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Intuitionistic fuzzy soft game theory

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Abstract

In this article, we define two person intuitionistic fuzzy soft games, which can be applied to problems containing vagueness and uncertainty. We then give one solution method of the game which is intuitionistic fuzzy soft saddle point. Finally we give an application which shows that the method can be successfully applied to a real life problem and extend the two person intuitionistic fuzzy soft game (tp ifs-games) to n-person ifs-games. We also introduce intuitionistic fuzzy soft upper and intuitionistic lower values of a two person intuitionistic fuzzy soft game.

Keywords: soft set, fuzzy soft set, intuitionistic fuzzy soft set, intuitionistic fuzzy soft payoff

1. Introduction

Game theory was established as a field in its own right after the 1944 publication of the volume theory of games and economic behavior by Von Neumann and the economist Oskar Morgenstern (Neumann & Morgenstern, 1944). This book provided much of the basic terminology and problem set up that is still in use today. The objective of study in game theory is the game, which is a formal model of an interactive situation. Different games have been introduced since from the beginning of the existence of human life. But later, we realize that principles used in these games are also applicable in real life situation and using these principles we solve many problems. It typically involves several players. A game with only one player is usually called a decision problem. The formal definition pays out the players, their preferences, their information, the strategic actions available to them, and how these influence the outcome.

After Molodtsov works on soft set theory, it has been progressing rapidly and is finding applications in a wide variety of fields, for example theory of soft sets (Maji, 2003; Molodtsov, 1999), soft decision making (Maji, 2002), fuzzy soft sets (Maji, 2001), intuitionistic fuzzy soft sets (Maji, 2001, 2004; Mukherjee, 2008; Jiang, 2011), intuitionistic fuzzy soft matrix (Mao, 2013).

In 1965, Zadeh (Zadeh, 1965) developed the theory of fuzzy sets that is the most appropriate theory till date for dealing with uncertainties. In recent years, many interesting applications of game theory have been expanded by embedding the ideas of fuzzy sets. The two person zero sum games with fuzzy pay offs and fuzzy goals game theory have been studied by many authors (Maeda, 2003; Deli, 2013).

In this paper we concentrate on intuitionistic fuzzy soft set. Intuitionistic fuzzy set was introduced by K.T. Atanassov (Atanassov, 1986) as extensions of the standard fuzzy sets. Later Maji (Maji, 2001, 2004) introduced the concept of intuitionistic fuzzy soft set. In this work we propose a game model for dealing with uncertainties. The proposed new game is called intuitionistic fuzzy soft game (ifs-game) as it is based on soft set theory and intuitionistic fuzzy set theory. We then give one solution method of the game which is intuitionistic fuzzy soft saddle point. Finally we give an application which shows that the method can be successfully applied to a real life problem and extend the two persons intuitionistic fuzzy soft games (tp ifs-games) to n-person ifs-games. We also introduce intuitionistic fuzzy soft upper and intuitionistic lower values of a two person intuitionistic fuzzy soft game.

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2. Preliminaries

In this section we present the basic definitions and results of soft sets (Molodtsov, 1999), fuzzy sets (Zadeh, 1965), fuzzy soft soft sets (Maji, 2001) and intuitionistic fuzzy soft sets (Maji, 2001, 2004) which are useful for future discussions. More detailed explanations related to this subsection may be found in (Maji, 2003; Mukherjee, 2008).

Definition 2.1 (Molodtsov, 1999) Let U be an initial universe and E be a set of parameters. Let P(U) denotes power set of U and $A \subseteq E$. Then the pair (F, A) is called a soft set over U, where F is a mapping given by $F:A \rightarrow P(U)$

Definition 2.2 (Zadeh, 1965) Let X be a nonempty set. Then a fuzzy set A is a set having the form $A=\{(x,\mu_A(x)):x \in X\}$, where the function $\mu_A:X \rightarrow [0,1]$ is called the membership function and $\mu_A(x)$ is called the degree of membership of each element $x \in X$.

Definition 2.3 (Atanassov, 1986) Let X be a nonempty set. An intuitionistic fuzzy set A in X is an object $A=\{(x,\mu_A(x),\gamma_A(x)): x \in X\}$, where the functions $\mu_A:X \rightarrow [0,1]$ and $\gamma_A:X \rightarrow [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\gamma_A(x)$) of each element $x \in X$ to the set A respectively and $0 \le \mu_A(x)+\gamma_A(x) \le 1$ for each $x \in X$.

Definition 2.4 (Maji, 2001) Let U be an initial universe and E be a set of parameters. Let I^U be the set of all fuzzy subsets of U and $A \subseteq E$. Then the pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F:A \rightarrow I^U$.

Definition 2.5 (Maji, 2001; Maji, 2004) Let U be an initial universe and E be a set of parameters. Let IF^U be the set of all intuitionistic fuzzy subsets of U and $A \subseteq E$. Then the pair (F, A) is called an intuitionistic fuzzy softset over U, where F is a mapping given by $F:A \rightarrow IF^U$.

Example 2.6 (Maji, 2001, 2004; Mukherjee, 2008) Consider an intuitionistic fuzzy soft set (F, A) where U is a set of five houses under the consideration of a decision maker to purchase, which is denoted by $U=\{h_1,h_2,h_3,h_4,h_5\}$ and A is a parameter set where $A=\{e_1,e_2,e_3,e_4,e_5\}$ ={expensive, beautiful, wooden, in good repair, in the green surroundings}. The intuitionistic fuzzy soft set (F, A) describes the "attractiveness of the houses" to the decision maker.

$$F(e_1)=\{(h_1,0.6,0.1),(h_2,0.8,0.05),(h_3,0.6,0.2),(h_4,0.65,0.15),(h_5,0.56,0.2)\}$$

$$F(e_2)=\{(h_1,0.7,0.15),(h_2,0.6,0.15),(h_3,0.5,0.2),(h_4,0.7,0.15),(h_5,0.6,0.25)\}$$

$$F(e_3)=\{(h_1,0.75,0.1),(h_2,0.5,0.2),(h_3,0.6,0.1),(h_4,0.68,0.1),(h_5,0.7,0.1)\}$$

$$F(e_4)=\{(h_1,0.8,0.01),(h_2,0.65,0.2),(h_3,0.66,0.2),(h_4,0.69,0.1)(h_5,0.72,0.1)\}$$

$$F(e_5)=\{(h_1,0.77,0.05),(h_2,0.6,0.2),(h_3,0.6,0.2),(h_4,0.63,0.15),(h_5,0.7,0.1)\}$$

The intuitionistic fuzzy soft set (F, A) is a parameterized family $\{F(e_i,i=1,2,3,4,5)\}$ of interval valued intuitionistic fuzzy sets on U and

$$\begin{aligned}(F, A)=\{&\text{expensive houses}=\{(h_1,0.6,0.1),(h_2,0.8,0.05),(h_3,0.6,0.2),(h_4,0.65,0.15),(h_5,0.56,0.2)\},\\ &\text{beautiful houses}=\{(h_1,0.7,0.15),(h_2,0.6,0.15),(h_3,0.5,0.2),(h_4,0.7,0.15),(h_5,0.6,0.25)\}\\ &\text{wooden houses}=\{(h_1,0.75,0.1),(h_2,0.5,0.2),(h_3,0.6,0.1),(h_4,0.68,0.1),(h_5,0.7,0.1)\}\\ &\text{in good repair houses}=\{(h_1,0.8,0.01),(h_2,0.65,0.2),(h_3,0.66,0.2),(h_4,0.69,0.1)(h_5,0.72,0.1)\}\\ &\text{in the green surroundings}=\{(h_1,0.77,0.05),(h_2,0.6,0.2),(h_3,0.6,0.2),(h_4,0.63,0.15),(h_5,0.7,0.1)\}\end{aligned}$$

3. Two Person Intuitionistic Fuzzy Soft Games

In this section we construct two person intuitionistic fuzzy soft games with intuitionistic fuzzy soft payoffs. In the soft game (Cagman & Deli, 2013) the strategy sets and the soft payoffs are crisp. In fuzzy soft game the strategy sets are crisp but the fuzzy soft payoffs are fuzzy subsets of U. Here, we extend the definitions and results on game theory defined in (Cagman & Deli, 2013; Deli, 2013; Neumann & Morgenstern, 1944) to intuitionistic fuzzy soft game by using intuitionistic fuzzy soft set.

Definition 3.1 Let E be a set of strategy and X, Y $\subseteq$ E. A choice of behavior in an intuitionistic fuzzy soft game is called an if-action. The elements of X $\times$ Y are called action pairs. That is X $\times$ Y is the set of available if-actions.

Definition 3.2 Let U be a set of alternatives, IF (U) be all intuitionistic fuzzy sets over U, E be a set of strategies, X, Y $\subseteq E$. Then a set valued function $\gamma_{X\times Y}$: $X \times Y \rightarrow IF(U)$ is called an intuitionistic fuzzy soft payoff (ifs-payoff) function. For each $(x,y) \in X \times Y$, the value $\gamma_{X\times Y}(x,y)$ is called an ifs-payoff, where $\gamma_{X\times Y}(x,y)=(\mu_{X\times Y}(x,y),\delta_{X\times Y}(x,y))$, 1st coordinate is the membership value and the 2nd co-ordinate is the non-membership value.

Definition 3.3 Let $X\times Y$ be a set of if-action pairs. Then, an if-action $(x^*,y^*) \in X\times Y$ is called an optimal if-action if $\gamma_{X\times Y}(x^*,y^*) \supseteq \gamma_{X\times Y}(x,y)$ for all $(x,y) \in X\times Y$

Definition 3.4 Let $X\times Y$ be a set of if-action pairs and $(x_i,y_j),(x_r,y_s) \in X\times Y$. Then,

a) if $\gamma_{X\times Y}(x_i,y_j) \supseteq \gamma_{X\times Y}(x_r,y_s)$,we says that a player strictly prefers if-action pair (x_i,y_j) over if-action (x_r,y_s) .
b) if $\gamma_{X\times Y}(x_i,y_j) = \gamma_{X\times Y}(x_r,y_s)$,we says that a player is indifferent between the two if-actions.
c) if $\gamma_{X\times Y}(x_i,y_j) \supseteq \gamma_{X\times Y}(x_r,y_s)$,we says that a player either prefers (x_i,y_j) to (x_r,y_s) or is indifferent between the two if-actions.

Definition 3.5 Let $\gamma^k_{X\times Y}$ be an intuitionistic fuzzy soft payoff for player k, (k=1, 2), and $(x_i,y_j),(x_r,y_s) \in X\times Y$. Then, player k is called if-rational; if the player's intuitionistic fuzzy soft payoff satisfies the following conditions:

a) Either $\gamma^k_{X\times Y}(x_i,y_j) \supseteq \gamma^k_{X\times Y}(x_r,y_s)$ or

$\gamma^k_{X\times Y}(x_r,y_s) \supseteq \gamma^k_{X\times Y}(x_i,y_j)$

b) If $\gamma^k_{X\times Y}(x_i,y_j) \supseteq \gamma^k_{X\times Y}(x_r,y_s)$ and

$\gamma^k_{X\times Y}(x_r,y_s) \supseteq \gamma^k_{X\times Y}(x_i,y_j)$, then

$\gamma^k_{X\times Y}(x_i,y_j) = \gamma^k_{X\times Y}(x_r,y_s)$

Definition 3.6 Let X and Y be a set of strategies of Player 1 and 2, respectively, U be a set of alternatives and $\gamma^k_{X\times Y}$: $X\times Y \to IF(U)$ be an intuitionistic fuzzy soft payoff (ifs-payoff) function for player k, (k=1,2). Then, for each player k, a two person intuitionistic fuzzy soft game (tp ifs-game) is defined by an intuitionistic fuzzy soft set over U as

$$\Gamma^k_{X\times Y}=\{((x,y),\gamma^k_{X\times Y}(x,y)): (x,y) \in X\times Y\}$$

where $\gamma^k_{X\times Y}(x,y)=(\mu^k_{X\times Y}(x,y),\delta^k_{X\times Y}(x,y))$, 1st co-ordinate is the membership value and the 2nd co-ordinate is the non-membership value.

The tp ifs-game is played as follows: At a certain time Player 1 chooses a strategy $x_i \in X$, simultaneously Player 2 chooses a strategy $y_j \in Y$ and once this is done each player k (k=1,2) receives the intuitionistic fuzzy soft payoff $\gamma^k_{X\times Y}(x_i,y_j)$.

If $X=\{x_1,x_2,x_3,..........,x_m\}$ and $Y=\{y_1,y_2,y_3,..........,y_n\}$, then the intuitionistic fuzzy soft payoffs of $\Gamma^k_{X\times Y}$ can be arranged in the form of the m x n matrix shown in Table 1. Now, we can give an example for tp ifs-game.

Example 3.7 Let $U=\{u_1,u_2,u_3,u_4,u_5,u_6\}$ be a set of alternatives, IF(U) be all intuitionistic fuzzy sets over U, $E=\{x_1,x_2,x_3,x_4,x_5\}$ be a set of strategies and $X=\{x_1,x_2,x_4\}$ and $Y=\{x_1,x_2\}$ be a set of strategies of Player 1 and 2, respectively.

If Player 1 constructs a tp ifs-games as follows,

$$\Gamma^1_{X\times Y}=\{((x_1,x_1),\{0.7,0.2/u_1,0.6,0.3/u_2,0.4,0.5/u_5\}),\ ((x_1,x_2),\{0.2,0.7/u_1,0.3,0.7/u_2,0.8,0.1/u_3,0.1,0.7/u_4,0.9,0.1/u_5\}),\ ((x_3,x_1),\{0.8,0.2/u_1,0.1,0.7/u_3\}),((x_3,x_2),\{0.5,0.4/u_1,0.3,0.6/u_2,0.8,0.1/u_3,0.7,0.2/u_5\}),\ ((x_5,x_1),\{0.5,0.4/u_3,0.7,0.3/u_5,0.3,0.6/u_4\}),((x_5,x_2),\{0.5,0.4/u_1,0.6,0.3/u_2,0.5,0.5/u_3,0.7,0.3/u_4,0.3,0.6/u_5\})\}$$

then the fuzzy soft payoffs of the game can be arranged as in Table 2.

Let us explain some element of this game; if Player 1 selects X_3 and Player 2 selects X_2, then the value of game will be an intuitionistic fuzzy soft payoff $\gamma^1_{X\times Y}(x_3,x_2)=\{0.5,0.4/u_1,0.3,0.6/u_2,0.8,0.1/u_3,0.7,0.2/u_5\}$. In this case, Player 1 wins the set of alternatives $\{0.5,0.4/u_1,0.3,0.6/u_2,0.8,0.1/u_3,0.7,0.2/u_5\}$ and Player 2 loses the same set of alternatives.

Similarly, if Player 2 constructs a tp ifs-game as follows,

$$\Gamma^2_{X\times Y}=\{((x_1,x_1),\{0.7,0.3/u_3,0.6,0.3/u_4,0.4,0.5/u_6\}),\ ((x_1,x_2),\{0.2,0.7/u_6\}),\ ((x_3,x_1),\{0.8,0.2/u_2,0.1,0.8/u_4,0.3,0.6/u_5,0.8,0.1/u_6\}),((x_3,x_2),\{0.5,0.4/u_4,0.3,0.7/u_6\}),\ ((x_5,x_1),\{0.5,0.4/u_1,0.7,0.3/u_2,0.3,0.6/u_6\}),((x_5,x_2),\{0.5,0.4/u_6\})\}$$

then the intuitionistic fuzzy soft payoffs of the game can be arranged as in Table 3.

Let us explain some element of this tp ifs-game; if Player 1 selects X_3 and Player 2 selects X_2, then the value of game will be intuitionistic fuzzy soft payoff $\gamma^2_{X\times Y}(x_3,x_2)=\{0.5,0.4/u_4,0.3,0.7/u_6\}$. In this case, Player 1 wins the set of alternatives $\{0.5,0.4/u_4,0.3,0.7/u_6\}$and Player 2 loses $\{0.5,0.4/u_4,0.3,0.7/u_6\}$

Table 1. Two person intuitionistic fuzzy soft game

$\Gamma^k_{X\times Y}$	y_1	y_2	….	y_n
x_1	$\gamma^k_{X\times Y}(x_1,y_1)$	$\gamma^k_{X\times Y}(x_1,y_2)$	….	$\gamma^k_{X\times Y}(x_1,y_n)$
x_2	$\gamma^k_{X\times Y}(x_2,y_1)$	$\gamma^k_{X\times Y}(x_2,y_2)$	….	$\gamma^k_{X\times Y}(x_2,y_n)$
⋮	⋮	⋮	⋱	⋮
x_m	$\gamma^k_{X\times Y}(x_m,y_1)$	$\gamma^k_{X\times Y}(x_m,y_2)$	….	$\gamma^k_{X\times Y}(x_m,y_n)$

Table 2. Fuzzy soft payoffs of the game

$\Gamma^1_{X\times Y}$	x_1	x_2
x_1	$\{0.7,0.2/u_1,0.6,0.3/u_2,0.4,0.5/u_5\}$	$\{0.2,0.7/u_1,0.3,0.7/u_2,0.8,0.1/u_3,0.1,0.7/u_4,0.9,0.1/u_5\}$
x_3	$\{0.8,0.2/u_1,0.1,0.7/u_3\}$	$\{0.5,0.4/u_1,0.3,0.6/u_2,0.8,0.1/u_3,0.7,0.2/u_5\}$
x_5	$\{0.5,0.4/u_3,0.7,0.3/u_5,0.3,0.6/u_4\}$	$\{0.5,0.4/u_1,0.6,0.3/u_2,0.5,0.5/u_3,0.7,0.3/u_4,0.3,0.6/u_5\}$

Table 3. Intuitionistic fuzzy soft payoffs of the game

$\Gamma^2_{X\times Y}$	x_1	x_2
x_1	$\{0.7,0.3/u_3,0.6,0.3/u_4,0.4,0.5/u_6\}$	$\{0.2,0.7/u_6\}$
x_3	$\{0.8,0.2/u_2,0.1,0.8/u_4,0.3,0.6/u_5,0.8,0.1/u_6\}$	$\{0.5,0.4/u_4,0.3,0.7/u_6\}$
x_5	$\{0.5,0.4/u_1,0.7,0.3/u_2,0.3,0.6/u_6\}$	$\{0.5,0.4/u_6\}$

Definition 3.8 A tp ifs-game is called a two person empty intersection intuitionistic fuzzy soft game if intersection of the ifs-payoff of players is empty set for each if-action pairs.

Definition 3.9 Let $\gamma^k_{X\times Y}$ be an intuitionistic fuzzy soft payoff function of a tp ifs-game $\Gamma^k_{X\times Y}$. Then the following properties hold

a) $\bigcup_{i=1}^{m}\gamma^k_{X\times Y}(x_i,y_j)=\gamma^k_{X\times Y}(x,y)=\{\max\mu^K_{x\times Y}(x_i,y_j),\min\delta^K_{X\,xY}(x_i,y_j)\}/u_t$,where i=1,2,.....,m

b) $\bigcap_{j=1}^{n}\gamma^k_{X\times Y}=\gamma^k_{X\times Y}(x,y)=\{\min\mu^K_{x\times Y}(x_i,y_j),\max\delta^K_{X\,xY}(x_i,y_j)\}/u_t$,where j=1,2,.....,n

then $\gamma^k_{X\times Y}(x,y)$ is called an intuitionistic fuzzy soft saddle point value and (x,y) is called a fuzzy soft saddle point of player k's in the tp ifs-game.

We will take Γ^k_A , Γ^k_B , Γ^k_C,………… etc. for two person intuitionistic fuzzy soft game and $\gamma^k_A,\gamma^k_B,\gamma^k_C$,…….etc. for their intuitionistic fuzzy soft payoffs respectively. We now give two examples:

Example 3.10 Let $U=\{u_1,u_2,u_3,u_4,u_5,u_6,u_7\}$ be a set of alternatives, $X=\{x_1,x_2,x_3\}$ and $Y=\{y_1,y_2\}$ be the strategies for Player 1 and 2, respectively. Then, tp ifs-game of Player 1 is given as in Table 4**.**

Clearly,

$\bigcup_{i=1}^{3}\gamma^1_{X\times Y}(x_i,y_1)=\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}=$

$\{\max\mu^1_{X\times Y}(x_i,y_j),\min\delta^1_{X\times Y}(x_i,y_j)\}/u_t$,

where i=1,2,3 and j=1,2,....,10.

Similarly,

$\bigcup_{i=1}^{3}\gamma^1_{X\times Y}(x_i,y_2)=\{0.9,0.1/u_1,0.6,0.3/u_2,0.8,0.2/u_3,0.9,0.1/u_4,0.8,0.2/u_5\}$

and

$\bigcap_{j=1}^{2}\gamma^1_{X\times Y}(x_i,y_j)=\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}=\{\min\mu^1_{X\times Y}(x_i,y_j),$

$\max\delta^1_{X\times Y}(x_i,y_j)\}/u_t$, where j=1,2and j=1,2,....,10.

Similarly,

$\bigcap_{j=1}^{2}\gamma^1_{X\times Y}(x_2,y_j)=\{0.5,0.4/u_1\}$

$\bigcap_{j=1}^{2}\gamma^1_{X\times Y}(x_3,y_j)=\{0.2,0.6/u_1,0.1,0.8/u_4\}$

Therefore, $\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$ is an intuitionistic fuzzy soft saddle point value of the tp ifs-game, since the intersection of the first row is equal to the union of the first column. So, the value of the tp ifs-game is $\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$.

Note that every tp ifs-game has not an intuitionistic fuzzy soft saddle point (for instance, in the above example, if $\{0.8,0.1/u_1,0.4,0.5/u_2\}$ is replaced with $\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$ in intuitionistic fuzzy soft payoff $\gamma^1_{X\times Y}(x_1,y_1)$, then an intuitionistic fuzzy soft saddle point of the game cannot be found).

Table 4. tp ifs-game of Player 1

$\Gamma^1_{X\times Y}$	y_1	y_2
x_1	$\{0.8,0.1/u_1, 0.4,0.5/u_2, 0.6,0.3/u_4\}$	$\{0.9,0.1/u_1, 0.6,0.3/u_2, 0.6,0.4/u_3, 0.9,0.1/u_4, 0.5,0.4/u_5\}$
x_2	$\{0.7,0.2/u_1, 0.1,0.7/u_2\}$	$\{0.5,0.4/u_1, 0.8,0.2/u_3, 0.8,0.2/u_5\}$
x_3	$\{0.2,0.6/u_1, 0.1,0.8/u_4\}$	$\{0.5,0.4/u_1, 0.5,0.5/u_3, 0.7,0.3/u_4\}$

Example 3.11 Let U = $\{u_1,u_2,u_3,u_4,..........,u_{10}\}$ be a set of alternatives. $X=\{x_1,x_2,x_3,x_4\}$ and $Y=\{y_1,y_2,y_3\}$ be the strategies for Player 1 and 2 respectively. Then, two person fuzzy soft game of Player 1 is given by the Table 5.

Table 5. Two person fuzzy soft game of Player 1

$\Gamma^1_{X\times Y}$	y_1	y_2	y_3
x_1	$\{(0.4,0.5)/u_2, (0.6,0.4)/u_4, (0.7,0.2)/u_7\}$	$\{(0.3,0.6)/u_4\}$	$\{(0.5,0.4)/u_4\}$
x_2	$\{(0.3,0.6)/u_5\}$	$\{(0.6,0.3)/u_7\}$	$\{(0.4,0.5)/u_4, (0.5,0.4)/u_7\}$
x_3	$\{(0.3,0.7)/u_2, (0.4,0.5)/u_4, (0.6,0.3)/u_5, (0.2,0.8)/u_7, (0.8,0.1)/u_8, (0.5,0.4)/u_{10}\}$	$\{(0.7,0.3)/u_4, (0.8,0.2)/u_8\}$	$\{(0.6,0.4)/u_7, (0.3,0.6)/u_8$
x_4	$\{(0.2,0.8)/u_2, (0.5,0.4)/u_4, (0.6,0.3)/u_5, (0.7,0.2)/u_7, (0.8,0.2)/u_8\}$	$\{(0.2,0.8)/u_1, (0.6,0.3)/u_4, (0.8,0.2)/u_7, (0.7,0.3)/u_8\}$	$\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.2)/u_8\}$

$$\vee_{i=1}^{4} f_{s_1}(x_i,y_1)=\{(0.4,0.5)/u_2, (0.6,0.4)/u_4, (0.6,0.3)/u_5, (0.8,0.1)/u_8, (0.7,0.2)/u_7, (0.5,0.4)/u_{10}\}$$
$$\vee_{i=1}^{4} f_{s_1}(x_i,y_2)=\{(0.2,0.8)/u_1, (0.7,0.3)/u_4, (0.8,0.2)/u_7, (0.8,0.2)/u_8\}$$
$$\vee_{i=1}^{4} f_{s_1}(x_i,y_3)=\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$$
$$\wedge_{j=1}^{3} f_{s_1}(x_1,y_j)=\{(0.3,0.6)/u_4\}$$
$$\wedge_{j=1}^{3} f_{s_1}(x_2,y_j)=\{(0.0,1.0)$$
$$\wedge_{j=1}^{3} f_{s_1}(x_3,y_j)=\{(0.3,0.6)/u_8\}$$
$$\wedge_{j=1}^{3} f_{s_1}(x_4,y_j)=\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$$

Therefore, $\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$ is an intuitionistic fuzzy soft saddle point value of the tp ifs-game, since the intersection of the fourth row is equal to the union of the third column. So, the value of the tp ifs-game is $\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$.

Note that every tp ifs-game has not a fuzzy soft saddle point (For instance, in the above example, if $\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$ is replaced with $\{(0.5,0.4)/u_4, (0.6,0.4)/u_7, (0.7,0.3)/u_8\}$ in intuitionistic fuzzy soft payoff $\gamma^1_{X\times Y}(x_1,y_1)$, then an intuitionistic fuzzy soft saddle point of the game cannot be found).

4. n-Person Intuitionistic Fuzzy Soft Games

In many applications the intuitionistic fuzzy soft games can be often played between more than two players. Therefore, tp ifs-games can be extended to n-person intuitionistic fuzzy soft games.

From now on, X_n^x will be used for $X_1 \times X_2 \times X_3 \times ....\times X_n$.

Definition 4.1 Let U be a set of alternatives, $I^{F(U)}$ be all intuitionistic fuzzy sets over U, E be a set of strategies, and $X_1, X_2, X_3, ..., X_n \subseteq E$, X_k is the set of strategies of player k, (k=1,2,3,…..,n). Then, for each player k, an n-person intuitionistic fuzzy soft game is defined by a fuzzy soft set over U as

$$\overset{*}{\Gamma}{}^{k}_{X_n^\times} = \{((x_1, x_2, x_3, ..., x_n), (\mu^k_{X_n^\times}(x_1, x_2, x_3, ..., x_n), \delta^k_{X_n^\times}(x_1, x_2, x_3, ..., x_n))) : (x_1, x_2, x_3, ..., x_n) \in X_n^\times\}$$

Where $\mu^k_{X_n^\times}$ and $\delta^k_{X_n^\times}$ are intuitionistic fuzzy soft payoff functions respectively for membership and non-membership of player k.

The np ifs-game is played as follows: At a certain time Player 1 chooses a strategy $x_1 \in X_1$ and simultaneously each player k (k=2, 3, ……, n) chooses a strategy $x_k \in X_k$ and once this is done each player k receives the intuitionistic fuzzy soft payoff functions

$$\mu^k_{X_n^\times}(x_1, x_2, x_3, ..., x_n) \text{ and } \delta^k_{X_n^\times}(x_1, x_2, x_3, ..., x_n).$$

Definition 4.2 Let $\overset{*}{\Gamma}{}^{k}_{X_n^\times}$ be an np ifs-game with its intuitionistic fuzzy soft payoff functions $\mu^k_{X_n^\times}$ and $\delta^k_{X_n^\times}$ for k=1, 2, ……., n. Then, a strategy $x_k \in X_k$ is called intuitionistic fuzzy soft dominated to another strategy $x \in X_k$, if

$$\mu^k_{X_n^\times}(x_1,x_2,......,x_{k-1},x_k,x_{k+1},......,x_n) \supseteq \mu^k_{X_n^\times}(x_1,x_2,......,x_{k-1},x,x_{k+1},......,x_n)$$

$$\delta^k_{X_n^\times}(x_1,x_2,......,x_{k-1},x_k,x_{k+1},......,x_n) \subseteq \delta^k_{X_n^\times}(x_1,x_2,......,x_{k-1},x,x_{k+1},......,x_n)$$

For each strategy $x_i \in X_i$ of player i (i=1,2,………,k-1,k+1,…….,n), respectively.

Definition 4.3 Let $\mu^k_{X_n^\times}$ and $\delta^k_{X_n^\times}$ be intuitionistic fuzzy soft payoff functions respectively for membership and non-membership of and ifs-game $\Gamma^k_{X_n^\times}$. If for each player k (k=1, 2, ….., n) the following properties hold:

$$\mu^k_{X_n^\times}(x_1^*,x_2^*,.....,x_{k-1}^*,x_k^*,x_{k+1}^*,.........,x_n^*) \supseteq \mu^k_{X_n^\times}(x_1^*,x_2^*,.....,x_{k-1}^*,x,x_{k+1}^*,.........,x_n^*)$$

$$\delta^k_{X_n^\times}(x_1^*,x_2^*,.....,x_{k-1}^*,x_k^*,x_{k+1}^*,.........,x_n^*) \subseteq \delta^k_{X_n^\times}(x_1^*,x_2^*,.....,x_{k-1}^*,x,x_{k+1}^*,.........,x_n^*)$$

For each $x \in X_k$, then $(x_1^*,x_1^*,.......,x_1^*) \in X_n^\times$ is called an intuitionistic fuzzy soft Nash equilibrium of an np ifs-game.

Definition 4.4 Let $\Gamma_{X\times Y}$ be a tp ifs-game with its ifs-payoff function $\gamma_{X\times Y}$ where $\gamma_{X\times Y}(x,y)=\{\mu_{X\times Y}(x,y),\delta_{X\times Y}(x,y)\}$ where 1st co-ordinate is the membership value and the 2nd co-ordinate is the non-membership value. Then

a) Intuitionistic fuzzy soft upper value of the tp ifs-game, denoted by

$$V^* = \{\bigcap_{y\in Y}^{n}(\bigcup_{x\in X}^{n}\gamma_{X\times Y}(x,y)) = \{\bigcap_{y\in Y}^{n}(\bigcup_{x\in X}^{n}\mu_{X\times Y}(x,y)), \bigcup_{y\in Y}^{n}(\bigcap_{x\in X}^{n}\delta_{X\times Y}(x,y))\}$$

b) Intuitionistic fuzzy soft lower value of the tp ifs-game, denoted by

$$V_* = \{\bigcup_{x\in X}(\bigcap_{y\in Y}(\gamma_{X\times Y}(x,y))\} = \{(\bigcup_{x\in X}(\bigcap_{y\in Y}\mu_{X\times Y}(x,y)), (\bigcap_{x\in X}(\bigcup_{y\in Y}^{n}\delta_{X\times Y}(x,y))\}$$

c) If intuitionistic fuzzy soft upper (ifs-upper) and intuitionistic fuzzy soft lower (ifs-lower) value of a tp ifs-game are equal, then they are called value of the tp ifs-game denoted by V i.e $V = V^* = V_*$

A game with saddle point is that in which the players use pure strategies i.e they choose the same course of action throughout the game. The point of intersection of their pure strategies used is known as a saddle point. The gain at the saddle point gives the value of the game.

A game with optimal pure strategies is sometimes called “strictly determined”. Thus for strictly determined games V= V* =V* (If ifs upper and ifs lower value of a tp ifs-game are equal, they are called value of the tp ifs-game, denoted by V, i.e V= V* =V*.

Thus a saddle point is that at which the element of pay-off matrix is both the smallest in its row and the greatest in the column. Not all the rectangular games involve a saddle point, but if a game has a saddle point, then the pure strategy corresponding to a saddle point is the best strategies and the number at that point is the value of the game. If a matrix involves more than one saddle point then there exists more than one optimal solution of the game.

Example 4.5 Let us consider the table 4 of the example 3.10. Here we have $V^*=\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$ is the ifs-upper value and ifs lower value is $V_*=\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$. Hence $V^*=V_*$

It shows that the value of tp ifs-game is $\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}$

Theorem 4.6 V_* and V^* be a ifs- lower and ifs-upper value of a tp ifs-game respectively. Then $V_*\subseteq V^*$

Proof: Assuming V^* be a ifs-upper value and V_* be a ifs-lower value of a tp ifs-game and $X=\{x_1,x_2,x_3,.......,x_m\}$ and $Y=\{y_1,y_2,y_3,.......,y_n\}$ are sets of strategies for player 1 and player 2 respectively. We choose $x_i^*\in X$ and $y_j^*\in Y$. Then

$$V_*=\bigcup_{x\in X}(\bigcap_{y\in Y}(\gamma_{X\times Y}(x,y))$$
$$=\{(\bigcup_{x\in X}(\bigcap_{y\in Y}\mu_{X\times Y}(x,y)),(\bigcap_{x\in X}(\bigcup_{y\in Y}^{n}\delta_{X\times Y}(x,y))\}$$
$$\subseteq\{\bigcap_{y\in Y}\mu_{X\times Y}(x^*,y),\bigcup_{y\in Y}\delta_{X\times Y}(x^*,y)\}$$
$$\subseteq\{\mu_{X\times Y}(x^*,y^*),\delta_{X\times Y}(x^*,y^*)\}$$
$$\subseteq(\bigcup_{x\in X}\{\mu_{X\times Y}(x,y^*)\},\{\bigcap_{x\in X}\delta_{X\times Y}(x,y^*)\}$$
$$\subseteq\{(\bigcap_{y\in Y}(\bigcup_{x\in X}\mu_{X\times Y}(x,y)),\bigcup_{y\in Y}(\bigcap_{x\in X}\delta_{X\times Y}(x,y))\}$$
$$=V^*$$

Example 4.7 In example 3.10, $V^*=\{0.8,0.1/u_1,0.4,0.5/u_2,0.6,0.3/u_4\}\subseteq V_*$

Theorem 4.8 Let $\gamma_{X\times Y}(x,y)$ be an ifs- saddle point value, V^* be an ifs-upper value and V_* be an ifs-lower value of a ifs-game then $V_*\subseteq\gamma_{X\times Y}(x,y)\subseteq V^*$

Proof: Assume that $\gamma_{s_k}(x^*,y^*)=\{\mu_{s_k}(x^*,y^*),\delta_{s_k}(x^*,y^*)\}$ be an ifs-saddle point value; V_* be an ifs-lower value and V^* be an ifs-upper value of a tp ifs-game and $X=\{x_1,x_2,x_3,.......,x_m\}$ and $Y=\{y_1,y_2,y_3,.......,y_n\}$ are sets of strategies for player 1 and player 2 respectively.

We choose $x_i^*\in X$ and $y_j^*\in Y$ then

$$\bigcup_{i=1}^{m}\gamma_{s_k}(x_i,y_j)=\bigcap_{j=1}^{n}\gamma_{s_k}(x_i,y_j)=\gamma_{s_k}(x^*,y^*)$$

Since $\gamma_{S_k}(x^*,y^*)$ is an ifs-saddle point value. Now

$$V_*=\left\{\bigcup_{x\in X}(\bigcap_{y\in Y}(\gamma_{XxY}(x,y)))\right\}$$
$$=\left\{\bigcup_{x\in X}(\bigcap_{y\in Y}\mu_{XxY}(x,y),\bigcap_{x\in X}(\bigcup_{y\in Y}(\delta_{X\times Y}(x,y))\right\}$$
$$\subseteq\bigcup_{i=1}^{m}(\gamma_{S_k}(x_i,y_i))$$
$$=\bigcup_{i=1}^{m}\left\{\mu_{S_k}(x_i,y_j)\right\},\{\bigcap_{i=1}^{m}\delta_{S_k}(x_i,y_j)\}$$
$$=\gamma_{S_k}(x^*,y^*)$$
$$=(\mu_{S_k}(x^*,y^*),\delta_{S_k}(x^*,y^*))\ \ldots\ldots\ldots\ldots\ldots(1)$$

On the other hand,

$$\gamma_{S_k}(x^*,y^*)=\bigcap_{j=1}^{n}\gamma_{S_k}(x_i,y_j)$$
$$=\left\{\bigcap_{j=1}^{n}\left\{\mu_{S_k}(x_i,y_j)\},\{\bigcup_{j=1}^{m}\delta_{S_k}(x_i,y_j)\right\}\right\}$$
$$\subseteq\left\{(\bigcap_{y\in Y}(\bigcup_{x\in X}\mu_{XxY}(x,y))),(\bigcap_{y\in Y}(\bigcup_{x\in X}\delta_{XxY}(x,y)))\right\}$$
$$=\bigcap_{y\in Y}(\bigcup_{x\in X}(\gamma_{XxY}(x,y))$$
$$=V^*\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

From (1) and (2) $V_* \subseteq \gamma_{XxY}(x,y) \subseteq V^*$

Note:

1) If (x,y) be an ifs-saddle point, V_* be an ifs-lower value and V^* be an ifs-upper value of a tp ifs-game. If $V^* = V_* = V$ then $\gamma_{XxY}(x,y)$ is exactly V.

2) In every tp ifs-game the ifs-lower value V_* cannot be equal to the ifs-upper value V^*. In (0.8,0.1/u_1,0.4,0.5/u_2, 0.6,0.3/ u_4) is replaced with (0.1,0.1/ u_1,0.4,0.5/ u_2,0.6,0.3/ u_4) in ifs-pay off $\gamma^1_{XxY}(x_i,y_j)$, then the ifs-lower value V_* cannot be equal to the ifs-upper value V^*.

3) Here we give a real example based on financial problem that can be solved by using ifs saddle point method reference to example 3.10.
   Suppose there are two car manufacturing companies, say Player 1 and Player 2, who wish to increase the sale of production of cars throughout the country. For that, they decided to give advertisement. To attract the customers for buying cars, the two companies consider a set of parameters, where(i=1,2,3,….,7) stands for "size", "capacity", "family members", "styling", "fuel efficiency", "price" and "comfort".

The parameters can be characterized by a set of strategy and the strategies (j=1,2,3) stand for "social networking site", "newspaper" and "sales person".

Suppose $X = \{x_1,x_2,x_3\}$ and $Y = \{x_1 = y_1, x_2 = y_2\}$ are strategies of Player 1 and Player 2, respectively. Then a tp ifs-game of player 1 is given as reference to Table 4. Here, optimal strategy of the game is (x_1, y_1) since

$$\bigcup_{i=1}^{3}\gamma^1_{X\times Y}(x_i,y_1) = \bigcap_{j=1}^{2}\gamma^1_{X\times Y}(x_1,y_j)$$

Therefore the value of the tp ifs-game is $\{(0.8,0.1)/u_1,(0.4,0.5)/u_2,(0.6,0.3)/u_4\}$

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