ON GENERALIZATION OF PRIMENESS IN MODULE CATEGORY

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ABSTRACT

Prime ideals play an important role in the study of ring structures, especially in commutative algebras.

Motivating the definition of prime submodules of Sanh et al. in 2008, we introduced and investigated the class of IFP and nearly prime submodules. Using our notions, we generalized the Anderson's Theorem, following that for a finitely generated, quasi-projective, fully IFP module M, which is a self-generator, if every minimal prime submodule over a proper fully invariant submodule U of M is finitely generated, then there are finitely many minimal prime submodules over U.

The main result in this thesis is that a finitely generated right *R*-module is Noetherian if and only if every nearly prime submodule is finitely generated. This can be considered as a generalization of Cohen's Theorem in commutative rings.

KEY WORDS: IFP MODULES / FULLY IFP MODULES
ANDERSON'S THEOREM / NEARLY PRIME SUBMODULES
COHEN'S THEOREM / NOETHERIAN MODULES

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CHAPTER I
INTRODUCTION

Throughout this thesis, all rings are associative with identity and all modules are unitary right R-modules, i.e., for every $m \in M$ we have $m \cdot 1 = m$ with the unit element 1 of the ring R. For special cases, we describe with a precision.

1.1 On Ring and Ideal Theory

Ring theory was firstly founded by the works of Richard Dedekind [26] and David Hilbert [46, 47] at the end of the 19$^{\text{th}}$ century with topics of algebraic integers and commutative rings. The pioneers were Sir William Hamilton [43] with noncommutative algebra of quaternions in 1843, Sir Arthur Cayley [18] with theory of matrices in 1855, Benjamin Peirce [80] with linear associative algebra in 1881. Then the examination of specific rings was continually enriched and followed by abstract theory. In 1908, Wedderburn [90] invented hypercomplex number then contributed to structure theorems for finite dimensional algebras. In 1914, Fraenkel [32] formulated the first abstract definition of ring. The concept of ideals was defined and widely used in the works of Cartan, Molien and Frobenius, but their important applications were found out and developed throughout the works of Wedderburn, Noether and Artin (for instance, see [46,47,77,78] and [79]).

The basic of modern theory of ring was mainly developed in 1920s by Emmy Noether [77,78] and Emil Artin [62,63]. Module structures and techniques which were first used in algebraic number theory have been found useful for further theoretic investigations then shown by Noether to have close connections with both theory of algebras and theory of representations. In the development of noncommutative ring theory, the thinking of representations played a major role and led to an emphasis on irreducible representation as basic building blocks. The Fundamental Lemma of Ring Representation Theory (see [4], Proposition 4.10) said that every ring is isomorphic to a subring of the endomorphism ring of an abelian group. In this direction, one of the most important achievement was first built for semisimple rings by the famous Wedderburn-Artin theorem (see [62], Chapter 1), which is the cornerstone of noncommutative ring theory. In a very natural sense, the most perfect objects in noncommutative ring theory are the division rings, i.e., (non-zero) rings in which each non-zero element has an inverse. From division rings, we can build up matrix rings and form finite direct product of such matrix rings. According to the Wedderburn-Artin theorem, the rings obtained in this way comprise exactly the all importance class of semisimple rings.

The theory of ring structures is enriched by the study of specific ring structures which are constructively conditioned. Finiteness conditions such as Ascending Chain Condition (ACC) and Descending Chain Condition (DCC) are powerful tools in studying some classes of rings. The ascending chain condition (for a class of ideals) which is also called the maximum condition (for a lattice of ideals) was introduced by Dedekind in his researches in algebraic number fields. Then Noether [79] applied it to her researches in her commutative rings which were defined to be Noetherian rings as well as in studying abstract ring with chain conditions. The descending chain condition (also called minimum condition) was applied by many authors that we can mention to Wedderburn in studying structure of algebras, Krull [60,61] with his works in abelian groups with operators and Noether's characterization of Dedekind rings. In 1944, by the joint work of Artin, Nesbitt and Thrall [5], the first systematic study of Artinian modules and rings was offered in the book "Rings with minimum conditions".

Prime ideals play an important role in the structure of ring theory and of major researches. Under suitable chain conditions for a ring, the knowledge of the prime spectrum can lead to the understanding of the whole ring structure. Prime, strongly prime, semiprime, strongly semiprime modules appear very prolific and have been strongly developed in research. In 1929, Krull [60] proved the existence of minimal prime ideals in the commutative case, that every prime ideal contains a minimal prime ideal.

The concepts of semiprime ideals were introduced in the commutative case by Krull in 1929, and by Nagata [74] in 1951 in the noncommutative case. Krull proved that a commutative ring is semiprime if and only if it has no non-zero nilpotent elements. In 1945, Jacobson [53] initiated the general notion of the radical of an arbitrary ring R by definition, the (Jacobson) radical of R, denoted by $Rad(R)$, is the intersection of the maximal left (or maximal right) ideals of R. For rings satisfying an one-sided minimum condition, the Jacobson radical agrees with the classical Wedderburn radical for left Artinian rings. In fact, Wedderburn radical is defined only for certain classes of rings but Jacobson radical is defined for all rings. Although, these are several other kinds of radicals which can be defined for arbitrary rings and which can provide alternative generalizations of the Wedderburn radical but we focus our attention on the Jacobson radical. In 1947, he created the theory of primitive ring and characterized primitive ring by the celebrate Jacobson's Density Theorem. The family of prime ideals was shown very large and profound in structure. The constructing of prime ideals in commutative rings by the Prime Ideal Principle was contributed by many authors such as Krull, Cohen, Kaplansky, Herstein, Issacs, McAdam, Anderson, Lam and Reyes. (See [22, 23, 57, 60–65]).

In 1949, McCoy ([73]) introduced the McCoy radical of a ring as the set of its elements not contained in any multiplicative system, and also proved that the McCoy radical of a ring R equals the intersection of the prime ideals of R. Then McCoy radical coincides with Baer lower radical. Baer lower radical is the smallest ideal N such that R/N has no non-zero nilpotent ideals. In 1951, Levitzky and Nagata independently gave a criterion for semiprime ideals in noncommutative rings. Nagata also studied the finiteness of the set of minimal prime ideals along with its ring structure satisfying the ACC on semiprime ideals.

1.2 On Primeness of Modules and Submodules

Prime submodules and prime modules have been appeared in many contexts. Modifying the structure of prime ideals, many authors want to transfer this notion to right or left modules over an arbitrary associative ring. By an adaptation of basic properties of prime ideals, some authors introduced the notion of prime submodules and prime modules and studied their structures. However, these notions are valid in some cases of modules over a commutative ring such as multiplication modules, but for the case of non-commutative rings, nearly we could not find something similar to the structure of prime ideals.

In 1961, Andrunakievich and Dauns [24], [68] first introduced and investigated prime modules following that, a left R-module M is called *prime* if for every ideal I of R, and every element $m \in M$ with $Im = 0$, implies that either $m = 0$ or $IM = 0$.

In 1975, Beachy and Blair [12,13] proposed another definition of primeness, for which a left R-module M is called a *prime module* if $(0 :_R M) = (0 :_R N)$ for every non-zero submodule N of M.

In 1978, Dauns [24, 68] defined that a module M is a *prime module* if $(0 :_R M) = A(M)$, where $A(M) = \{a \in R \mid aRm = 0, m \in M\}$. For the class of submodules, he also created the definitions of prime submodules and semiprime submodules. A submodule P of a left R-module M is called a *prime submodule* if for any element $r \in R$ and any element $m \in M$ such that $rRm \subset P$, then either $m \in P$ or $r \in (P :_R M)$, and a submodule N of M is called a *semiprime submodule* if $N \neq M$ and for any elements $r \in R$ and $m \in M$ such that $r^n m \in N$, then $rm \in N$.

Following Bican [16], we say that a left R-module M is *B-prime* if and only if M is cogenerated by each of its non-zero submodules. It is easy to see that B-prime implies prime. In [91], it is pointed out that M is B-prime if and only if $L \cdot Hom_R(M, N) \neq 0$ for every pair L, N of non-zero submodules of M.

In studying noncommutative ring structures, Goodearl and Warfield in 1983 [40] and McConnel and Robson in 1987 [72] called a left R-module M

a prime module if for any non-zero submodule N of M, $ann_R(M) = ann_R(X)$. The authors also defined that a left R-module M is called a prime module if it is a fully faithful module over the ring $R/r_R(M)$, i.e., every submodule and the module itself are faithful as modules over the ring $R/r_R(M)$.

In 1983, Wisbauer [91, 92] investigated the structure of Wisbauer category $\sigma[M]$. He called it *the full subcategory of Mod-R* whose objects are M-generated modules, i.e., modules which are isomorphic to submodules of M-generated modules. Following him, M is called a *strongly prime module* if M is subgenerated by any of its non-zero submodules, i.e., for any non-zero submodule N of M, the module M belongs to $\sigma[N]$, or equivalently, for any $x, y \in M$, there exists a set of elements $\{a_1, \cdots, a_n\} \subset R$ such that $ann_R\{a_1x, \cdots, a_nx\} \subset ann_R\{y\}$.

In 1984, Lu [69] defined that for a left R-module M and a submodule X of M, an element $r \in R$ is called a *prime* to X if $rm \in X$ implies $m \in X$. In this case, $X = \{m \in M \mid rM \subset X\} = (X : r)$. Then X is called a *prime submodule* of M if for any $r \in R$, the homothety $h_r : M/X \to M/X$ defined by $h_r(\overline{m}) = \overline{m}r$, where $\overline{m} \in M/X$ is either injective or zero. This implies that $(0 : M/X)$ is a prime ideal of R and the submodule X is called a *prime submodule* if for $r \in R$ and $m \in M$ with $rm \in X$ implies either $m \in X$ or $r \in (X : M)$.

In 1993, McCasland and Smith [68, 70, 71] gave a definition that a submodule P of a right R-module M is called a *prime submodule* if for any ideal I of R and any submodule X of M with $IX \subset P$, then either $IM \subset P$ or $X \subset P$.

In 2002, Ameri [2] and Gaur, Maloo, Parkash [34, 35] examined the structure of prime submodules in multiplication modules over commutative rings. Following them, we say a left R-module M a *multiplication module* if every submodule X is of the form IM for some ideal I of R and M is called a *weak multiplication module* if every prime submodule of M is of the form IM for some ideal I of R. Although, multiplicative ideal theory of rings was first introduced by Dedekind and Noether in the 19^{th} century, multiplication modules over commutative rings were newly created by Barnard [9] in 1980 to obtain a module structure which behaves like rings. The structure of multiplication modules over noncommutative rings was first studied by Tuganbaev [89] in 2003.

In 2004, Behboodi and Koohy [14] defined weakly prime submodules. Following them, a submodule P of a module M is called a *weakly prime submodule* if for any ideals I, J of R and any submodule X of M with $IJX \subset P$, then either $IX \subset P$ or $JX \subset P$.

In 2008, Sanh [82] proposed a new definition of prime submodules. Let R be a ring and M, a right R-module, and S, its endomorphism ring. A fully invariant submodule X of M is called a *prime submodule* if for any ideal I of S and any fully invariant submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. A fully invariant submodule is called *semiprime* if it is an intersection of prime submodules. A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Consequently, the ring R is semiprime ring if R_R is a semiprime module. By symmetry, the ring R is a semiprime ring if $_RR$ is a semiprime left R-module.

In 2008, Sanh et al. [83] introduced the concepts of M-annihilator and of Goldie modules as a generalization the concept of Goldie ring. Following that definition, a right R-module M is called a *Goldie module* if M has finite Goldie dimension and satisfies the ascending chain condition for M-annihilators. A ring R is a *right Goldie ring* if R_R is Goldie as a right R-module. It is equivalent to say that a ring R is a *right Goldie ring* if it has finite right Goldie dimension and satisfies the ascending chain condition for right annihilators. By using some properties of prime modules and Goldie modules, our research group studied the class of prime Goldie modules.

1.3 On Rings with Insertion Factor Property

With the help of [66, Proposition 3.2.1], the set of nilpotent elements form an ideal that coincides with the prime radical in a commutative ring. This property is also possessed by certain noncommutative rings, which is called *2-primal*. Shin [86] proved that given a ring R, $Rad(R)$ coincides with the set of all nilpotent elements of R if and only if R/P is a domain for every minimal prime ideal P of R (i.e. P is completely prime): Birkenmeier et al. [17] called such rings 2-primal; while Hirano [49] used the term *N-ring* for this concept.

A well-known property between "commutative" and "2-primal" is the *insertion-of-factors-property* (or simply *IFP*) due to Bell [15]. A right (or left) ideal I of a ring R is said to have the IFP if $ab \in I$ implies $aRb \subset I$ for $a, b \in R$. A ring R is called IFP if the zero ideal of R has the IFP.

Many researchers studied IFP rings and called it by many terms. For example, Shin [86] used the term SI for the IFP; while Habeb [42] used the term zero insertive (or simply *zi*) for it, in the study of QF-3 rings. IFP rings are also known as semicommutative in Narbonne's paper [75]. They also investigated the relationship between IFP rings and others, such as: Reduced rings $\Rightarrow$ Symmetric rings $\Rightarrow$ Reversible rings $\Rightarrow$ IFP rings $\Rightarrow$ Abelian rings.

Weakly semicommutative rings, homomorphically rings, g-IFP rings, nil-IFP rings, central semicommutative rings were introduced as the generalization of IFP rings. The property and the relationship between these kinds of rings and others as Baer rings, Armendariz rings, polynomial rings were also studied. Especially, homomorphically rings was applied to generalize the Anderson's theorem on noncommutative ring.

In 2003, N. K. Kim and Y. Lee [58] extended the class of *semicommutative* rings in "Extensions of reversible rings". They showed that reversible rings are semicommutative rings, but the converse is not true (i.e. there is a nonreversible semicommutative ring).

In 2006, M. Baser and N. Agayev [10] extended semicommutative rings to semicommutative modules. They also investigated the relationship among reduced, semicommutative and principally quasi-Baer module.

In 2007, N. Agayev and A. Harmanci [1] continued to give properties of semicommutative rings and modules related to Baer and Armendariz modules.

During the same year, other researchers, L. Liang, L. Wang and Z. Liu [67] introduced the concept of weakly semicommutative rings which are generalization of semicommutative rings. They also studied the relationship between semicommutative rings and weakly semicommutative rings. From their results, they showed that weakly semicommutative rings may not be semicommutative rings.

Also in this year, S. U. Hwang, Y. C. Jeon, and K. S. Park [52] gave the definition of g-IFP rings and studied the basic properties. Furthermore, they showed that from any IFP rings there can be constructed a right g-IFP ring but not IFP.

In commutative algebra, Anderson's Theorem [3] is stated that there are only finitely many prime ideals minimal over I whenever every prime ideal minimal over I is finitely generated. In order to extend the class of rings that satisfies this condition to noncommutative rings, in 2008, C. Huh, N. K. Kim, and Y. Lee [50] introduced concept homomorphically IFP rings.

As a generalization of α-rigid rings as well as an extension of semicommutative rings, the notion of an α-semicommutative ring with the endomorphism α of ring R was introduced by M. Baser, A. Hamrnci and T. K. Kwak [11], in 2008. In that paper, various results of semicommutative rings was extended to α-semicommutative rings and their related properties were obtained.

Continuing this work, in 2010, M. Baser and T. K. Kwak [10] investigated more properties of α-semicommutative rings and gave the relationship between extended Armendariz rings and α-semicommutative rings and several known results were obtained as consequences of their results.

Let R be a ring with identity, M is a right R-module and $S = End(M)$. Then M is a left S-module, right R-module and $S - R-$bimodule. From this, in 2009, N. Agayev, T. Ozen and A. Harmanci [1] studied on a class of semicommutative modules. They introduced S-semicommutative, as well as S-Baer, S-quasi-Baer and S-principally quasi -Baer modules and studied the relations between those modules. Furthermore, in 2011, they continued to give one more a generalization of the class of semicommutative rings, which was called central semicommutative.

In 2010, A. O. Atagun [6] studied properties of IFP ideals in near-rings which is extended to the ideals in near-rings. The relation between prime ideals and IFP-ideals are investigated.

In 2012, J. Baek, W. Chin, J. Choi, T. Eom, J.C. Jeon, and Y. Lee [8] studied the properties of IFP on nilpotent elements by giving the concept of *nil-IFP* rings that was also a generalization of NI-rings. The class of minimal non-commutative nil-IFP rings were determined definitely, up to isomorphism where the minimal means having smallest cardinality.

For the structure of this thesis, Chapter I dealt with the early history of commutative and noncommutative ring theory. The notion of primeness in module category and the insertion factor property on rings of different authors are also presented in this chapter. All the essential basic notions, examples and their properties are given in Chapter II. Our recent results on primeness in module category are also provided. Chapter III dealt with the modules which have Insertion Factor Property (briefly, IFP Modules). In this chapter, we introduce the definition of IFP modules and fully IFP modules. The equivalently properties of IFP module are provided. Then we generalize the Anderson's Theorem to fully IFP modules. Chapter IV provides the definition of nearly prime submodules as a generalization of prime submodule. The relation of nearly prime and nearly strongly prime are given in this chapter. There are also given important result that can be considered as a characterization of Noetherian modules. Finally, we review and conclude the results in Chapter V.

CHAPTER II

BASIC KNOWLEDGE

Let R be an arbitrary ring and Mod-R, the category of all right R-modules. The notation M_R indicates a right R-module M. The set $\operatorname{Hom}(M, N)$ denotes the set of right R-module homomorphisms between two right R-modules M and N and if further emphasis is needed, the notation $\operatorname{Hom}_R(M, N)$ is used. For a right R-module M, we denote $S = \operatorname{End}_R(M)$ for its endomorphism ring. A submodule X of M is denoted by writing $X \leq M$. Also $I \leq R_R$ means that I is a right ideal of R and $I \leq {}_RR$ that I is a left ideal. The notation $I \leq R$ is reserved for two-sided ideals. The symbol $\subsetneq$ is reserved for proper inclusion with $\subset$ indicating the inclusion. As usual, $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ represent the sets of natural, integers, rational, real and complex numbers. For a prime number p, the Prüfer group Z_{p^∞} denotes the abelian group $\{q \in \mathbb{Q} \mid qp^n \in \mathbb{Z} \text{ for some } n \in \mathbb{N}\}$. The result in this chapter can be found in [4], [20], [56], [62], [63], [83], [85], [82], [91].

2.1 Preliminaries

Before dealing with deeper results on the structure of rings with the help of module theory, we provide first some essential elementary definition, examples and properties.

Definition 2.1.1. A right R-module M is said to be *finitely generated* if there exists a finite set of generators for M, or equivalently, if there exists an epimorphism $R^n \longrightarrow M$ for some $n \in \mathbb{N}$. In particular, M is *cyclic* if it is generated by a single element, or equivalently, if there exists an epimorphism $R \longrightarrow M$. It follows that M is cyclic if and only if $M \cong R/I$ for some right ideal I of R.

For example, let M be a right R-module and $m \in M$. Then m generates a cyclic submodule mR of M. There is an epimorphism $f : R \longrightarrow mR$ given by $f(r) = mr$ and $Ker(f) = \{r \in M \mid mr = 0\}$, which is a right ideal of R. Hence $mR \cong R/Ker(f)$

Lemma 2.1.2 ([87]). *Let X be a submodule of a right R-module M.*

(1) If M is finitely generated, then so is M/X.

(2) If X and M/X are finitely generated, then so is M.

Theorem 2.1.3 ([56]). *A right R-module M is finitely generated if and only if for any family $\{A_i : i \in I\}$ of submodules $A_i \subsetneq M$ with $\sum\limits_{i\in I} A_i = M$, there exits a finite subfamily $\{A_i : i \in I_0\}$ where $I_0 \subset I$ and I_0 is finite, such that $\sum\limits_{i\in I_0} A_i = M$.*

Definition 2.1.4. A right R-module M is said to be *finitely cogenerated* if and only if for any family $\{A_i : i \in I\}$ of submodules $A_i \subsetneq M$ with $\bigcap\limits_{i\in I} A_i = 0$, there exists a finite subfamily $\{A_i : i \in I_0\}$ where $I_0 \subset I$ and I_0 is finite, such that $\bigcap\limits_{i\in I_0} A_i = 0$.

For example, the $\mathbb{Z}$-module $\mathbb{Z}_{\mathbb{Z}}$ is finitely generated but not finitely cogenerated. On the other hand, the *Prüfer group* Z_{p^∞} is finitely cogenerated. A vector space V over a field K is finitely cogenerated if and only if it is finite dimensional.

Definition 2.1.5. A ring R is called *simple* (or *irreducible*) if $R \neq 0$ and R has precisely two nontrivial ideals 0 and R. A simple ring can have many nontrivial left or right ideals.

A module M is said to be *simple* if $M \neq 0$ and the only submodules of M are 0 and M. Every simple module is cyclic; in fact, it is generated by any nonzero element $x \in M$. If M is a simple module, then $S = End_R(M)$ is a division ring. A *maximal submodule* X of M is a proper submodule of M such that for any submodule Y of M, if $X \subsetneq Y \subsetneq M$ then either $Y = X$ or $Y = M$.

The submodule X is maximal if and only if M/X is simple. A right

R-module M is simple if and only if $M \cong R/I$ for some maximal right ideal I of R.

Definition 2.1.6. A submodule X of M is called a *direct summand* of M if there exists a submodule Y of M such that $X + Y = M$ and $X \cap Y = 0$. We denote $M = X \oplus Y$ and say that M is a direct sum of X and Y.

The direct summands of R_R correspond to idempotent elements of R (i. e. $e \in R$ such that $e^2 = e$). For any such element $e \in R$, we get a direct sum decomposition $R = eR \oplus (1-e)R$. Conversely, if $R = I \oplus J$ with any right ideals I and J, then we can write $1 = e + f$ with $e \in I$, $f \in J$ which gives $e - e^2 = fe \in I \cap J = 0$, so $e = e^2$. For each $a \in I$, we similarly get $a = ea \in eR$. So $I = eR$ and $J = (1-e)R$.

Definition 2.1.7. A right R-module M is called *indecomposable* if it cannot be decomposed into a direct sum of non-zero submodules, that is, if $M = X \oplus Y$ then either $X = 0$ or $Y = 0$. A short exact sequence $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ is called *split exact* if $Im(f)$ is a direct summand of B. The following theorem gives some properties of split exact sequences.

Theorem 2.1.8 ([87], page 9)**.** *Let* $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ *be a short exact sequence. Then the following properties are equivalent:*

(1) The sequence splits;

(2) There exists a homomorphism $f' : B \longrightarrow A$ *such that* $f' \circ f = 1_A$*;*

(3) There exists a homomorphism $g' : C \longrightarrow B$ *such that* $g \circ g' = 1_C$*.*

Definition 2.1.9. A ring R is called a *reduced* ring if it has no non-zero nilpotent elements.

Example 2.1.10.

(1) The ring of integers $\mathbb{Z}$ is reduced ring.

(2) Every integral domain is a reduced ring.

(3) Subrings, products and localizations of reduced rings are again reduced rings.

Definition 2.1.11. A ring R is called an *abelian ring* if its idempotents are central.

Definition 2.1.12. A module M_R is said to be *reduced* if for any $m \in M$ and any $a \in R, ma = 0$ implies $mR \cap Ma = 0$.

Definition 2.1.13.

(1) A ring R is called a *Baer ring* if the right (left) annihilator of every nonempty subset is generated by an idempotent.

(2) A ring R is called a *quasi-Baer* if the right (left) annihilator of each right (left) ideal of R is generated (as a right (left) ideal) by an idempotent.

(3) A ring R is called a *right (left) principally quasi-Baer* (or, simply, *right (left) p.q.-Baer*) ring if the right (left) annihilator of a principally right (left) ideal of R is generated by an idempotent. R is called a *p.q.-Baer* ring if it is both right and left p.q.-Baer.

(4) A ring R is called a *right (left) p.q.-ring* if the right (left) annihilator of an element of R is generated by an idempotent. R is called *p.q.-ring* if it is both right and left p.q.-ring.

Definition 2.1.14.

(1) A right R-module M is called a *Baer-module* if for any subset X of $M, r_R(X) = eR$ where $e^2 = e \in R$.

(2) A right R-module M is called *quasi-Baer* if for any submodule N of $M, r_R(N) = eR$ where $e^2 = e \in R$.

(3) A right R-module M is called *p.q.-module* if for any $m \in M, r_R(m) = eR$ where $e^2 = e \in R$.

Theorem 2.1.15 ([3], Anderson's Theorem). *Let R be a commutative ring with identity, and let $I \neq R$ be an ideal of R. If every prime ideal minimal over I is finitely generated, then there are only finitely many prime ideals minimal over I.*

2.2 Generators and Cogenerators

Generators and cogenerators are notions in category of modules. They play an important role in Module Theory and in some categories. Below we will review these notions.

Definition 2.2.1.

(a) A module B_R is called a *generator* for Mod-R, if

$$\forall M \in \text{Mod-}R\ [M = \sum_{\varphi \in Hom_R(B,M)} \text{Im}\varphi].$$

(b) A module C_R is called a *cogenerator* for Mod-R, if

$$\forall M \in \text{Mod-}R\ [0 = \bigcap_{\varphi \in Hom_R(M,C)} \text{Ker}\varphi].$$

For any module B and M,

$$\text{Im}(B, M) = \sum_{\varphi \in Hom_R(B,M)} \text{Im}\varphi$$

is itself, as a sum of submodules in M, a submodule of M. The property that B is a generator for Mod-R means that for any right R-module M, $\text{Im}(B, M)$ is as large as possible for every M and so equals M.

For any modules C and M,

$$\text{Ker}(M, C) = \bigcap_{\varphi \in Hom_R(M,C)} \text{Ker}\varphi$$

is itself, as an intersection of submodules of M, a submodule of M. The property that C_R is a cogenerator for Mod-R means that $\text{Ker}(M, C)$ is as small as possible for every M and so equals 0.

An R-module M is called a *self-generator (self-cogenerator)* if it generates all its submodules (cogenerator all its factor modules). A submodule of an M-generated module is called M-subgenerated if it is isomorphic to a submodule of an M-generated module.

Following [91], a subcategory $\mathcal{C}$ of $Mod - R$ is *subgenerated* by M, or that M is a *subgenerator* for $\mathcal{C}$, if every object in $\mathcal{C}$ is M-subgenerated. We denote by $\sigma[M]$ the full subcategory of $Mod - R$ whose objects are all M-subgenerated R-modules.

Corollary 2.2.2.

(a) If B is a generator and A is a module such that $\mathrm{Im}(A, B) = B$, *then A is also a generator;*

(b) Every module M such that there is an epimorphism from M to R_R is also a generator;

(c) If C is a cogenerator and D is a module such that $\mathrm{Ker}(C, D) = 0$, *then D is also a cogenerator.*

Generators and cogenerators can be characterized in the following theorem by properties of homomorphisms.

Theorem 2.2.3.

(a) B is a generator $\Leftrightarrow \forall \mu \in \mathrm{Hom}_R(M, N)$, $\mu \neq 0$, $\exists \varphi \in \mathrm{Hom}_R(B, M)$: $\mu\varphi \neq 0$.

(b) C is a cogenerator $\Leftrightarrow \forall \lambda \in \mathrm{Hom}_R(L, M)$, $\lambda \neq 0$, $\exists \varphi \in \mathrm{Hom}_R(M, C)$: $\varphi\lambda \neq 0$.

2.3 Injectivity, Projectivity and Generalizations

Injective modules may be regarded as modules that are "complete" in the following algebraic sense: Any "partial" homomorphism (from a submodule of a module B) into an injective module A can be "completed" to a "full" homomorphism (from all of B) into A.

Injective modules first appear in the context of abelian groups. The general notion for modules was first investigated by Baer in 1940. The theory of these modules was studied long before the dual notion of projective modules was considered. The "injective" and "projective" terminology invented in 1956 by Cartan and Eilenberg.

Definition 2.3.1. Let M be a right R-module.

(1) A submodule N of M is called *essential* or *large* in M if for any submodule X of M, $X \cap M = 0 \Rightarrow X = 0$. If N is essential in M and we denote $N \subsetneq^{*} M$.

(2) A submodule N of M is called *superfluous* or *small* in M if for any submodule X of M, $N + X = M$, then $X = M$. In this case we write $N \subsetneq^{\circ} M$.

(3) A right ideal I of a ring R is called an *essential right ideal* of R if it is essential in R_R as a right R-module. Similarly, a right ideal I of a ring R is called a *superfluous right ideal* of R if it is superfluous in R_R as a right R-module.

(4) A homomorphism $\alpha : M_R \to N_R$ is called *essential* if $\mathrm{Im}\alpha \subsetneq^{*} N$. The homomorphism α is called *superfluous* if $\mathrm{Ker}\alpha \subsetneq^{\circ} M$.

Remark From the definition, we have the following:

(1) $A \subsetneq^{\circ} M \Leftrightarrow \forall U \subsetneqq M$, $A + U \subsetneqq M$.

(2) $A \subsetneq^{*} M \Leftrightarrow \forall U \subsetneq M$, $U \neq 0 \Rightarrow U \cap A \neq 0$.

(3) $M \neq 0$ and $A \subsetneq^{\circ} M \Rightarrow A \neq M$.

(4) $M \neq 0$ and $A \subsetneq^{*} M \Rightarrow A \neq 0$.

Example 2.3.2.

(1) For any module M, we have $0 \subsetneq^{\circ} M$, $M \subsetneq^{*} M$.

(2) A module M is called semisimple if every submodule is a direct summand. If M is a semisimple module, then only 0 is co-essential in M and only M is essential in M.

(3) In any free $\mathbb{Z}$-module (free abelian group), only 0 is co-essential.

(4) Every finitely generated submodule of $\mathbb{Q}_{\mathbb{Z}}$ is superfluous in $\mathbb{Q}_{\mathbb{Z}}$.

Lemma 2.3.3 ([56], Lemma 5.1.3)**.**

(1) $A \hookrightarrow B \hookrightarrow M \hookrightarrow N$, $B \overset{\circ}{\hookrightarrow} M \Rightarrow A \overset{\circ}{\hookrightarrow} N$.

(2) $A_i \overset{\circ}{\hookrightarrow} M, i = 1, 2, \cdots n \Rightarrow \sum\limits_{i=1}^{n} A_i \overset{\circ}{\hookrightarrow} N$.

(3) $A \overset{\circ}{\hookrightarrow} M$ *and* $\varphi \in Hom_R(M, N) \Rightarrow \varphi(A) \overset{\circ}{\hookrightarrow} N$.

(4) If $\alpha : A \to B$ *and* $\beta : B \to C$ *are superfluous epimorphisms, then* $\beta\alpha$ *is also a superfluous epimorphism.*

Lemma 2.3.4 ([56], Lemma 5.1.4)**.** *For* $a \in M_R$*, the submodule* aR *of* M *is not superfluous in* M *if and only if there exists a maximal submodule* $C \hookrightarrow M$ *such that* $a \notin C$*.*

Lemma 2.3.5 ([56], Lemma 5.1.5)**.**

(1) $A \hookrightarrow B \hookrightarrow M \hookrightarrow N$ *and* $A \overset{*}{\hookrightarrow} N \Rightarrow B \overset{*}{\hookrightarrow} M$.

(2) $A_i \overset{*}{\hookrightarrow} M, i = 1, 2, \cdots n \Rightarrow \bigcap\limits_{i=1}^{n} A_i \overset{*}{\hookrightarrow} N$.

(3) $B \overset{*}{\hookrightarrow} N$ *and* $\varphi \in Hom_R(M, N) \Rightarrow \varphi^{-1}(B) \overset{*}{\hookrightarrow} M$.

(4) If $\alpha : A \to B$ *and* $\beta : B \to C$ *are essential monomorphisms, then* $\beta\alpha$ *is also a essential monomorphism.*

Lemma 2.3.6 ([56], Lemma 5.1.6)**.** *Let* $A \hookrightarrow M_R$*. Then* $A \overset{*}{\hookrightarrow} M_R \Leftrightarrow \forall m \in M, m \neq 0 \Rightarrow \exists r \in R : 0 \neq mr \in A$*.*

Definition 2.3.7. Let M and U be two right R-modules. A right R-module U is said to be *M-injective* if for every monomorphism $\alpha : L \to M$ and every homomorphism $\psi : L \to U$, there exists a homomorphism $\bar{\psi} : M \to U$ such that $\bar{\psi}\alpha = \psi$.

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "\alpha"] \arrow[d, "\psi"'] & M \arrow[dl, dotted, "\bar{\psi}"] \\
 & U &
\end{tikzcd}$$

A right R-module E is *injective* if it is M-injective, for all right R-module M. A right R-module M is called *quasi-injective* (or *self-injective*) if it is M-injective.

The following Theorem gives us a characterization of injective module:

Theorem 2.3.8 ([56], Lemma 5.3.1)**.** *Let M be a right R-module. The following conditions are equivalent:*

(1) M is injective;

(2) Every monomorphism $\varphi : M \to B$ splits (i.e. $Im(\varphi)$ is a direct summand of B);

(3) For every monomorphism $\alpha : A \to B$, $\mathrm{Hom}(\alpha, 1_M) : Hom_R(B, M) \to Hom_R(A, M)$ is an epimorphism.

A powerful test of injectivity is given by Baer's Criterion which guarantees the equivalence between injectivity and R-injectivity.

Theorem 2.3.9 ([91], 16.4)**.** *For a right R-module E, the following conditions are equivalent:*

(1) E is an injective R-module;

(2) E is R-injective;

(3) For every right ideal I of R and every homomorphism $h : I \to E$, there exists $y \in E$ with $h(a) = ya$, for all $a \in I$.

Definition and basic properties of projective module are dual to those of injective module.

Definition 2.3.10. A right R-module P is said to be *M-projective* if for every epimorphism $\beta : M \to N$ and every homomorphism $\varphi : P \to N$, there exists a homomorphism $\bar{\varphi} : P \to M$ such that $\beta\bar{\varphi} = \varphi$.

$$\begin{tikzcd} & P \arrow[dl, dotted, "\bar{\varphi}"'] \arrow[d, "\varphi"] & \\ M \arrow[r, "\beta"'] & N \arrow[r] & 0 \end{tikzcd}$$

Now we have the following fundamental characterizations of projective modules.

Theorem 2.3.11 ([56], Lemma 5.3.1)**.** *The following properties of a right R-module P are equivalent :*

(1) P is projective;

(2) Every epimorphism $\varphi : M \to P$ splits (i.e. $\mathrm{Ker}(\varphi)$ *is a direct summand of M);*

(3) For every epimorphism $\alpha : B \to C$, $\mathrm{Hom}(1_P, \beta)$ *: $Hom_R(P, B) \to Hom_R(P, C)$ is an epimorphism.*

Theorem 2.3.12 ([56], Lemma 5.4.1)**.** *A module is projective if and only if it is isomorphic to a direct summand of a free module.*

Proposition 2.3.13 ([4], Proposition 16.10)**.** *Let M be a right R-module and $(U_\alpha)_{\alpha \in A}$ be a set of right R-modules with index $\alpha \in A$. Then*

(1) A direct sum $\bigoplus_A U_\alpha$ is M-projective if and only if each U_α is M-projective.

(2) A direct product $\prod_A U_\alpha$ *is* M*-injective if and only if each* U_α *is* M*-injective.*

Proposition 2.3.14 ([4], Corollary 16.11)**.** *Let* $(U_\alpha)_{\alpha\in A}$ *be a set of right* R*-modules with index* $\alpha \in A$*. Then*

(1) A direct sum $\bigoplus_A U_\alpha$ *is projective if and only if each* U_α *is projective.*

(2) A direct product $\prod_A U_\alpha$ *is injective if and only if each* U_α *is injective.*

2.4 Noetherian and Artinian modules and rings

In the 1920s, Emmy Noether [77, 78] provided the appropriate notions and interpretations and thereby sowed the seeds for the further development. As finiteness assumptions she introduced maximal and minimal condition which can also be formulated as chain conditions. In other parts of algebra these have turned out to be just as significant and natural. These conditions are now about to be provided so that in the following considerations we can always refer back to them.

Definition 2.4.1.

(1) A right R-module M_R is called *Noetherian* if every nonempty set of its submodules has a maximal element. Dually, a module M_R is called *Artinian* if every set of its submodules has a minimal element.

(2) A ring R is called *right Noetherian* (resp. *right Artinian*) if module R_R is Noetherian (resp. Artinian)

(3) A chain of submodules of M_R

$$\cdots \subsetneq A_{i-1} \subsetneq A_i \subsetneq A_{i+1} \subsetneq \cdots$$

(finite or infinite) is called *stationary* if it contains only a finite number of distinct A_i, i.e, there is $n_0 \in \mathbb{N}$ such that $A_n = A_{n+1}$ for any $n \geq n_0$.

Remarks

(a) Clearly, the definitions above are preserved by isomorphisms.

(b) Noetherian modules are called *module with ascending chain condition* or *module with ACC on submodules.*

(c) Artinian modules are called *module with descending chain condition* or *module with DCC on submodules.*

(d) A module M is of finite length if there exists a finite chain $0 = A_0 \subset A_1 \subset \cdots \subset A_n = M$ such that A_{i+1}/A_i is simple.

Theorem 2.4.2 ([56], Theorem 6.1.2)**.** *Let M be a right R-module and A, its submodule.*

I. *The following statements are equivalent:*

(1) M is Artinian;

(2) A and M/A are Artinian;

(3) Every descending chain $A_1 \supset A_2 \supset \cdots \supset A_{n-1} \supset A_n \supset \cdots$ of submodules of M is stationary;

(4) Every factor module of M is finitely cogenerated;

(5) For every family $\{A_i \mid i \in I\} \neq \phi$ of submodules of M, there exists a finite subfamily $\{A_i \mid i \in I_0\}$ (i.e. $I_0 \subset I$ and finite) such that

$$\bigcap_{i \in I} A_i = \bigcap_{i \in I_0} A_i.$$

II. *The following conditions are equivalent:*

(1) M is Noetherian;

(2) A and M/A are Noetherian;

(3) Every ascending chain $A_1 \subset A_2 \subset \cdots \subset A_{n-1} \subset A_n \subset \cdots$ *of submodules of* M *is stationary;*

(4) Every submodule of M *is finitely generated;*

(5) For every family $\{A_i \mid i \in I\} \neq \phi$ *of submodules of* M, *there exists a finite subfamily* $\{A_i \mid i \in I_0\}$ *(i.e.* $I_0 \subset I$ *and finite) such that*

$$\sum_{i\in I} A_i = \sum_{i\in I_0} A_i.$$

III. *The following conditions are equivalent:*

(1) M *is Artinian and Noetherian;*

(2) M *is a module of finite length.*

The condition (I)(3) in Theorem 2.4.2 is called *descending chain condition*, brief *DCC*. The condition (II)(3) in Theorem 2.4.2 is called *ascending chain condition*, brief *ACC*. Thus, Theorem 2.4.2 asserts that a module M if Noetherian if it satisfies ACC, and Artinian if it satisfies DCC.

Corollary 2.4.3 ([56], Corollary 6.1.3)**.**

(1) If M *is a finite sum of Noetherian submodules, then it is Noetherian; If* M *is a finite sum of Artinian submodules, then it is Artinian.*

(2) If the ring R *is right Noetherian (resp. right Artinian), then every finitely generated right* R*-module* M_R *is Noetherian (resp. Artinian).*

(3) Every factor ring of right Noetherian (resp. Artinian) ring is again right Noetherian (resp. Artinian).

Let M_R be a right R-module and φ an endomorphism of M. Then $\varphi^n (n \in \mathbb{N})$ is also an endomorphism of M. We have:

$$Ker(\varphi) \subset Ker(\varphi^2) \subset Ker(\varphi^3) \subset \dots,$$

$$Im(\varphi) \supset Im(\varphi^2) \supset Im(\varphi^3) \supset \ldots.$$

For Noetherian (resp. Artinian) module, the first (resp. the second) chain is stationary. It follows the interesting results:

Theorem 2.4.4. *Let φ be an endomorphism of the module M. Then*

(1) M is Artinian $\Rightarrow \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 : M = Im(\varphi^n) + Ker(\varphi^n)$.

(2) M is Artinian and φ is an monomorphism $\Rightarrow \varphi$ is an automorphism.

(3) M is Noetherian $\Rightarrow \exists n_0 \in \mathbb{N} \quad \forall n \geq n_0 : 0 = Im(\varphi^n) \cap Ker(\varphi^n)$.

(4) M is Noetherian and φ is an epimorphism $\Rightarrow \varphi$ is an automorphism.

In the next part, we will provide some examples.

(1) Any finite dimensional vector space is a module of finite length. So any finite dimensional vector space is Noetherian and Artinian.

(2) Infinite dimensional vector space V_K is neither Artinian nor Noetherian.

(3) Module $\mathbb{Z}_{\mathbb{Z}}$ is Noetherian but not Artinian. Note that the ring $\mathbb{Z}$ is right and left Noetherian but it is not Artinian. Conversely, every right Artinian with identity is right Noetherian.

2.5 Radicals and Socles

In mathematics, more specifically ring theory, a branch of abstract algebra, the Jacobson radical of a ring R is the ideal consisting of those elements in R that annihilate all simple right R-modules. It happens that substituting "left" in place of "right" in the definition yields the same ideal, and so the notion is left-right symmetric. The radical of a module extends the definition of the

Jacobson radical to include modules. It plays a prominent role in many ring and module theoretic results.

The concept dual to that of the radical is the socle. The socle of a module M over a ring R is defined to be the sum of all non-zero minimal submodules of M.

Definition 2.5.1. Let M be a right R-module. The *Jacobson radical* of M, denoted by $\mathrm{Rad}(M)$, is defined to be the intersection of all maximal submodules of M.

In case $M = R$, we have $\mathrm{Rad}(R_R) = \mathrm{Rad}(_R R)$ (by [56], Theorem 9.3.2). So we define $\mathrm{Rad}(R) := \mathrm{Rad}(R_R) = \mathrm{Rad}(_R R)$

The *prime radical* or *lower nil radical*, denoted by $P(R)$, is defined to be the intersection of all prime ideals of R.

We now review the relation between the concepts of nilpotent ideal, nil ideal and the concepts of radical.

Proposition 2.5.2 ([62], Proposition 10.16)**.** *For any ring R, the following are equivalent:*

(1) R is a semiprime ring;

(2) $P(R) = 0$;

(3) R has no non-zero nilpotent ideal;

(4) R has no non-zero nilpotent left ideal.

Proposition 2.5.3 ([40])**.** *Let R be a ring. Then any semiprime ideal of R will contain all nilpotent one-sided ideals of R.*

Since the prime radical of R is a semiprime ideal of R, we have:

Corollary 2.5.4 ([40])**.** *The prime radical of R contains all nilpotent one-sided ideal of R.*

Proposition 2.5.5 ([87], Proposition XV.1.4). *If R satisfies ACC on two-sided ideals, then the prime radical of R is a nilpotent ideal.*

Proposition 2.5.6 ([40], Corollary 4.14). *For a right or a left Artinian ring R, the Jacobson radical coincides with the prime radical.*

Theorem 2.5.7 ([40], Theorem 3.11). *Let R be a right or left Noetherian ring and let N be the prime radical of R. Then N is a nilpotent ideal of R containing all the nilpotent right or left ideals of R.*

The following theorem due to Hopkins and Levitzki.

Theorem 2.5.8 ([40], Theorem 4.15). *If R is a right Artinian ring, then R is also right Noetherian and* $\mathrm{Rad}(R)$ *is nilpotent.*

In this case, we have $\mathrm{Rad}(R) = P(R)$.

Corollary 2.5.9 ([56], Corollary 9.3.7). *For any ring R, we have the following:*

(1) The sum of two nilpotent right (left or two-sided) ideals is again nilpotent.

(2) If R_R is Noetherian, then every two-sided nil ideal is nilpotent.

Proposition 2.5.10 ([56]). *Let I be a nil ideal of R.*

(1) If J/I is a nil ideal of R/I, then J is a nil ideal of R.

(2) An arbitrary sum of nil ideals is nil.

Definition 2.5.11. Let R be an arbitrary ring, then its *nil radical* $\mathrm{Nil}(R)$ is the sum of all nil two-sided ideals of R.

From proposition 2.5.10, we see that $\mathrm{Nil}(R/\mathrm{Nil}(R)) = 0$.

Theorem 2.5.12 ([56], Theorem 9.3.8). *Every (one-sided or two-sided) nil ideal is contained in* $\mathrm{Rad}(R)$.

Proposition 2.5.13 ([64], Proposition 10.27). *For any ring R, we have $P(R) \subset \mathrm{Nil}(R) \subset \mathrm{Rad}(R)$. If R is left Artinian, then $P(R) = \mathrm{Nil}(R) = \mathrm{Rad}(R)$.*

The following theorem is called Levitzki's Theorem.

Theorem 2.5.14 ([64], Theorem 10.30). *Let R be a right Noetherian ring. Then every nil one-sided ideal N of R is nilpotent. We have $P(R) = \mathrm{Nil}(R)$, and this is the largest nilpotent right (resp. left) ideal of R.*

Now, we will review the concept of locally nilpotent.

Definition 2.5.15. Let I be a right ideal of a ring R. I is called *locally nilpotent* if for any finite subset $\{s_1, \cdots, s_n\} \subset I$, there exists an integer k such that any product of k elements from $\{s_1, \cdots, s_n\}$ is zero.

Proposition 2.5.16 ([64], Proposition 10.31). *Let I, J be locally nilpotent one-sided ideals in R. Then $I + J$ is locally nilpotent right ideals.*

Definition 2.5.17. The *Levitzki radical* of a ring R, denoted by L-rad(R), is the sum of all locally nilpotent ideals of R. It is the largest locally nilpotent ideal of R, and contains every locally nilpotent one-sided ideal of R.

Moreover, we have:

Proposition 2.5.18 ([64]). *$P(R) \subset L\text{-rad}(R) \subset \mathrm{Nil}(R) \subset \mathrm{Rad}(R)$.*

2.6 Primeness in module category

In this section, before stating our new results we are interested to list some basic properties from [40].

Definition 2.6.1. A proper ideal P in a ring R is called a *prime ideal* of R if for any ideals I, J of R with $IJ \subset P$, then either $I \subset P$ or $J \subset P$. An ideal I of a ring R is called *strongly prime* if for any $a, b \in R$ with $ab \in I$, then either $a \in I$ or $b \in I$. A ring R is called a *prime ring* if 0 is a prime ideal. (Note that a prime ring must be non-zero).

Proposition 2.6.2 ([40], Proposition 3.1)**.** *For a proper ideal P of a ring R, the following conditions are equivalent:*

(1) P is a prime ideal;

(2) If I and J are any ideals of R properly containing P, then $IJ \not\subseteq P$;

(3) R/P is a prime ring;

(4) If I and J are any right ideals of R such that $IJ \subset P$, then either $I \subset P$ or $J \subset P$;

(5) If I and J are any left ideals of R such that $IJ \subset P$, then either $I \subset P$ or $J \subset P$;

(6) If $x, y \in R$ with $xRy \subset P$, then either $x \in P$ or $y \in P$.

By induction, it follows from Proposition 2.6.2 that if P is a prime ideal in a ring R and $J_1, \ldots, J_n$ are right ideals of R such that $J_1 \cdots J_n \subset P$, then some $J_i \subset P$. By a *maximal ideal* in a ring is meant a maximal proper ideal, i.e., an ideal which is a maximal element in the collection of proper ideals.

Proposition 2.6.3 ([40], Proposition 3.2)**.** *Every maximal ideal of a ring R is a prime ideal.*

Proposition 2.6.3 together with Zorn's Lemma guarantees that every non-zero ring has at least one prime ideal.

Definition 2.6.4. A prime ideal P in a ring R is called a *minimal prime ideal* if it does not properly contain any other prime ideals. For instance, if R is a prime ring, then 0 is the unique minimal prime ideal of R.

Proposition 2.6.5 ([40], Proposition 3.3)**.** *Any prime ideal P in a ring R contains a minimal prime ideal.*

Theorem 2.6.6 ([40], Theorem 3.4)**.** *In a right or left Noetherian ring R, there exist only finitely many minimal prime ideals, and there is a finite product of minimal prime ideals (repetitions allowed) that equals zero.*

Definition 2.6.7. An ideal P in a ring R is called a *semiprime ideal* if it is an intersection of prime ideals. A ring R is called a *semiprime ring* if 0 is a semiprime ideal.

Remark In $\mathbb{Z}$, the intersection of any infinite number of prime ideals is 0. The intersection of any finite list $p_1\mathbb{Z}, \ldots, p_k\mathbb{Z}$ of prime ideals, where $p_1, \ldots, p_k$ are distinct prime integers, is the ideal $p_1 \cdots p_k\mathbb{Z}$. Hence the non-zero semiprime ideals of $\mathbb{Z}$ are $n\mathbb{Z}$, where n is any square-free positive integer.

It follows from [40, Proposition 3.6] that an ideal I in a commutative ring R is semiprime if and only if, whenever $x \in R$ and $x^2 \in I$, then $x \in I$. The example of a matrix ring over a field shows that this criterion fails in the noncommutative case. However, there is an analogous criterion due to Levitzki-Nagata, as we will see in the next theorem.

Theorem 2.6.8 ([40], Theorem 3.7). *An ideal I in a ring R is semiprime if and only if*

$$(\star) \qquad \textit{whenever } x \in R \textit{ with } xRx \subset I, \textit{ then } x \in I.$$

The reader should be aware that many authors define semiprime ideals by the condition $(\star)$ in Theorem 2.6.8. From that view point, the theorem then says that an ideal is semiprime if and only if it is an intersection of prime ideals.

Corollary 2.6.9 ([40], Corollary 3.8). *For an ideal I in a ring R, the following conditions are equivalent:*

(1) I is a semiprime ideal;

(2) If J is any ideal of R such that $J^2 \subset I$, then $J \subset I$;

(3) If J is any ideal of R such that $J \supsetneqq I$, then $J^2 \not\subseteq I$;

(4) If J is any right ideal of R such that $J^2 \subset I$, then $J \subset I$;

(5) If J is any left ideal of R such that $J^2 \subset I$, then $J \subset I$.

Corollary 2.6.10 ([40], Corollary 3.9)**.** *Let I be a semiprime ideal in a ring R. If J is a right or a left ideal of R such that $J^n \subset I$ for some positive integer n, then $J \subset I$.*

Definition 2.6.11. An element x in a ring R is called a *nilpotent element* if $x^n = 0$ for some $n \in \mathbb{N}$. A right or a left ideal I in a ring R is called a *nilpotent ideal* if $I^n = 0$ for some $n \in \mathbb{N}$. More generally, I is called a *nil ideal* if each of its elements is nilpotent. The *prime radical* $P(R)$ of a ring R is the intersection of all the prime ideals of R.

Remarks ([40, page 53])

(a) In Noetherian rings, all nil one-sided ideals are nilpotent.

(b) If R is the zero ring, it has no prime ideals, and so $P(R) = R$. If R is non-zero, it has at least one maximal ideal, which is prime by Proposition 2.6.3. Thus, the prime radical of a non-zero ring is a proper ideal.

(c) A ring R is semiprime if and only if $P(R) = 0$. In any case, $P(R)$ is the smallest semiprime ideal of R, and because $P(R)$ is semiprime, it contains all nilpotent one-sided ideals of R.

Now, let R be a semiprime ring and let A and B be right ideals of R with $AB = 0$, then $(BA)^2 = 0$ and $(A \cap B)^2 = 0$, so that $BA = 0$ and $A \cap B = 0$. Thus if I is an ideal of R then $Ir(I) = 0$, hence $r(I)I = 0$. Similarly, $Il(I) = 0$. Therefore $l(I) = r(I)$. If I is a right annihilator, then $I = r(l(I)) = l(r(I))$ is also a left annihilator, and in these circumstances we call I an *annihilator ideal*. We have the following lemmas

Lemma 2.6.12 ([91], Proposition 3.13)**.** *For a ring R with identity, the following conditions are equivalent:*

(1) R is a semiprime ring (i.e. $P(R) = 0$);

(2) 0 is the only nilpotent ideal in R;

(3) For ideals I, J in R with $IJ = 0$ implies $I \cap J = 0$.

Lemma 2.6.13. *Let R be a semiprime ring with the ACC (equivalently DCC) for annihilators ideals, then R has only finite number of minimal prime ideals. If $P_1, \cdots, P_n$ are the minimal prime ideals of R then $P_1 \cap \cdots \cap P_n = 0$. Also a prime ideal of R is minimal if and only if it is an annihilator ideal.*

Proposition 2.6.14 ([40], page 54)**.** *In any ring R, the prime radical equals the intersection of the minimal prime ideals of R.*

Definition 2.6.15. Let X be a subset of a right R-module M. The *right annihilator* of X is the set $r_R(X) = \{r \in R : xr = 0 \text{ for all } x \in X\}$ is a right ideal of R. If X is a submodule of M, then $r_R(X)$ is a two-sided ideal of R. Annihilators of subsets of left R-modules are defined analogously, and are left ideals of R. If $M = R$, then the *right annihilator* of $X \subset R$ is

$$r_R(X) = \{r \in R \mid xr = 0 \text{ for all } x \in X\}$$

as well as a *left annihilator* of X is

$$l_R(X) = \{r \in R \mid rx = 0 \text{ for all } x \in X\}.$$

A right annihilator is a right ideal of R which is of the form $r_R(X)$ (or simply $r(X)$) for some subset X of R and a left annihilator is a left ideal of the form $l_R(X)$ (or simply $l(X)$).

We now give the following basic properties of right and left annihilators which have important consequences.

Properties 2.6.16 ([20])**.** *Let R be a ring and let X, Y be subsets of R. Then we have the following properties:*

(1) $X \subset Y$ implies that $r(X) \supset r(Y)$ and $l(X) \supset l(Y)$;

(2) $X \subset l(r(X)) \cap r(l(X))$;

(3) $r(l(r(X))) = r(X)$ *and* $l(r(l(X))) = l(X)$.

From these relationships, it follows easily that the ACC for right annihilators is equivalent to the DCC for left annihilators.

Definition 2.6.17. Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. A submodule X of M is called a *fully invariant submodule* of M if for any $f \in S$, we have $f(X) \subset X$.

By definition, the class of all fully invariant submodules of M is nonempty and closed under intersections and sums. Indeed, if X and Y are fully invariant submodules of M, then for every $f \in S$, we have $f(X+Y) = f(X)+f(Y) \subset X+Y$ and $f(X \cap Y) \subset f(X) \cap f(Y) \subset X \cap Y$. In general, if $\{X_i : i \in I\}$ where I is an index set, is a family of fully invariant submodules of M, then $\sum\limits_{i \in I} X_i$ and $\bigcap\limits_{i \in I} X_i$ are fully invariant submodules of M. Especially, a right ideal I of a ring R is a fully invariant submodule of R_R if it is a two-sided ideal.

Now, let $I, J \subset S$ and $X \subset M$. For convenience, we denote $I(X) = \sum\limits_{f \in I} f(X)$, $Ker(I) = \bigcap\limits_{f \in I} Ker(f)$, and $IJ = \{\sum\limits_{1 \le i \le n} x_i y_i \mid x_i \in I, y_i \in J, 1 \le i \le n, n \in \mathbb{N}\}$. With these notations, we can see that for any right R-module M and any right ideal I of R, the set MI is a fully invariant submodule of M. We now are ready to define prime submodules that was introduced by Sanh in [82].

Definition 2.6.18. Let M be a right R-module and X, a fully invariant proper submodule of M. Then X is called a *prime submodule* of M (we say that X is prime in M) if for any ideal I of S, and any fully invariant submodule U of $M, I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. A fully invariant submodule X of M is called *strongly prime* if for any $f \in S$ and any $m \in M, f(m) \in X$ implies $f(M) \subset X$ or $m \in X$.

The following theorem gives some characterizations of prime submodules similar to that of prime ideals and we use it as a tool for checking the primeness.

Theorem 2.6.19 ([82]). *Let M be a right R-module and P, a proper fully invariant submodule of M. Then the following conditions are equivalent:*

(1) P is a prime submodule of M;

(2) For any right ideal I of S and any submodule U of M, if $I(U) \subset P$, then either $I(M) \subset P$ or $U \subset P$;

(3) For any $\varphi \in S$ and any fully invariant submodule U of M, if $\varphi(U) \subset P$, then either $\varphi(M) \subset P$ or $U \subset P$;

(4) For any left ideal I of S and any subset A of M, if $IS(A) \subset P$, then either $I(M) \subset P$ or $A \subset P$;

(5) For any $\varphi \in S$ and any $m \in M$, if $\varphi(S(m)) \subset P$, then either $\varphi(M) \subset P$ or $m \in P$.

Moreover, if M is quasi-projective, then the above conditions are equivalent to:

(6) M/P is a prime module.

In addition, if M is quasi-projective and a self-generator, then the above conditions are equivalent to:

(7) If I is an ideal of S and U, a fully invariant submodule of M such that $I(M)$ and U properly contain P, then $I(U) \not\subset P$.

Example 2.6.20.

(1) Let $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ be the additive group of integers modulo 4. Then $X =< 2 >$ is a prime submodule of $\mathbb{Z}_4$.

(2) If M is a semisimple module having only one homogeneous component, then 0 is a prime submodule. Especially, if M is simple, then 0 is a prime submodule.

Definition 2.6.21. A prime submodule P of a right R-module M is called a *minimal prime submodule* if it is minimal in the class of prime submodules of M.

The following proposition gives us a property similar to that of rings (see Proposition 2.6.5).

Proposition 2.6.22 ([82]). *If P is a prime submodule of a right R-module M, then P contains a minimal prime submodule of M.*

Lemma 2.6.23 ([82]). *Let M be a right R-module and $S = \mathrm{End}_R(M)$. Suppose that X is a fully invariant submodule of M. Then the set $I_X = \{f \in S \mid f(M) \subset X\}$ is a two-sided ideal of S.*

Theorem 2.6.24 ([82]). *Let M be a right R-module, $S = End_R(M)$ and X, a fully invariant submodule of M. If X is a prime submodule of M, then I_X is a prime ideal of S. Conversely, if M is a self-generator and if I_X is a prime ideal of S, then X is a prime submodule of M.*

Definition 2.6.25. A fully invariant submodule X of a right R-module M is called a *semiprime submodule* if it is an intersection of prime submodules of M. A right R-module M is called a *prime module* if 0 is a prime submodule of M. A ring R is a prime ring if R_R is a prime module.

A right R-module M is called a *semiprime module* if 0 is a semiprime submodule of M. Consequently, the ring R is a semiprime ring if R_R is a semiprime module. By symmetry, the ring R is semiprime if ${}_RR$ is a semiprime left R-module.

Example 2.6.26.

(1) Every semisimple module with only one homogeneous component is a prime module. Especially, every simple module is prime.

(2) Every semisimple module is semiprime.

(3) As a $\mathbb{Z}$-module, the module $\mathbb{Z}_4$ is not semiprime.

Theorem 2.6.27 ([82]). *Let M be a prime module. Then its endomorphism ring S is a prime ring. Conversely, if M is a self-generator and S is a prime ring, then M is a prime module.*

Lemma 2.6.28 ([82]). *Let M be a quasi-projective module, P be a prime submodule of M, $A \subset P$ be a fully invariant submodule of M. Then P/A is a prime submodule of M/A.*

Lemma 2.6.29 ([82]). *Let M be a quasi-projective module and A a fully invariant submodule of M. If $\bar{P} \subset M/A$ is a prime submodule of M/A, then $\nu^{-1}(\bar{P})$ is a prime submodule of M.*

For a right R-module M, let $P(M)$ be the intersection of all prime submodules of M. By our definition, M is a semiprime module if $P(M) = 0$. We want to get some properties similar to that of prime radical of rings and at first step, the following theorem is true for quasi-projective modules.

Theorem 2.6.30 ([82]). *Let M be a quasi-projective module. Then $M/P(M)$ is a semiprime module, that is, $P(M/P(M)) = 0$.*

Theorem 2.6.31 ([82]). *If M is a semiprime module, then S is a semiprime ring.*

In ([82, Theorem 2.9]), it was proved that if M is a semiprime module, then its endomorphism ring S is a semiprime ring. For the converse part, M needs to be a self-generator and finitely generated module as follows:

Proposition 2.6.32 ([83]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If S is a semiprime ring, then M is a semiprime module.*

Theorem 2.6.33 ([82]).

(1) If M is a prime module, then so is M^n for any $n \in \mathbb{N}$.

(2) If M is a semiprime module, then so is M^n for any $n \in \mathbb{N}$.

Proposition 2.6.34 ([84]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then:*

(1) If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.

(2) If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.

Theorem 2.6.35 ([84]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Let X be a fully invariant submodule of M. Then the following conditions are equivalent:*

(1) X is a semiprime submodule of M;

(2) If J is any ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(3) If J is any ideal of S properly containing X, then $J^2(M) \not\subset X$;

(4) If J is any right ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

(5) If J is any left ideal of S such that $J^2(M) \subset X$, then $J(M) \subset X$;

Corollary 2.6.36 ([84]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator and X, a semiprime submodule of M. If J is a right or left ideal of S such that $J^n(M) \subset X$ for some positive integer n, then $J(M) \subset X$.*

Finally, applications are made to semiprime modules as the following theorem shows.

Theorem 2.6.37 ([82]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then the following conditions are equivalent:*

(1) M is semisimple;

(2) M is semiprime Artinian;

(3) M is semiprime and satisfies the DCC on M-cyclic submodules.

We next mention the concept of Goldie dimension (also known as uniform dimension) of a module.

Definition 2.6.38. A nonzero module M is said to be *uniform* if any two non-zero submodules of M have non-zero intersection, i.e., if every non-zero submodule of M is essential in M.

Let M be a right R-module. Then M is said to have *finite Goldie dimension* if M does not contain a direct sum of a infinite number of non-zero submodules.

It is easy to show that M has finite Goldie dimension if M is Noetherian or Artinian. A ring R is said to have *finite right Goldie dimension* if R has finite Goldie dimension as a right R-module. A ring R is called a *right Goldie ring* if it has finite right Goldie dimension and satisfies the ACC for right annihilators. A right Noetherian ring is right Goldie, but the converse is not true.

The next lemma gives the basic properties of modules of finite Goldie dimension.

Lemma 2.6.39 ([20], Lemma 1.9)**.** *Let M be a non-zero right R-module.*

(1) If M has finite Goldie dimension, then every non-zero submodule of M contains a uniform submodule of M and there is a finite number of uniform submodules of M whose sum is direct and essential in M.

(2) Suppose that M has uniform submodules $U_1, \ldots, U_n$ such that the sum $U_1 + \cdots + U_n$ is direct and essential in M, then M has finite Goldie dimension and the positive integer n is independent of the choice of U_i. We call n the Goldie dimension of M and it is denoted by $dim(M)$.

Let M be a module of finite Goldie dimension. Then by definition, submodules of M also have finite Goldie dimension, but it is not always true that arbitrary factor modules of M have finite Goldie dimension. For example, $\mathbb{Q}$ has Goldie dimension 1 as a $\mathbb{Z}$-module but $\mathbb{Q}/\mathbb{Z}$ does not have finite Goldie dimension. If V is a vector space, then V has finite Goldie dimension if and only if V has finite dimension in the usual sense of linear algebra and in these circumstances, the two dimensions are equal.

Proposition 2.6.40 ([83])**.** *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then M has finite Goldie dimension if and* *only if S has finite right Goldie dimension. Moreover, in this case, $dim(M) = dim(S_S)$.*

Definition 2.6.41. Let M be a right R-module. A submodule X of M is called an *M-annihilator* if $X = Ker(I) = \bigcap_{f \in I} Ker(f)$ for some subset I of S.

We call M a *Goldie module* if M has finite Goldie dimension and satisfies the ACC on M-annihilators.

A ring R is called a *right Goldie ring* if R_R is a Goldie module, or equivalently, if R has finite right Goldie dimension and satisfies the ACC on right annihilators.

Lemma 2.6.42 ([83]). *Let M be a right R-module and $S = End_R(M)$, its endomorphism ring. If M satisfies the ACC (resp. DCC) on M-annihilators, then S satisfies the ACC (resp. DCC) on right annihilators.*

Theorem 2.6.43 ([83]). *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If M is a Goldie module, then S is a right Goldie ring.*

Proposition 2.6.44 ([83]). *Let M be a right R-module with finite Goldie dimension and $f \in S$, a monomorphism. Then $f(M)$ is an essential submodule of M.*

Definition 2.6.45. The *right singular ideal* of a ring R is denoted and defined by

$$Z_R(R) = \{x \in R \mid xK = 0 \text{ for some essential right ideal } K \text{ of } R\}.$$

In other words, if $x \in R$, then $x \in Z_R(R)$ if and only if $r_R(x)$ is an essential right ideal of R. If $Z_R(R) = 0$, then R is called a *right nonsingular ring.*

Let M be a right R-module. An element $x \in M$ is called a *singular element* of M if the right ideal $r_R(x)$ is essential in R_R. The set of all singular elements of M is a submodule of M and it is called the *singular submodule* of M. Denoted it by $Z(M)$. If $Z(M) = M$, then M is a *singular module* and if $Z(M) = 0$, then M is *nonsingular.* A ring R is *right nonsingular* if the right R-module R_R is a nonsingular module.

Theorem 2.6.46 ([63], Lemma 7.2)**.** *Let M be a right R-module.*

(1) $Z(M) \cdot soc(R_R) = 0$, where $soc(R_R)$ denotes the socle of R_R.

(2) If $f : M \to N$ is any R-homomorphism, then $f(Z(M)) \subset Z(N)$.

(3) If $X \subset M$, then $Z(X) = X \cap Z(M)$.

Proposition 2.6.47 ([83])**.** *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. If X is an essential submodule of M, then $I_X = \{f \in S \mid f(M) \subset X\}$ is an essential right ideal of S.*

Proposition 2.6.48 ([83])**.** *Let M be a nonsingular right R-module with finite Goldie dimension. Then M satisfies the ACC and DCC on M-annihilators. Especially, if R is a right nonsingular ring with finite Goldie dimension, then R satisfies the ACC and DCC on right annihilators.*

Proposition 2.6.49 ([83])**.** *Let M be a nonsingular right R-module with the ACC on M-annihilators and let $f \in S$ be such that $f(M)$ is an essential submodule of M. Then f is a monomorphism.*

Definition 2.6.50. Let M be a right R-module. The set of all prime submodules of M is called the *prime spectrum* of M and denoted by Spec(M). Recall that the set of all prime ideals of R is called the prime spectrum of R and denoted by Spec(R) or X^R. The topological structure on Spec(R) will help us to determine the topological on $Spec(M)$. There are some useful facts about the topological on Spec(R).

Let R be a ring. Denote $Spec(R)$ (or X^R) for the set of all prime ideals of R. For any ideal I of R, we define:

$V^R(I) = \{P \in Spec(R) \mid I \subset P\}$

Proposition 2.6.51. *We have the following properties*

(1) $V^R(0) = X^R$ and $V^R(R) = \phi$;

(2) If $\{E_i\}_{i\in J}$ is any family of ideals of R, then $\bigcap\limits_{i\in J} V^R(E_i) = V^R(\bigcup\limits_{i\in J}(E_i)$;

(3) If I, J are ideals of R, then $V^R(I) \cup V^R(J) = V^R(IJ) = V^R(I \cap J)$.

Let $\Gamma(R) = \{V^R(I) \mid I \text{ is an ideal of } R\}$. From $(1) - (3)$, there exists a topology, say Γ^R, on *Spec*(R) having $\Gamma(R)$ as the family of all closed sets. This topology is called the *Zariski topology* on *Spec*(R). With this topology, we have the following lemma:

Lemma 2.6.52 ([88]). *Let R be a ring. Then we have the following:*

(1) X^R is compact;

(2) A subset Y of X^R is irreducible if and only if $J^\star(Y)$ is a prime ideal of R, where $J^\star(Y)$ denote the intersection of all elements in Y.

CHAPTER III
ON MODULES WITH INSERTION FACTOR PROPERTY

3.1 IFP Modules and Their Endomorphism Rings

In this section we define the IFP modules that we shall study, and we investigate some of their basic properties.

Definition 3.1.1. A submodule X of a right R-module M is said to have *"insertion factor property"* (briefly, an IFP-submodule) if for any endomorphism φ of M and any element $m \in M$, if $\varphi(m) \in X$, then $\varphi Sm \subset X$. A right ideal I of R is an IFP-right ideal if it is an IFP submodule of R_R, that is for any $a, b \in R$, if $ab \in I$, then $aRb \subset I$. A right R-module M is called an IFP-module if 0 is an IFP-submodule of M. A ring is IFP if 0 is an IFP ideal.

By definition, we can see that any intersection of a family of IFP-submodules is again IFP. Clearly, every ideal in a commutative ring is IFP.

The following examples are due to [1].

Example 3.1.2.

1. There exists a semicommutative R-module M such that it is not IFP. Let F be a field, $R = \begin{pmatrix} F & 0 \\ 0 & F \end{pmatrix}$ and $M = \begin{pmatrix} F & 0 \\ F & 0 \end{pmatrix}$.
It is easy to check that M is a semicommutative module.
Now, let $m = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in M$; $f, g \in S$ is defined by

$$f\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}; g\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix} \text{ where } \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \in M.$$

Then we have:

$$f(m) = f\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \text{ and } fg(m) = fg\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix} \neq \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}.$$

That is, M is not IFP.

2. There is a module M which is IFP but not semicommutative. Let $\mathbb{Z}$ denote the ring of intergers, $M = \mathbb{Z}\times\mathbb{Z}$, $R = End_{\mathbb{Z}}(\mathbb{Z}\times\mathbb{Z})$ and $S = End_R(\mathbb{Z}\times\mathbb{Z})$. Then M is an IFP but not semicommutative. Indeed, if we let $f, g \in R$ be defined by $(a,b)f = (a,0)$ and $(a,b)g = (b,0)$ where $(a,b) \in \mathbb{Z}\times\mathbb{Z}$, then $(1,0)f = (0,0)$ but $(0,1)gf \neq (0,0)$.

Proposition 3.1.3. *Let X be a submodule of a right R-module M. If X is an IFP-submodule and M is quasi-projective, then M/X is an IFP-module. Conversely, if M/X is IFP and X is fully invariant, then X is an IFP-submodule of M.*

Proof. Suppose that X is an IFP-submodule of M and $\bar{\varphi}(\bar{m}) = 0$, where $\bar{\varphi} \in \bar{S} = End(M/X)$ and $\bar{m} \in M/X$. By the quasi-projectivity of M, there is a $\varphi \in S$ such that $\nu\varphi = \bar{\varphi}\nu$, where $\nu : M \longrightarrow M/X$ is the natural epimorphism. It follows that $\varphi(m) \in X$. Let $\bar{\xi}$ be any endomorphism of M/X. Then as above, there is $\xi \in S$ such that $\bar{\xi}\nu = \nu\xi$. By assumption, $\varphi\xi(m) \in X$. This leads to $\bar{\varphi}\bar{\xi}(\bar{m}) = 0$, proving that M/X is an IFP-module.

Suppose that X is fully invariant submodule of M, with M/X is IFP. Let $\varphi(m) \in X$, where $\varphi \in S$ and $m \in M$. Since X is fully invariant submodule of M, there is $\bar{\varphi} \in \bar{S}$ such that $\bar{\varphi}\nu = \nu\varphi$. It follows that $\bar{\varphi}(\bar{m})$. By assumption, we get $\bar{\varphi}\bar{\xi}(\bar{m}) = 0$, for any $\xi \in S$, where $\bar{\xi}\nu = \nu\xi$. This leads to the fact that $\varphi S(m) \subset X$, proving our proposition. □

Let X be a submodule of M. Define $I_X = \{f \in S \mid f(M) \subset X\}$. Then we can see that I_X is a right ideal of S. Moreover, if X is fully invariant in M, then I_X is a two-sided ideal of S. The following lemma is useful.

Lemma 3.1.4. *If X is an IFP submodule of M, then I_X is an IFP right ideal of S. The converse is true if M is a self-generator.*

Proof. Let $\varphi\psi \in I_X$. Then, $\varphi(\psi(m)) \in X$ for any $m \in M$. By hypothesis, $\varphi\xi(\psi(m)) \in X$, for any $\xi \in S$ and any $m \in M$. It follows that $\varphi S\psi \subset I_X$, showing that I_X is IFP.

Conversely, let $\varphi(m) \in X$, where $\varphi \in S$, and $m \in M$. We have $\varphi(m)R \subset X$ and it would imply that $\varphi(mR) \subset X$. Since M is a self-generator, $mR = \sum_{g\in A} g(M)$ for some subset A of S. And hence, $\varphi(mR) = \varphi(\sum_{g\in A} g(M)) = \sum_{g\in A} \varphi g(M)$. This implies that $\varphi g \subset I_X$. It follows from hypothesis, $\varphi S g \subset I_X$. This implies that $\varphi Sg(M) \subset X$, showing that $\varphi S(m) \subset X$. It means that X is an IFP submodule. □

We now study the relationship between an IFP module and its endomorphism ring by following Theorem.

Theorem 3.1.5. *Let M be a right R-module and S its endomorphism ring. If M is an IFP-module, then S is an IFP-ring. The converse is true if M is a self-generator.*

Proof. Let $\varphi\psi = 0 \in S$. Then $\varphi(\psi(m)) = 0$ for all $m \in M$. If M is IFP, then for any $\xi \in S$, we have $\varphi\xi(\psi(m)) = 0$ for all $m \in M$. It follows that $\varphi S\psi = 0$, showing that S is an IFP-ring.

Conversely, since $I_0 = \{f \in S \mid f(M) = 0 \subset M\} = 0$, is an IFP-ideal, it follows that 0 is an IFP-submodule of M by Lemma 3.1.4, proving our theorem. □

The following Theorem gives some characterizations of IFP modules.

Theorem 3.1.6. *Let M be a right R-module and $S = End(M)$. The following conditions are equivalent:*

(1) M is an IFP-module;

(2) For any $m \in M$, $l_S(m)$ is an ideal of S;

(3) For any $\varphi \in S, Ker(\varphi)$ is a fully invariant submodule of M;

If M is quasi-projective, then the above conditions are equivalent to:

(4) For any $\varphi \in S, Ker(\varphi)$ is an IFP-ideal of S;

(5) $M/Ker(I)$ is an IFP-module for any subset I of S;

If M is a self-generator, then the above conditions (1), (2) and (3) are equivalent to:

(6) For any $m \in M$, $l_S(m)$ is an IFP-ideal of S;

(7) $S/l_S(A)$ is an IFP-ring for any subset $A \subset M$.

Proof. (1) $\Rightarrow$ (2). Let $\varphi \in l_S(m)$ and $\xi \in S$, where $m \in M$. Then $\varphi(m) = 0$. By (1), we have $\varphi\xi(m) = 0$. It follows that $\varphi\xi \in l_S(m)$, proving that $l_S(m)$ is a two-sided ideal of S.

(2) $\Rightarrow$ (1). Let $\varphi(m) = 0$ where $\varphi \in S$ and $m \in M$. Since $l_S(m)$ is an ideal, for any $\xi \in S$, we have $\varphi\xi \in l_S(m)$. This shows that M is an IFP-module.

(1) $\Rightarrow$ (3). Let $\varphi \in S$. For any $m \in Ker(\varphi)$, we get $\varphi(m) = 0$. By assumption, $\varphi\xi(m) = 0$ for all $\xi \in S$. This shows that $\xi(Ker(\varphi)) \subset Ker(\varphi)$, i.e., $Ker(\varphi)$ is a fully invariant submodule of M.

(3) $\Rightarrow$ (1). With $\varphi(m) = 0$ we have $m \in Ker(\varphi)$, which is fully invariant in M. Thus for any $\xi \in S$, $\xi(m) \in Ker(\varphi)$, and hence $\varphi\xi(m) = 0$, proving that M is an IFP-module.

(1) $\Rightarrow$ (4). Let $\varphi \in S, m \in M$ such that $\psi(m) \in Ker(\varphi)$. Then $(\varphi\psi)(m) = 0$. By (1), we get that $\varphi\psi\xi(m) = 0$ for all $\xi \in S$. This shows that $\psi S(m) \subset Ker(\varphi)$, showing that $Ker(\varphi)$ is an IFP-submodule of M.

(4) $\Rightarrow$ (5). We note that $Ker(I) = \bigcap_{f \in I} Ker(f)$, and each $Ker(f)$ is an IFP-submodule of M, and hence $Ker(I)$ is an IFP-submodule of M. Since M is quasi-projective, by applying Proposition 3.1.3, we can see that $M/Ker(I)$ is an IFP-module.

(5) $\Rightarrow$ (1). This part is clear by taking $I = \{1_M\}, 1_M$ is the identity map of M.

(1) $\Rightarrow$ (6). Let $m \in M$ and $\varphi\psi \in l_S(m)$, where $\varphi, \psi \in S$. Then $\varphi(\psi(m)) = 0$. By assumption, $\varphi S\psi(m) = 0$. It follows that $\varphi S\psi \subset l_S(m)$, as required.

(6) $\Rightarrow$ (7). Since $l_S(A) = \bigcap\limits_{a \in A} l_S(a)$ for any subset A of M, we see that $l_S(A)$ is an IFP ideal of S, and therefore $S/l_S(A)$ is an IFP-ring.

(7) $\Rightarrow$ (1). Taking $A = M$, then it is clear that S is an IFP-ring. Since M is a self-generator, by applying Theorem 3.1.5 we can see that M is an IFP-module. $\square$

The following Corollary is a direct consequence of the above Theorem.

Corollary 3.1.7. *For a ring R, the following conditions are equivalent:*

(1) R is an IFP-ring;

(2) For any $a \in R, l_R(a)$ is an ideal of R;

(3) For any $a \in R, r_R(a)$ is an ideal of R;

(4) For any $a \in R, l_R(a)$ is an IFP-ideal of R;

(5) For any $a \in R, r_R(a)$ is an IFP-ideal of R;

(6) For any $a \in R, R/r_R(a)$ is an IFP-ring;

(7) For any $a \in R, R/l_R(a)$ is an IFP-ring.

3.2 Generalizing Anderson's Theorem

The goal of this section is to prove a generalization of Anderson's Theorem to fully IFP modules.

Definition 3.2.1. A module M is called *fully IFP* if M/U is IFP for every proper fully invariant submodule U of M. A ring is called *fully IFP* if R/I is IFP for every proper ideal I of R.

Clearly, every fully IFP-module is an IFP-module.

Due to [50], fully IFP rings is called homomorphically IFP rings. Next, we study the relationship between a fully IFP module and its endomorphism ring.

Theorem 3.2.2. *Let M be a right R-module. If M is a fully IFP, then S is a fully IFP ring. Conversely, if S is a fully IFP and M is a self-generator, then M is a fully IFP module.*

Proof. Suppose that M is a fully IFP module and J is a proper ideal of S. This shows that $J(M)$ is a fully invariant submodule of M. By assumption, $M/J(M)$ is IFP. By Proposition 3.1.3, we can see that $I_{J(M)}$ is an IFP right ideal of S. Hence S/J is IFP, proving that S is a fully IFP ring.

Conversely, suppose that S is a fully IFP ring. Let U be a proper fully invariant submodule of M, then I_U is a proper ideal of S. By assumption, S/I_U is IFP. Since M is a self-generator, M/U is IFP. Thus M is a fully IFP module. The proof of our theorem is now complete. $\square$

Recall from [55] that a module N is called *M-generated* if there is an epimorphism $M^{(I)} \to N$ for some index set I. If I is finite, then N is called *finitely M-generated.* From this, we can see that if M is quasi-projective and X is finitely M-generated, then $I_X = \{f \in S \mid f(M) \subset X\}$ is a finitely generated right ideal of S.

Proposition 3.2.3 ([85], Proposition 1.1)**.** *Let M be a quasi-projective, finitely generated right R-module which is a self-generator. Then we have the following:*

(1) If X is a minimal prime submodule of M, then I_X is a minimal prime ideal of S.

(2) If P is a minimal prime ideal of S, then $X := P(M)$ is a minimal prime submodule of M and $I_X = P$.

For following Theorem, we refer to Huh et al. [50].

Theorem 3.2.4 ([50], Theorem 3)**.** *Let R be a homomorphically IFP ring and I be a proper ideal of R. If every prime ideal minimal over I is finitely generated then there are only finitely many prime ideals minimal over I.*

Applying this result we can prove the following theorem as a generalization of Anderson's theorem for modules.

Theorem 3.2.5. *Let M be a quasi-projective, finitely generated, fully IFP which is a self-generator. Assume that U is a proper fully invariant of M. If every prime submodule minimal over U is finitely generated, then there are only finitely many prime submodules minimal over U.*

Proof. Since M is a fully IFP module, then S is a fully IFP ring. By assumption, I_U is a proper ideal of S. By Theorem 3.2.5, it is easy to see that there are only finitely many prime ideals minimal over I_U. Applying Proposition 3.2.4, we can see that there are only finitely many prime submodules minimal over U. This completes the proof. □

The following Corollary is a direct consequence of the above Theorem.

Corollary 3.2.6. *Let M be a quasi-projective, finitely generated, fully IFP which is a self-generator. If every minimal prime submodule of M is finitely generated, then there are only finitely many minimal prime submodules of M.*

CHAPTER IV
ON NEARLY PRIME SUBMODULES

In this chapter, we introduce the notion of nearly prime submodules as a generalization of prime submodules.

4.1 On nearly prime submodules

In the following definition, instead of requiring the submodule X to be fully invariant, we reduce this by a weaker condition that X is invariant under φS.

Definition 4.1.1. A proper submodule X of a right R-module M is called a *nearly prime submodule* if for any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$, then either $m \in X$ or $\varphi(M) \subset X$. Especially, a proper right ideal P of R is a *nearly prime right ideal* if for $a, b \in R$ such that $aRb \subset P$ and $aRP \subset P$, then either $a \in P$ or $b \in P$.

From our definitions, any prime submodule of a right R-module M is nearly prime.

In the following Theorem and its corollary, we can see that a proper right ideal P of R is nearly prime if for any right ideals $A, B \subset R$ such that $AP \subset P$ and $AB \subset P$, then either $A \subset P$ or $B \subset P$. Note that Koh gave this definition and used the terminology *prime right ideals.*

Theorem 4.1.2. *Let X be a proper submodule of M. Then the following conditions are equivalent:*

(1) X is a nearly prime submodule of M;

(2) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$ and $I(X) \subset X$, then either $I(M) \subset X$ or $U \subset X$;

(3) For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$ and $\varphi S(X) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$.

Proof.

(1) $\Rightarrow$ (2). Let $I \subset S$, $U \subset M$ such that $I(U) \subset X$ and $I(X) \subset X$. Suppose that $I(M) \not\subset X$, then we can find $\varphi \in I$ such that $\varphi(M) \not\subset X$. Since $I(U) = IS(U) \subset X$, then for any $u \in U$, we have $\varphi S(u) \subset X$. By assumption, $u \in X$, proving that $U \subset X$.

(2) $\Rightarrow$ (3). Let $\varphi \in S, U$, a fully invariant with $\varphi(U) \subset X$ and $\varphi S(X) \subset X$. We can see that $\varphi S(U) \subset X$ and $\varphi S(X) \subset X$ and by (2), we have $(\varphi S)(M) \subset X$ or $U \subset X$. This shows that $\varphi(M) \subset X$ or $U \subset X$.

(3) $\Rightarrow$ (1). Let $\varphi \in S$, $m \in M$ with $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$. From $\varphi S(m) \subset X$, we have $\varphi S(mR) \subset X$. Hence $mR \subset X$ or $\varphi(M) \subset X$, by assumption. This shows that either $m \in X$ or $\varphi(M) \subset X$. $\square$

Corollary 4.1.3. *Let I be a proper right ideal of a ring R. Then the following conditions are equivalent:*

(1) I is a nearly prime right ideal of R;

(2) For any right ideal A, B of R, if $AB \subset I$ and $AI \subset I$, then either $A \subset I$ or $B \subset I$;

(3) For any $a \in R$ and any ideal B of R, if $aB \subset I$ and $aRI \subset I$, then either $a \in I$ or $B \subset I$.

The following example is due to Reyes [81].

Example 4.1.1. *Let D be a division ring and let R be the following subring of $\mathbb{M}_3(k)$:*

$$R := \begin{pmatrix} D & D & D \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix} .$$

Let $P \subset R$ be the right ideal consisting of matrices in R whose first row is zero, i.e., $P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}$. *Now we assume that $aRb \subset P$ and $aRP \subset P$ for arbitrary $a, b \in R$. Since $aRP \subset P$, we get $a_{11} + a_{12} = 0$ and $a_{11} + a_{13} = 0$. And hence*

$$aRb = \begin{pmatrix} a_{11}b_{11}D & a_{11}b_{12}D & a_{11}b_{13}D \\ 0 & a_{22}b_{22}D & 0 \\ 0 & 0 & a_{33}b_{33}D \end{pmatrix} \subset P.$$

This would imply that either $a \in P$ or $b \in P$, so P is a nearly prime right ideal of R.

Proposition 4.1.4. *Any maximal submodule of a right R-module M is nearly prime.*

Proof. Let $\varphi(U) \subset X$ where U is a submodule of M and $\varphi \in S$ with $\varphi S(X) \subset X$. Suppose that $U \not\subset X$. Then there is an $u \in U$ such that $X + uR = M$. It follows that $\varphi(M) = \varphi(X) + \varphi(uR) = \varphi(X) + \varphi(u)R \subset X$ since $\varphi(U) \subset X$. This shows that X is nearly prime. $\square$

Note that, in general, a maximal submodule of a right R-module M needs not to be fully invariant. Therefore the class of nearly prime submodules of a given right R-module M is larger than that of prime submodules. As a consequence, every maximal right ideal is a nearly prime right ideal.

Let X be a submodule of M. Then the set $I_X = \{f \in S \mid f(M) \subset X\}$ is a right ideal of S. In the following Proposition, we consider the relation between X and I_X.

Proposition 4.1.5. *Let M be a right R-module which is a self-generator and X, a submodule of M. If X is a nearly prime submodule, then $I_X = \{f \in S \mid f(M) \subset X\}$ is a nearly prime right ideal of S. Conversely, if I_X is a nearly prime right ideal of S, then X is a nearly prime submodule.*

Proof. Since $\varphi S I_X \subset I_X$, we have $\varphi S I_X(M) \subset I_X(M)$, and since M is a self-generator, we have $I_X(M) = X$ and hence $\varphi S(X) \subset X$. Take any $\psi \in S$. If

$\varphi S\psi \subset I_X$, then $\varphi S\psi(M) \subset X$. It follows that $\varphi S\psi(m) \subset X$, for all $m \in M$. Hence $\varphi(M) \subset X$ or $\psi(m) \in X$, showing that $\varphi \in I_X$ or $\psi \in I_X$.

Conversely, suppose that I_X is nearly prime. Let $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$. We have $\varphi S(m)R \subset X$ and it would imply that $\varphi S(mR) \subset X$. Since $mR = \sum_{g\in A} g(M)$ for some subset A of S, $\varphi S(mR) = \varphi S(\sum_{g\in A} g(M)) = \sum_{g\in A} \varphi Sg(M) \subset X$. This would imply that $\varphi Sg \subset I_X$. It follows from the hypothesis that $\varphi \in I_X$ or $g \in I_X$. This shows that $\varphi(M) \subset X$ or $g(M) \subset X$ for all $g \in A$. Hence $\varphi(M) \subset X$ or $m \in X$, proving our proposition. □

Following [7], a proper submodule X of a right R-module M is called a *nearly strongly prime submodule* if for any $\varphi \in S$ and $m \in M$, if $\varphi(m) \in X$ and $\varphi(X) \subset X$, then either $m \in X$ or $\varphi(M) \subset X$.

The following Proposition give the relationship between a nearly prime submodule and nearly strongly prime submodule.

Proposition 4.1.6. *Let M be an R-module. If a submodule X of M is nearly prime and IFP, then it is a nearly strongly prime submodule.*

Proof. The proof is immediate. □

4.2 A characterization of Noetherian modules

The following Theorem shows how nearly prime submodules control the structure of a finitely generated module.

Theorem 4.2.1. *Let M be a finitely generated right R-module. Then M is a Noetherian module if and only if every nearly prime submodule of M is finitely generated.*

Proof. We use a mild modification of the argument as that given in [59] and we present it here for the sake of completeness. Clearly, if M is Noetherian, then every nearly prime submodule is finitely generated. We now assume that every nearly prime submodule is finitely generated and suppose on the contrary that there is a submodule A of M which is not finitely generated. Consider the set $\mathcal{F} = \{X \subset M \mid A \subset X \text{ and } X \text{ is not finitely generated}\}$ and any chain $X_1 \subset X_2 \subset \ldots$ in $\mathcal{F}$. Then $\bigcup\limits_{i\in\mathbb{N}} X_i$ is not finitely generated and hence it is a proper submodule of M, since M is finitely generated. By Zorn's Lemma, the set $\mathcal{F}$ has a maximal element, A_0 says. We now prove that A_0 is nearly prime. Suppose that there are $\varphi \in S, m \in M$ such that $\varphi Sm \subset A_0$ with $\varphi SA_0 \subset A_0$ but $\varphi(M) \not\subset A_0$ and $m \notin A_0$. Then $A_0 + \varphi(M)$ contains properly A_0, and hence it is finitely generated, that is $A_0+\varphi(M) = x_1R+x_2R+\cdots+x_nR$ for some $x_1, x_2, \ldots, x_n \in M$. Let $K = \{a \in M \mid \varphi(a) \in A_0\}$. By assumption $A_0 \subset K$ and $m \in K$. Since $m \notin A_0$, K contains properly $A_0 + mR$, and hence it is finitely generated. Since each $x_i \in A_0 + \varphi(M)$ we can write $x_i = b_i + \varphi(m)$ where $b_i \in A_0$ and $m_i \in M$. By definition, $\varphi(K) \in A_0$. It follows that $b_1R + b_2R + \cdots + b_nR + \varphi(K) \subset A_0$. We now prove that $A_0 \subset b_1R + \cdots + b_nR + \varphi(K)$. For this, take any $w \in A_0$. Then $w \in A_0 + \varphi(M)$ and we can write:

$$\begin{aligned} w &= \sum_{i=1}^{n} x_i r_i = \sum_{i=1}^{n}(b_i + \varphi(m_i))r_i \\ &= \sum_{i=1}^{n} b_i r_i + \sum_{i=1}^{n} \varphi(m_i r_i) = \sum_{i=1}^{n} b_i r_i + \varphi(\sum_{i=1}^{n} m_i r_i) \end{aligned}$$

From this we can see that $\sum\limits_{i=1}^{n} m_i r_i \in K$, and hence $w \in b_1R + \cdots + b_nR + \varphi(K)$. This proves that $A_0 = b_1R + \cdots + b_nR + \varphi(K)$. Since K is finitely generated, it would imply that $\varphi(K)$ is finitely generated and hence A_0 is finitely generated, which is a contradiction. Thus every submodule of M is finitely generated and we now can conclude that M is Noetherian. $\square$

The following Theorem was in [59, Theorem 1] and can be considered as a Corollary of our Theorem.

Corollary 4.2.2 ([59]). *Let R be an associative ring with identity. Then R is a right Noetherian ring if and only if every nearly prime right ideal of R is finitely generated.*

Recall that a right R-module M is called a *duo module* if every submodule of M is fully invariant and note that every prime submodule is nearly prime. Therefore we can get the following Corollary.

Corollary 4.2.3. *Let M be a finitely generated duo right R-module. Then M is Noetherian if and only if every prime submodule is finitely generated.*

For $M = R_R$, the next Corollary follows consequently.

Corollary 4.2.4 ([19])**.** *If R is a right duo ring and suppose that every prime ideal in R is finitely generated, then R is right Noetherian.*

The following Corollary is a famous Theorem in Commutative Algebra due to I. S. Cohen, in [23, Theorem 2].

Corollary 4.2.5 (Cohen Theorem)**.** *A commutative ring R with identity is Noetherian if and only if every prime ideal of R is finitely generated.*

CHAPTER V
CONCLUSION

In 2008, Sanh et al. [82] introduced the new notion of prime and semiprime submodules. Following that, a prime submodule X of a right R-module M is a proper fully invariant submodule of M with the property that for any ideal I of $S = End_R(M)$ and any fully invariant submodule U of M, $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. We can say that this new approach is nontrivial, creative and well-posed. We already got many results using those new notions that are unparalleled. As an extension of our work, we want to generalize the notion of a prime submodule. To do that, there are several ways but we put our attention to replace a weaker condition that X is invariant under φS instead of requiring the submodule X to be fully invariant, and we called it nearly prime submodule. Using this new definition, we proved many beautiful properties of nearly prime submodules which are similar to that of prime submodules and also prime ideals.

The important point to note here is a characterization of Noetherian modules by the class of nearly prime submodules. Theorem 4.2.1 in Chapter IV provided a natural and intrinsic generalization of the Cohen Theorem, which is a famous Theorem in Commutative Algebra. All the results presented in Chapter IV have proven that we are on the right track.

Also in this study, we introduced the concept of IFP modules which is a generalization of IFP rings. Then we investigated the basic properties of IFP modules, and further examined the relationship between IFP modules and their endomorphism rings. After that we also gave the concept of fully IFP. In order to effectively apply the fully IFP modules, we provided and proved Theorem 3.2.6 as a generalization of Anderson's Theorem for modules.

The study has its limitation, the Conjectures that we propose as follows have not solved yet. Their solutions require more investigations and further results must be deeper and wider than the fundamental theorem that we got.

Conjecture 1 Let M be finitely generated a right R-module. Then M is a cyclic module if and only if every nearly prime submodule of M is cyclic.

Conjecture 2 Let M be a right R-module. If every submodule of M is prime then M is semiprimitive (i.e. $Rad(M) = 0$).

With a little bitterness, we recognize that a pure mathematical research requires much more subtle consideration and spiritual cultivation, even algebra always renders far more than one can require. We hope that this work will be a firm foundation for a strongly significant theory.

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