque problem on the plane ($\mathbb{R}^2$) is a problem of seeking the shortest curve or union of curves that intersects all the lines passing through the given shape (e.g., square, circle, polygons, etc.). We shall use the term "general curves" for unions of curves. Let $P(\mathbb{R}^2)$ be the class of all subsets of $\mathbb{R}^2$ (the power set of $\mathbb{R}^2$). Let $C(\mathbb{R}^2)$ be the class of all convex sets in $\mathbb{R}^2$ and $C'(\mathbb{R}^2)$ be the class of all non-convex sets in $\mathbb{R}^2$. Then $C(\mathbb{R}^2)$ and $C'(\mathbb{R}^2)$ are subsets of $P(\mathbb{R}^2)$. Given that $S \in P(\mathbb{R}^2)$, let $\overline{S}$ be the closure of S, ∂S the boundary of S and $P(S)$ the power set of S. For example, if S is a square, then ∂S is a boundary of the square; see Figure 1.1. On the plane, some straight lines intersect S but some do not. The problem is to find a general curve $\widetilde{S}$ inside S that satisfies:

1) for any straight line l on the plane, l intersects $\widetilde{S}$ if and only if l intersects ∂S, and

2) among all general curves inside S that satisfy 1), $\widetilde{S}$ has the shortest length.

Imagine that there is an observer on the plane. The observer's vision is represented by straight lines from the eyes. Similarly in the opaque problem, if there are many observers surrounding the shape, the straight lines on the plane represent the vision of each observer. See Figure 1.2. In this study, we simplify this problem by considering the specific set of straight lines on the plane through one observer. That means we fix a point A on the plane outside S and consider the straight lines that pass through A. See Figure 1.3. Then our problem is to find a general curve $\widetilde{S}$ inside S that satisfies:

1) for any straight line l that passes through A, l intersects $\widetilde{S}$ if and only if l intersects ∂S, and

2) among all general curves inside S that satisfy 1), $\widetilde{S}$ has the shortest length.

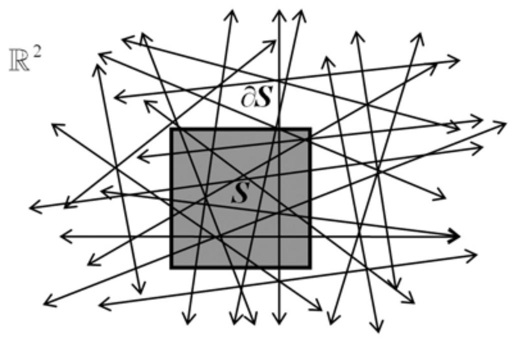


Figure 1.1: S is a square on the plane.

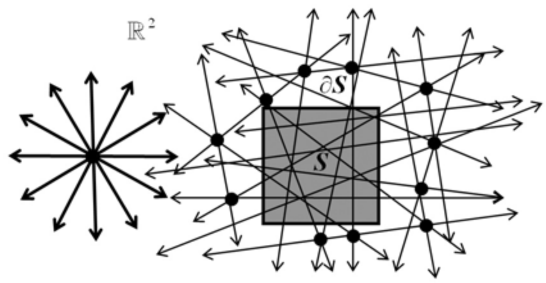


Figure 1.2: Each • is an observer and there are many observers surrounding the square.

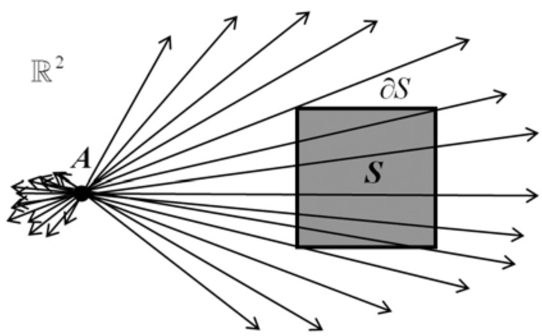


Figure 1.3: One observer: a point A on the plane.

In this study we consider separately the convex and non-convex shape problems in which the boundary of the region in question is a simple closed curve (a closed curve which does not intersect itself). We consider the case of solutions being connected general curves and the case of solutions not necessarily connected, each among general curves of the same type. For the convex shape problems, we prove that the connected general curve solutions are in the form of line segments but the not-necessarily-connected solutions do not exist. For the non-convex shape problems, we prove that the solutions are in the form of arcs of circles with centers at A, which are subcurves of the boundaries of the given shapes. See Figure 1.4.

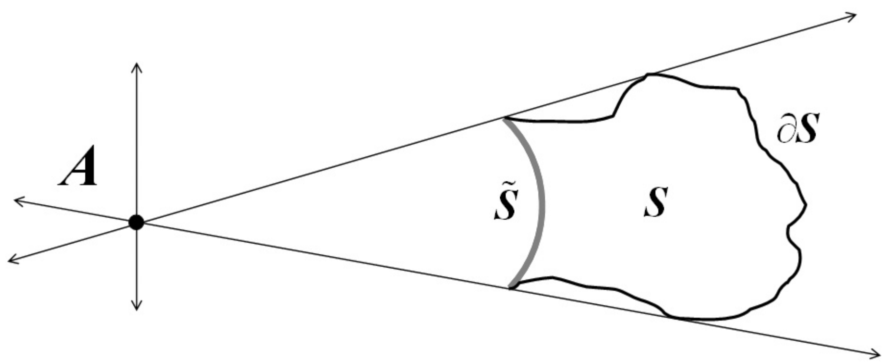


Figure 1.4: Example of the non-convex shape which has the best solution.

1.1 Objectives

The main objectives of this study are

1. Finding solutions of the convex and non-convex shape problems with one observer.

2. Studying properties of solutions of the convex and non-convex shape problems with one observer.

1.2 Expected Benefits

In this study, we expect to

1. Obtain solutions of the problems with one observer and their properties.

2. Apply the results to more general cases that relate to the opaque problem.

1.3 Organization of the Study

This study will be organized into five chapters. First, an introduction to this study is given in Chapter 1. Chapter 2 describes a literature review. The construction and basic concepts relating to this study are presented in Chapter 3. The results and the proofs are shown in Chapters 4 and 5. In particular, we show in Chapter 4 that connected solutions exist for convex shapes, while not-necessarily-connected solutions do not, except for line segments. In any case, the infimum of the measure of candidates can be computed. Finally, we conclude our results in Chapter 6.

CHAPTER II

LITERATURE REVIEW

In 1992, the report of Kenneth A. Brakke stated in [1], the opaque square problem:

"What is the shortest length fence that can block any line of sight across a square plot of ground?"

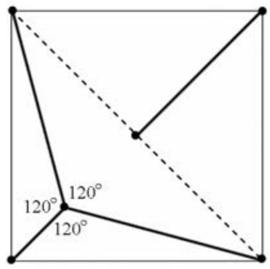


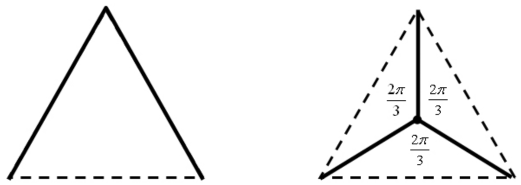
Figure 2.1: The best known opaque square solution (total length $= \frac{2+\sqrt{3}}{\sqrt{2}}$) [1].

The best known solution (not proved minimal) is shown in Figure 2.1 from [2]. It has straight fence from three corners meeting at a point at angles of $2\pi/3$ plus fence from the fourth corner to the center. It has length $\frac{2+\sqrt{3}}{\sqrt{2}} \approx 2.64$, see [3].

In 1997, Bernd Kawohl ([3] and [4]) proved that the solution in Figure 2.1 is a general curve, which has the shortest length in the class of all doubly connected curves (two connected components) and satisfies the property that every straight line intersecting the square also intersects one of its components.

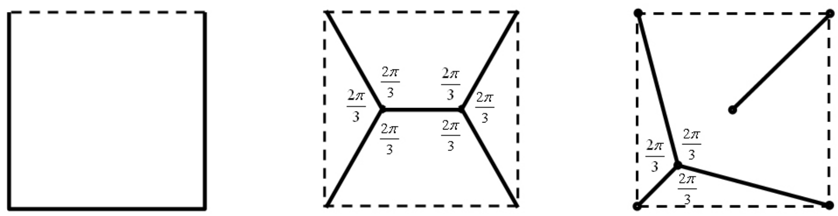
Figures 2.2-2.6 [5] show opaque regular polygons (polygons with all their sides equal in length), in which figures (a) show the shortest known connected curves, figures (b) show the shortest known connected union of curves, and figures (c) show the shortest known union of curves.

Note that in figures 2.2(b) and (c), 2.3(b), and 2.6(c), the shortest known union of curves that connect all the vertices of the given polygon are called Steiner



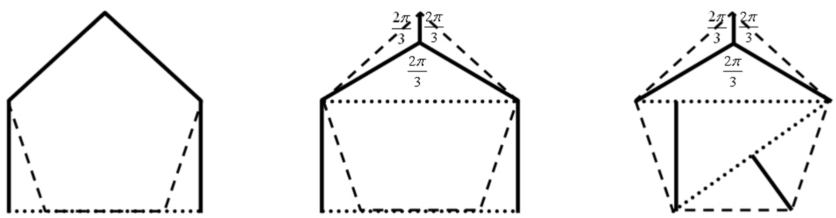
(a) $\frac{2}{3} = 0.66666...$ (b) $\frac{\sqrt{3}}{3} = 0.577350269...$ (c) $\frac{\sqrt{3}}{3} = 0.577350269...$

Figure 2.2: An equilateral triangle with sidelength $\frac{1}{3}$; see [5].



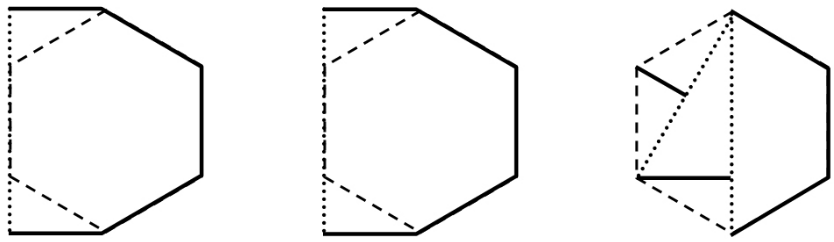
(a) $\frac{3}{4} = 0.75$ (b) $\frac{1+\sqrt{3}}{4} = 0.683012702...$ (c) 0.659739609...

Figure 2.3: A square with sidelength $\frac{1}{4}$; see [5].



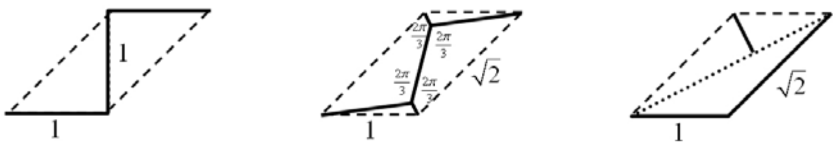
(a) 0.780422607... (b) 0.778231365... (c) 0.705577112...

Figure 2.4: A 5-polygon with sidelength $\frac{1}{5}$; see [5].



(a) $\frac{3+\sqrt{3}}{6} = 0.788675135...$ (b) $0.788675135...$ (c) $\frac{7+\sqrt{3}}{12} = 0.727670901...$

Figure 2.5: A 6-polygon with sidelength $\frac{1}{6}$; see [5].



(a) $\frac{3}{2+2\sqrt{2}} = 0.621320344...$ (b) $0.602538433...$ (c) $0.592620968...$

Figure 2.6: A parallelogram; see [5].

span of the vertices. See [5].

CHAPTER III
BASIC KNOWLEDGE

In this chapter, we review some basic concepts and facts that will be used throughout the thesis. We also define our measure and discuss some properties for our problem.

3.1 Compactness

We now discuss some properties of compactness. All of these properties are found in [6].

Definition 3.1 *Let X be a topological space. A subset $Y \subset X$ is said to be compact if whenever Y is contained in a union of open sets U_i (called an open cover of Y), Y is contained in the union of a finite subcollection of these open sets (called a finite subcover)*

The next proposition allows us to deduce that certain sets are compact by knowing that they are closed subsets of a compact set.

Proposition 3.2 *Let X be compact and let Y be closed in X. Then Y is compact.*

The next theorem classifies closed segments as compact sets in the usual topology of the real line.

Theorem 3.3 *(Heine-Borel) The closed interval $[a, b]$ is compact.*

As a corollary, we can now characterize compact sets in the line.

Corollary 3.4 *$A \subset \mathbb{R}$ is compact iff it is closed and bounded.*

The characterization of compact sets in $\mathbb{R}^n$ is shown in the next theorem.

Theorem 3.5 *(Heine-Borel) A subset of $\mathbb{R}^n$ is compact iff it is closed and bounded.*

The next proposition is an important property of a closed bounded set.

Proposition 3.6 *A compact subset X of $\mathbb{R}$ has a largest element M and a smallest element m such that $m \leq a \leq M$ for all $a \in X$.*

The next proposition shows that compactness is preserved by continuous maps.

Proposition 3.7 *Let $f : X \to Y$ be continuous and suppose that X is compact. Then the image set $f(X)$ is compact.*

The next proposition is an application of the previous two propositions.

Proposition 3.8 *Let $f : X \to Y$ be continuous and suppose that X is compact. Then f assumes a maximum (and a minimum) on X; that is, there are $x, y \in X$ with $f(x) \leq f(z) \leq f(y)$ for all $z \in X$.*

3.2 Measurable spaces and integration

We now discuss some definitions and theorems in Real Analysis. All of these properties are from references [7], [8], and [9].

Let H be a (non-empty) set. By a **σ-algebra** we mean a collection $\mathcal{M}$ of subsets of H having the following properties:

(i) $\emptyset$ is in $\mathcal{M}$,

(ii) if A is in $\mathcal{M}$, then A^c is also in $\mathcal{M}$, and

(iii) if $\{E_i\}$ is a sequence of elements of M, then $\bigcup_{i=1}^{\infty} E_i$ is also an element of $\mathcal{M}$.

Then a set H together with a σ-algebra in H is called a **measurable space**, and the elements of $\mathcal{M}$ are called its **measurable sets**. If $(H, \mathcal{M})$ and $(G, \mathcal{M}_G)$ are measurable spaces, and $f : H \to G$ is a map, we define f to be measurable if for every V in $\mathcal{M}_G$ the set $f^{-1}(V)$ is in $\mathcal{M}_G$. A **positive measure** on $\mathcal{M}$ (or on H) is a map $\mu : \mathcal{M} \to [0, \infty]$ which is countably additive. In other words $\mu(\emptyset) = 0$, and if $\{E_i\}$ is a sequence of measurable sets which are mutually disjoint ($A_n \cap A_m = \emptyset$ if $n \neq m$), then $\mu\left(\bigcup\limits_{i=1}^{\infty} E_i\right) = \sum\limits_{i=1}^{\infty} \mu(E_i)$.

Theorem 3.9 *Let f be measurable and nonnegative on H, and let $\{E_i\}$ be a sequence of disjoint measurable subsets of H. If $E = \bigcup\limits_{i=1}^{\infty} E_i$, then*

$$\int_E f d\mu = \sum_{i=1}^{\infty} \int_{E_i} f d\mu.$$

3.3 Measure for unions of curves

Let ρ be the usual metric on $\mathbb{R}^2$. Thus ρ is defined by

$$\rho(X, Y) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2},$$

where $X = (x_1, y_1)$ and $Y = (x_2, y_2)$ are points in the plane $\mathbb{R}^2$. The pair $(\mathbb{R}^2, \rho)$ is called a metric space. In this study, any curve γ in $\mathbb{R}^2$ is a continuous map of a compact interval into $\mathbb{R}^2$. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve, where $a, b \in \mathbb{R}$ and $a < b$. Then γ is said to be closed if $\gamma(a) = \gamma(b)$. The length $\ell(\gamma)$ of γ is defined by

$$\ell(\gamma) = \sup \sum_{i=1}^{k} \rho(\gamma(t_{i-1}), \gamma(t_i)),$$

where the supremum is taken over all partitions $a = t_0 < t_1 < ... < t_k = b$ of $[a, b]$.

Let γ be a simple rectifiable curve on the plane with $\ell(\gamma) = c$. Then we can parametrize γ by its length, $\gamma : [0, c] \to \mathbb{R}^2$. Next, we will define the measure of subsets of the image $\{\gamma\}$ of γ by using the Lebesgue outer measure. The length of a bounded interval I (open, closed, half-open) with endpoints a and b ($a < b$) is defined by $|I| = b - a$. If I is (a, ∞), $(-\infty, b)$ or $(-\infty, \infty)$, then $|I| = \infty$. For each subset E of $\{\gamma\}$, we define $\mu_\gamma(E)$ by

$$\mu_\gamma(E) = \mu^*(\gamma^{-1}[E]),$$

where μ^* is the Lebesgue measure on $\mathbb{R}$. Then μ_γ is well-defined and

$$\mu_\gamma(\{\gamma\}) = |[0, \ell(\gamma)]| = \ell(\gamma).$$

Furthermore, μ_γ is a positive measure on the set of subsets of $\{\gamma\}$. Next, the author will show that if E is a subset of the images of two distinct curves, γ and α, then the measures of E on each curve are equal, $\mu_\gamma(E) = \mu_\alpha(E)$. Let α be a simple rectifiable curve on the plane parametrized by arclength, with $\ell(\alpha) = d$. Suppose that E is also a subset of $\{\alpha\}$. We prove that $\mu_\gamma(E) = \mu_\alpha(E)$. Let (A_k) be a sequence of countable unions of disjoint closed intervals in $[0, c]$ containing $\gamma^{-1}[E]$ such that $\mu^*(A_k)$ converges to $\mu_\gamma(E)$. Then $\gamma^{-1}[E] \subseteq A_k = \bigcup_{i=1}^{\infty} I^\gamma_{k,i} \subseteq [0, c]$. Let (B_k) be a sequence of countable unions of disjoint closed intervals in $[0, d]$ containing $\alpha^{-1}[E]$, $B_k = \bigcup_{i=1}^{\infty} I^\alpha_{k,i}$ such that for all i, $\mu^*(I^\alpha_{k,i}) \leq \mu^*(I^\gamma_{k,i})$. From the countably additive property and that for all i, $\mu^*(I^\alpha_{k,i}) \leq \mu^*(I^\gamma_{k,i})$,

$$\begin{aligned} \sum_{i=1}^{\infty}\mu^*(I^\alpha_{k,i}) &\leq \sum_{i=1}^{\infty}\mu^*(I^\gamma_{k,i}) \\ \mu^*(B_k) &\leq \mu^*(A_k). \end{aligned}$$

Then $\mu^*(B_k)$ converges to $\mu_\alpha(E)$ as $\mu^*(A_k)$ converges to $\mu_\gamma(E)$. Since E is a subset of $\gamma[A_k]$ and $\alpha[B_k]$, $\gamma(I^\gamma_{k,i}) \cap E = \alpha(I^\alpha_{k,i}) \cap E \subseteq \alpha[I^\alpha_{k,i}]$ and for all i, the length of $\gamma[I^\gamma_{k,i}] \cap E$ is less than or equal to the length of $\alpha[I^\alpha_{k,i}]$,

$$\begin{aligned} \mu^*(I^\gamma_{k,i} \cap \gamma^{-1}(E)) &\leq \mu^*(I^\alpha_{k,i}) \\ \sum_{i=1}^{\infty}\mu^*(I^\gamma_{k,i} \cap \gamma^{-1}(E)) &\leq \sum_{i=1}^{\infty}\mu^*(I^\alpha_{k,i}) \\ \mu^*(\gamma^{-1}(E)) &\leq \mu^*(B_k) \\ \mu_\gamma(E) &\leq \mu^*(B_k). \end{aligned}$$

Since for all $\varepsilon > 0$ there exists a positive integer N such that $k > N$ implies $|\mu^*(A_k) - \mu_\gamma(E)| < \varepsilon$ and $|\mu^*(B_k) - \mu_\alpha(E)| < \varepsilon$. We obtain that

$$\begin{aligned} |\mu_\gamma(E) - \mu_\alpha(E)| &= |\mu_\gamma(E) - \mu^*(A_k) + \mu^*(A_k) - \mu^*(B_k) + \mu^*(B_k) - \mu_\alpha(E)| \\ &\leq |\mu_\gamma(E) - \mu^*(A_k)| + |\mu^*(A_k) - \mu^*(B_k)| + |\mu^*(B_k) - \mu_\alpha(E)| \\ &< |\mu^*(A_k) - \mu^*(B_k)| + 2\varepsilon. \end{aligned}$$

Since $\mu_\gamma(E) \leq \mu^*(B_k)$, $|\mu^*(A_k) - \mu^*(B_k)| < \varepsilon$ for large k and we have that

$$|\mu_\gamma(E) - \mu_\alpha(E)| < 3\varepsilon.$$

Since ε is arbitrary, $\mu_\gamma(E) = \mu_\alpha(E)$. Thus, we may write $\mu(E)$ in place of $\mu_\gamma(E)$.

Let $V(\mathbb{R}^2)$ be the set of compact countable unions of images of simple rectifiable curves in $\mathbb{R}^2$ such that any pair of distinct curves in the union intersect in at most one point. Let G be a compact countable union of curves in $V(\mathbb{R}^2)$. Then G is of the form $\bigcup_{i=1}^{\infty}\{\gamma_i\}$, where γ_i and γ_j intersect in at most one point, for all $i \neq j$. We define $m : V(\mathbb{R}^2) \to [0, \infty]$ by

$$m(G) = m\left(\bigcup_{i=1}^{\infty}\{\gamma_i\}\right) = \sum_{i=1}^{\infty}\ell(\gamma_i) = \sum_{i=1}^{\infty}\mu(\{\gamma_i\}).$$

We will show that m is well-defined. Suppose that $G = \bigcup_{i=1}^{\infty}\{\gamma_i\} = \bigcup_{j=1}^{\infty}\{\alpha_j\}$, we will show that $\sum_{i=1}^{\infty}\ell(\gamma_i) = \sum_{j=1}^{\infty}\ell(\alpha_j)$. For a fixed i,

$$\begin{aligned}\{\gamma_i\} &= \{\gamma_i\}\bigcap G \\ &= \{\gamma_i\}\bigcap\left[\bigcup_{j=1}^{\infty}\{\alpha_j\}\right] \\ &= \bigcup_{j=1}^{\infty}\left[\{\gamma_i\}\bigcap\{\alpha_j\}\right].\end{aligned}$$

Then, since μ is a positive measure,

$$\begin{aligned}\sum_{i=1}^{\infty}\ell(\gamma_i) &= \sum_{i=1}^{\infty}\ell\left(\bigcup_{j=1}^{\infty}\left[\{\gamma_i\}\bigcap\{\alpha_j\}\right]\right) = \sum_{i=1}^{\infty}\mu\left(\bigcup_{j=1}^{\infty}\left[\{\gamma_i\}\bigcap\{\alpha_j\}\right]\right) \\ &= \sum_{i=1}^{\infty}\left[\sum_{j=1}^{\infty}\mu\left(\{\gamma_i\}\bigcap\{\alpha_j\}\right)\right] \\ &= \sum_{j=1}^{\infty}\left[\sum_{i=1}^{\infty}\mu\left(\{\gamma_i\}\bigcap\{\alpha_j\}\right)\right] \\ &= \sum_{j=1}^{\infty}\mu\left(\bigcup_{i=1}^{\infty}\left[\{\gamma_i\}\bigcap\{\alpha_j\}\right]\right) \\ &= \sum_{j=1}^{\infty}\mu(\alpha_j) \\ &= \sum_{j=1}^{\infty}\ell(\alpha_j).\end{aligned}$$

It is obvious that countable additivity holds for m. We will use this function m to measure the size of unions of curves in $V(\mathbb{R}^2)$.

The next lemma gives a way to find the length of a curve by using its polar equation.

Lemma 3.10 *[10] If a curve γ is defined by a polar equation $r = r(\theta)$ on $[a, b]$ where $r(\theta)$ is smooth, then the length of γ is*

$$\int_a^b \sqrt{[r(\theta)]^2 + [r'(\theta)]^2} d\theta.$$

Next, we define some terms that will be needed in finding solutions of our problems. Let $P(\mathbb{R}^2)$ be the class of all subsets of $\mathbb{R}^2$ (the power set of $\mathbb{R}^2$). Let $C(\mathbb{R}^2)$ be the class of convex sets in $\mathbb{R}^2$ and $C'(\mathbb{R}^2)$ be the class of non-convex sets in $\mathbb{R}^2$. Given that S is an element in $P(\mathbb{R}^2)$, let $\overline{S}$ be the closure of S. Let ∂S be the boundary of S and $P(S)$ be the power set of S. In what follows, we consider the problem described in the introduction, with only one observer located at a given point A and a given region denoted by S on the plane. Assume that the convex hull of $\overline{S}$ does not contain A.

Definition 3.11 *(The Two-Condition Property) Let L be the set of straight lines l in $\mathbb{R}^2$ such that $A \in l$. A general curve $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ satisfies the two-condition property if it satisfies*

*(**C1**) for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial(S) \neq \emptyset$, and*

*(**C2**) for all G in $V(\mathbb{R}^2) \cap P(\overline{S})$ that satisfies **C1**, $m(\widetilde{S}) \leq m(G)$.*

The next definition defines the first and last lines from A intersecting a curve not passing through A. Let $\gamma : [a, b] \to \mathbb{R}^2$ be a curve such that $\gamma(t) \neq A$ for all $t \in [a, b]$, and let $\{\gamma\} = \{\gamma(t) : a \leq t \leq b\}$. By the Heine-Borel Theorem, the closed and bounded interval $[a, b]$ on $\mathbb{R}$ is compact. Since γ is a continuous map on $[a, b]$, $\{\gamma\}$ is compact. For each $X \in \{\gamma\}$, let $l_X \in L$ be such that $X \in l_X$. Fix $u \in [a, b]$, then $\gamma(u) \in \{\gamma\}$. We define a continuous function $f_u : [a, b] \to [-\pi, \pi)$ by

$$\begin{aligned} f_u(t) &= \text{the signed angle at } A \text{ measured clockwise from } l_{\gamma(u)} \text{ to } l_{\gamma(t)} \\ &= \angle(l_{\gamma(u)}, l_{\gamma(t)}), \end{aligned}$$

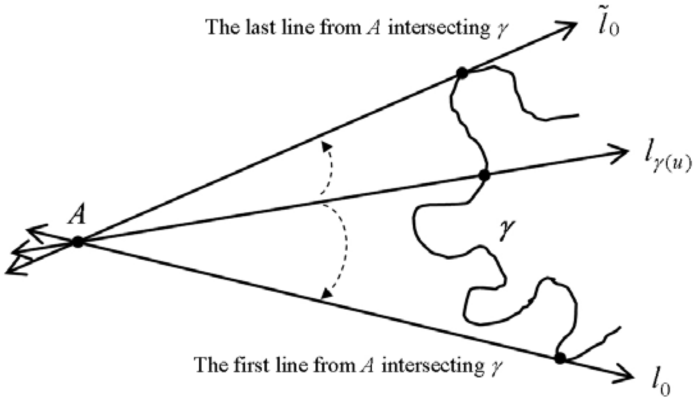


Figure 3.1: The first and last lines intersecting γ.

for all $t \in [a, b]$. By the preservation of compact sets under continuous maps, $f_u[[a, b]] \subseteq [-\pi, \pi)$ is compact and we get the following definition.

Definition 3.12 *Let $\gamma : [a, b] \longrightarrow \mathbb{R}^2$ be a curve such that $\gamma(t) \neq A$ for all $t \in [a, b]$. Fix $u \in [a, b]$. The first line in L intersecting the curve γ is the line in L that intersects γ and makes the biggest signed angle with $l_{\gamma(u)}$ measured in the clockwise direction. The last line in L intersecting the curve γ is the line in L that intersects γ and makes the smallest signed angle with $l_{\gamma(u)}$ measured in the clockwise direction. See Figure 3.1.*

The next definition deals with tangent lines to a curve.

Definition 3.13 *Let X be any point in a curve γ. Then a line l passing through X is a tangent to the curve γ at X if there is an open disk $D(X, \varepsilon)$ of radius $\varepsilon > 0$ with center at X such that a part of l lies entirely in a half disk of $D(X, \varepsilon)$.*

CHAPTER IV
THE CONVEX SHAPES

In this chapter, we study the solutions of the convex shape problems and their properties. We assume throughout that A is the origin on the plane and S is a convex set not containing A. It is easily seen that the first and last lines are independent of the choices of $u \in [a, b]$. Our first lemma is the properties of the first and the last lines intersecting S.

Lemma 4.1 *Let S be a convex set not containing A on the plane which is not a line segment. Then l_0 is the first or last line in L that intersects ∂S if and only if either*

1) there is a point Z in ∂S such that $l_0 \cap \partial S = \{Z\}$ or

2) l_0 contains an edge of ∂S, i.e., a line segment contained in $\partial(S)$.

Proof. Let S be a convex set not containing A on the plane. Assume that S is not a line segment.

($\Rightarrow$) Without lost of generality, suppose that $l_0 \in L$ is the first line in L that intersects ∂S, i.e., $l_0 \cap \partial S \neq \emptyset$. We will show that either there is a point Z in ∂S such that $l_0 \cap \partial S = \{Z\}$ or l_0 contains an edge of ∂S by showing that if l_0 does not contain any edge of ∂S then there is a point Z of ∂S such that $l_0 \cap \partial S = \{Z\}$. Suppose that l_0 does not contain any edge of ∂S. Since $l_0 \cap \partial S \neq \emptyset$, we choose $Z \in l_0 \cap \partial S$. If $l_0 \cap \partial S \neq \{Z\}$, then there is a point V in $\partial S \backslash \{Z\}$ such that $V \in l_0 \cap \partial S$. Since S is a convex set, $\overline{VZ} \subseteq S$ and $\overline{VZ} \subseteq l_0$. But l_0 does not contain any edge of ∂S, so $\overline{VZ} \nsubseteq \partial S$. This implies that l_0 decomposes S into two parts and l_0 is not the first line in L intersecting ∂S, a contradiction. So $l_0 \cap \partial S = \{Z\}$ for some point Z in ∂S.

($\Leftarrow$) Let l_0 be any line in L. Suppose that either there is a point Z in ∂S such that $l_0 \cap \partial S = \{Z\}$ or l_0 contains an edge of ∂S. We will show that l_0 is the first (or the last) line in L that intersects ∂S. Suppose that there is a point Z

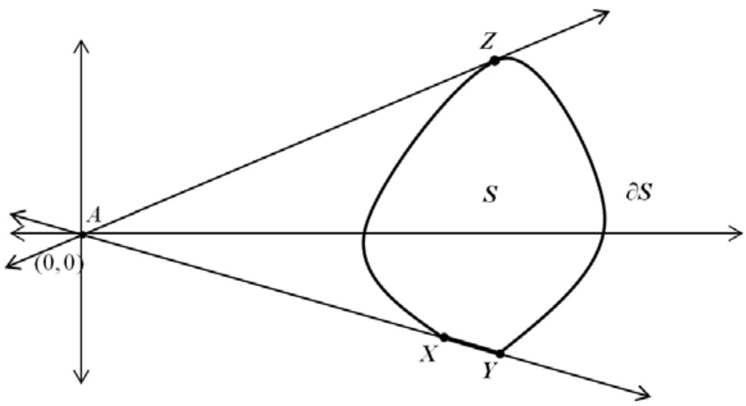


Figure 4.1: The convex set S.

in ∂S such that $l_0 \cap \partial S = \{Z\}$. Then l_0 does not contain any edge of ∂S. From Definition 3.12, we suppose that the size of angle measured clockwise from the first to the last lines from A intersecting ∂S is ω degrees. Since S is a convex set and A is outside S, $\omega < 180°$. We choose $\delta = \frac{180° - \omega}{2} > 0$. Then l_0 is the first (or the last) line in L that intersects ∂S if and only if for all l in L such that $\angle(l_0, l) \leq \delta$ measured in clockwise direction from l_0, $l \cap \partial S = \emptyset$. By contradiction, suppose that l_0 is not the first (or the last) line in L that intersects ∂S then there is l in L such that $\angle(l_0, l) \leq \delta$ in clockwise of l_0, $l \cap \partial S \neq \emptyset$. Then there is a point V in ∂S such that $V \in l \cap \partial S$. Since S is a convex set, $\overline{ZV} \subseteq S$. Since l_0 is not the first (or the last) line that intersects ∂S, l_0 decomposes S into two parts and there is X in ∂S staying in opposite part of l_0 with V such that $X \notin \overline{ZV}$ and $\overline{ZV} \nsubseteq \overline{VX}$. Therefore $\overline{VX} \cap l_0 \neq \emptyset$ and $Z \notin \overline{VX} \cap l_0$. Since S is a convex set, $\overline{VX} \subseteq S$ and then l_0 intersects ∂S more than one point, $l_0 \cap \partial S \neq \{Z\}$, contradiction. So l_0 is the first (or the last) line in L that intersects ∂S.

Suppose that l_0 contains an edge $\overline{XY}$ of ∂S. Then there is no point Z in ∂S such that $l_0 \cap \partial S = \{Z\}$. By contradiction, suppose that l_0 is not the first (or the last) line in L that intersects ∂S. Then there is l in L such that $\angle(l_0, l) \leq \delta$ in clockwise of l_0, $l \cap \partial S \neq \emptyset$. Then l_0 decomposes S into two parts. We choose a point in ∂S from each part and construct line segments from each point to any points in open line segment $\overline{XY}$. Since S is a convex set, all line segments are in

S and then all points in open line segment $\overline{XY}$ are not the boundary points of S. This contradicts with $\overline{XY} \subseteq \partial S$. So l_0 is the first (or the last) line in L that intersects ∂S. ∎

4.1 Connected solutions for convex sets

In this section, we consider a convex set S with one observer when solution candidates are connected general curves in $V(\mathbb{R}^2) \cap P(\overline{S})$ that is enclosed by ∂S. Then the solution $\widetilde{S}$ that we find in this part is in fact a curve. Let l_0 and $\widetilde{l_0}$ be the first and the last lines in L that intersect ∂S, respectively. From the first of the two-condition property, for all $l \in L$, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$, so $\widetilde{S}$ is a curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ that connects l_0 and $\widetilde{l_0}$. Since the shortest curve that connects any two points on the plane is a line segment and S is convex, $\widetilde{S}$ is the shortest line segment that connects X and Y for some $X \in l_0 \cap \partial S$ and $Y \in \widetilde{l_0} \cap \partial S$. So $\widetilde{S} = \overline{XY}$ corresponds to the second condition property, $m(\widetilde{S}) = m(\overline{XY}) \leq m(G)$, where $G \in V(\mathbb{R}^2) \cap P(\overline{S})$ that satisfies the first condition property, for all $l \in L$, $l \cap G \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$.

4.1.1 S is a line segment.

Let $B = (x_1, y_1)$ and $C = (x_2, y_2)$ be extreme points of S.

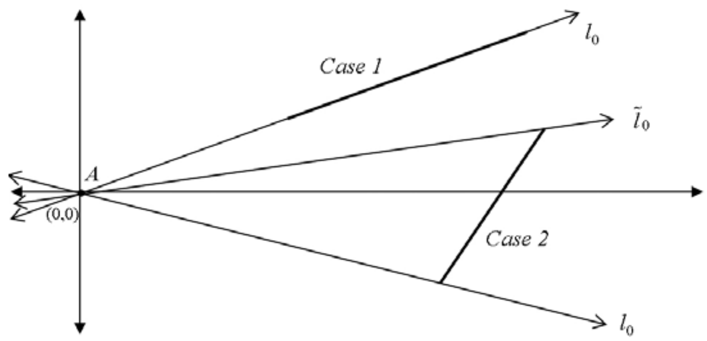


Figure 4.2: S is a line segment.

Case 1. There is $l_0 \in L$ such that $\partial S \subseteq l_0$. See Figure 4.2. Then for all $X \in \partial S$, $\{X\} \cap l_0 \neq \emptyset$ if and only if $\partial S \cap l_0 \neq \emptyset$. So $\widetilde{S} = \{X\}$, for all $X \in \partial S$.

Case 2. $\partial S \nsubseteq l$, for all $l \in L$. See Figure 4.2. We choose $\widetilde{S} = \partial S$ and we will show that $\widetilde{S}$ satisfies the two-condition property.

1) For all $l \in L$, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$. This is obvious.

2) Let $G \in V(\mathbb{R}^2) \cap P(\overline{S})$ that satisfies **C1**: for all $l \in L$, $l \cap G \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$. We will show that $m(\widetilde{S}) = m(\overline{XY}) \leq m(G)$. Suppose not. Since we consider only members of $V(\mathbb{R}^2) \cap P(\overline{S})$, G is the set of points that are elements in ∂S. Since the answer must be a connected curve in $V(\mathbb{R}^2) \cap P(\overline{S})$, G is a line segment. If $G \neq \partial S$, then there is a point D in ∂S such that $D \notin G$. Let l_D be a line in L such that $D \in l_D \cap \partial S$. Then $l_D \cap G = \emptyset$ but $l_D \cap \partial S \neq \emptyset$. This contradicts with the supposition. So G is the solution $\widetilde{S}$ of itself.

4.1.2 S is a triangle.

Let B, C and D be extreme points of ∂S.

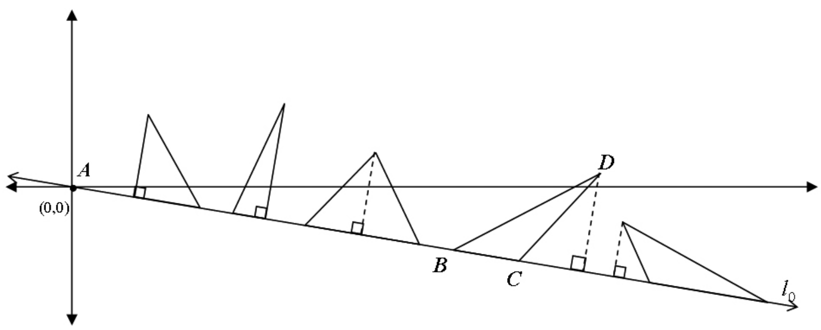


Figure 4.3: There is a line in L that contains two extreme points of ∂S.

Case 1. There is a line in L that contains two extreme points of ∂S. See Figure 4.3. That means this line contains an edge of ∂S. From Lemma 4.1, this line is the first (or the last) line in L that intersects ∂S. Suppose this line is l_0 and $l_0 \cap \partial S = \overline{BC}$. Then the last line from A ($\widetilde{l_0}$) intersects ∂S only at a point D. So $\widetilde{S}$ is the shortest line segment that connects D and l_0. Let P_D be the perpendicular from D to l_0. From the perpendicular is the shortest line segment that connects the point to the line, P_D is $\widetilde{S}$. If $P_D \notin V(\mathbb{R}^2) \cap P(\overline{S})$, then $\triangle BCD$ is an obtuse angle and either $B\widehat{C}D$ or $D\widehat{B}C$ is an obtuse angle. Suppose $B\widehat{C}D$ is an obtuse angle. Then $\overline{CD}$ is the shortest line segment that contains in

$V(\mathbb{R}^2) \cap P(\overline{S})$ and connects D and l_0. So $\overline{CD}$ is $\widetilde{S}$. Similarly, we get $\overline{BD}$ is $\widetilde{S}$, if $D\widehat{B}C$ is an obtuse angle. Hence

$$\widetilde{S} = \begin{cases} P_D \text{ if } P_D \in V(\mathbb{R}^2) \cap P(\overline{S}), \\ \overline{CD} \text{ if } P_D \notin V(\mathbb{R}^2) \cap P(\overline{S}) \text{ and } B\widehat{C}D \text{ is an obtuse angle}, \\ \overline{BD} \text{ if } P_D \notin V(\mathbb{R}^2) \cap P(\overline{S}) \text{ and } D\widehat{B}C \text{ is an obtuse angle}. \end{cases}$$

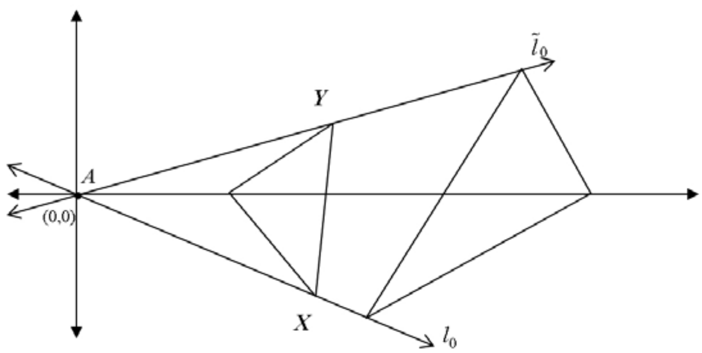


Figure 4.4: There are no lines in L that contain any two extreme points of ∂S.

Case 2. There are no lines in L that contain any two extreme points of ∂S. See Figure 4.4. From Lemma 4.1, we get $l_0 \cap \partial S = \{X\}$ and $\widetilde{l_0} \cap \partial S = \{Y\}$ for some extreme points X and Y of ∂S. Then $\widetilde{S}$ is the curve that contains in $V(\mathbb{R}^2) \cap P(\overline{S})$ and connects X and Y. Since the shortest curve that connects any two points is a line segment, so $\widetilde{S}$ is $\overline{XY}$. Since X and Y are extreme points of ∂S, $\overline{XY} \subseteq \partial S$.

4.1.3 S is quadrilateral.

Let B, C, D and E be extreme points of ∂S such that B is adjacent to C and E.

Case 1. There is only one line in L that contains edges of ∂S. See Figure 4.5. From Lemma 4.1, this line is the first or the last line in L that intersects ∂S. Suppose this line is l_0 and $l_0 \cap \partial S = \overline{BC}$. Then $\widetilde{l_0} \cap \partial S = \{Z\}$, for some $Z \in \{D, E\}$ and we can find $\widetilde{S}$ by consider $\triangle BCZ$.

Case 2. There are no lines in L that contain edges of ∂S. See Figure 4.6. From Lemma 4.1, we obtain that $l_0 \cap \partial S = \{X\}$ and $\widetilde{l_0} \cap \partial S = \{Y\}$ for some

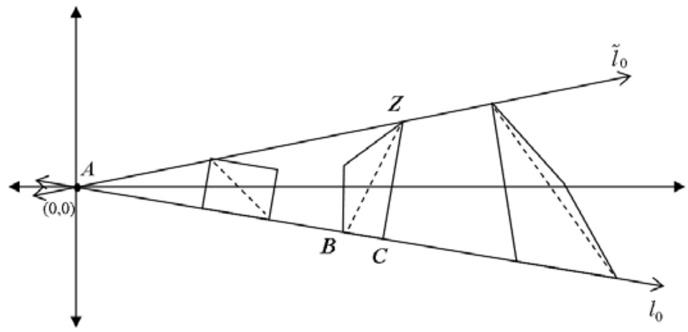


Figure 4.5: There is only one line in L that contains an edge of ∂S.

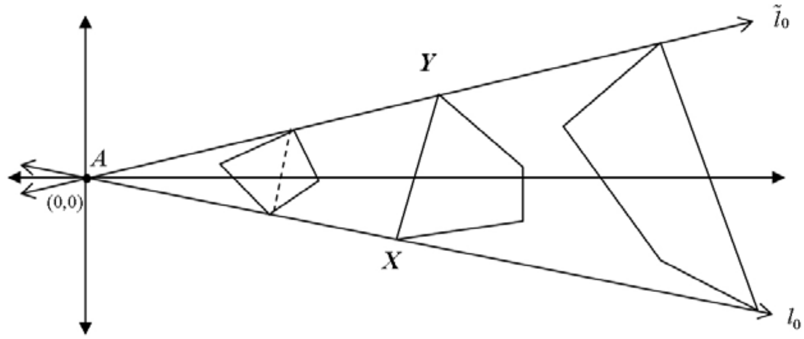


Figure 4.6: There are no lines in L that contain edges of ∂S.

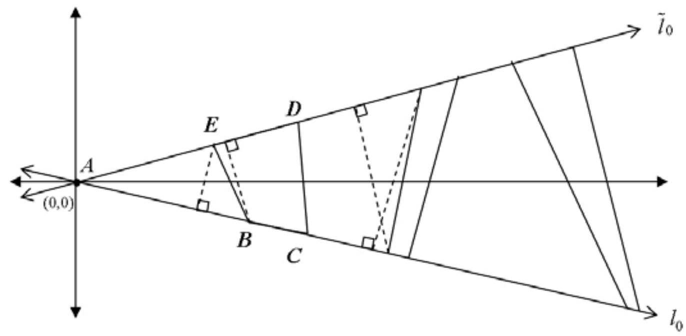


Figure 4.7: There are only two lines in L such that each line contains one edge of ∂S.

extreme points X and Y of ∂S. Then $\widetilde{S}$ is the curve that contains in $V(\mathbb{R}^2) \cap P(\overline{S})$ and connects X and Y. Since the shortest curve that connects any two points is a line segment, so $\widetilde{S}$ is $\overline{XY}$.

Case 3. There are only two lines in L such that each line contains one edge of ∂S and the distinct extreme points of $\partial(S)$. See Figure 4.7. From Lemma 4.1, we get these two lines are l_0 and $\widetilde{l}_0$. Suppose that $l_0 \cap \partial S = \overline{BC}$ and $\widetilde{l}_0 \cap \partial S = \overline{DE}$. Then $\widetilde{S}$ is the shortest line segment that connects $\overline{BC}$ and $\overline{DE}$. Suppose that $|\overline{AB}| < |\overline{AC}|$ and $|\overline{AE}| < |\overline{AD}|$. We will show that

$$\widetilde{S} = \begin{cases} P_B \text{ if } P_B \in V(\mathbb{R}^2) \cap P(\overline{S}), \\ P_E \text{ if } P_E \in V(\mathbb{R}^2) \cap P(\overline{S}), \\ \overline{BE} \text{ if } A\widehat{B}E \text{ and } A\widehat{E}B \text{ are acute angles, } P_B \text{ and } P_E \text{ are not in } V(\mathbb{R}^2) \cap P(\overline{S}), \\ \overline{BD} \text{ if } A\widehat{E}B \text{ is an obtuse angle, } P_B \text{ and } P_E \text{ are not in } V(\mathbb{R}^2) \cap P(\overline{S}), \\ \overline{CE} \text{ if } A\widehat{B}E \text{ is an obtuse angle, } P_B \text{ and } P_E \text{ are not in } V(\mathbb{R}^2) \cap P(\overline{S}), \end{cases}$$

where P_B is a perpendicular from B to $\widetilde{l}_0$, P_C is a perpendicular from C to $\widetilde{l}_0$, P_D is a perpendicular from D to l_0 and P_E is a perpendicular from E to l_0.

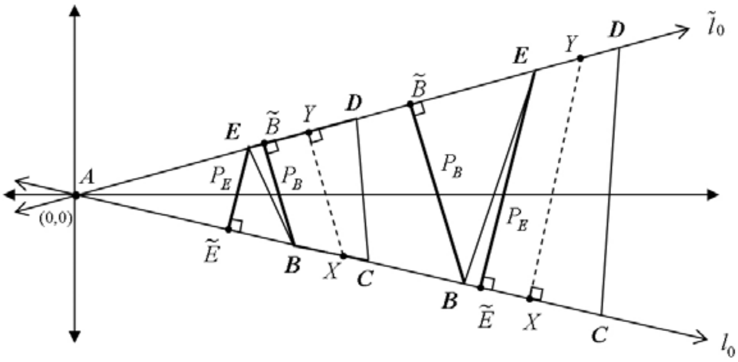


Figure 4.8: Either $P_B \in V(\mathbb{R}^2) \cap P(\overline{S})$ or $P_E \in V(\mathbb{R}^2) \cap P(\overline{S})$.

First, we will show that if $P_B \in V(\mathbb{R}^2) \cap P(\overline{S})$ then $P_E \notin V(\mathbb{R}^2) \cap P(\overline{S})$ and $m(P_B) \leq |\overline{XY}| = m(|\overline{XY}|)$ for all $X \in \overline{BC}$ and $Y \in \overline{DE}$. See Figure 4.8. Suppose that $P_B \in V(\mathbb{R}^2) \cap P(\overline{S})$ and intersects with $\widetilde{l}_0$ at $\widetilde{B} \in \overline{DE}$. Then $A\widehat{E}B$ is an obtuse angle and $A\widehat{B}E$ is an acute angle. If P_E is also in $V(\mathbb{R}^2) \cap P(\overline{S})$, then $A\widehat{E}B$ must be an acute angle. Therefore, $P_E \notin V(\mathbb{R}^2) \cap P(\overline{S})$. Then $P_E \notin V(\mathbb{R}^2) \cap P(\overline{S})$. Let X and Y be any points in $\overline{BC}$ and $\overline{DE}$, respectively. Then $\overline{XY}$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$.

Case I. $\overline{XY} \perp \widetilde{l}_0$. Then $\overline{XY} // P_B$ and we obtain

$$\begin{aligned} \frac{\left|\overline{B\widetilde{B}}\right|}{\left|\overline{AB}\right|} &= \frac{\left|\overline{XY}\right|}{\left|\overline{AX}\right|}, \\ \left|\overline{B\widetilde{B}}\right| &= \frac{\left|\overline{XY}\right| \times \left|\overline{AB}\right|}{\left|\overline{AX}\right|}. \end{aligned}$$

Since $\left|\overline{AB}\right| \leq \left|\overline{AX}\right|$, $m(P_B) = \left|\overline{B\widetilde{B}}\right| \leq \left|\overline{XY}\right| = m(\overline{XY})$.

Case II. $\overline{XY} \not\perp \widetilde{l}_0$. Let P_X be a perpendicular from X to $\widetilde{l}_0$ at $\widetilde{X}$. Then $m(P_X) = \left|\overline{X\widetilde{X}}\right| < \left|\overline{XY}\right| = m(\overline{XY})$ and we obtain that $m(P_B) = \left|\overline{B\widetilde{B}}\right| \leq \left|\overline{X\widetilde{X}}\right| < m(\overline{XY})$. Therefore, $m(P_B) \leq m(\overline{XY})$ for all $X \in \overline{BC}$ and $Y \in \overline{DE}$.

On the other hand, if $P_E \in V(\mathbb{R}^2) \cap P(\overline{S})$ then $P_B \notin V(\mathbb{R}^2) \cap P(\overline{S})$ and for all $X \in \overline{BC}$ and $Y \in \overline{DE}$, $m(P_B) \leq m(\overline{XY}) = \left|\overline{XY}\right|$. Hence

$$\widetilde{S} = \begin{cases} P_B, & \text{if } P_B \in V(\mathbb{R}^2) \cap P(\overline{S}), \\ P_E, & \text{if } P_E \in V(\mathbb{R}^2) \cap P(\overline{S}). \end{cases}$$

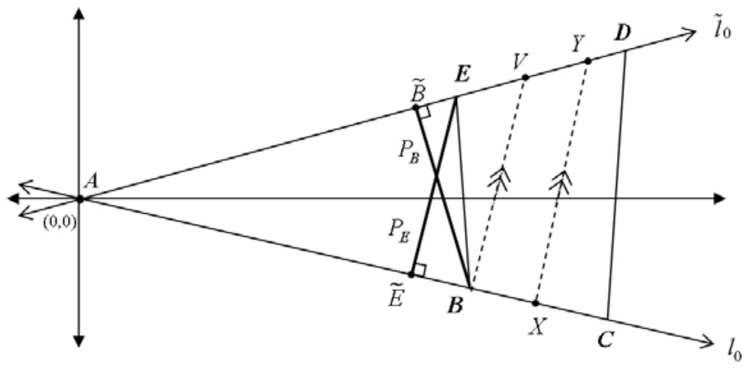


Figure 4.9: $A\widehat{B}E$ and $A\widehat{E}B$ are acute angles.

Next, we will show that if $A\widehat{B}E$ and $A\widehat{E}B$ are acute angles, then for all $X \in \overline{BC}$ and $Y \in \overline{DE}$, $m(\overline{BE}) \leq m(\overline{XY})$. See Figure 4.9. Suppose that $A\widehat{B}E$ and $A\widehat{E}B$ are acute angles. Then P_B and $P_E \notin V(\mathbb{R}^2) \cap P(\overline{S})$. Let $X \in \overline{BC}$ and $Y \in \overline{DE}$. We will show that $m(\overline{BE}) \leq m(\overline{XY})$.

Case I. $\overline{BE}//\overline{XY}$. Then we obtain

$$\begin{aligned}\frac{|\overline{BE}|}{|\overline{AB}|} &= \frac{|\overline{XY}|}{|\overline{AX}|},\\ |\overline{BE}| &= \frac{|\overline{XY}|\times|\overline{AB}|}{|\overline{AX}|}.\end{aligned}$$

Since $|\overline{AB}| \leq |\overline{AX}|$, so $m(\overline{BE}) = |\overline{BE}| \leq |\overline{XY}| = m(\overline{XY})$.

Case II. $\overline{BE} \not\parallel \overline{XY}$. Since $X \in \overline{BC}$ and $Y \in \overline{DE}$, there is either a point V in $\overline{DE}$ such that $\overline{BV}//\overline{XY}$ or a point U in $\overline{BC}$ such that $\overline{EU}//\overline{XY}$. If there is a point V in $\overline{DE}$ such that $\overline{BV}//\overline{XY}$, then we obtain

$$\begin{aligned}\frac{|\overline{BV}|}{|\overline{AB}|} &= \frac{|\overline{XY}|}{|\overline{AX}|}\\ |\overline{BV}| &= \frac{|\overline{XY}|\times|\overline{AB}|}{|\overline{AX}|}.\end{aligned}$$

Since $|\overline{AB}| \leq |\overline{AX}|$, so $m(\overline{BV}) = |\overline{BV}| \leq |\overline{XY}| = m(\overline{XY})$. Since $A\widehat{E}B$ is an acute angle, $V\widehat{E}B$ is an obtuse angle. So $m(\overline{BE}) < |\overline{BV}| \leq m(\overline{XY})$. Similarly, we get that $m(\overline{BE}) < |\overline{EU}| \leq m(\overline{XY})$ if there is $U \in \overline{BC}$ such that $\overline{EU}//\overline{XY}$. Hence, $\widetilde{S}$ is $\overline{BE}$ if $A\widehat{B}E$ and $A\widehat{E}B$ are acute angles, P_B and $P_E \notin V(\mathbb{R}^2)\cap P(\overline{S})$.

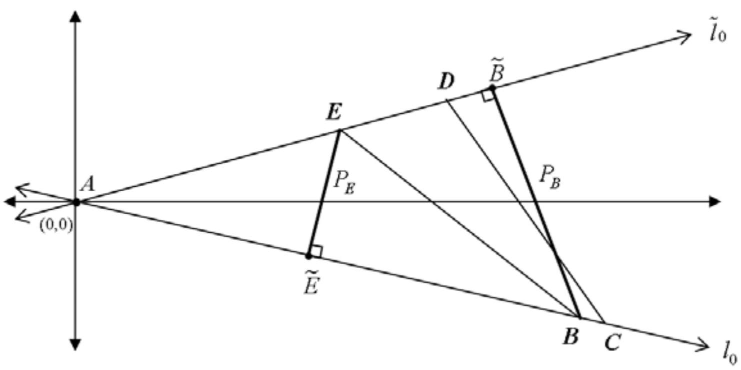


Figure 4.10: $A\widehat{E}B$ is an obtuse angle, P_B and P_E are not in $V(\mathbb{R}^2)\cap P(\overline{S})$.

For the last condition, if $A\widehat{E}B$ is an obtuse angle, P_B and P_E are not in $V(\mathbb{R}^2)\cap P(\overline{S})$, then for all $X \in \overline{BC}$ and $Y \in \overline{DE}$, $m(\overline{BD}) \leq m(\overline{XY})$, see Figure 4.10. Suppose that $A\widehat{E}B$ is an obtuse angle, P_B and $P_E \notin V(\mathbb{R}^2)\cap P(\overline{S})$. We will show that for all $X \in \overline{BC}$ and $Y \in \overline{DE}$, $m(\overline{BD}) \leq m(\overline{XY})$. Let $\widetilde{B}$ be an intersection point between P_B and $\widetilde{l}_0$. By assumptions, $\widetilde{B} \notin \overline{DE}$ and $A\widehat{D}B$ is an obtuse angle. Let $X \in \overline{BC}$ and $Y \in \overline{DE}$. Choose $V \in \overline{AD}$ such that $\overline{BV}//\overline{XY}$. Then

$$\begin{aligned} \frac{|\overline{BV}|}{|\overline{AB}|} &= \frac{|\overline{XY}|}{|\overline{AX}|}, \\ |\overline{BV}| &= \frac{|\overline{XY}| \times |\overline{AB}|}{|\overline{AX}|}. \end{aligned}$$

Since $|\overline{AB}| \leq |\overline{AX}|$, $m(\overline{BV}) = |\overline{BV}| \leq |\overline{XY}| = m(\overline{XY})$. Since $A\widehat{D}B$ is an obtuse angle and $V \in \overline{AD}$, $V\widehat{D}B$ is an obtuse angle and $m(\overline{BD}) \leq |\overline{BV}| \leq m(\overline{XY})$. Similarly, if $A\widehat{B}E$ is an obtuse angle, P_B and $P_E \notin V(\mathbb{R}^2) \cap P(\overline{S})$, then for all $X \in \overline{BC}$ and $Y \in \overline{DE}$, $m(\overline{CE}) \leq m(\overline{XY})$. Hence,

$$\widetilde{S} = \begin{cases} \overline{BD}, \text{ if } A\widehat{E}B \text{ is an obtuse angle, } P_B \text{ and } P_E \text{ are not in } V(\mathbb{R}^2) \cap P(\overline{S}), \\ \overline{CE}, \text{ if } A\widehat{B}E \text{ is an obtuse angle, } P_B \text{ and } P_E \text{ are not in } V(\mathbb{R}^2) \cap P(\overline{S}). \end{cases}$$

4.1.4 S is a convex n-gon, where $n \geq 5$.

Let $E_1, E_2, ..., E_n$ be extreme points of S, with $E_1 = E_{n+1}$.

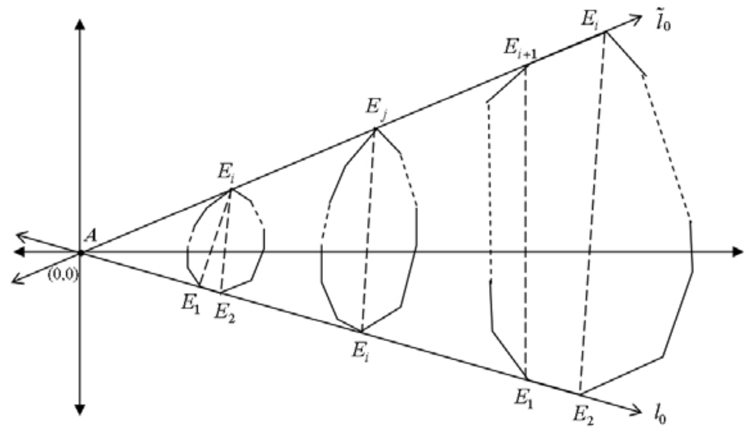


Figure 4.11: Convex n-gons S, where $n \geq 5$.

Case 1. There is only one line in L that contains an edge of ∂S. See Figure 4.11. Suppose that $\overline{E_1E_2} \subseteq l_0$ and $\widetilde{l}_0 \cap \partial S = \{E_i\}$ for some $i \in [3, n]$. Similar to the case 1 of quadrilateral, $\widetilde{S}$ is the line segment that connects E_i and X for some $X \in \overline{E_1E_2}$. Then we can find $\widetilde{S}$ by considering $\triangle E_1E_2E_i$.

Case 2. There are no lines in L that contain edges of ∂S. See Figure 4.11. From Lemma 4.1, we get $l_0 \cap \partial S = \{E_i\}$ and $\widetilde{l}_0 \cap \partial S = \{E_j\}$ for some $i, j \in [1, n]$ and $i \neq j$. Then $\widetilde{S}$ is the curve that connects E_i and E_j. Similar to the case 2 of quadrilateral, $\widetilde{S}$ is $\overline{E_i E_j}$.

Case 3. There are two lines in L such that each line contains one edge of ∂S. See Figure 4.11. From Lemma 4.1, we get that these two lines are l_0 and $\widetilde{l}_0$. Suppose that $l_0 \cap \partial S = \overline{E_1 E_2}$ and $\widetilde{l}_0 \cap \partial S = \overline{E_i E_{i+1}}$, for some $i \in [3, n]$. Then $\widetilde{S}$ is contained in the quadrilateral $E_1 E_2 E_i E_{i+1}$, that is similar to the case 3 of quadrilateral.

For the case S is a circle, l_0 and $\widetilde{l}_0$ are the tangent lines of ∂S. Let X and Y be points of tangency of ∂S with l_0 and $\widetilde{l}_0$, respectively. See Figure 4.12. We show that $\widetilde{S}$ is $\overline{XY}$. Let $l \in L$.

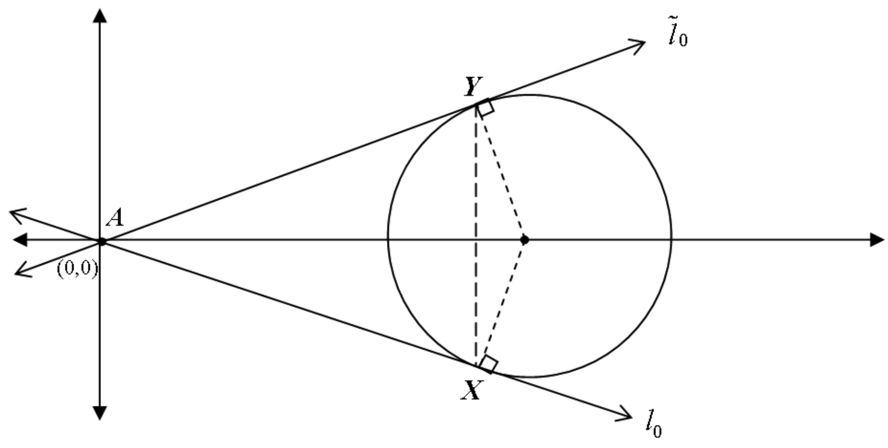


Figure 4.12: S is a circle.

1) Since $\overline{XY}$ connects l_0 and $\widetilde{l}_0$, so $l \cap \overline{XY} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$.

2) Let $G \in V(\mathbb{R}^2) \cap P(\overline{S})$ that satisfies **C1**: $l \cap G \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$. We show that $m(\widetilde{S}) = m(\overline{XY}) \leq m(G)$. Then G is a curve that connects X and Y. Since the shortest curve that connects any two points is a line segment, so $m(\widetilde{S}) = m(\overline{XY}) \leq m(G)$.

4.2 General, not necessarily connected solutions for convex sets

In this section, we study a convex set S which is not a line segment (the interior of S is not empty). We prove that solutions in the form of finite union of curves does not exist and then we study properties of the candidates for solutions.

4.2.1 Non-existence of finite solutions

In this part we prove the following theorem, which implies that there are no solutions in the form of finite union of curves.

Theorem 4.2 *Assume that S is a convex set with nonempty interior. For any curve in $\overline{S}$, where $S \in C(\mathbb{R}^2)$, we can find a general curve in $V(\mathbb{R}^2 \cap P(\overline{S})$ that intersects all lines in L intersecting the curve and the length of this general curve is shorter than the length of the curve.*

Proof. Let $S \in C(\mathbb{R}^2)$ and $\gamma : [a, b] \longrightarrow \mathbb{R}^2$ be a curve in $\overline{S}$, where $a, b \in \mathbb{R}$ and $a < b$. Let $\gamma(t) = \gamma_t$, for all $t \in [a, b]$ and $\{\gamma\} = \{\gamma(t) : a \leq t \leq b\}$. We denote the line from A passing through a point X by l_X. We show that there is a general curve $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ satisfying

1) for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$, and

2) $m(\widetilde{S}) < m(\gamma)$.

If there is a line l in L such that $\{\gamma\} \subseteq l$, then we can choose $\widetilde{S}$ be a point on $\{\gamma\}$ and this $\widetilde{S}$ satisfies 1) and 2). Next, we consider the case that for all l in L, $\{\gamma\} \nsubseteq l$. Let $L_\gamma = \{l \in L | l \cap \{\gamma\} \neq \emptyset\}$. Suppose $u, v \in [a, b]$ such that l_u and l_v in L_γ are the first and the last lines from A that intersect γ, respectively, and for all t in $(u, v), \gamma(t) \notin l_u$ and l_v. Then $\gamma_u = \gamma(u) \in l_u \cap \{\gamma\}$ and $\gamma_v = \gamma(v) \in l_v \cap \{\gamma\}$. Let $\widetilde{\gamma}$ be a curve in $\mathbb{R}^2$ which is a subcurve of γ and $\widetilde{\gamma} : [u, v] \longrightarrow \mathbb{R}^2$. Then $\widetilde{\gamma}_t = \widetilde{\gamma}(t) = \gamma(t) = \gamma_t$, for all $t \in [u, v]$ and $m(\widetilde{\gamma}) \leq m(\gamma)$. We consider $\widetilde{\gamma}$.

Case 1. $\widetilde{\gamma}$ is not a line segment. See Figure 4.13. We choose $\widetilde{S} = \overline{\widetilde{\gamma}_u \widetilde{\gamma}_v}$. Since S is a convex set, $\widetilde{S} = \overline{\widetilde{\gamma}_u \widetilde{\gamma}_v} \in V(\mathbb{R}^2) \cap P(\overline{S})$. Since $\widetilde{S}$ is a line segment that connects l_u and l_v, $\widetilde{S}$ satisfies 1). Since $\widetilde{\gamma}$ is not a line segment, $m(\widetilde{S}) \leq m(\overline{\widetilde{\gamma}_u \widetilde{\gamma}_v}) < m(\widetilde{\gamma}) \leq m(\gamma)$.

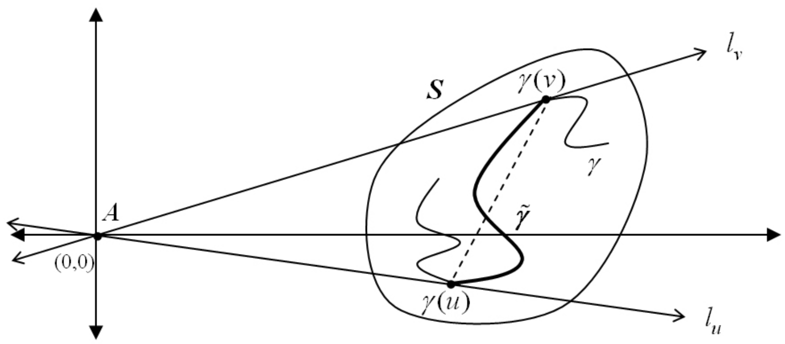


Figure 4.13: $\widetilde{\gamma}$ is not a line segment.

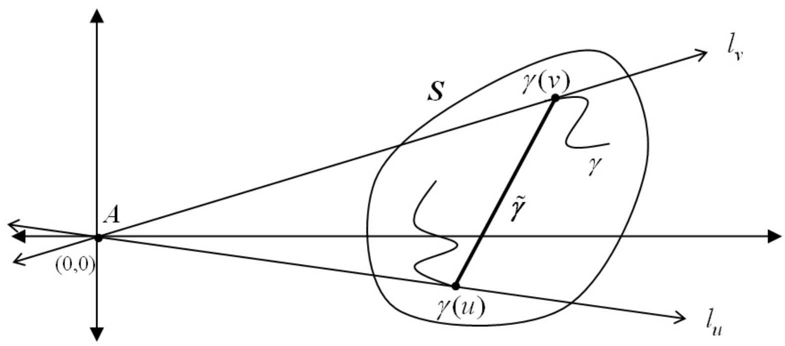


Figure 4.14: $\widetilde{\gamma}$ is a line segment.

Case 2. $\widetilde{\gamma}$ is a line segment. Then $\widetilde{\gamma} = \overline{\widetilde{\gamma}_u\widetilde{\gamma}_v}$. See Figures 4.14 and 4.15. First, we claim that there is a line segment $\overline{CE}$, which is a subset of $\{\widetilde{\gamma}\}$, that satisfies

1) $l_C, l_E \in L_\gamma$ such that $C \in l_C \cap \{\gamma\}, E \in l_E \cap \{\gamma\}$ and $\overline{CE}$ is not perpendicular to l_C and l_E, and

2) P_C or P_E is in $V(\mathbb{R}^2) \cap P(\overline{S})$, where P_C and P_E are the perpendiculars from C to l_E, respectively.

Let P_A be a perpendicular from A to $\widetilde{\gamma}$ and $\widetilde{A}$ an intersection point between P_A and $\widetilde{\gamma}$. If $\widetilde{A} \notin \{\widetilde{\gamma}\}$, then there is $z \in (u, v)$ such that $\widetilde{\gamma}(z) \neq \widetilde{A}$. On the other hand, if $\widetilde{A} \in \{\widetilde{\gamma}\}$, then there is $z \in (u, v)$ such that $\widetilde{\gamma}(z) \neq \widetilde{A}$ because $\widetilde{\gamma}$ is continuous. Choose a point C, which is $\widetilde{\gamma}(z)$ in $\{\widetilde{\gamma}\}$ such that $z \in (u, v)$ and

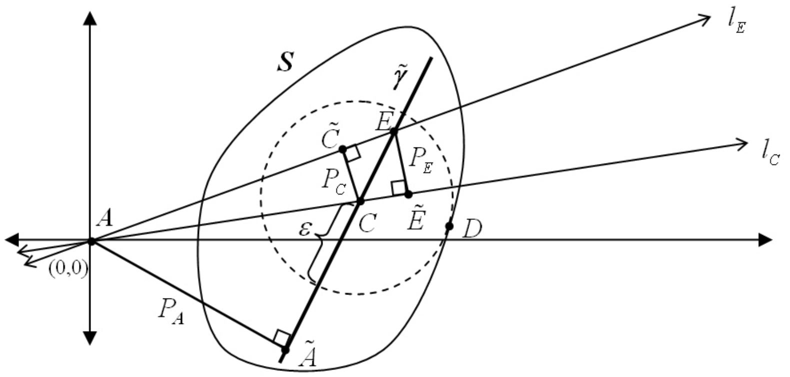


Figure 4.15: $\widetilde{\gamma}$ is a line segment.

$\widetilde{\gamma}(z) \neq \widetilde{A}$. Let D be a point in ∂S such that D is not a point on $\{\gamma\}$. We choose $\varepsilon = \min\left\{m(\overline{CD}), m(\overline{C\widetilde{\gamma}_u}), m(\overline{C\widetilde{\gamma}_v}), m(\overline{C\widetilde{A}})\right\} > 0$. We construct an open disk $D(C, \varepsilon)$ with center at C and radius ε. Since C is not the point $\widetilde{A}$ and $\widetilde{\gamma}$ is a continuous mapping, there is E in $D(C,\varepsilon)\backslash\{C\}$ such that E is a point on $\{\widetilde{\gamma}\}$ and $A\widehat{C}E$ is an obtuse angle. Therefore, the line segment $\overline{CE}$ is a subset of $\{\widetilde{\gamma}\}$. Let l_C and l_E be a line in L_γ such that C is a point in $l_C \cap \{\gamma\}$ and E is a point in $l_E \cap \{\gamma\}$, respectively. Since $A\widehat{C}E$ is an obtuse angle and C is not $\widetilde{A}$, then l_C and l_E are not perpendicular to $\overline{CE}$. Let P_C and P_E be the perpendiculars from C to l_E at a point $\widetilde{E}$ and from E to l_C at a point $\widetilde{C}$, respectively. If $\widetilde{E}$ is not in $D(C,\varepsilon)$, then $\varepsilon \leq m(\overline{C\widetilde{E}}) < m(\overline{CE}) < \varepsilon$, a contradiction. Therefore, $\widetilde{E}$ is not a point in $D(C,\varepsilon)$. Since $\triangle ACE$ is an obtuse triangle, then $\widetilde{C} \in \overline{AE}$ and $\widetilde{E} \in \overline{AC}$. Since $m(\overline{C\widetilde{E}}) < m(\overline{CE}) < \varepsilon$, then $\widetilde{C} \in D(C,\varepsilon)$. So $\widetilde{C}$ and $\widetilde{E}$ are points in $D(C,\varepsilon)$ and are on opposite sides of $\overline{CE} \subseteq \{\widetilde{\gamma}\}$. In other word, if $\{\widetilde{\gamma}\}$ is a subset of ∂S, then P_C and P_E is in $V(\mathbb{R}^2)\cap P(\overline{S})$. Hence there is $\overline{CE} \subseteq \{\widetilde{\gamma}\}$ such that l_C and l_E are not perpendicular to $\overline{CE}$, and P_C or P_E is in $V(\mathbb{R}^2) \cap P(\overline{S})$.

Next, we suppose that $\overline{CE}$ is a subset of $\{\widetilde{\gamma}\}$ such that l_C and l_E are not perpendicular to $\overline{CE}$, and P_C is in $V(\mathbb{R}^2) \cap P(\overline{S})$. Choose $\widetilde{S} = \overline{\widetilde{\gamma}_u C} \cup P_C \cup \overline{E\widetilde{\gamma}_v}$. Then $\widetilde{S} \in V(\mathbb{R}^2)\cap P(\overline{S})$ and we get

$$m(\widetilde{S}) = m\left(\overline{\widetilde{\gamma}_u C} \cup P_C \cup \overline{E\widetilde{\gamma}_V}\right) \leq m(\overline{\widetilde{\gamma}_u C}) + m(P_C) + m(\overline{E\widetilde{\gamma}_v}).$$

Since $\widetilde{S}$ is the finite union of line segments,

$$m(\widetilde{S}) = m(\overline{\widetilde{\gamma}_u C}) + m(P_C) + m(\overline{E\widetilde{\gamma}_v}).$$

Since $m(P_C) < m(\overline{CE})$,

$$m(\widetilde{S}) < m(\overline{\widetilde{\gamma}_u C}) + m(\overline{CE}) + m(\overline{E\widetilde{\gamma}_v}) = m(\widetilde{\gamma}) \leq m(\gamma).$$

Since P_C is a perpendicular from C to l_E, for all l in L, $l \cap \overline{CE} \neq \emptyset$ if and only if $l \cap P_C \neq \emptyset$. So for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$.

From these two cases, we have that for any curve γ in $V(\mathbb{R}^2) \cap P(\overline{S})$, where $S \in C(\mathbb{R}^2)$, there is a general curve $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$, and $m(\widetilde{S}) < m(\gamma)$. ■

4.2.2 The union of perpendiculars from boundary points of S to lines in L

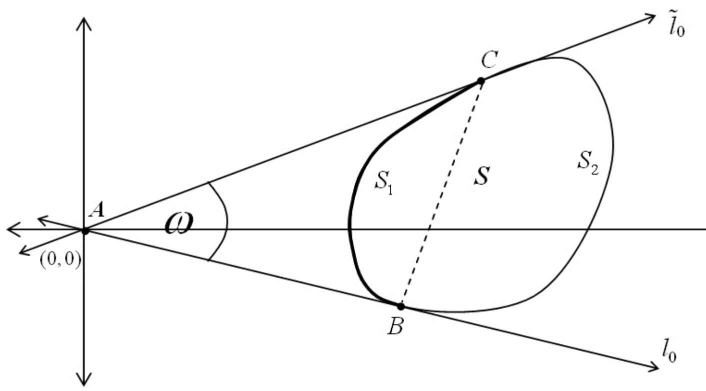


Figure 4.16: $S \in C(\mathbb{R}^2)$ and $\angle(l_0, \widetilde{l}_0) = \omega$.

Let $l_0, \widetilde{l}_0$ be the first and the last lines from A in L that intersect ∂S. Let ω be the angle between l_0 and $\widetilde{l}_0$, i.e., $\angle(l_0, \widetilde{l}_0) = \omega$. Let B be a point in ∂S such that $|\overline{AB}| = \min\{|\overline{AX}| \,|X \in l_0 \cap \partial S\}$ and C be a point in ∂S such that $|\overline{AC}| = \min\{|\overline{AY}| \,|Y \in \widetilde{l}_0 \cap \partial S\}$. We divide S into two parts by the line segment $\overline{BC}$. Then ∂S is divided into two curves. Let S_1 and S_2 be these two curves, then $\partial S = S_1 \cup S_2$ and B, C are the endpoints of each S_1 and S_2. See Figure 4.16. The distance from A to each point in S_1 is always less than or equal to the distance from A to a point in S_2, i.e., $\left|\overline{AX}\right| \le \left|\overline{AY}\right|$, for all $X \in S_1$ and $Y \in S_2$. We then describe S_1, without loss of generality, by a polar equation

$$r = r(\theta),\ 0 \le \theta \le \omega$$

and the parametric equations

$$\begin{aligned} x &= r(\theta)\cos\theta = r\cos\theta,\ \text{and} \\ y &= r(\theta)\sin\theta = r\sin\theta,\ 0 \le \theta \le \omega. \end{aligned}$$

Next, we consider a special general curve. Let X be a point in $S_1 \backslash \{B, C\}$ and l_X a line in L that intersects S_1 at X. We divide ω into n equal parts by the line l_i from A that intersect S, where $i = 0, 1, 2, ..., n$. Then we construct a perpendicular from the intersection point of l_i and S_1 to l_{i-1} for all $i = 1, 2, ..., n$. Let D_n be a union of all these perpendiculars. We will prove the following theorem.

Theorem 4.3 $\lim\limits_{n\to\infty} m(D_n) = \int_0^\omega r d\theta$.

Case 1. l_X is not perpendicular to l_X^*, where l_X^* is a tangent line of ∂S at X.

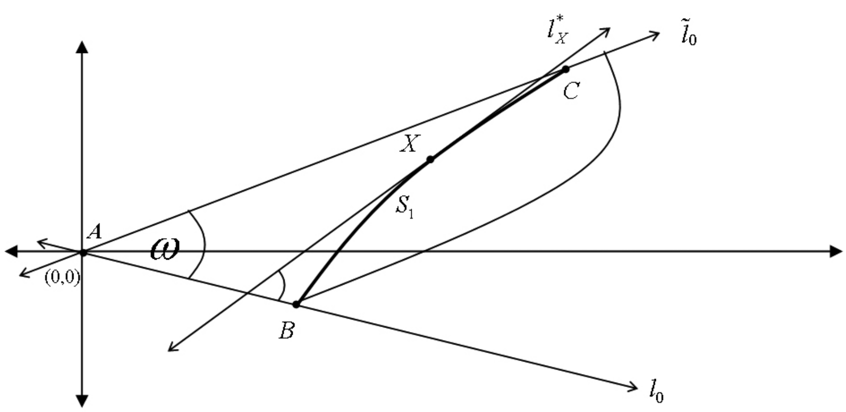


Figure 4.17: $\angle(l_0, l_X^*) < 90^{\circ}$, for all $X \in S_1 \backslash \{B, C\}$.

Case 1.1. $\angle(l_0, l_X^*) < 90^{\circ}$, for all X in $S_1 \backslash \{B, C\}$. See Figure 4.17. Consider subdivisions of the interval $0 \le \theta \le \omega$ by values $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_n = \omega$ and $\triangle\theta_i = \theta_i - \theta_{i-1}$. See Figure 4.18. Suppose $\theta_i = \frac{i\omega}{n}$, for all

$i = 1, 2, ..., n$. Then $\triangle\theta_i = \frac{\omega}{n}$, for all $i = 1, 2, ..., n$. Let l_{θ_i} be a line in L such that $l_{\theta_i} \cap \partial S \neq \emptyset$ and $\angle(l_0, l_{\theta_i}) = \theta_i$, for all $i = 1, 2, ..., n$. Then there is a point X_i in S_1 such that X_i is a point in $l_{\theta_i} \cap \partial S$, for all $i = 1, 2, ..., n$. Since X_i is a point in S_1, for all $i = 1, 2, ..., n$, we can write the coordinates of X_i as

$$(x_i, y_i) = (r(\theta_i)\cos\theta_i, r(\theta_i)\sin\theta_i)$$

for all $i = 1, 2, ..., n$.

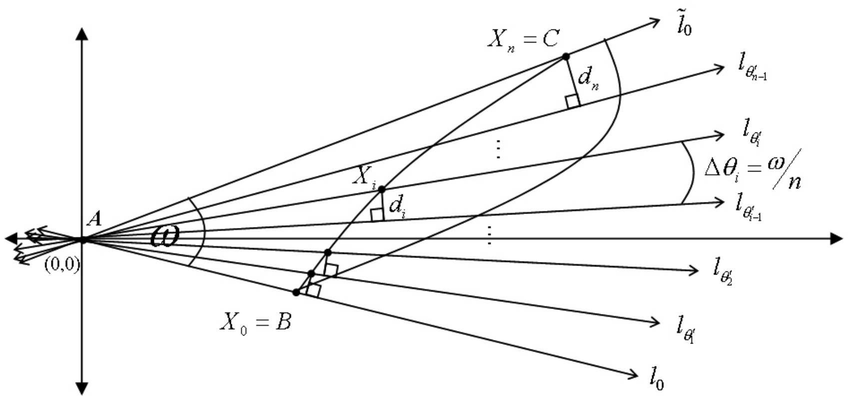


Figure 4.18: Subdivisions of the interval $0 \leq \theta \leq \omega$.

Let us define the equation of l_{θ_i} by $a_i x + b_i y + c_i = 0$, for all $i = 1, 2, ..., n$. Since l_{θ_i} intersects y-axis at the origin $A\,(0, 0)$, we obtain that

$$\begin{aligned} a_i x + b_i y &= 0 \\ y &= m_i x, \text{ where } m_i = -\frac{a_i}{b_i} \text{ is a slope of } l_{\theta_i} \\ 0 &= m_i x - y, \text{ for all } i = 1, 2, ..., n. \end{aligned}$$

Let d_i be a perpendicular from X_i to l_{θ_i}, for all $i = 1, 2, ..., n$. See Figure 4.18. From this equation, we get that

$$m(d_i) = \frac{|m_{i-1}x_i - y_i|}{\sqrt{m_{i-1}^2 + 1}}, \text{ for all } i = 1, 2, ..., n.$$

Since $\angle(l_0, l_{\theta_{i-1}}) = \theta_{i-1}$, we obtain that $m_{i-1} = \tan(\theta_{i-1})$, for all $i = 1, 2, ..., n$.

Then

$$\begin{aligned} m(d_i) &= \frac{|\tan(\theta_{i-1})x_i - y_i|}{\sqrt{\tan^2(\theta_{i-1})+1}} \\ &= \frac{|\tan(\theta_{i-1})x_i - y_i|}{|\sec(\theta_{i-1})|}, \text{ since } \tan^2(\theta_{i-1})+1 = \sec^2(\theta_{i-1}) \\ &= \frac{\left|\frac{\sin(\theta_{i-1})}{\cos(\theta_{i-1})}x_i - y_i\right|}{\left|\frac{1}{\cos(\theta_{i-1})}\right|} \\ &= |x_i \sin(\theta_{i-1}) - y_i \cos(\theta_{i-1})|, \text{ for all } i = 1, 2, ..., n. \end{aligned}$$

Since $(x_i, y_i) = (r(\theta_i)\cos\theta_i, r(\theta_i)\sin\theta_i)$, we have

$$\begin{aligned} m(d_i) &= |r(\theta_i)[\cos(\theta_i)\sin(\theta_{i-1}) - \sin(\theta_i)\cos(\theta_{i-1})]| \\ &= |r(\theta_i)[\sin(\theta_{i-1} - \theta_i)]| \\ &= |r(\theta_i)[-\sin(\theta_i - \theta_{i-1})]| \\ &= |r(\theta_i)[\sin(\triangle\theta_i)]|, \text{ for all } i = 1, 2, ..., n. \end{aligned}$$

Let $D_n = \bigcup\limits_{i=1}^{n} d_i$. See Figure 4.18. Then $m(D_n) = \sum\limits_{i=1}^{n} m(d_i)$. Since d_i is a perpendicular from X_i fo $l_{\theta_{i-1}}$, for all $i = 1, 2, ..., n$, we have for any line l in L, $l \cap D_n \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$. We take limit as n goes to infinity and obtain

$$\begin{aligned} \lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} \sum_{i=1}^{n} m(d_i) \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} |r(\theta_i)[\sin(\triangle\theta_i)]|. \end{aligned}$$

Since $\triangle\theta_i = \frac{\omega}{n}$, for all $i = 1, 2, ..., n$, we obtain that

$$\begin{aligned} \lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} \left[\left|\sin\left(\frac{\omega}{n}\right)\right| \sum_{i=1}^{n} |r(\theta_i)|\right] \\ &= \lim_{n\to\infty} \left[\left|\frac{\sin\left(\frac{\omega}{n}\right)}{\frac{\omega}{n}}\right| \times \frac{\omega}{n} \times \sum_{i=1}^{n} |r(\theta_i)|\right] \\ &= \lim_{n\to\infty} \left[\left|\frac{\sin\left(\frac{\omega}{n}\right)}{\frac{\omega}{n}}\right| \sum_{i=1}^{n} \left[|r(\theta_i)|\frac{\omega}{n}\right]\right]. \end{aligned}$$

Let $\theta_i^* \in [\theta_{i-1}, \theta_i]$. Since $\triangle\theta_i = \frac{\omega}{n}$, for all $i = 1, 2, ..., n$,

$$h = \max\{\triangle\theta_i = \theta_i - \theta_{i-1} | i = 1, 2, ..., n\} = \frac{\omega}{n}$$

and h goes to 0 as n goes to infinity. So

$$\lim_{n\to\infty} m(D_n) = \lim_{h\to 0}\left|\frac{\sin(h)}{h}\right| \times \lim_{n\to\infty}\sum_{i=1}^{n}\left[|r(\theta_i)| \triangle \theta_i\right].$$

Since $\lim_{x\to 0}\frac{\sin(x)}{x} = 1$,

$$\lim_{n\to\infty} m(D_n) = \lim_{n\to\infty}\sum_{i=1}^{n}\left[|r(\theta_i)| \triangle \theta_i\right],\ \theta_i^* = \theta_i \in [\theta_{i-1}, \theta_i].$$

From the definition of definite integrals and that r is a positive-definite continuous mapping, we obtain that

$$\lim_{n\to\infty} m(D_n) = \int_0^{\omega} |r(\theta)|\, d\theta = \int_0^{\omega} r d\theta.$$

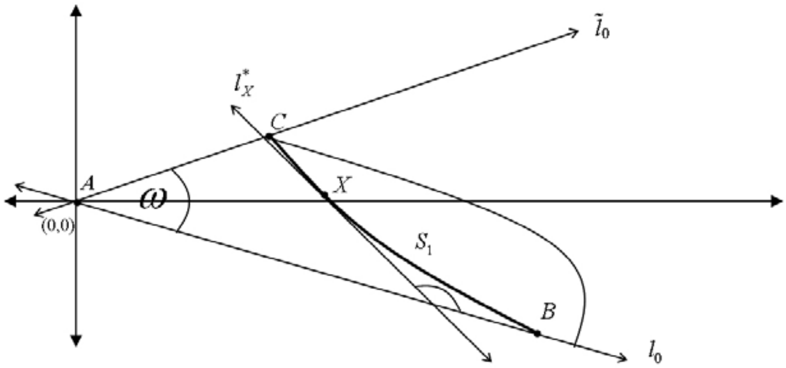


Figure 4.19: $\angle(l_0, l_X^*) > 90^{\text{o}}$, for all $X \in S_1 \setminus \{B, C\}$.

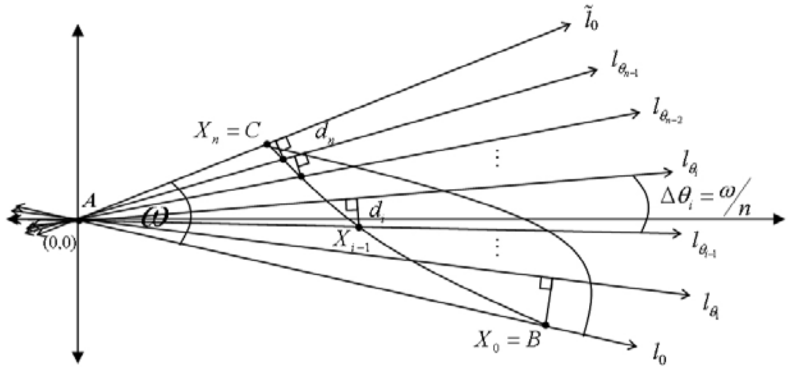


Figure 4.20: Subdivisions of the interval $0 \le \theta \le \omega$.

Case 1.2. $\angle(l_0, l_X^*) > 90^o$, for all X in $S_1 \backslash \{B, C\}$. See Figure 4.19. Similar to the case 1.1, but we let d_i be a perpendicular from X_{i-1} to l_i, for all $i = 1, 2, ..., n$. See Figure 4.20. Then

$$\begin{aligned} m(D_n) &= m\left(\bigcup_{i=1}^{n} d_i\right) \\ &= \sum_{i=1}^{n} \frac{|m_i x_{i-1} - y_{i-1}|}{\sqrt{m_i^2 + 1}} \\ &= \sum_{i=1}^{n} \frac{|\tan(\theta_i) x_{i-1} - y_{i-1}|}{\sqrt{\tan^2(\theta_i) + 1}} \\ &= \sum_{i=1}^{n} |x_{i-1}\sin(\theta_i) - y_{i-1}\cos(\theta_i)| \\ &= \sum_{i=1}^{n} |r(\theta_{i-1})\,[\sin(\theta_i - \theta_{i-1})]| \end{aligned}$$

and we take limit as n goes to infinity to get

$$\begin{aligned} \lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} |r(\theta_{i-1})\,[\sin(\triangle\theta_i)]|\right] \\ &= \lim_{n\to\infty}\left[\sum_{i=1}^{n} |r(\theta_i^*)| \triangle \theta_i\right], \ \ \theta_i^* = \theta_{i-1} \in [\theta_{i-1}, \theta_i]. \end{aligned}$$

By the definition of definite integrals, we get that

$$\lim_{n\to\infty} m(D_n) = \int_0^{\omega} |r(\theta)|\, d\theta = \int_0^{\omega} r d\theta.$$

Case 1.3. There is $\widetilde{X} \in S_1 \backslash \{B, C\}$ such that $l_{\widetilde{X}}^*$ is perpendicular to l_0 and for all Y in $S_1 \backslash \left\{B, C, \widetilde{X}\right\}$, $\angle(l_Y^*, l_0) < 90^o$. See Figure 4.21. In this case, we get the same result to the case 1.1.

Case 1.4. There is $\widetilde{X} \in S_1 \backslash \{B, C\}$ such that $l_{\widetilde{X}}^*$ is perpendicular to l_0 and for all Y in $S_1 \backslash \left\{B, C, \widetilde{X}\right\}$, $\angle(l_Y^*, l_0) > 90^o$. See Figure 4.22. In this case, we get the same result to the case 1.2.

Case 2. There is $\widetilde{X} \in S_1 \backslash \{B, C\}$ such that $l_{\widetilde{X}}^*$ is perpendicular to $l_{\widetilde{X}}$, where $l_{\widetilde{X}}^*$ is a tangent line of ∂S at $\widetilde{X}$. See Figure 4.23. First, we show that $\widetilde{X}$ is unique. By contradiction, suppose that $\widetilde{Y} \in S_1 \backslash \left\{B, C, \widetilde{X}\right\}$ satisfying this case. Since S is convex, $\overline{\widetilde{X}\widetilde{Y}}$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$. We consider $\triangle A\widetilde{X}\widetilde{Y}$. If $A\widehat{\widetilde{X}}\widetilde{Y}$ and

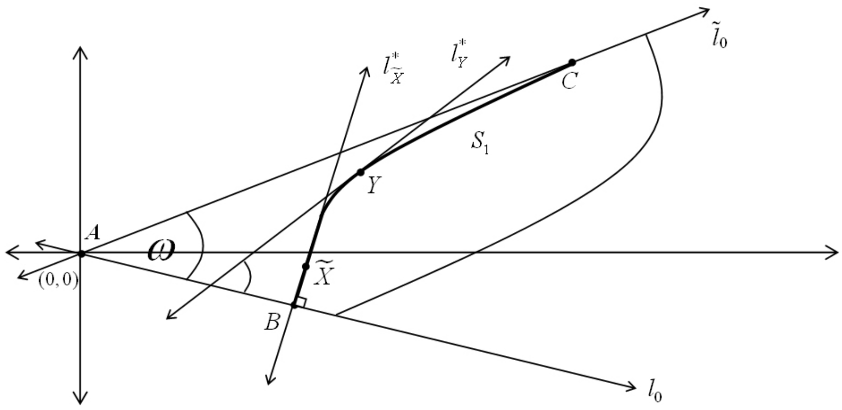


Figure 4.21: There is $\widetilde{X} \in S_1 \backslash \{B, C\}$ such that $l^*_{\widetilde{X}} \perp l_0$ and $\angle(l^*_Y, l_0) < 90^{\mathrm{o}}$.

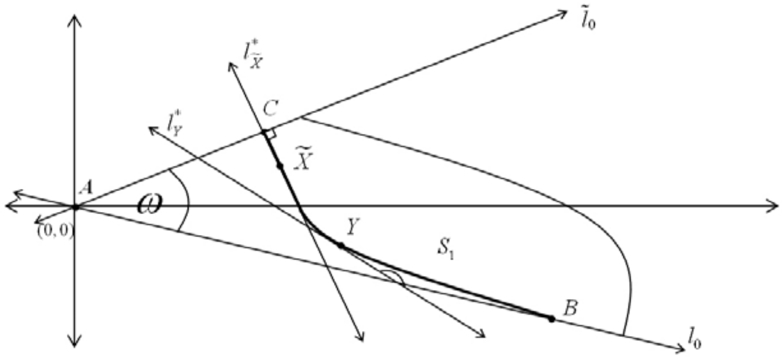


Figure 4.22: There is $\widetilde{X} \in S_1 \backslash \{B, C\}$ such that $l^*_{\widetilde{X}} \perp l_0$ and $\angle(l^*_Y, l_0) > 90^{\mathrm{o}}$.

$A\widehat{\widetilde{Y}}\widetilde{X}$ are acute angles, then $l^*_{\widetilde{X}}$ and $l^*_{\widetilde{Y}}$ are not tangent lines of ∂S. If $A\widehat{\widetilde{X}}\widetilde{Y}$ is an acute angle and $A\widehat{\widetilde{Y}}\widetilde{X}$ is an obtuse angle, then $l^*_{\widetilde{X}}$ is not a tangent line of ∂S. If $A\widehat{\widetilde{Y}}\widetilde{X}$ is an acute angle and $A\widehat{\widetilde{X}}\widetilde{Y}$ is an obtuse angle, then $l^*_{\widetilde{Y}}$ is not a tangent line of ∂S. This contradicts with the assumption, so $\widetilde{X}$ is unique.

Let $\angle(l_{\widetilde{X}}, l_0) = \beta$. Then $\angle(l_{\widetilde{X}}, \widetilde{l_0}) = \omega - \beta$. Next, we divide S_1 into two parts by $\widetilde{X}$ and then S_1 is the union of curves $B\widetilde{X}$ ($\gamma_{B\widetilde{X}}$) and $\widetilde{X}C$ ($\gamma_{\widetilde{X}C}$). Then $\gamma_{B\widetilde{X}}$ corresponds to case 1.2 and $\gamma_{\widetilde{X}C}$ corresponds to case 1.1. Consider $m(D_{B\widetilde{X}})$ of $\gamma_{B\widetilde{X}}$ and $m(D_{\widetilde{X}C})$ of $\gamma_{\widetilde{X}C}$. Similar to cases 1.1 and 1.2, we obtain that

$$\lim_{n\to\infty} m(D_{B\widetilde{X}}) = \int_0^\beta r d\theta$$

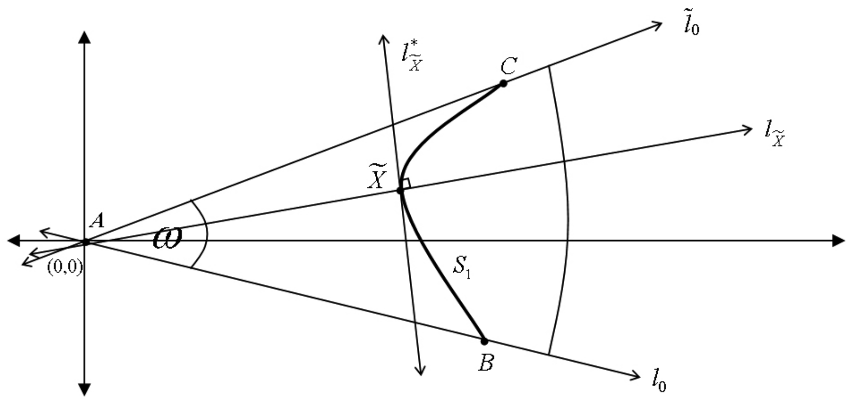


Figure 4.23: There is $\widetilde{X} \in S_1 \setminus \{B, C\}$ such that $l^*_{\widetilde{X}} \perp l_{\widetilde{X}}$.

and

$$\lim_{n\to\infty} m(D_{\widetilde{X}C}) = \int_\beta^\omega r d\theta.$$

Since for all l in L, $l \cap D_{B\widetilde{X}} \neq \emptyset$ if and only if $l \cap \gamma_{B\widetilde{X}} \neq \emptyset$ and $l \cap D_{\widetilde{X}C} \neq \emptyset$ if and only if $l \cap \gamma_{\widetilde{X}C} \neq \emptyset$, we have that $l \cap \partial S \neq \emptyset$ if and only if $l \cap \left(D_{B\widetilde{X}} \cup D_{\widetilde{X}C}\right) \neq \emptyset$. Let $D_n = D_{B\widetilde{X}} \cup D_{\widetilde{X}C}$. Then

$$\begin{aligned} \lim_{n\to\infty} m(D_n) &= \lim_{n\to\infty} \left(m(D_{B\widetilde{X}}) + m(D_{\widetilde{X}C})\right) \\ &= \int_0^\beta r d\theta + \int_\beta^\omega r d\theta \\ &= \int_0^\omega r d\theta. \end{aligned}$$

4.2.3 The infimum of the length of the candidates

First, we consider $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that $\widetilde{S}$ is in the form of finite union of curves in $V(\mathbb{R}^2) \cap P(\overline{S})$. Let

$$\widetilde{P} = \left\{ \widetilde{S} \;\middle|\; \begin{array}{l} \widetilde{S} \text{ is the finite union of curves in } V(\mathbb{R}^2) \cap P(\overline{S}) \text{ such} \\ \text{that for all } l \text{ in } L, l \cap \widetilde{S} \neq \emptyset \text{ if and only if } l \cap \partial S \neq \emptyset \end{array} \right\}$$

and $\ell[\widetilde{P}]$ the set of the length of $\widetilde{S}$ in $\widetilde{P}$, i.e., $\ell[\widetilde{P}] = \left\{ m(\widetilde{S}) \mid \widetilde{S} \in \widetilde{P} \right\}$. Putting $\omega = \angle(l_0, \widetilde{l}_0)$, we prove the following.

Theorem 4.4 $\inf \ell[\widetilde{P}] = \int_0^\omega r d\theta$, *where r is as defined in 4.2.2*

Proof. To show $\inf \ell[\widetilde{P}] = \int_0^\omega r d\theta$, we prove that

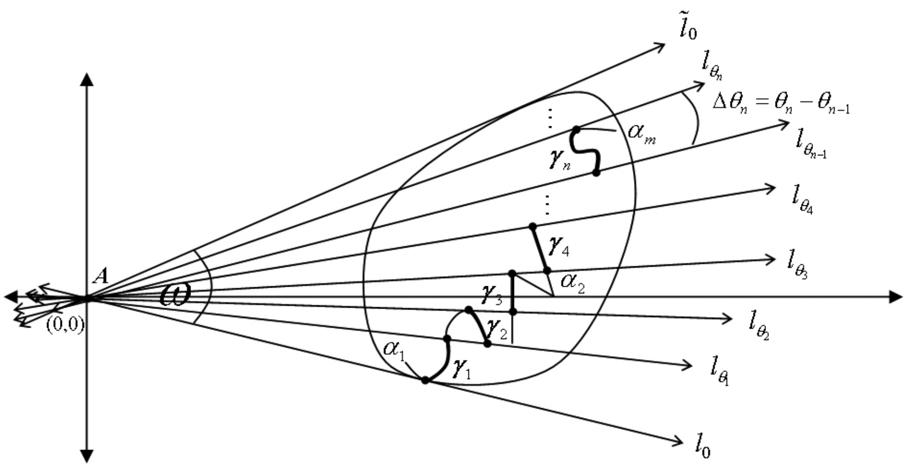


Figure 4.24: Example of $G = \bigcup_{i=1}^{\infty} \gamma_i \subseteq \widetilde{S} = \bigcup_{i=1}^{\infty} \alpha_i$.

(1) $\int_0^{\omega} r d\theta$ is a lower bound of $\ell[\widetilde{P}]$, and

(2) for all $\varepsilon > 0$ there is $\widetilde{S} \in \widetilde{P}$ such that $m(\widetilde{S}) < \int_0^{\omega} r d\theta + \varepsilon$.

To obtain (1), let $\widetilde{S} \in \widetilde{P}$. Then $m(\widetilde{S}) \in \ell[\widetilde{P}]$. We show that $\int_0^{\omega} r d\theta \leq m(\widetilde{S})$. We can write $\widetilde{S}$ in the form of countable union of curves, $\widetilde{S} = \bigcup_{i=1}^{m} \alpha_i$. Since $\widetilde{S} \in \widetilde{P}$, the first and the last lines from A that intersect $\widetilde{S}$ are l_0 and $\widetilde{l}_0$, respectively. Then for each point X in $\widetilde{S}$ there is l_θ in L such that X is in $l_\theta \cap \widetilde{S}$ and $\angle(l_\theta, l_0) = \theta \in [0, \omega]$. We can divide $\widetilde{S}$ into countable parts by using l_θ such that in each part there is a curve γ_i in $\widetilde{S}$ connecting two consecutive lines. See Figure 4.24. Let $G = \bigcup_{i=1}^{n} \{\gamma_i\}$. Then G is a subset of $\widetilde{S}$ and for all l in $L, l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap G \neq \emptyset$. Therefore, $m(G) \leq m(\widetilde{S})$.

Let l_{γ_i} and $\widetilde{l}_{\gamma_i}$ be the first and the last lines from A intersecting γ_i for all $i = 1, 2, 3, ..., n$. Given that β_i is a line segment connecting the intersection points between l_{γ_i} and $\widetilde{l}_{\gamma_i}$ with γ_i for all $i = 1, 2, 3, ..., n$ and $\widetilde{G}$ is the union of β_i, $\widetilde{G} = \bigcup_{i=1}^{n} \beta_i$. Since S is a convex set, $\widetilde{G}$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$. Since $m(\beta_i) \leq m(\gamma_i)$, for all i, $m(\widetilde{G}) \leq m(G) \leq m(\widetilde{S})$. Let θ_i be the measure of angle determined by l_0 and l_{γ_i}, $\angle(l_0, l_{\gamma_i}) = \theta_i$, and $\widetilde{\theta}_i$ be the measure of angle determined by l_0 and $\widetilde{l}_{\gamma_i}$, $\angle(l_0, \widetilde{l}_{\gamma_i}) = \widetilde{\theta}_i$, for all $i = 1, 2, 3, ..., n$. Since β_i is smooth on $\left[\theta_i, \widetilde{\theta}_i\right]$, the length

$m(\beta_i)$ of β_i from θ_i to $\widetilde{\theta}_i$ is

$$m(\beta_i) = \int_{\theta_i}^{\widetilde{\theta}_i} \left[(R_i')^2 + (R_i)^2\right]^{\frac{1}{2}} d\theta,$$

where $R_i = R_i(\theta)$ is a polar equation of β_i for all $i = 1, 2, 3, ..., n$. Then

$$m(\widetilde{G}) = \sum_{i=1}^{n} \int_{\theta_i}^{\widetilde{\theta}_i} \left[(R_i')^2 + (R_i)^2\right]^{\frac{1}{2}} d\theta.$$

From properties of integration and $R_i \leq [(R_i')^2 + (R_i)^2]^{\frac{1}{2}}$ for all $i = 1, 2, 3, ..., n$, we have

$$\int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta \leq \int_{\theta_i}^{\widetilde{\theta}_i} \left[(R_i')^2 + (R_i)^2\right]^{\frac{1}{2}} d\theta$$

and

$$\sum_{i=1}^{n} \int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta \leq m(\widetilde{G}).$$

From polar equation $R_i = R_i(\theta)$, where $\theta_i \leq \theta \leq \widetilde{\theta}_i$, for all $i = 1, 2, 3, ..., n$, the parametric equations are

$$\begin{aligned} x &= R_i(\theta)\cos\theta = R_i\cos\theta, \\ \text{and } y &= R_i(\theta)\sin\theta = R_i\sin\theta, \end{aligned}$$

where $\theta_i \leq \theta \leq \widetilde{\theta}_i$ and for all $i = 1, 2, 3, ..., n$. Since any line from A meet S_1 before $\widetilde{G}$, at the same value of θ the distance from A to the point in $\widetilde{G}$, $(R_i\cos\theta, R_i\sin\theta)$, is greater than or equal to the distance from A to the point in S_1, $(r\cos\theta, r\sin\theta)$. This implies that

$$r = r(\theta) \leq R_i(\theta) = R_i,$$

$\theta_i \leq \theta \leq \widetilde{\theta}_i$ and for all $i = 1, 2, 3, ..., n$. From the property of integration, we obtain that

$$\int_{\theta_i}^{\widetilde{\theta}_i} r d\theta \leq \int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta, \quad \text{for all } i = 1, 2, 3, ..., n$$

and hence

$$\int_0^{\omega} r d\theta = \sum_{i=1}^{n} \int_{\theta_i}^{\widetilde{\theta}_i} r d\theta \leq \sum_{i=1}^{n} \int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta \leq m(\widetilde{G}) \leq m(G) \leq m(\widetilde{S}).$$

Hence $\int_0^{\omega} r d\theta \leq m(\widetilde{S})$, for all $\widetilde{S} \in \widetilde{P}$.

For (2), let $\varepsilon > 0$ be given. We show that there exists $m(\widetilde{S})$ in $\ell[\widetilde{P}]$ such that $m(\widetilde{S})$ is less than $\int_0^\omega r d\theta + \varepsilon$. Next, we construct $\widetilde{S} \in \widetilde{P}$ to prove (2). First, we construct the union of perpendicular from boundary points of S to lines in L, D_n, that depends on the case that ∂S corresponds to. Suppose that ∂S corresponds to the case 1.1 of Theorem 4.3, $\angle(l_0, l_X^*) < 90^\circ$ for all $X \in S_1 \setminus \{B, C\}$, and $D_n = \bigcup_{i=1}^{n} d_i$. See Figure 4.18. We consider the cases that can happen.

Case 1. $d_n \notin V(\mathbb{R}^2) \cap P(\overline{S})$. Since ∂S is the boundary of a convex set, $\overline{CX_{n-1}}$ is a line segment in $V(\mathbb{R}^2) \cap P(\overline{S})$. Since $\lim_{n\to\infty} \frac{\omega}{n} = 0$ and X_n is C, X_{n-1} goes to C as n goes to ∞. Therefore, $\lim_{n\to\infty} m(\overline{CX_{n-1}}) = 0$. Since $\lim_{n\to\infty} \sum_{i=1}^{n} m(d_i) = \int_0^\omega r d\theta$,

$$\begin{aligned} \lim_{n\to\infty} \left[\sum_{i=1}^{n-1} m(d_i) + m(\overline{CX_{n-1}}) \right] &= \lim_{n\to\infty} \left[\sum_{i=1}^{n-1} m(d_i) \right] + \lim_{n\to\infty} m(\overline{CX_{n-1}}) \\ &= \int_0^\omega r d\theta. \end{aligned}$$

Case 2. $d_1 \notin V(\mathbb{R}^2) \cap P(\overline{S})$. Since ∂S is the boundary of a convex set, $\overline{BX_1} \in V(\mathbb{R}^2) \cap P(\overline{S})$. Since $\lim_{n\to\infty} \frac{\omega}{n} = 0$ and $X_0 = B$, X_1 goes to B as n goes to ∞. Therefore $\lim_{n\to\infty} m(\overline{BX_1}) = 0$. Since $\lim_{n\to\infty} \sum_{i=1}^{n} m(d_i) = \int_0^\omega r d\theta$,

$$\lim_{n\to\infty} \left[\sum_{i=2}^{n} m(d_i) + m(\overline{BX_1}) \right] = \int_0^\omega r d\theta.$$

Case 3. $d_1, d_n \notin V(\mathbb{R}^2) \cap P(\overline{S})$. From cases 1 and 2, we get that

$$\lim_{n\to\infty} \left[m(\overline{BX_1}) + \sum_{i=2}^{n-1} m(d_i) + m(\overline{CX_{n-1}}) \right] = \int_0^\omega r d\theta.$$

From these three cases, we let S_n be the series such that $S_n = m(\overline{BX_1}) + \sum_{i=2}^{n-1} m(d_i) + m(\overline{CX_{n-1}})$. Then the sequence (S_n) converges to $\int_0^\omega r d\theta$. From the definition of convergent sequence, (S_n) converges to $\int_0^\omega r d\theta$, if for every $\varepsilon > 0$, there exists N in $\mathbb{N}$ such that whenever $n \geq N$ it follows that

$$\begin{aligned} \left| S_n - \int_0^\omega r d\theta \right| &< \varepsilon, \\ \text{or } \int_0^\omega r d\theta - \varepsilon &< S_n < \int_0^\omega r d\theta + \varepsilon. \end{aligned}$$

We choose $n \geq N$ and we get the union of curves $\overline{BX_1} \bigcup \left[\bigcup_{i=2}^{n-1} d_i\right] \bigcup \overline{CX_{n-1}}$. Then we consider d_i such that $d_i \notin V(\mathbb{R}^2) \cap P(\overline{S})$. See Figrue 4.25. From the construction of d_i, $X_i \in l_i \cap S_1$ is the endpoint of d_i. Let $\widetilde{X}_i$ be an endpoint of d_i in l_{i-1}. Then $d_i = \overline{X_i\widetilde{X}_i}$. Since $d_i \notin V(\mathbb{R}^2) \cap P(\overline{S})$, $\widetilde{X}_i \notin S$ and there is a point E_2 in S_2 such that $E_2 \in S_2 \cap d_i$. Since ∂S corresponds to case 1.1 and $d_i \notin V(\mathbb{R}^2) \cap P(\overline{S})$, S_2 divides d_i into two parts such that the first part $\widetilde{d}_{1,i} = \overline{X_iE_2}$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$ but the second part $\widetilde{d}_{2,i} = \overline{E_2\widetilde{X}_i}$ does not. Let l_E be a line in L such that $E_2 \in l_E \cap S_2$ and let E_1 be a point in S_1 such that $E_1 \in l_E \cap S_1$. Let F be a point in S_2 such that $F \in S_2 \cap l_{i-1}$. Next, we find a finite union of curves $\widetilde{D}_i$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that the length is less than or equal to $m\left(\widetilde{d}_{2,i}\right) = \left|\widetilde{d}_{2,i}\right|$ and for all l in L, $l \cap \widetilde{d}_{2,i} \neq \emptyset$ if and only if $l \cap \widetilde{D}_i \neq \emptyset$.

Case 1. $m\left(\overline{E_1F}\right) = \left|\overline{E_1F}\right| \leq \left|\widetilde{d}_{2,i}\right| = m\left(\widetilde{d}_{2,i}\right)$. We choose $\widetilde{D}_i$ is $\overline{E_1F}$. Since $\overline{E_1F}$ connects l_E and l_{i-1}, we have for all l in L, $l \cap \widetilde{d}_{2,i} \neq \emptyset$ if and only if $l \cap \widetilde{D}_i \neq \emptyset$. Then $\widetilde{d}_{1,i} \cup \widetilde{D}_i$ is a finite union of curves in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that $m\left(\widetilde{d}_{1,i} \cup \widetilde{D}_i\right) \leq m\left(\widetilde{d}_{1,i} \cup \widetilde{d}_{2,i}\right) = m(d_i)$ and for all l in L, $l \cap d_i \neq \emptyset$ if and only if $l \cap \left(\widetilde{d}_{1,i} \cup \widetilde{d}_{2,i}\right) \neq \emptyset$.

Case 2. $m\left(\overline{E_1F}\right) = \left|\overline{E_1F}\right| > \left|\widetilde{d}_{2,i}\right| = m\left(\widetilde{d}_{2,i}\right)$. Since S is convex, $\overline{E_2F}$ is a line segment in $V(\mathbb{R}^2) \cap P(\overline{S})$ and we can construct a line segment W in $V(\mathbb{R}^2) \cap P(\overline{S})$ from X_{i-1} to a point in l_E such that W is parallels to $\overline{E_2F}$ and connects l_E and l_{i-1}. Then for all l in L, $l \cap \widetilde{d}_{2,i} \neq \emptyset$ if and only if $l \cap W \neq \emptyset$. In this case, we consider l_{i-1} as the x-axis (transformation of the curves). We define the equation of W and $\overline{E_2F}$ by

$$y = Mx + C_1 \text{ and } y = Mx + C_2,$$

respectively. Then $C_2 < C_1 < 0$. Let ψ be the angle measured counterclockwise from l_{i-1} to l_E, $\angle(l_E, l_{i-1}) = \psi$. Similar to the case 1.1, $\angle(l_0, l_X^*) < 90^{\circ}$ for all $X \in S_1 \backslash \{B, C\}$, we consider subdivisions of the interval $0 \leq \theta \leq \psi$ by values $0 = \theta_0 < \theta_1 < \theta_2 < ... < \theta_k = \psi$ and $\triangle\theta_j = \theta_j - \theta_{j-1}$. Suppose $\theta_j = \frac{j\psi}{k}$, for all $j = 1, 2, ..., k$. Then $\triangle\theta_j = \frac{\psi}{k}$, for all $j = 1, 2, ..., k$. Let $l_{\theta_j} \in L$ such that $l_{\theta_j} \cap W \neq \emptyset$ and $\angle(l_{i-1}, l_{\theta_j}) = \theta_j$, for all $j = 1, 2, ..., k$. Then there is Y_j in W such that $Y_j \in l_{\theta_j} \cap W$, for all $j = 1, 2, ..., k$. Since $Y_j \in W$, for all $j = 1, 2, ..., k$,

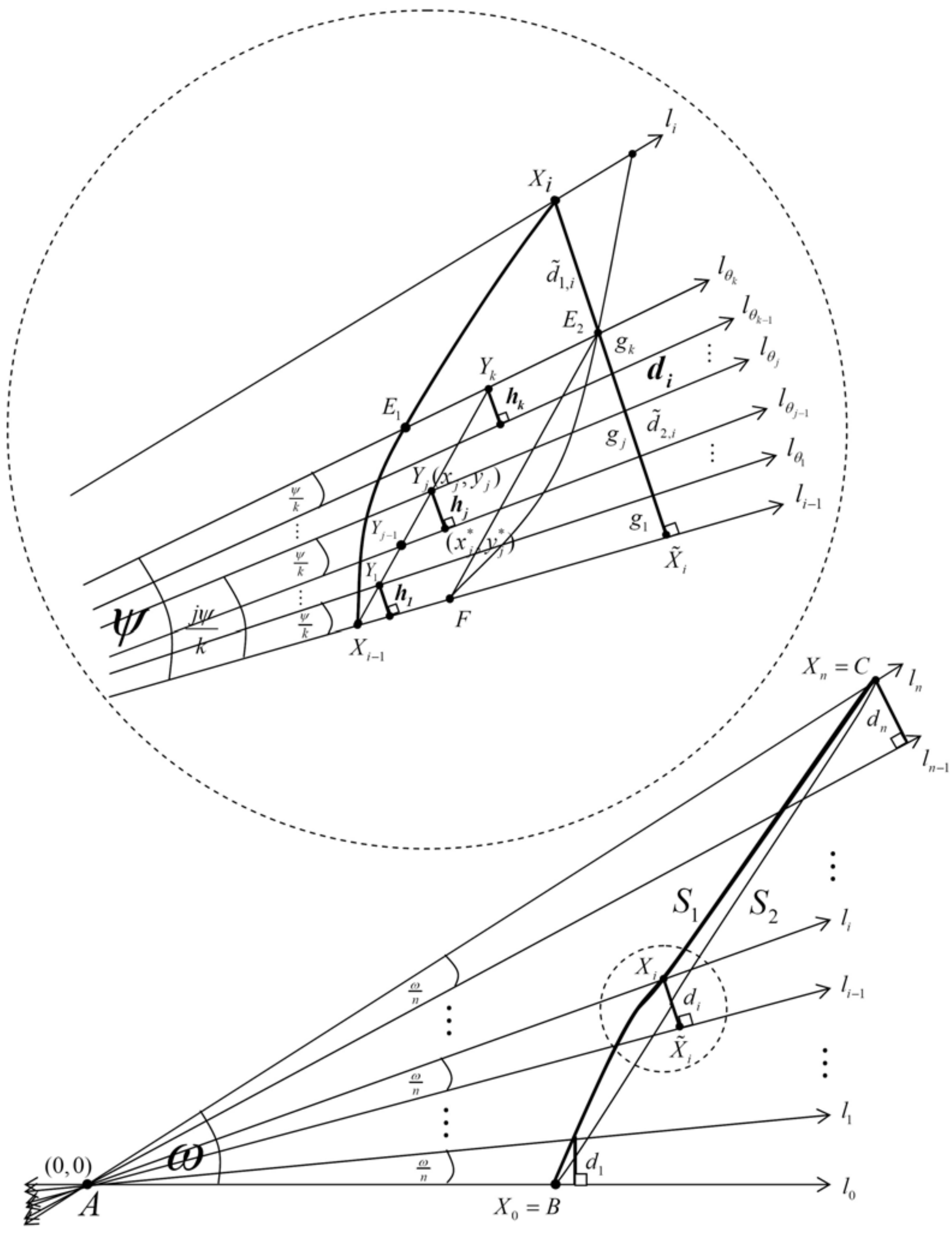


Figure 4.25: d_i is not in $V(\mathbb{R}^2) \cap P(\overline{S})$ and $m(\overline{E_1F}) > m(\widetilde{d_{2,i}})$.

we can define (x_j, y_j) to be the coordinate of Y_j, for all $j = 1, 2, ..., k$. We define the equation of l_{θ_j} by $a_jx + b_jy + c_j = 0$, for all $j = 1, 2, ..., k$. Since l_{θ_j} intersects the y-axis at the origin $A(0,0)$, we get that

$$\begin{aligned} a_jx + b_jy &= 0 \\ y &= m_jx \\ 0 &= m_jx - y, \end{aligned}$$

where $m_j = -\frac{a_j}{b_j} = tan\frac{j\psi}{k}$ is a slope of l_{θ_j}, for all $j = 1, 2, ..., k$. Let h_j be a perpendicular from Y_j (x_j, y_j) to $l_{\theta_{j-1}}$, for all $j = 1, 2, ..., k$. Then the equation of h_j is defined by

$$\begin{aligned} (y - y_j) &= \frac{-(x - x_j)}{m_{j-1}} \\ y &= \frac{-(x - x_j)}{m_{j-1}} + y_j. \end{aligned}$$

We find the intersection point Y_j (x_j, y_j) between W and l_{θ_j} by using $y = Mx + C_1$ and $y = m_jx$. We get that

$$\begin{aligned} m_jx &= Mx + C_1 \\ x &= \frac{C_1}{m_j - M}. \end{aligned}$$

Therefore,

$$x_j = \frac{C_1}{m_j - M} \quad \text{and} \quad y_j = \frac{m_jC_1}{m_j - M}.$$

Then we find the intersection point $\left(x_j^*, y_j^*\right)$ between h_j and $l_{\theta_{j-1}}$. We obtain that

$$\begin{aligned} m_{j-1}x &= \frac{-(x - x_j)}{m_{j-1}} + y_j \\ (m_{j-1})^2 x + x &= x_j + m_{j-1}y_j \\ x_j^* &= \frac{x_j + m_{j-1}y_j}{(m_{j-1})^2 + 1}. \\ \text{and } y_j^* &= \frac{m_{j-1}(x_j + m_{j-1}y_j)}{(m_{j-1})^2 + 1}. \end{aligned}$$

We substitute x_j and y_j into the above equation and obtain that

$$\begin{aligned} x_j^* &= \frac{x_j + m_{j-1}y_j}{(m_{j-1})^2 + 1} = \frac{C_1 + m_{j-1}m_jC_1}{(m_j - M)\left((m_{j-1})^2 + 1\right)} \\ \text{and} \quad y_j^* &= \frac{m_{j-1}(x_j + m_{j-1}y_j)}{(m_{j-1})^2 + 1} = \frac{m_{j-1}(C_1 + m_{j-1}m_jC_1)}{(m_j - M)\left((m_{j-1})^2 + 1\right)}. \end{aligned}$$

From the equation of $\overline{E_2F}$, we suppose that

$$f(x,y) = Mx - y + C_2 = 0.$$

We substitute $\left(x_j^*, y_j^*\right)$ into $f(x,y)$,

$$\begin{aligned} f(x_j^*, y_j^*) &= Mx_j^* - y_j^* + C_2 \\ &= M\left[\frac{C_1 + m_{j-1}m_jC_1}{(m_j - M)\left((m_{j-1})^2 + 1\right)}\right] - \left[\frac{m_{j-1}\left(C_1 + m_{j-1}m_jC_1\right)}{(m_j - M)\left((m_{j-1})^2 + 1\right)}\right] + C_2 \\ &= C_1\left(1 + m_{j-1}m_j\right)\left[\frac{M - m_{j-1}}{(m_j - M)\left((m_{j-1})^2 + 1\right)}\right] + C_2. \qquad (4.1) \end{aligned}$$

Since $\angle(l_{i-1}, l_{\theta_j}) = \theta_j = \frac{j\psi}{k}$ for all $j = 1, ..., k$, we have

$$\begin{aligned} m_{j-1} &= \tan\frac{(j-1)\psi}{k} \\ &= \frac{\tan\frac{j\psi}{k} - \tan\frac{\psi}{k}}{1 + \tan\frac{j\psi}{k}\tan\frac{\psi}{k}} \\ &= \frac{m_j - \tan\frac{\psi}{k}}{1 + m_j\tan\frac{\psi}{k}}. \end{aligned}$$

We obtain that

$$\begin{aligned} &f(x_j^*, y_j^*) \\ &= C_1\left(1 + \left(\frac{m_j - \tan\frac{\psi}{k}}{1 + m_j\tan\frac{\psi}{k}}\right)m_j\right)\left[\frac{M - \frac{m_j - \tan\frac{\psi}{k}}{1 + m_j\tan\frac{\psi}{k}}}{(m_j - m)\left(\left(\frac{m_j - \tan\frac{\psi}{k}}{1 + m_j\tan\frac{\psi}{k}}\right)^2 + 1\right)}\right] + C_2 \\ &= C_1\left(\frac{1 + (m_j)^2}{1 + m_j\tan\frac{\psi}{k}}\right)\left[\frac{\begin{pmatrix} M + Mm_j\tan\frac{\psi}{k} \\ -m_j + \tan\frac{\psi}{k}\end{pmatrix}\left(1 + m_j\tan\frac{\psi}{k}\right)^2}{(m_j - M)\left(1 + m_j\tan\frac{\psi}{k}\right)\begin{pmatrix} (m_j)^2 + \tan^2\frac{\psi}{k} \\ +1 + \left(m_j\tan\frac{\psi}{k}\right)^2\end{pmatrix}}\right] + C_2 \\ &= C_1\left(1 + (m_j)^2\right)\left[\frac{\left(M + Mm_j\tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)}{(m_j - M)\left((m_j)^2 + \tan^2\frac{\psi}{k} + 1 + \left(m_j\tan\frac{\psi}{k}\right)^2\right)}\right] + C_2 \\ &= C_1\left(1 + (m_j)^2\right)\left[\frac{\left(M + Mm_j\tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)}{(m_j - M)\left((m_j)^2 + 1\right)\left(1 + \tan^2\frac{\psi}{k}\right)}\right] + C_2 \\ &= C_1\left[\frac{\left(M + Mm_j\tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)}{(m_j - M)\left(1 + \tan^2\frac{\psi}{k}\right)}\right] + C_2. \qquad (4.2) \end{aligned}$$

If we want $\left(x_j^*, y_j^*\right)$ to be a point inside S for all $j = 1, ..., k$, then

$$f(x_j^*, y_j^*) = C_1 \left[\frac{\left(M + Mm_j \tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)}{(m_j - M)\left(1 + \tan^2\frac{\psi}{k}\right)} \right] + C_2 \le 0.$$

Next, we find $K \in \mathbb{N}$ such that $Mx_j^* - y_j^* + C_2 \le 0$, for all $j = 1, 2, ..., K$. First, we extend the formula for $f(x_j^*, y_j^*)$ to all $j \in \mathbb{R}$ and find the derivative of this function with respect to j. We get that

$$\begin{aligned}
\frac{d\left(f(x_j^*, y_j^*)\right)}{dj} &= \frac{C_1}{\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{\begin{array}{c}(m_j - M)\left(m_j' M \tan\frac{\psi}{k} - m_j'\right) \\ -\left(M + Mm_j \tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)(m_j')\end{array}}{(m_j - M)^2} \right] \\
&= \frac{C_1 m_j'}{\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{\begin{array}{c}(m_j - M)\left(M \tan\frac{\psi}{k} - 1\right) \\ -\left(M + Mm_j \tan\frac{\psi}{k} - m_j + \tan\frac{\psi}{k}\right)\end{array}}{(m_j - M)^2} \right] \\
&= \frac{C_1 m_j'}{\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{\begin{array}{c}m_j M \tan\frac{\psi}{k} - m_j - M^2 \tan\frac{\psi}{k} + M \\ -M - Mm_j \tan\frac{\psi}{k} + m_j - \tan\frac{\psi}{k}\end{array}}{(m_j - M)^2} \right] \\
&= \frac{C_1 m_j'}{\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{-M^2 \tan\frac{\psi}{k} - \tan\frac{\psi}{k}}{(m_j - M)^2} \right],
\end{aligned}$$

where

$$m_j' = \frac{d\left(\tan\frac{j\psi}{k}\right)}{dj} = \frac{\psi}{k}\sec^2\left(\frac{j\psi}{k}\right) = \frac{\psi}{k}\left(1 + \tan^2\left(\frac{j\psi}{k}\right)\right) = \frac{\psi}{k}\left(1 + (m_j)^2\right).$$

Thus

$$\begin{aligned}
\frac{d\left(f(x_j^*, y_j^*)\right)}{dj} &= \frac{C_1 \psi \left(1 + (m_j)^2\right)}{k\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{-M^2 \tan\frac{\psi}{k} - \tan\frac{\psi}{k}}{(m_j - M)^2} \right] \\
&= \frac{(-C_1)\,\psi \left(1 + (m_j)^2\right)}{k\left(1 + \tan^2\frac{\psi}{k}\right)} \left[\frac{(M^2 + 1)\tan\frac{\psi}{k}}{(m_j - M)^2} \right].
\end{aligned}$$

Since $C_1 < 0$, we have

$$\frac{d\left(f(x_j^*, y_j^*)\right)}{dj} > 0.$$

Therefore, $f(x_j^*, y_j^*) = Mx_j^* - y_j^* + C_2$ is an increasing function with respect to j. From Equation 4.1, if we fix $j = 1$, we get that

$$\begin{aligned} f(x_1^*, y_1^*) &= C_1\left(1 + m_{1-1}m_1\right)\left[\frac{M - m_{1-1}}{(m_1 - M)\left((m_{1-1})^2 + 1\right)}\right] + C_2 \\ &= C_1\left(1 + \tan\frac{(1-1)\psi}{k}\tan\frac{\psi}{k}\right)\left[\frac{M - \tan\frac{(1-1)\psi}{k}}{\left(\tan\frac{\psi}{k} - M\right)\left(\left(\tan\frac{(1-1)\psi}{k}\right)^2 + 1\right)}\right] + C_2 \\ &= C_1\left[\frac{M}{\tan\frac{\psi}{k} - M}\right] + C_2. \end{aligned}$$

We take a limit as k goes to infinity,

$$\begin{aligned} \lim_{k\to\infty}\left(Mx_1^* - y_1^* + C_2\right) &= \lim_{k\to\infty}\left(C_1\left[\frac{M}{\tan\frac{\psi}{k} - M}\right] + C_2\right) \\ &= -C_1 + C_2. \end{aligned}$$

Since $C_2 < C_1 < 0$, $\lim_{k\to\infty}(Mx_1^* - y_1^* + C_2) = -C_1 + C_2 < 0$. From Equation 4.2, if $j = k$, we get

$$\begin{aligned} f(x_k^*, y_k^*) &= C_1\left[\frac{\left(M + Mm_k\tan\frac{\psi}{k} - m_k + \tan\frac{\psi}{k}\right)}{(m_k - M)\left(1 + \tan^2\frac{\psi}{k}\right)}\right] + C_2 \\ &= C_1\left[\frac{\left(M + M\left(\tan\frac{k\psi}{k}\right)\tan\frac{\psi}{k} - \left(\tan\frac{k\psi}{k}\right) + \tan\frac{\psi}{k}\right)}{\left(\left(\tan\frac{k\psi}{k}\right) - M\right)\left(1 + \tan^2\frac{\psi}{k}\right)}\right] + C_2. \end{aligned}$$

We take a limit as k goes to infinity,

$$\begin{aligned} \lim_{k\to\infty} f(x_k^*, y_k^*) &= \lim_{k\to\infty} C_1\left[\frac{\left(M + M\left(\tan\frac{k\psi}{k}\right)\tan\frac{\psi}{k} - \left(\tan\frac{k\psi}{k}\right) + \tan\frac{\psi}{k}\right)}{\left(\left(\tan\frac{k\psi}{k}\right) - M\right)\left(1 + \tan^2\frac{\psi}{k}\right)}\right] + C_2 \\ &= C_1\left[\frac{(M - \tan\psi)}{(\tan\psi - M)}\right] + C_2 \\ &= -C_1 + C_2 < 0. \end{aligned}$$

Since $Mx_j^* - y_j^* + C_2$ is an increasing function with respect to j and

$$\lim_{k\to\infty} f(x_k^*, y_k^*) = -C_1 + C_2 < 0,$$

it follows that $f(x_k^*, y_k^*) < 0$ for all j in $\mathbb{N}$ and there exists $K \in \mathbb{N}$ such that $Mx_j^* - y_j^* + C_2 \leq 0$, for all $j = 1, 2, ..., K$. Let $\widetilde{D}_i = \bigcup_{j=1}^{K} h_j$. So $\widetilde{D}_i$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$

and for all l in L, $l \cap \widetilde{d}_{2,i} \neq \emptyset$ if and only if $l \cap \widetilde{D}_i \neq \emptyset$. Let g_j be a line segment such that $g_j \subseteq \widetilde{d}_{2,i}$ and for all l in L, $l \cap g_j \neq \emptyset$ if and only if $l \cap h_j \neq \emptyset$. Then $\widetilde{d}_{2,i} = \bigcup_{j=1}^{K} g_j$ and

$$m(h_j) = |h_j| < |g_j| = m(g_j),$$

for all $j = 1, 2, ..., K$. So

$$m\left(\widetilde{D}_i\right) = \sum_{j=1}^{K} m(h_j) = \sum_{j=1}^{K} |h_j| < \sum_{j=1}^{K} |g_j| = \sum_{j=1}^{K} m(g_j) = m(\widetilde{d}_{2,i}).$$

So we can find a finite union of curves $\widetilde{D}_i$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ where length is less than or equal to $m\left(\widetilde{d}_{2,i}\right) = \left|\widetilde{d}_{2,i}\right|$ and for all l in L, $l \cap \widetilde{d}_{2,i} \neq \emptyset$ if and only if $l \cap \widetilde{D}_i \neq \emptyset$. Then $\widetilde{d}_{1,i} \cup \widetilde{D}_i$ is a finite union of curves in $\overline{S}$ such that

$$m\left(\widetilde{d}_{1,i} \cup \widetilde{D}_i\right) \leq m\left(\widetilde{d}_{1,i} \cup \widetilde{d}_{2,i}\right) = m(d_i)$$

and for all l in L, $l \cap d_i \neq \emptyset$ if and only if $l \cap \left(\widetilde{d}_{1,i} \cup \widetilde{d}_{2,i}\right) \neq \emptyset$.

In these two cases, we let $R_i = \widetilde{d}_{1,i} \cup \widetilde{D}_i$. Then R_i is a finite union of curves in $\overline{S}$ and $m(R_i) \leq m(d_i)$. We choose $\widetilde{S} = \overline{BX_1} \bigcup \left[\bigcup_{i=2}^{n-1} (R_i)\right] \bigcup \overline{CX_{n-1}}$. Then $\widetilde{S}$ is in $\widetilde{P}$ and

$$\begin{aligned}
m\left(\widetilde{S}\right) &= m\left(\overline{BX_1} \bigcup \left[\bigcup_{i=2}^{n-1} (R_i)\right] \bigcup \overline{CX_{n-1}}\right) \\
&= m\left(\overline{BX_1}\right) + m\left(\bigcup_{i=2}^{n-1} (R_i)\right) + m\left(\overline{CX_{n-1}}\right) \\
&= m\left(\overline{BX_1}\right) + \sum_{i=2}^{n-1} m(R_i) + m\left(\overline{CX_{n-1}}\right) \\
&\leq m\left(\overline{BX_1}\right) + \sum_{i=2}^{n-1} m(d_i) + m\left(\overline{CX_{n-1}}\right) \\
&= S_n \\
m\left(\widetilde{S}\right) &< \int_0^{\omega} r d\theta + \varepsilon.
\end{aligned}$$

We get the same result in the other case that ∂S corresponds to. From these two cases, we can conclude that $\int_0^{\omega} r d\theta$ is an infimum of $\ell[\widetilde{P}]$. ■

Now we consider the case

$$\widetilde{P} = \left\{ \widetilde{S} \;\middle|\; \begin{array}{c} \widetilde{S} \text{ is the countable union of curves in } V(\mathbb{R}^2) \cap P(\overline{S}) \\ \text{such that for all } l \text{ in } L, l \cap \widetilde{S} \neq \emptyset \text{ if and only if } l \cap \partial S \neq \emptyset \end{array} \right\}.$$

Each element $\widetilde{S}$ of $\widetilde{P}$ can be written in the form of countable union of curves, $\widetilde{S} = \bigcup_{i=1}^{\infty} \alpha_i$. We show that $\int_0^{\omega} r d\theta \leq m(\widetilde{S})$. Since $\widetilde{S} \in \widetilde{P}$, the first and the last lines from A that intersect $\widetilde{S}$ are l_0 and $\widetilde{l}_0$, respectively. Since for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$, for each point X in $\widetilde{S}$ there is l_θ in L such that X is in $l_\theta \cap \widetilde{S}$ and $\angle(l_\theta, l_0) = \theta \in [0, \omega]$. We can divide $\widetilde{S}$ into countable parts by using l_θ such that in each part there is a curve γ_i in $\widetilde{S}$ connecting two consecutive lines. Let $G = \bigcup_{i=1}^{\infty} \gamma_i$. Then G is a subset of $\widetilde{S}$ and for all l in $L, l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap G \neq \emptyset$. Therefore, $m(G) \leq m(\widetilde{S})$.

Let l_{γ_i} and $\widetilde{l}_{\gamma_i}$ be the first and the last lines from A intersecting γ_i for all i in $\mathbb{N}$. Given that β_i is a line segment connecting the intersection points between l_{γ_i} and $\widetilde{l}_{\gamma_i}$ with γ_i for all i in $\mathbb{N}$ and $\widetilde{G}$ is the union of β_i, $\widetilde{G} = \bigcup_{i=1}^{n} \beta_i$. Since S is a convex set, $\widetilde{G}$ is in $V(\mathbb{R}^2) \cap P(\overline{S})$. Since for all i, $m(\beta_i) \leq m(\gamma_i)$, $m(\widetilde{G}) \leq m(G) \leq m(\widetilde{S})$. Let θ_i be the measure of angle determined by l_0 and l_{γ_i}, $\angle(l_0, l_{\gamma_i}) = \theta_i$, and $\widetilde{\theta}_i$ be the measure of angle determined by l_0 and $\widetilde{l}_{\gamma_i}$, $\angle(l_0, \widetilde{l}_{\gamma_i}) = \widetilde{\theta}_i$, for all i in $\mathbb{N}$. Since β_i is smooth on $\left[\theta_i, \widetilde{\theta}_i\right]$, the length $m(\beta_i)$ of β_i from θ_i to $\widetilde{\theta}_i$ is

$$m(\beta_i) = \int_{\theta_i}^{\widetilde{\theta}_i} \left[(R_i')^2 + (R_i)^2\right]^{\frac{1}{2}} d\theta,$$

where $R_i = R_i(\theta)$ is a polar equation of β_i for all i in $\mathbb{N}$. From the property of integration and $R_i \leq [(R_i')^2 + (R_i)^2]^{\frac{1}{2}}$ for all i in $\mathbb{N}$, we have that

$$\sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta \leq \sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} \left[(R_i')^2 + (R_i)^2\right]^{\frac{1}{2}} d\theta = m(\widetilde{G}).$$

Since any lines from A meet S_1 before $\widetilde{G}$, we have that in the same θ the distance from A to the point in $\widetilde{G}$, $(R_i \cos\theta, R_i \sin\theta)$, is greater that or equal to the distance from A to the point in S_1, $(r\cos\theta, r\sin\theta)$. This implies that

$$r = r(\theta) \leq R_i(\theta) = R_i,$$

$\theta_i \leq \theta \leq \widetilde{\theta}_i$ and for all i in $\mathbb{N}$. From the property of integration, we obtain that

$$\sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} r d\theta \leq \sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} R_i d\theta \leq m(\widetilde{G}) \leq m(G) \leq m(\widetilde{S}). \tag{4.3}$$

Next, we show that $\int_0^{\omega} r d\theta = \sum\limits_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} r d\theta$. In the extended real line $[-\infty, \infty]$, its topology τ is generated by the open sets $(a, b), [-\infty, a), (a, \infty]$ and any union of these sets. In the interval $[0, \infty]$, its topology $\tau_{[0,\infty]}$ is generated by the intersection between $[0, \infty]$ and the elements in τ. Since $r : [0, \omega] \to [0, \infty]$ is polar equation of S_1, r is continuous. Let $\mathcal{M}$ be a power set of $[0, \omega]$. We show that $\mathcal{M}$ is a σ-algebra in $[0, \omega]$.

(i) $[0, \omega] \in \mathcal{M}$.

(ii) Let $E \in \mathcal{M}$. Since $E \subseteq [0, \omega]$, $E^c \subseteq [0, \omega]$ and $E^c \in \mathcal{M}$.

(iii) Let $E = \bigcup\limits_{i=1}^{\infty} E_i$, where $E_i \in \mathcal{M}$ for $i \in \mathbb{N}$. Then $E_i \subseteq [0, \omega]$ for $i \in \mathbb{N}$. So $E = \bigcup\limits_{i=1}^{\infty} E_i \subseteq [0, \omega]$ and $E \in \mathcal{M}$.

So $\mathcal{M}$ is a σ-algebra in $[0, \omega]$ and $[0, \omega]$ is called a measurable space and the members in $\mathcal{M}$ are called the measurable sets. Let V be an open set in $[0, \infty]$, then V is in $\tau_{[0,\infty]}$. Since r is continuous, $r^{-1}(V) \subseteq [0, \omega]$ and $r^{-1}(V) \in \mathcal{M}$. So r is called measurable. Define the sequence E_i by

$$E_i = \begin{cases} \{\omega\}, \ i = 1 \\ [\theta_i, \widetilde{\theta}_i), \ i = 2, 3, 4, ... \end{cases}$$

Since $[\theta_i, \widetilde{\theta}_i) \subseteq [0, \omega]$, $\{E_i\}$ be a sequence of disjoint measurable subsets of $[0, \omega]$ and $\bigcup\limits_{i=1}^{\infty} E_i = [0, \omega] \in \mathcal{M}$. Since $r : [0, \omega] \to [0, \infty]$ is measurable and Theorem 3.9 in real analysis, we obtain that

$$\int_{[0,\omega]} r d\theta = \sum_{i=1}^{\infty} \int_{E_i} r d\theta$$

and then

$$\int_0^{\omega} r d\theta = \sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} r d\theta.$$

From equation 4.3, we obtain that

$$\int_0^{\omega} r d\theta = \sum_{i=1}^{\infty} \int_{\theta_i}^{\widetilde{\theta}_i} r d\theta \leq m(\widetilde{S}).$$

Hence $\int_0^{\omega} r d\theta \leq m(\widetilde{S})$, for all $\widetilde{S} \in \widetilde{P}$.

CHAPTER V

THE NON-CONVEX SHAPES

In this chapter, we consider the non-convex shape problem where the boundary of the give region is a simple closed curve (a closed curve which does not intersect itself). We consider the case of candidates being connected general curves and the case of candidates not necessary being connected curves. Let S be a non-convex set such that ∂S is a simple closed curve.

5.1 Connected solutions of non-convex sets

We consider a non-convex boundary ∂S with one observer when solution candidates are connected general curves in $V(\mathbb{R}^2) \cap P(\overline{S})$. Then the solution $\widetilde{S}$ that we find in this part is a connected general curve. Let l_0 and $\widetilde{l}_0$ be the first and the last lines in L that intersect ∂S, respectively. From **C1**: for all $l \in L$, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$, so $\widetilde{S}$ is the shortest curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ that connects some points in $l_0 \cap \partial S$ and $\widetilde{l}_0 \cap \partial S$. From the preservation of compact set $\bar{S}$ to the set of curves in $V(\mathbb{R}^2) \cap P(\overline{S})$, that connects the first and last lines intersecting S, we obtain that the shortest curve $\widetilde{S}$ exists and

$$m(\widetilde{S}) = \min\{m(\gamma) : \gamma \text{ is a curve from } X \text{ to } Y \text{ where } X \in l_0 \cap \partial S \text{ and } Y \in \widetilde{l}_0 \cap \partial S\}.$$

5.2 General, not necessarily connected solutions for non-convex sets

Let S be a non-convex. Let $l_0, \widetilde{l}_0$ be the first and the last lines from A in L that intersect ∂S. Let ω be the angle between l_0 and $\widetilde{l}_0$, $\angle(l_0, \widetilde{l}_0) = \omega$. Let B be a point in ∂S such that $\left|\overline{AB}\right| = \min\left\{\left|\overline{AX}\right| | X \in l_0 \cap \partial S\right\}$ and C be a point in ∂S such that $\left|\overline{AC}\right| = \min\left\{\left|\overline{AY}\right| | Y \in \widetilde{l}_0 \cap \partial S\right\}$. Then ∂S is divided into two curves with the same endpoints, B and C. Let S_1 and S_2 be these two curves, then $\partial S = S_1 \cup S_2$ and B, C are the endpoints of S_1 and S_2.

Proposition 5.1 *If S is a non-convex n-gon, then there is no solution in the form of finite union of curves in $\overline{S}$.*

Proof. Suppose that S is a non-convex n-gon and let $\gamma : [a,b] \to \mathbb{R}^2$ be a curve in $\overline{S}$. To prove this, we will show that there is a general curve $\widetilde{S}$ in $V(\mathbb{R}^2)\cap P(\overline{S})$ satisfying

1) for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$, and

2) $m(\widetilde{S}) < m(\gamma)$.

Let $L_\gamma = \{l \in L | l \cap \{\gamma\} \neq \emptyset\}$. Suppose $u, v \in [a,b]$ such that l_u and l_v in L_γ are the first and the last lines from A that intersect γ, respectively. Let $\widetilde{\gamma}$ be a subcurve of γ such that $\widetilde{\gamma} : [u,v] \longrightarrow \mathbb{R}^2$. Then $m(\widetilde{\gamma}) \leq m(\gamma)$. We consider $\widetilde{\gamma}$.

Case 1. $\partial S \cap \{\widetilde{\gamma}\} = \emptyset$. Let $D \in \partial S$. We choose a point C in $\{\widetilde{\gamma}\} \backslash \{\widetilde{\gamma}_u, \widetilde{\gamma}_v\}$ and $\varepsilon = \min\left\{m(\overline{CD}), m(\overline{C\widetilde{\gamma}_v}), m(\overline{C\widetilde{\gamma}_v})\right\} > 0$. Let $D(C,\varepsilon)$ be a disk with center at C radius ε and β be a subcurve of $\widetilde{\gamma}$ such that $\beta = D(C,\varepsilon) \cap \{\widetilde{\gamma}\}$. Therefore, $D(C,\varepsilon)$ is a convex subset of S. From Theorem 4.2, we can find a general curve $\widetilde{\beta}$ in $\overline{S}$ satisfying

1) for all l in L, $l \cap \{\widetilde{\beta}\} \neq \emptyset$ if and only if $l \cap \{\beta\} \neq \emptyset$, and

2) $m(\widetilde{\beta}) < m(\beta)$.

We choose $\widetilde{S} = [\{\widetilde{\gamma}\} \backslash \{\beta\}] \cup \widetilde{\beta}$. Then $\widetilde{S}$ is a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$ and $m(\widetilde{S}) < m(\widetilde{\gamma}) \leq m(\gamma)$.

Case 2. $\partial S \cap \{\widetilde{\gamma}\} \neq \emptyset$. Let α be a curve in $\overline{S}$ such that $\{\alpha\} \subseteq \{\widetilde{\gamma}\}$.

Case 2.1. $\{\alpha\} \cap \partial S = \emptyset$, then we can consider α similar to case 1 and we get the same result.

Case 2.2. $\{\alpha\} \subseteq \{\widetilde{\gamma}\} \cap \partial S$. Since S is a non-convex n-gon, There is a subcurve of α which is a line segment and does not contain in any lines from A intersecting ∂S. From the case 2 of Theorem 4.2, we can find a general curve $\widetilde{\alpha}$ in $\overline{S}$ that satisfies

1) for all l in L, $l \cap \{\widetilde{\alpha}\} \neq \emptyset$ if and only if $l \cap \{\alpha\} \neq \emptyset$, and

2) $m(\widetilde{\alpha}) < m(\alpha)$.

Then we choose $\widetilde{S} = [\{\widetilde{\gamma}\} \backslash \{\alpha\}] \cup \widetilde{\alpha}$. So $\widetilde{S}$ is a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$ and $m(\widetilde{S}) \leq m(\gamma)$. ■

In the next proposition, we show that for any curve γ in $V(\mathbb{R}^2) \cap P(S)$, which is not contained in S_1, we can find a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ that intersects all the lines from A intersecting γ but its length is shorter.

Proposition 5.2 *For any curve γ in $V(\mathbb{R}^2) \cap P(S)$, which is not contained in S_1, there is a general curve $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$ and $m(\widetilde{S}) < m(\gamma)$.*

Proof. Let γ be a curve in $V(\mathbb{R}^2) \cap P(S)$ which is not contained in S_1. Then there is a curve $\widetilde{\gamma}$ which is contained in $\{\gamma\} \setminus S_1$. We consider the curve $\widetilde{\gamma}$.

Case 1. $\{\widetilde{\gamma}\}$ is a subset of an interior of S. We obtain the same result with the case 1 of the proof of Proposition 5.1.

Case 2. $\{\widetilde{\gamma}\}$ is a subset of S_2. Then there is an interior point X of S which is a point in the line from A intersecting $\widetilde{\gamma}$ and $|\overline{AX}| < |\overline{AY}|$ where Y is a point in $l_X \cap \widetilde{\gamma}$. From the preservation of compact set ∂S, we construct an open disk $D(X, \varepsilon)$ which is a subset of S. Then we can find a line segment in this disk such that intersects all the lines intersecting a curve, which is subset of $\{\widetilde{\gamma}\}$, and its length is less than this curve.

From these two cases, there is a general curve $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \{\gamma\} \neq \emptyset$ and $m(\widetilde{S}) < m(\gamma)$. ■

Lemma 5.3 *If S_1 is an arc of a circle with center at A, then S_1 is a connected solution of S.*

Proof. Suppose S_1 is an arc of a circle with center at A. From Proposition 5.2, the connected solution of S must be a subset of S_1. Since the solution must satisfy **C1**, S_1 is a connected solution of S. ■

Next, we study the properties of existence of solutions of the non-convex shapes. First, we consider S_1, where any $l \in L$ intersects S_1 in at most a singleton. Let l_X be a straight line in L that intersects S_1 at a point X in S and let l_X^* be a tangent line of S_1 at X. Then we describe S_1 by a polar equation

$$r = g(\theta), \ 0 \leq \theta \leq \omega$$

and the parametric equations

$$x = g(\theta)\cos\theta = r\cos\theta, \text{ and}$$
$$y = g(\theta)\sin\theta = r\sin\theta, \ 0 \le \theta \le \omega.$$

Lemma 5.4 *Let γ be a curve such that A is not a point on γ and any $l \in L$ intersects γ in at most a singleton. If for all X on a curve γ, any lines from A intersecting γ meet γ before the perpendicular line to l_X at X, then γ is an arc of a circle with center at A.*

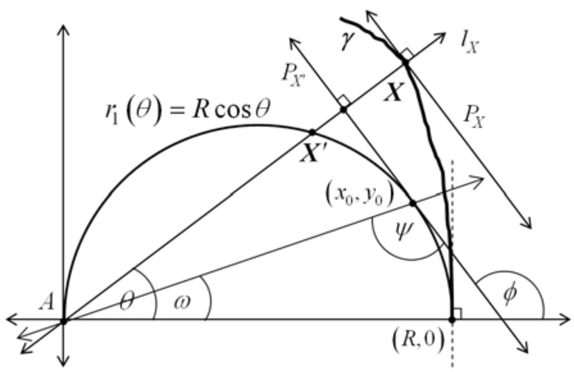


Figure 5.1: The curve r_n, where $r_1(\theta) = R\cos\theta$

Proof. Let γ be a curve such that A is not a point on γ and any $l \in L$ intersects γ in at most a singleton. The polar equation of γ is defined by $r(\theta)$, for all θ in $\left[0, \frac{\pi}{2}\right]$. Let X be a point on the curve γ and positive values of θ be an angle measured counterclockwise from the polar axis to l_X. Suppose that any lines from A intersecting γ meet γ before the line P_X which is perpendicular to l_X at X. We prove that γ is an arc of a circle with center at $A(0,0)$. Let $R = r(0)$. We consider the polar curve

$$r_1(\theta) = R\cos\theta,$$

for all θ in $\left[0, \frac{\pi}{2}\right]$. This curve is a circular arc of radius $\frac{R}{2}$ with center at $\left(\frac{R}{2}, 0\right)$. See Figure 5.1. Let X' be a point in $\{r_1\} \setminus \{A\}$ which has rectangular coordinates $(r_1(\theta)\cos\theta, r_1(\theta)\sin\theta)$. Then X' is an intersection point of l_X to the curve r_1

and the tangent line to $\{r_1\} \setminus \{A\}$ at X' is not perpendicular to l_X except the point $(R, 0)$. So $r(\theta) \geq r_1(\theta)$. Let $P_{X'}$ be a perpendicular line to l_X at some points on l_X such that $P_{X'}$ is a tangent line of the curve r_1 at a point (x_0, y_0). Suppose that an angle ω is measured counterclockwise from the polar axis to the rotated axis passing through the point (x_0, y_0), its expression in polar coordinates

$$x_0 = r_1(\omega)\cos\omega \text{ and } y_0 = r_1(\omega)\sin\omega.$$

We can define the equation of l_X and $P_{X'}$ by

$$\begin{aligned} y &= x\tan\theta, \\ y - y_0 &= \frac{-(x - x_0)}{\tan\theta} \end{aligned}$$

respectively. Then the intersection point of l_X and $P_{X'}$ is

$$\left(\frac{x_0 + y_0\tan\theta}{1+\tan^2\theta}, \frac{\tan\theta\,(x_0 + y_0\tan\theta)}{1+\tan^2\theta}\right).$$

Since the tangent line to the curve γ at a point X on it must be perpendicular to l_X, the distance form A to X is greater than the distance from A to this intersection point and we obtain that

$$\begin{aligned} r^2(\theta) &\geq \left[\frac{x_0 + y_0\tan\theta}{1+\tan^2\theta}\right]^2 + \left[\frac{\tan\theta\,(x_0 + y_0\tan\theta)}{1+\tan^2\theta}\right]^2 \\ &= \frac{(x_0 + y_0\tan\theta)^2(1+\tan^2\theta)}{(1+\tan^2\theta)^2} \\ &= \frac{(x_0 + y_0\tan\theta)^2}{1+\tan^2\theta} \\ &= \left[\frac{x_0 + y_0\tan\theta}{\sec\theta}\right]^2 \\ r^2(\theta) &\geq [x_0\cos\theta + y_0\sin\theta]^2 \end{aligned}$$

and then

$$\begin{aligned} r(\theta) &\geq x_0\cos\theta + y_0\sin\theta \\ &= r_1(\omega)\cos\omega\cos\theta + r_1(\omega)\sin\omega\sin\theta \\ r(\theta) &\geq r_1(\omega)\cos(\theta - \omega). \end{aligned}$$

From $r_1(\theta) = R\cos\theta$, we have

$$r(\theta) \geq R\cos\omega\cos(\theta - \omega).$$

Next, we find the angle ω in the form of θ. Let ϕ be an angle of inclination of tangent line $P_{X'}$ to the curve r_1 at (x_0, y_0) and ψ be an angle measured counter-clockwise from the radial line $r_1(\omega)$ to $P_{X'}$. Then

$$\theta + 90^\circ = \phi = \omega + \psi.$$

Since $r_1(\theta)$ is differentiable, the slope of $P_{X'}$ is equal to $\tan\phi$ and we can find the angle ϕ by the formula

$$\begin{aligned}
\tan\phi &= \frac{dy}{dx} \\
\tan\phi &= \frac{\frac{d(r_1(\theta)\sin\theta)}{d\theta}}{\frac{d(r_1(\theta)\cos\theta)}{d\theta}}.
\end{aligned}$$

We obtain that

$$\begin{aligned}
\tan\phi &= \frac{r_1(\omega)\cos\omega + r_1'(\omega)\sin\omega}{-r_1(\omega)\sin\omega + r_1'(\omega)\cos\omega} \\
\tan\phi &= \frac{R\cos\omega\cos\omega + (-R\sin\omega)\sin\omega}{-R\cos\omega\sin\omega + (-R\sin\omega)\cos\omega} \\
\frac{\sin\phi}{\cos\phi} &= \frac{\cos(2\omega)}{-\sin(2\omega)} \\
\cos\phi\cos 2\omega + \sin\phi\sin 2\omega &= 0 \\
\cos(\phi - 2\omega) &= 0.
\end{aligned}$$

Since the angle ψ is greater than 90° and $\phi = \omega + \psi$, ϕ is greater than 2ω and

$$\begin{aligned}
\phi - 2\omega &= 90^\circ \\
(\theta + 90^\circ) - 2\omega &= 90^\circ \\
\omega &= \frac{\theta}{2}.
\end{aligned}$$

So

$$\begin{aligned}
r(\theta) &\geq R\cos\frac{\theta}{2}\cos\left(\theta - \frac{\theta}{2}\right) \\
r(\theta) &\geq R\cos^2\frac{\theta}{2}.
\end{aligned}$$

Next, we assume that

$$r_2(\theta) = R\cos^2\frac{\theta}{2}$$

and consider r_2 by the same process. We get that

$$r(\theta) \geq R\cos^2\frac{\omega}{2}\cos(\theta-\omega).$$

Since $r_2(\theta)$ is differentiable, the slope of $P_{X'}$ is equal to $\tan\phi$ and we can find the angle ϕ by

$$\begin{aligned}
\tan\phi &= \frac{r_2(\omega)\cos\omega + r_2'(\omega)\sin\omega}{-r_2(\omega)\sin\omega + r_2'(\omega)\cos\omega} \\
\tan\phi &= \frac{R\cos^2\frac{\omega}{2}\cos\omega + \left(-R\cos\frac{\omega}{2}\sin\frac{\omega}{2}\right)\sin\omega}{-R\cos^2\frac{\omega}{2}\sin\omega + \left(-R\cos\frac{\omega}{2}\sin\frac{\omega}{2}\right)\cos\omega} \\
\frac{\sin\phi}{\cos\phi} &= \frac{\cos\left(\frac{3\omega}{2}\right)}{-\sin\left(\frac{3\omega}{2}\right)} \\
\cos\phi\cos\frac{3\omega}{2} + \sin\phi\sin\frac{3\omega}{2} &= 0 \\
\cos\left(\phi - \frac{3\omega}{2}\right) &= 0.
\end{aligned}$$

Since the angle ψ is greater than $90°$ and $\phi = \omega + \psi$, ϕ is greater than $\frac{3\omega}{2}$ and

$$\begin{aligned}
\phi - \frac{3\omega}{2} &= 90° \\
(\theta + 90°) - \frac{3\omega}{2} &= 90° \\
\omega &= \frac{2\theta}{3}.
\end{aligned}$$

So

$$\begin{aligned}
r(\theta) &\geq R\cos^2\frac{\theta}{3}\cos\left(\theta - \frac{2\theta}{3}\right) \\
r(\theta) &\geq R\cos^3\frac{\theta}{3}.
\end{aligned}$$

By Mathematical Induction, suppose $r(\theta) \geq r_n(\theta)$, where $r_n(\theta) = R\cos^n\frac{\theta}{n}$, is true. We show that $r(\theta) \geq r_{n+1}(\theta)$, where $r_{n+1}(\theta) = R\cos^{n+1}\frac{\theta}{n+1}$. From this supposition, we let

$$r_n(\theta) = R\cos^n\frac{\theta}{n}$$

and consider r_n by the same process. We get that

$$r\left(\theta\right) \geq R\cos^n \frac{\omega}{n}\cos\left(\theta-\omega\right).$$

Since $r_n\left(\theta\right)$ is differentiable, the slope of $P_{X'}$ is equal to $\tan\phi$ and we can find the angle ϕ by

$$\begin{aligned}
\tan\phi &= \frac{r_n\left(\omega\right)\cos\omega + r_n'\left(\omega\right)\sin\omega}{-r_n\left(\omega\right)\sin\omega + r_n'\left(\omega\right)\cos\omega}\\
\frac{\sin\phi}{\cos\phi} &= \frac{R\cos^n\frac{\omega}{n}\cos\omega + \left(-R\cos^{n-1}\frac{\omega}{n}\sin\frac{\omega}{n}\right)\sin\omega}{-R\cos^n\frac{\omega}{n}\sin\omega + \left(-R\cos^{n-1}\frac{\omega}{n}\sin\frac{\omega}{n}\right)\cos\omega}\\
\frac{\sin\phi}{\cos\phi} &= \frac{R\cos^n\frac{\omega}{n}\left(\cos\frac{\omega}{n}\cos\omega - \sin\frac{\omega}{n}\sin\omega\right)}{R\cos^n\frac{\omega}{n}\left(-\cos\frac{\omega}{n}\sin\omega - \sin\frac{\omega}{n}\cos\omega\right)}\\
\frac{\sin\phi}{\cos\phi} &= \frac{\cos\left(\frac{n+1}{n}\omega\right)}{-\sin\left(\frac{n+1}{n}\omega\right)}\\
\cos\phi\cos\frac{n+1}{n}\omega + \sin\phi\sin\frac{n+1}{n}\omega &= 0\\
\cos\left(\phi - \frac{n+1}{n}\omega\right) &= 0
\end{aligned}$$

Since the angle ψ is greater than $90°$ and $\phi = \omega + \psi$, ϕ is greater than $\frac{n+1}{n}\omega$ and

$$\begin{aligned}
\phi - \frac{n+1}{n}\omega &= 90°\\
\left(\theta + 90°\right) - \frac{n+1}{n}\omega &= 90°\\
\omega &= \frac{n\theta}{n+1}.
\end{aligned}$$

So

$$\begin{aligned}
r\left(\theta\right) &\geq R\cos^n\frac{\theta}{n+1}\cos\left(\theta - \frac{n\theta}{n+1}\right)\\
r\left(\theta\right) &\geq R\cos^{n+1}\frac{\theta}{n+1}
\end{aligned}$$

and

$$r_{n+1}\left(\theta\right) = R\cos^{n+1}\frac{\theta}{n+1},$$

for all n in $\mathbb{N}$.

Next, we consider the curve $r_n\left(\theta\right)$ for all θ in $\left[0,\frac{\pi}{2}\right]$ and n in $\mathbb{N}$. Let X be any point on the curve r_n and positive values of θ be an angle measured counterclockwise from the polar axis to the rotated axis passing through the

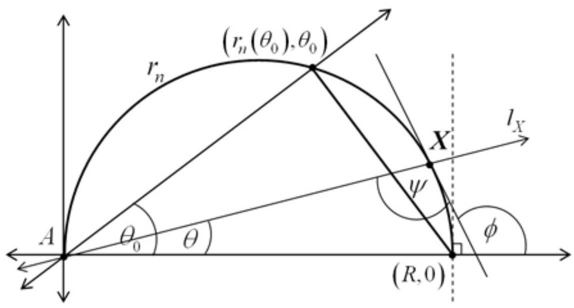


Figure 5.2: The curve r_n is concave down in $\left[0, \frac{\pi}{2}\right]$.

point X. Let ϕ be an angle of inclination of tangent line to the curve r_n at X and θ_0 be a value in $\left(0, \frac{\pi}{2}\right)$. We show that for any θ in $(0, \theta_0)$, l_X meets the line segment connecting the polar coordinates $(r_n(\theta_0), \theta_0)$ and $(R, 0)$ before the curve r_n. See Figure 5.2. To show this, we prove that the curve r_n is concave down in $\left[0, \frac{\pi}{2}\right)$. That means for the slopes of tangent lines $(\tan\phi)$ to increase as θ increases, the tangent lines must rotate counterclockwise as the curve r_n is travelled in the direction of increasing θ. We show that for any θ in $\left(0, \frac{\pi}{2}\right)$, $\frac{d(\tan\phi)}{d\theta} \geq 0$. We calculate the slope of tangent line $(\tan\phi)$ by the formula

$$\tan\phi = \frac{\cos\left(\frac{n+1}{n}\theta\right)}{-\sin\left(\frac{n+1}{n}\theta\right)}.$$

Then we find $\frac{d(\tan\phi)}{d\theta}$,

$$\begin{aligned}
\frac{d(\tan\phi)}{d\theta} &= \frac{\left(\frac{n+1}{n}\right)\sin^2\left(\frac{n+1}{n}\theta\right) + \left(\frac{n+1}{n}\right)\cos^2\left(\frac{n+1}{n}\theta\right)}{\sin^2\left(\frac{n+1}{n}\theta\right)} \\
\frac{d(\tan\phi)}{d\theta} &= \frac{n+1}{n\sin^2\left(\frac{n+1}{n}\theta\right)} \\
\frac{d(\tan\phi)}{d\theta} &> 0.
\end{aligned}$$

So the curve r_n is concave down in $\left[0, \frac{\pi}{2}\right)$ and then for any θ in $(0, \theta_0)$, l_X meets the line segment connecting the polar coordinates $(r_n(\theta_0), \theta_0)$ and $(R, 0)$ before the curve r_n.

To prove that γ is an arc of a circle with center at A, first we find the value of $\lim_{n\to\infty} r_n(\theta)$. See Figure 5.3. We consider the limit n goes to infinity of

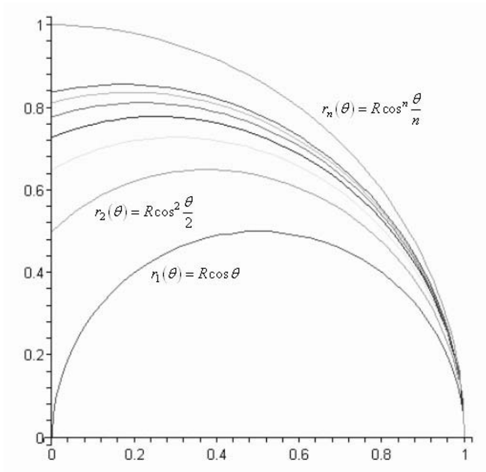


Figure 5.3: The curve r_n when n goes to infinity.

$\log\left(\cos^n \frac{\theta}{n}\right)$, where

$$\lim_{n\to\infty} \log\left(\cos^n \frac{\theta}{n}\right) = \lim_{n\to\infty} \frac{\log\left(\cos \frac{\theta}{n}\right)}{\frac{1}{n}}.$$

Since $\lim_{n\to\infty} \log\left(\cos \frac{\theta}{n}\right) = \lim_{n\to\infty} \frac{1}{n} = 0$, we can apply l'Hôpital's rule, obtaining

$$\begin{aligned}
\lim_{n\to\infty} \log\left(\cos^n \frac{\theta}{n}\right) &= \lim_{n\to\infty} \frac{\frac{1}{\cos\frac{\theta}{n}}\left(-\sin\frac{\theta}{n}\right)\left(-\frac{\theta}{n^2}\right)}{\left(-\frac{1}{n^2}\right)} \\
\log \lim_{n\to\infty} \left(\cos^n \frac{\theta}{n}\right) &= \lim_{n\to\infty} \frac{-\theta\sin\frac{\theta}{n}}{\cos\frac{\theta}{n}} \\
\log \lim_{n\to\infty} \left(\cos^n \frac{\theta}{n}\right) &= 0
\end{aligned}$$

and then

$$\lim_{n\to\infty} \cos^n \frac{\theta}{n} = 1.$$

So

$$r(\theta) \geq \lim_{n\to\infty} r_n(\theta) = \lim_{n\to\infty} R\cos^n \frac{\theta}{n} = R.$$

This implies that $r(\theta) \geq R$ for all $\theta \in [0, \frac{\pi}{2}$. It is obvious that the assertion also holds for negative values of θ. If γ is not an arc of a circle with center at A, then there are θ_1 and θ_2 such that $r(\theta_1) < r(\theta_2)$. This contradict with the result obtained from the above applying process starting at the point $r(\theta_2)$. So $r(\theta)$ is a positive constant and γ is an arc of a circle with center at A. ■

Theorem 5.5 *For a non-convex shape with non empty interior, a connected solution exists if and only if* S_1 *is an arc of a circle with center at* A*.*

Proof. ($\Leftarrow$) Suppose that S_1 is an arc of a circle with center at A, we show that S_1 is the solution. To show this, we prove that S_1satisfies the two-condition property:

1) for all l in L, $l \cap S_1 \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$, and

2) for all G in $V(\mathbb{R}^2) \cap P(\overline{S})$ satisfying 1), $m(S_1) \leq m(G)$.

Since S_1 is a curve connecting l_0 and $\widetilde{l}_0$, S_1 satisfies 1). From Proposition 5.2, S_1 is the solution of S.

($\Rightarrow$) By contrapositive, suppose S_1 is not an arc of a circle with center at A, we show that for any G in $V(\mathbb{R}^2) \cap P(\overline{S})$ we can find $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that $m\left(\widetilde{S}\right) \leq m\left(G\right)$. Suppose S_1 is a continuous mapping from $[0,1]$ to the plane and a is a number in the interval $(0,1)$.

Case 1. Let $K = \left\{t \in [a,1] \mid S_1|_{[a,t]} \text{ is an arc of a circle with center at } A\right\}$ so that the set K is bounded above and $c = \sup K$ exists. Then c is in K and c is not 1.

Case 2. Let $K = \left\{t \in [0,a] \mid S_1|_{[t,a]} \text{ is an arc of a circle with center at } A\right\}$ so that the set K is bounded above and $c = \inf K$ exists. Then c is in K and c is not 0.

From the supposition, case 1 or 2 can happen and we choose W is a point in $S_1 \backslash \{B, C\}$ such that $S_1(c) = W$. Then any curves containing W are not an arc of a circle with center at A. Suppose that α is a curve containing W which is not an arc of a circle with center at A. See Figure 5.4. From the preservation of compact set of $\mathrm{S}_2 \cup \{A\}$, there is a point W_0 in S_2 such that $\left|\overline{WW_0}\right| \leq \left|\overline{AT}\right|$ for all T in $S_2 \cup \{A\}$. Let $D(W, |\overline{WW_0}|)$ be a disk radius $|\overline{WW_0}|$ with center at W and W' be an interior point in $D(W, |\overline{WW_0}|)$ such that W' is a point in l_W and $\left|\overline{AW}\right| < \left|\overline{AW'}\right|$. Then the perpendicular line to l_W at W' intersects the boundary of $D(W, |\overline{WW_0}|)$ two points, U and V. We can construct a disk with center at W which is a subset of triangle AUV. There is a curve γ which is contained in α, lies entirely in this disk, and contains W. Then γ is not an arc of a circle with center at

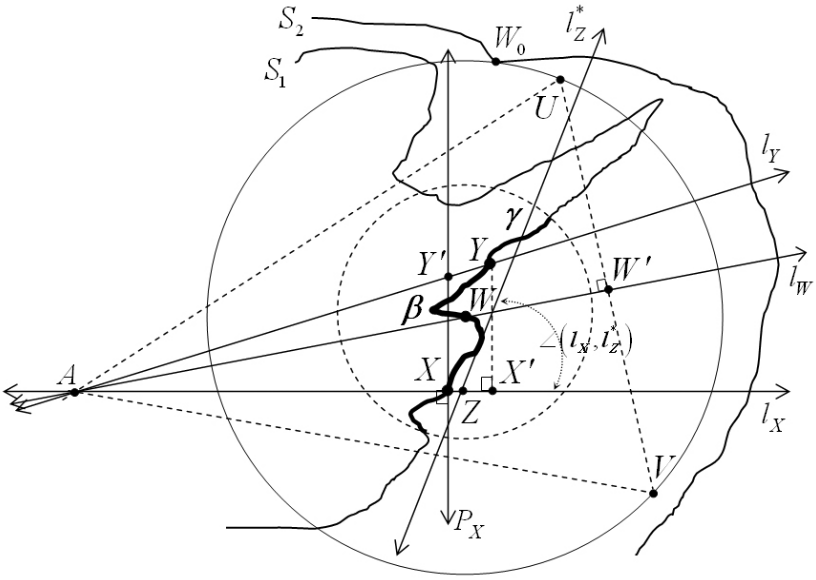


Figure 5.4: Finding the perpendicular line segment inside S for the forward prove of Theorem 5.5.

A. Let X be a point in $\{\gamma\}$, Y be a point in $\{\gamma\}\setminus\{X\}$ and Y' be an intersection point between l_Y and P_X, a perpendicular line to l_X at X. From Lemma 5.4, there are a point X in $\{\gamma\}$ and a point Y in $\{\gamma\}\setminus\{X\}$ such that $\left|\overline{AY}\right| > \left|\overline{AY'}\right|$. We consider a curve β that is contained in γ and X, Y are endpoints of β. Let X' be an intersection point between the perpendicular line from Y to l_X. Then X' is an interior point of S and $\left|\overline{AX}\right| < \left|\overline{AX'}\right|$. We can choose a point Z in a line segment $\overline{XX'}$ which is not X and X'. From the preservation of compact set $\{\beta\}$, at the point Z, we can find a line l_Z^* that intersects β and makes the smallest angle $\angle(l_X, l_Z^*)$ measured counterclockwise from l_X to l_Z^*. Since Y can be the intersection point between β and l_Z^*, and the angle $\angle\left(l_X, \overline{ZY}\right)$ measured counterclockwise from l_X to $\overline{ZY}$ is less than 90°, we have that $\angle(l_X, l_Z^*) < 90^{\circ}$. So we can construct the perpendicular line segment from this intersection point to l_X which is a subset of $\overline{S}$. Then we use the perpendicular line segment to construct a union of curves $\widetilde{S}$ such that $m\left(\widetilde{S}\right) \leq m(S_1)$. From this result and Proposition 5.2, we can conclude that for any G in $V(\mathbb{R}^2) \cap P(\overline{S})$ we can find $\widetilde{S}$ in $V(\mathbb{R}^2) \cap P(\overline{S})$ such that $m\left(\widetilde{S}\right) \leq m(G)$. ■

Next, we consider S_1, where any lines from A intersecting S and S_1 intersect in at least one point.

Corollary 5.6 *For a non-convex shape with nonempty interior, a solution exists if and only if there is $\widetilde{S}$, a countable union of arcs of circles with centers at A in $V(\mathbb{R}^2) \cap P(\overline{S})$, that satisfies*

(i) $\widetilde{S}$ is a subset of S_1,

(ii) for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$,

(iii) any lines from A intersecting $\widetilde{S}$ does intersect $\widetilde{S}$ in either one point or two points which are the endpoints of two distinct arcs, and

(iv) for any point X in $\widetilde{S}$, if l_X intersects $\widetilde{S}$ at X only (X is an interior point of $\widetilde{S}$), then for all Y in $l_X \cap \partial S$, $|\overline{AX}| \leq |\overline{AY}|$.

In fact, any $\widetilde{S}$ satisfying the above mentioned properties is a solution to the problem. See Figure 5.5.

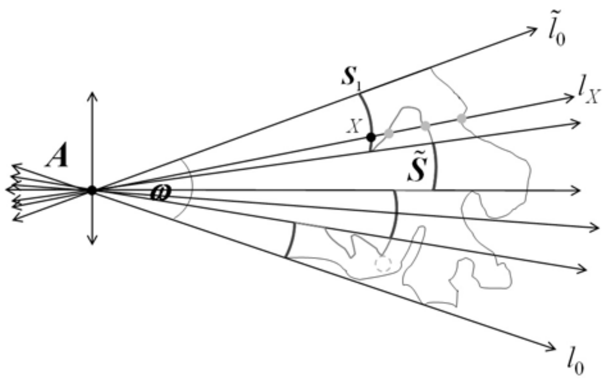


Figure 5.5: Example of $\widetilde{S}$ satisfying all properties of Corollary 5.6.

Proof. ($\Leftarrow$) Suppose there is $\widetilde{S}$, a countable union of arcs of circles with center at A in $V(\mathbb{R}^2) \cap P(\overline{S})$, that satisfies these four properties. We choose $\widetilde{S} = \bigcup_{i=1}^{\infty} \{\alpha_i\}$, where α_i is an arc of a circle with center at A that is contained in S_1 and satisfies other three properties. From $\widetilde{S}$ satisfies (ii), (iii) and (iv), we can suppose that for all l in L, $\widetilde{S}$ and l intersect in at most two points. Let l_{α_i} and

$\widetilde{l}_{\alpha_i}$ be the first and last lines in L intersecting α_i, respectively, for all $i \in \mathbb{N}$. We prove that $\widetilde{S}$ is the solution by showing that $\widetilde{S}$ satisfies the two-condition property. From assumption, $\widetilde{S}$ satisfies **C1**. Let G be a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ that satisfies **C1**. Then we can rewrite G in the form of compact countable union of sets, $G = \bigcup_{i=1}^{\infty} H_i$ such that H_i is an element in $V(\mathbb{R}^2) \cap P(\overline{S})$ and has l_{α_i}, $\widetilde{l}_{\alpha_i}$ to be the first and the last lines in L intersecting H_i, respectively, for all $i \in \mathbb{N}$. From the backward proof of Theorem 5.5, we get that $m(\alpha_i) \leq m(H_i)$ for all $i \in \mathbb{N}$. Therefore, $m(\widetilde{S}) \leq m(G)$.

($\Rightarrow$) By contrapositive, suppose that for all $\widetilde{S}$, which is a countable union of arcs of circles with center at A in $V(\mathbb{R}^2) \cap P(\overline{S})$, $\widetilde{S}$ satisfies either of the following

(i) $\widetilde{S}$ is not a subset of S_1

(ii) there is l in L such that $l \cap \widetilde{S} = \emptyset$ if and only if $l \cap \partial S \neq \emptyset$

(iii) there is a line from A that intersects $\widetilde{S}$ in more than two points

(iv) there is a point X in $\widetilde{S}$ such that l_X intersects $\widetilde{S}$ at X only and there is a point Y in $l_X \cap \partial S$, $|\overline{AX}| > |\overline{AY}|$.

We show that the solution does not exist by showing that for any general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$, we can find a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ which its length is less than the length of the given general curve and intersects all the lines intersecting the given general curve. Let G be a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$. Then we consider G in the following cases.

Case 1. G is not a subset of S_1. From Proposition 5.2, we can find a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ which its length is less than the length of G and intersects all the lines intersecting G.

Case 2. G is a subset of S_1 but G is not countable union of arcs of circles with center at A. If for any lines from A intersecting G, the number of intersection point between each line and G is one, from Theorem 5.5 we can find a general curve which its length is less than the length of G and intersects all the lines intersecting G. If there is a line from A, which intersects G more than one point, then there are two subcurves of G, γ and α, such that for all l in L, $l \cap \{\gamma\} \neq \emptyset$ if and only if $l \cap \{\alpha\} \neq \emptyset$. We obtain that $G \backslash \{\gamma\}$ (or $G \backslash \{\alpha\}$) is a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ which its length is less than the length of G and intersects all the lines intersecting G.

Case 3. G is a subset of S_1 and G is a countable union of arcs of circles with center at A. From the four properties in the supposition about a countable union of arcs of circles with center at A, we consider G in the following cases.

Case 3.1. There is l in L such that $l \cap G = \emptyset$ if and only if $l \cap \partial S \neq \emptyset$. Then G does not satisfy **C1** and it is not the candidate solution.

Case 3.2. There is a line from A that intersects G more than two points. Since G is a countable union of arcs of circles with center at A, there are three arcs of circles with center at A, γ, α and β, such that there is a line from S intersecting these three arcs. Then there are two arcs, γ and α, such that for all l in L, $l \cap \{\gamma\} \neq \emptyset$ if and only if $l \cap \{\alpha\} \neq \emptyset$. We obtain that $G \backslash \{\gamma\}$ (or $G \backslash \{\alpha\}$) is a general curve in $V(\mathbb{R}^2) \cap P(\overline{S})$ which its length is less than the length of G and intersects all the lines intersecting G.

Case 3.3. There is a point X in G such that l_X intersects G at X only and there is a point Y in $l_X \cap \partial S$, $|\overline{AX}| > |\overline{AY}|$. Since G is a countable union of arcs of circles with center at A, there are an arc of a circle γ with center at A containing X and a curve α such that

1) α and γ are contained in S_1,

2) the first and the last lines intersecting α and γ are the same, and

3) for all Z on γ, $|\overline{AZ}| > |\overline{AZ_0}|$, where Z_0 is a point in $l_Z \cap \{\alpha\}$.

Then we can choose a point W in the interior of S which is a point in line l_Z and $|\overline{AZ}| > |\overline{AZ_0}|$, for some Z on γ. From the preservation of compact set ∂S, we construct an open disk $D(W, \varepsilon)$ which is a subset of S. Then we can find a line segment in this disk such that intersects all the lines intersecting an arc β, which is contained in $\{\gamma\}$, and its length is less than β. This implies that we can find a general curve, the union of $G \backslash \{\beta\}$ and this line segment, in $V(\mathbb{R}^2) \cap P(\overline{S})$ which its length is less than the length of G and intersects all the lines intersecting G.

From these three cases, for any general curves in $V(\mathbb{R}^2) \cap P(\overline{S})$ we can find a union of curves which its length is less than the length of general curve and intersects all the lines intersecting the general curve. We can conclude that the solution does not exist. ■

CHAPTER VI

CONCLUSIONS

The purpose of this chapter is to accumulate all the main results that we have proved in Chapter 4 and Chapter 5. In Chapter 4, we let S be a convex set and consider the solution candidates in $V(\mathbb{R}^2) \cap P(\overline{S})$. For the case that solution candidates are connected general curves, the solution is the shortest line segment that connects the first and last lines from A intersecting S. For the case that solution candidates need not be connected general curves, first we prove that there are no solutions in the form of finite union of curves. Then we show that for the union $D_n = \bigcup_{i=1}^{n} d_i$ of perpendiculars from boundary points of S to lines in L, if we take a limit as n goes to infinity of the length $m(D)$ of this union, we obtain that

$$\lim_{n\to\infty} m(D) = \int_0^{\omega} r d\theta,$$

where $r = r(\theta)$ is the equation, in polar form, of S_1 (the front boundary of S as seen from A), and ω is the angle between the first and last lines intersecting S. From this result, we prove that the greatest lower bound of the length of the candidates is equal to $\int_0^{\omega} r d\theta$.

In Chapter 5, we let S be a non-convex set where its boundary is a simple closed curve (and hence S cannot have empty interior). For the case that solution candidates are connected general curves, the solution is the shortest curve that connects the first and last lines from A intersecting S. For the case that solution candidates need not be connected general curves, we obtain the following results.

Lemma 6.1 *If S_1 is an arc of a circle with center at A, then S_1 is a connected solution of S.*

Lemma 6.2 *Let γ be a curve in S such that any $l \in L$ intersects γ in at most a singleton. If for all X on γ, any line from A intersecting γ meets γ before the* *perpendicular line to l_X at X, then γ is an arc of a circle with center at A.*

The next theorem asserts that by the prescribed property of S_1, the solution is S_1 itself. We prove this theorem by using the previous lemmas.

Theorem 6.3 *A connected solution exists if and only if S_1 is an arc of a circle with center at A.*

From the previous theorem, we can make it more general in the following corollary.

Corollary 6.4 *A (not-necessarily-connected) solution exists if and only if there is $\widetilde{S}$, a countable union of arcs of circles with centers at A in $V(\mathbb{R}^2) \cap P(\overline{S})$, that satisfies*

(i) widetildeS is a subset of S_1,

(ii) for all l in L, $l \cap \widetilde{S} \neq \emptyset$ if and only if $l \cap \partial S \neq \emptyset$,

(iii) any lines from A intersecting $\widetilde{S}$ does intersect $\widetilde{S}$ in either one point or two points which are the endpoints of two distinct arcs, and

(iv) for any point X in $\widetilde{S}$, if l_X intersects $\widetilde{S}$ at X only (X is an interior point of $\widetilde{S}$), then for all Y in $l_X \cap \partial S$, $|\overline{AX}| \leq |\overline{AY}|$.

In fact, any $\widetilde{S}$ satisfying the above mentioned properties is a solution to the problem.

REFERENCES

[1] Kenneth A. Brakke. *The Opaque Cube Problem.* The American Mathematical Monthly; 1992.

[2] Martin Gardner. *Gardner's Workout: Training the Mind and Entertaining the Spirit.* AK Peters, Ltd.; 2001.

[3] Bernd Kawohl. *Some Nonconvex Shape Optimization Problems.* Springer Lecture Notes in Math; 2000.

[4] Bernd Kawohl. *Symmetry or Not?.* The Mathematical Intelligencer; 1998.

[5] V. Faber and J. Mycielski. *The Shortest Curve that Meets all the Lines that Meet a Convex Body.* The American Mathematical Monthly; 1986.

[6] Terry Lawson. *Topology: A Geometric Approach.* Oxford University Press Inc.; 2003.

[7] Serge Lang. *Real Analysis.* Addison-Wesley Publishing Company; 1983.

[8] Walter Rudin. *Principles of mathematical analysis.* McGraw-Hill Kogakusha, Ltd.; 1964.

[9] Walter Rudin. *Real and complex analysis.* New York: McGraw-Hill; 1987.

[10] Earl W. Swokowski. *Calculus with Analytic Geometry.* PWS Publishers; 1979.

[11] Pertti Mattila. *Geometry of Sets and Measures in Euclidean Spaces.* Cambridge University Press; 1995.

[12] Seymour Lipschutz. *General Topology.* Schaum Publishing Company; 1965.

[13] Stephen Abbott. *Understanding Analysis.* Springer-Verlag New York, Inc.; 2001.

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