Songklanakarin J. Sci. Technol.
40 (1), 9-29, Jan. - Feb. 2018

http://www.sjst.psu.ac.th

Original Article

Q-fuzzy sets in UP-algebras

Kanlaya Tanamoon, Sarinya Sripaeng, and Aiyared Iampan*

Department of Mathematics, School of Science University of Phayao, Mueang, Phayao, 56000 Thailand

Received: 21 March 2016; Revised: 29 July 2016; Accepted: 27 September 2016

Abstract

In this paper, we introduce the notions of Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) and a level subsets of a Q-fuzzy set are investigated, and conditions for a Q-fuzzy set to be a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu\cdot\delta$ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of $A\times B$, then either μ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of A or δ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of B.

Keywords: UP-algebra, Q-fuzzy UP-ideal, Q-fuzzy UP-subalgebra

1. Introduction and Preliminaries

The concept of a fuzzy subset of a set was first considered by Zadeh (1965). The fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

The concept of Q fuzzy sets is introduced by many researchers and was extensively investigated in many algebraic structures such as: Jun (2001) introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras. Roh *et al.* (2006) studied intuitionistic Q-fuzzy subalgebras of BCK/BCI-algebras. Muthuraj *et al.* (2010) introduced and investigated anti Q-fuzzy BG-ideals of BG-algebras. Mostafa *et al.* (2012) introduced the notions of Q-ideals and fuzzy Q-ideals in Q-algebras. Sitharselvam *et al.* (2012), Sithar Selvam *et al.* (2013) and Selvam *et al.* (2014) introduced and gave some properties anti Q-fuzzy KU-ideals, anti Q-fuzzy KU-subalgebras and anti Q-fuzzy R-closed KU-ideals of KU-algebras. The notion of anti Q-fuzzy R-closed PS-ideals of PS-algebras is introduced, and related properties are investigated Priya and Ramachandran (2014).

Iampan (2014) introduced a new algebraic structure, called a UP-algebra. In this paper, we introduce the notions of Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) and a level subsets of a Q-fuzzy set are investigated, and conditions for a Q-fuzzy set to be a Q-fuzzy UP-ideal (resp. Q fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu\cdot\delta$ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of $A\times B$, then either μ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of A or δ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of B. Before we begin our study, we will introduce the definition of a UP-algebras.

*Corresponding author
Email address: aiyared.ia@up.ac.th

Definition 1.1. (Iampan, 2014) An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

In (Iampan, 2014) there is given an example of a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of a UP-algebra.

Proposition 1.2. *(Iampan, 2014) In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

Definition 1.3. (Iampan, 2014) A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.4. *(Iampan, 2014) Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Definition 1.5. (Iampan, 2014) A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Proposition 1.6. *(Iampan, 2014) A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.7. *(Iampan, 2014) Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A.*

Lemma 1.8. *(Somjanta et al., 2015) Let f be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) $1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$, *and*

(2) $1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.

Definition 1.9. (Kim, 2006) A *Q-fuzzy set* in a nonempty set X (or a Q-fuzzy subset of X) is an arbitrary function $f\colon X \times Q \to [0,1]$ where Q is a nonempty set and $[0,1]$ is the unit segment of the real line.

Definition 1.10. A Q-fuzzy set f in A is called a *q-fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0,q) \geq f(x,q)$, and

(2) $f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y,q)\}$.

A Q-fuzzy set f in A is called a *Q-fuzzy UP-ideal* of A if it is a q-fuzzy UP-ideal of A for all $q \in Q$.

Example 1.11. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We define a Q-fuzzy set f in A as follows:

f	a	b
0	0.3	0.2
1	0.1	0.1

Using this data, we can show that f is a Q-fuzzy UP-ideal of A.

Example 1.12. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We define a Q-fuzzy set f in A as follows:

f	a	b
0	0.3	0.1
1	0.1	0.2

By Example 1.11, we have f is an a-fuzzy UP-ideal of A. Since $f(0,b) = 0.1 < 0.2 = f(1,b)$, we have Definition 1.10 (1) is false. Therefore, f is not a b-fuzzy UP-ideal of A. Hence, f is not a Q-fuzzy UP-ideal of A.

Definition 1.13. A Q-fuzzy set f in A is called a *q-fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$f(x \cdot y, q) \geq \min\{f(x,q), f(y,q)\}.$$

A Q-fuzzy set f in A is called a *Q-fuzzy UP-subalgebra* of A if it is a q-fuzzy UP-subalgebra of A for all $q \in Q$.

Example 1.14. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	1
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We defined a Q-fuzzy set f in A as follows:

f	a	b
0	0.4	0.7
1	0.2	0.1
2	0.3	0.5

Using this data, we can show that f is a Q-fuzzy UP-subalgebra of A.

Example 1.15. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	1
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We defined a Q-fuzzy set f in A as follows:

f	a	b
0	0.4	0.1
1	0.2	0.5
2	0.3	0.7

By Example 1.14, we have f is an a-fuzzy UP-subalgebra of A. Since $f(1 \cdot 1, b) = 0.1 < 0.5 = \min\{f(1, b), f(1, b)\}$, we have Definition 1.13 is false. Therefore, f is not a b-fuzzy UP-subalgebra of A. Hence, f is not a Q-fuzzy UP-subalgebra of A.

Definition 1.16. (Kim, 2006) Let f be a Q-fuzzy set in A. The Q-fuzzy set $\overline{f}$ defined by $\overline{f}(x, q) = 1 - f(x, q)$ for all $x \in A$ and $q \in Q$ is called the *complement* of f in A.

Remark 1.17. For all Q-fuzzy set f in A, we have $f = \overline{\overline{f}}$.

Definition 1.18. Let f be a Q-fuzzy set in A. For any $t \in [0, 1]$, the sets

$$U(f; t) = \{x \in A \mid f(x, q) \geq t \text{ for all } q \in Q\}$$

and

$$U^+(f; t) = \{x \in A \mid f(x, q) > t \text{ for all } q \in Q\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The sets

$$L(f; t) = \{x \in A \mid f(x, q) \leq t \text{ for all } q \in Q\}$$

and

$$L^-(f; t) = \{x \in A \mid f(x, q) < t \text{ for all } q \in Q\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively. For any $q \in Q$, the sets

$$U(f; t, q) = \{x \in A \mid f(x, q) \geq t\}$$

and

$$U^+(f; t, q) = \{x \in A \mid f(x, q) > t\}$$

are called a *q-upper t-level subset* and a *q-upper t-strong level subset* of f, respectively. The sets

$$L(f; t, q) = \{x \in A \mid f(x, q) \leq t\}$$

and

$$L^-(f;t,q) = \{x \in A \mid f(x,q) < t\}$$

are called a *q-lower t-level subset* and a *q-lower t-strong level subset* of f, respectively.

We can easily prove the following two remarks.

Remark 1.19. Let f be a Q-fuzzy set in A and for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$. Then the following properties hold:

(1) $L(f;t_1) \subseteq L(f;t_2)$,

(2) $U(f;t_2) \subseteq U(f;t_1)$,

(3) $L^-(f;t_1) \subseteq L^-(f;t_2)$, and

(4) $U^+(f;t_2) \subseteq U^+(f;t_1)$.

Remark 1.20. Let f be a Q-fuzzy set in A and for any $t_1, t_2 \in [0,1]$ with $t_1 \leq t_2$ and $q \in Q$. Then the following properties hold:

(1) $L(f;t_1,q) \subseteq L(f;t_2,q)$,

(2) $U(f;t_2,q) \subseteq U(f;t_1,q)$,

(3) $L^-(f;t_1,q) \subseteq L^-(f;t_2,q)$, and

(4) $U^+(f;t_2,q) \subseteq U^+(f;t_1,q)$.

Definition 1.21. (Iampan, 2014) Let $(A;\cdot,0)$ and $(A';\cdot',0')$ be UP-algebras. A mapping f from A to A' is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f\colon A \to A'$ is called a

(1) *UP-endomorphism* of A if $A' = A$,

(2) *UP-epimorphism* if f is surjective,

(3) *UP-monomorphism* if f is injective, and

(4) *UP-isomorphism* if f is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

Proposition 1.22. *(Iampan, 2014) Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then $f(0_A) = 0_B$.*

Definition 1.23. (Sithar Selvam et al., 2013) Let $f\colon A \to B$ be a function and μ be a Q-fuzzy set in B. We define a new Q-fuzzy set in A by μ_f as

$$\mu_f(x,q) = \mu(f(x),q) \text{ for all } x \in A \text{ and } q \in Q.$$

Definition 1.24. (Sithar Selvam et al., 2013) Let $f\colon A \to B$ be a bijection and μ_f be a Q-fuzzy set in A. We define a new Q-fuzzy set in B by μ as

$$\mu(y,q) = \mu_f(x,q) \text{ where } f(x) = y \text{ for all } y \in B \text{ and } q \in Q.$$

Definition 1.25. (Sithar Selvam et al., 2013) Let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. The *Cartesian product* $\mu \times \delta\colon (A \times B) \times Q \to [0,1]$ is defined by

$$(\mu \times \delta)((x,y),q) = \max\{\mu(x,q), \delta(y,q)\} \text{ for all } x \in A, y \in B \text{ and } q \in Q.$$

The *dot product* $\mu \cdot \delta \colon (A \times B) \times Q \to [0,1]$ is defined by

$$(\mu \cdot \delta)((x,y),q) = \min\{\mu(x,q), \delta(y,q)\} \text{ for all } x \in A, y \in B \text{ and } q \in Q.$$

2 Main Results

In this section, we study Q-fuzzy UP-ideals and Q-fuzzy UP-subalgebras of UP-algebras, and their properties are investigated. Relations between a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) and a level subsets of a Q-fuzzy set are investigated, and conditions for a Q-fuzzy set to be a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) are provided. Finally, prove that it is not true that if $\mu \cdot \delta$ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of $A \times B$, then either μ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of A or δ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of B.

Theorem 2.1. *Every q-fuzzy UP-ideal of A is a q-fuzzy UP-subalgebra of A.*

Proof. Let f be a q-fuzzy UP-ideal of A. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot y, q) &\geq \min\{f(x \cdot (y \cdot y), q), f(y,q)\} && \text{(Definition 1.10 (2))} \\ &= \min\{f(x \cdot 0, q), f(y,q)\} && \text{(Proposition 1.2 (1))} \\ &= \min\{f(0,q), f(y,q)\} && \text{(UP-3)} \\ &= f(y,q) && \text{(Definition 1.10 (1))} \\ &\geq \min\{f(x,q), f(y,q)\}. \end{aligned}$$

Hence, f is a q-fuzzy UP-subalgebra of A.

With Definition 1.10 and Theorem 2.1, we obtain the corollary.

Corollary 2.2. *Every Q-fuzzy UP-ideal of A is a Q-fuzzy UP-subalgebra of A.*

Theorem 2.3. *If f is a q-fuzzy UP-subalgebra of A, then $f(0,q) \geq f(x,q)$ for all $x \in A$.*

Proof. Assume that f is a q-fuzzy UP-subalgebra of A. By Proposition 1.2 (1), we have $f(0,q) = f(x \cdot x, q) \geq \min\{f(x,q), f(x,q)\} = f(x,q)$ for all $x \in A$.

With Definition 1.13 and Theorem 2.3, we obtain the corollary.

Corollary 2.4. *If f is a Q-fuzzy UP-subalgebra of A, then $f(0,q) \geq f(x,q)$ for all $x \in A$ and $q \in Q$.*

We can easily prove the following three lemmas.

Lemma 2.5. *Let f be a Q-fuzzy set in A and for any $t \in [0,1]$. Then the following properties hold:*

(1) $L(f;t) = U(\overline{f}; 1-t)$,

(2) $L^-(f;t) = U^+(\overline{f}; 1-t)$,

(3) $U(f;t) = L(\overline{f}; 1-t)$, *and*

(4) $U^+(f;t) = L^-(\overline{f}; 1-t)$.

Lemma 2.6. *Let f be a Q-fuzzy set in A and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:*

(1) $L(f;t,q) = U(\overline{f};1-t,q)$,

(2) $L^-(f;t,q) = U^+(\overline{f};1-t,q)$,

(3) $U(f;t,q) = L(\overline{f};1-t,q)$, *and*

(4) $U^+(f;t,q) = L^-(\overline{f};1-t,q)$.

Lemma 2.7. *Let f be a Q-fuzzy set in A and for any $t \in [0,1]$ and $q \in Q$. Then the following properties hold:*

(1) $L(f;t) = \bigcap_{q\in Q} L(f;t,q)$,

(2) $L^-(f;t) = \bigcap_{q\in Q} L^-(f;t,q)$,

(3) $U(f;t) = \bigcap_{q\in Q} U(f;t,q)$, *and*

(4) $U^+(f;t) = \bigcap_{q\in Q} U^+(f;t,q)$.

Lemma 2.8. *(Malik and Arora, 2014) For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \frac{b+a}{2} < b$.*

Theorem 2.9. *Let f be a Q-fuzzy set in A. Then the following statements hold:*

(1) $\overline{f}$ is a Q-fuzzy UP-ideal of A if and only if the following condition $(\star)$ holds: for any $t \in [0,1]$ and $q \in Q$, $L(f;t,q)$ is either empty or a UP-ideal of A,

(2) $\overline{f}$ is a Q-fuzzy UP-ideal of A if and only if the following condition $(\star)$ holds: for any $t \in [0,1]$ and $q \in Q$, $L^-(f;t,q)$ is either empty or a UP-ideal of A,

(3) f is a Q-fuzzy UP-ideal of A if and only if the following condition $(\star)$ holds: for any $t \in [0,1]$ and $q \in Q$, $U(f;t,q)$ is either empty or a UP-ideal of A, and

(4) f is a Q-fuzzy UP-ideal of A if and only if the following condition $(\star)$ holds: for any $t \in [0,1]$ and $q \in Q$, $U^+(f;t,q)$ is either empty or a UP-ideal of A.

Proof. (1) Assume that $\overline{f}$ is a Q-fuzzy UP-ideal of A. Then $\overline{f}$ is a q-fuzzy UP-ideal of A for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(f;t,q) \neq \emptyset$ and let $x \in L(f;t,q)$. Then $f(x,q) \leq t$. Now,

$$\begin{aligned}
\overline{f}(0,q) &= \overline{f}(x\cdot 0,q) && \text{(UP-3)}\\
&\geq \min\{\overline{f}(x\cdot(x\cdot 0),q), \overline{f}(x,q)\} && \text{(Definition 1.10 (2))}\\
&= \min\{\overline{f}(x\cdot 0,q), \overline{f}(x,q)\} && \text{(UP-3)}\\
&= \min\{\overline{f}(0,q), \overline{f}(x,q)\} && \text{(UP-3)}\\
&= \overline{f}(x,q). && \text{(Definition 1.10 (1))}
\end{aligned}$$

Then $1 - f(0,q) \geq 1 - f(x,q)$, so $f(0,q) \leq f(x,q) \leq t$. Hence, $0 \in L(f;t,q)$. Let $x, y, z \in A$ be such that $x\cdot(y\cdot z) \in L(f;t,q)$ and $y \in L(f;t,q)$. Then $f(x\cdot(y\cdot z),q) \leq t$ and $f(y,q) \leq t$. By Definition 1.10 (2), we have $\overline{f}(x\cdot z,q) \geq \min\{\overline{f}(x\cdot(y\cdot z),q), \overline{f}(y,q)\}$. Thus

$$\begin{aligned}
1 - f(x\cdot z,q) &\geq \min\{1 - f(x\cdot(y\cdot z),q), 1 - f(y,q)\}\\
&= 1 - \max\{f(x\cdot(y\cdot z),q), f(y,q)\}. && \text{(Lemma 1.8 (1))}
\end{aligned}$$

Then $f(x \cdot z, q) \leq \max\{f(x \cdot (y \cdot z), q), f(y, q)\} \leq t$. Hence, $x \cdot z \in L(f; t, q)$. Therefore, $L(f; t, q)$ is a UP-ideal of A.

Conversely, assume that the condition $(\star)$ holds and suppose that $\overline{f}(0, q) \geq \overline{f}(x, q)$ for all $x \in A$ and $q \in Q$ is false. Then there exist $x \in A$ and $q \in Q$ such that $\overline{f}(0, q) < \overline{f}(x, q)$. Thus $1 - f(0, q) < 1 - f(x, q)$, so $f(0, q) > f(x, q)$. Let $t = \frac{f(0,q)+f(x,q)}{2}$. Then $t \in [0, 1]$ and by Lemma 2.8, we have $f(0, q) > t > f(x, q)$. Thus $x \in L(f; t, q)$, so $L(f; t, q) \neq \emptyset$. By assumption, we have $L(f; t, q)$ is a UP-ideal of A. It follows that $0 \in L(f; t, q)$, so $f(0, q) \leq t$ which is a contradiction. Hence, $\overline{f}(0, q) \geq \overline{f}(x, q)$ for all $x \in A$ and $q \in Q$. Suppose that $\overline{f}(x \cdot z, q) \geq \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}$ for all $x, y, z \in A$ and $q \in Q$ is false. Then there exist $x, y, z \in A$ and $q \in Q$ such that $\overline{f}(x \cdot z, q) < \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z, q) &< \min\{1 - f(x \cdot (y \cdot z), q), 1 - f(y, q)\} \\ &= 1 - \max\{f(x \cdot (y \cdot z), q), f(y, q)\}. && \text{(Lemma 1.8 (1))} \end{aligned}$$

Then $f(x \cdot z, q) > \max\{f(x \cdot (y \cdot z), q), f(y, q)\}$. Let $g_0 = \frac{f(x \cdot z,q)+\max\{f(x \cdot (y \cdot z),q),f(y,q)\}}{2}$. Then $g_0 \in [0, 1]$ and by Lemma 2.8, we have $f(x \cdot z, q) > g_0 > \max\{f(x \cdot (y \cdot z), q), f(y, q)\}$. Thus $f(x \cdot (y \cdot z), q) < g_0$ and $f(y, q) < g_0$, so $x \cdot (y \cdot z) \in L(f; g_0, q)$ and $y \in L(f; g_0, q)$, so $L(f; g_0, q) \neq \emptyset$. By assumption, we have $L(f; g_0, q)$ is a UP-ideal of A. It follows that $x \cdot z \in L(f; g_0, q)$, so $f(x \cdot z, q) \leq g_0$ which is a contradiction. Hence, $\overline{f}(x \cdot z, q) \geq \min\{\overline{f}(x \cdot (y \cdot z), q), \overline{f}(y, q)\}$ for all $x, y, z \in A$ and $q \in Q$. Therefore, $\overline{f}$ is a q-fuzzy UP-ideal of A for all $q \in Q$. Consequently, $\overline{f}$ is a Q-fuzzy UP-ideal of A.

(2) Similarly to as in the proof of (1).

(3) Assume that f is a Q-fuzzy UP-ideal of A. Then f is a q-fuzzy UP-ideal of A for all $q \in Q$. Let $q \in Q$ and $t \in [0, 1]$ be such that $U(f; t, q) \neq \emptyset$ and let $x \in U(f; t, q)$. Then $f(x, q) \geq t$. Now,

$$\begin{aligned} f(0, q) &= f(x \cdot 0, q) && \text{(UP-3)} \\ &\geq \min\{f(x \cdot (x \cdot 0), q), f(x, q)\} && \text{(Definition 1.10 (2))} \\ &= \min\{f(x \cdot 0, q), f(x, q)\} && \text{(UP-3)} \\ &= \min\{f(0, q), f(x, q)\} && \text{(UP-3)} \\ &= f(x, q) && \text{(Definition 1.10 (1))} \\ &\geq t. \end{aligned}$$

Hence, $0 \in U(f; t, q)$. Let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f; t, q)$ and $y \in U(f; t, q)$. Then $f(x \cdot (y \cdot z), q) \geq t$ and $f(y, q) \geq t$. By Definition 1.10 (2), we have $f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y, q)\} \geq t$. Thus $x \cdot z \in U(f; t, q)$. Hence, $U(f; t, q)$ is a UP-ideal of A.

Conversely, assume that the condition $(\star)$ holds and suppose that $f(0, q) \geq f(x, q)$ for all $x \in A$ and $q \in Q$ is false. Then there exist $x \in A$ and $q \in Q$ such that $f(0, q) < f(x, q)$. Let $t = \frac{f(0,q)+f(x,q)}{2}$. Then $t \in [0, 1]$ and by Lemma 2.8, we have $f(0, q) < t < f(x, q)$. Thus $x \in U(f; t, q)$, so $U(f; t, q) \neq \emptyset$. By assumption, we have $U(f; t, q)$ is a UP-ideal of A. It follows that $0 \in U(f; t, q)$, so $f(0, q) \geq t$ which is a contradiction. Hence, $f(0, q) \geq f(x, q)$ for all $x \in A$ and $q \in Q$. Suppose that $f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y, q)\}$ for all $x, y, z \in A$ and $q \in Q$ is false. Then there exist $x, y, z \in A$ and $q \in Q$ such that $f(x \cdot z, q) < \min\{f(x \cdot (y \cdot z), q), f(y, q)\}$. Let $g_0 = \frac{f(x \cdot z,q)+\min\{f(x \cdot (y \cdot z),q),f(y,q)\}}{2}$. Then $g_0 \in [0, 1]$ and By Lemma 2.8, we have $f(x \cdot z, q) < g_0 < \min\{f(x \cdot (y \cdot z), q), f(y, q)\}$. Thus

$f(x \cdot (y \cdot z), q) > g_0$ and $f(y, q) > g_0$, so $x \cdot (y \cdot z) \in U(f; g_0, q)$ and $y \in U(f; g_0, q)$, so $U(f; g_0, q) \neq \emptyset$. By assumption, we have $U(f; g_0, q)$ is a UP-ideal of A. It follows that $x \cdot z \in U(f; g_0, q)$, so $f(x \cdot z, q) \geq g_0$ which is a contradiction. Hence, $f(x \cdot z, q) \geq \min\{f(x \cdot (y \cdot z), q), f(y, q)\}$ for all $x, y, z \in A$ and $q \in Q$. Therefore, f is a q-fuzzy UP-ideal of A for all $q \in Q$. Consequently, f is a Q-fuzzy UP-ideal of A.

(4) Similarly to as in the proof of (3).

Corollary 2.10. *Let f be a Q-fuzzy set in A. Then the following statements hold:*

(1) if $\overline{f}$ is a Q-fuzzy UP-ideal of A, then for any $t \in [0, 1]$, $L(f; t)$ is either empty or a UP-ideal of A,

(2) if $\overline{f}$ is a Q-fuzzy UP-ideal of A, then for any $t \in [0, 1]$, $L^-(f; t)$ is either empty or a UP-ideal of A,

(3) if f is a Q-fuzzy UP-ideal of A, then for any $t \in [0, 1]$, $U(f; t)$ is either empty or a UP-ideal of A, and

(4) if f is a Q-fuzzy UP-ideal of A, then for any $t \in [0, 1]$, $U^+(f; t)$ is either empty or a UP-ideal of A.

Proof. (1) Assume that $\overline{f}$ is a Q-fuzzy UP-ideal of A. By Theorem 2.9 (1), we have that for any $t \in [0, 1]$ and $q \in Q$, $L(f; t, q)$ is either empty or a UP-ideal of A. Let $t \in [0, 1]$. If $L(f; t, q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (1) that $L(f; t) = \bigcap_{q \in Q} L(f; t, q) = \emptyset$. If $L(f; t, q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.9 (1) that $L(f; t, q)$ is a UP-ideal of A for all $q \in Q$. By Lemma 2.7 (1) and Theorem 1.4, we have $L(f; t) = \bigcap_{q \in Q} L(f; t, q)$ is a UP-ideal of A.

(2) Similarly to as in the proof of (1).

(3) Assume that f is a Q-fuzzy UP-ideal of A. By Theorem 2.9 (3), we have that for any $t \in [0, 1]$ and $q \in Q$, $U(f; t, q)$ is either empty or a UP-ideal of A. Let $t \in [0, 1]$. If $U(f; t, q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (3) that $U(f; t) = \bigcap_{q \in Q} U(f; t, q) = \emptyset$. If $U(f; t, q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.9 (3) that $U(f; t, q)$ is a UP-ideal of A for all $q \in Q$. By Lemma 2.7 (3) and Theorem 1.4, we have $U(f; t) = \bigcap_{q \in Q} U(f; t, q)$ is a UP-ideal of A.

(4) Similarly to as in the proof of (3).

Theorem 2.11. *Let f be a Q-fuzzy set in A. Then the following statements hold:*

(1) $\overline{f}$ is a Q-fuzzy UP-subalgebra of A if and only if the following condition ($\star$) holds: for any $t \in [0, 1]$ and $q \in Q$, $L(f; t, q)$ is either empty or a UP-subalgebra of A,

(2) $\overline{f}$ is a Q-fuzzy UP-subalgebra of A if and only if the following condition ($\star$) holds: for any $t \in [0, 1]$ and $q \in Q$, $L^-(f; t, q)$ is either empty or a UP-subalgebra of A,

(3) f is a Q-fuzzy UP-subalgebra of A if and only if the following condition ($\star$) holds: for any $t \in [0, 1]$ and $q \in Q$, $U(f; t, q)$ is either empty or a UP-subalgebra of A, and

(4) f is a Q-fuzzy UP-subalgebra of A if and only if the following condition ($\star$) holds: for any $t \in [0, 1]$ and $q \in Q$, $U^+(f; t, q)$ is either empty or a UP-subalgebra of A.

Proof. (1) Assume that $\overline{f}$ is a Q-fuzzy UP-subalgebra of A. Then $\overline{f}$ is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $L(f;t,q) \neq \emptyset$ and let $x, y \in L(f;t,q)$. Then $f(x,q) \leq t$ and $f(y,q) \leq t$. Now,

$$\begin{aligned} \overline{f}(x \cdot y, q) &\geq \min\{\overline{f}(x,q), \overline{f}(y,q)\} \\ &= \min\{1 - f(x,q), 1 - f(y,q)\} \\ &= 1 - \max\{f(x,q), f(y,q)\}. \qquad \text{(Lemma 1.8 (1))} \end{aligned}$$

Then $f(x \cdot y, q) \leq \max\{f(x,q), f(y,q)\} \leq t$, so $x \cdot y \in L(f;t,q)$. Hence, $L(f;t,q)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and $q \in Q$ and let $t = \max\{f(x,q), f(y,q)\}$. Thus $f(x,q) \leq t$ and $f(y,q) \leq t$, so $x, y \in L(f;t,q) \neq \emptyset$. By assumption, we have $L(f;t,q)$ is a UP-subalgebra of A. It follows that $x \cdot y \in L(f;t,q)$. Thus $f(x \cdot y, q) \leq t = \max\{f(x,q), f(y,q)\}$, so

$$\begin{aligned} 1 - f(x \cdot y, q) &\geq 1 - \max\{f(x,q), f(y,q)\} \\ &= \min\{1 - f(x,q), 1 - f(y,q)\}. \qquad \text{(Lemma 1.8 (1))} \end{aligned}$$

Hence, $\overline{f}(x \cdot y, q) \geq \min\{\overline{f}(x,q), \overline{f}(y,q)\}$. Therefore, $\overline{f}$ is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Consequently, $\overline{f}$ is a Q-fuzzy UP-subalgebra of A.

(2) Similarly to as in the proof of the necessity of (1).

Conversely, assume that the condition $(\star)$ holds. Assume that there exist $x, y \in A$ and $q \in Q$ such that $\overline{f}(x \cdot y, q) < \min\{\overline{f}(x,q), \overline{f}(y,q)\}$. By Lemma 1.8 (1), we have $1 - f(x \cdot y, q) < \min\{1 - f(x,q), 1 - f(y,q)\} = 1 - \max\{f(x,q), f(y,q)\}$. Thus $f(x \cdot y, q) > \max\{f(x,q), f(y,q)\}$. Now $f(x \cdot y, q) \in [0,1]$, we choose $t = f(x \cdot y, q)$. Thus $f(x,q) < t$ and $f(y,q) < t$, so $x, y \in L^-(f;t,q) \neq \emptyset$. By assumption, we have $L^-(f;t,q)$ is a UP-subalgebra of A and so $x \cdot y \in L^-(f;t,q)$. Thus $f(x \cdot y, q) < t = f(x \cdot y, q)$ which is a contradiction. Hence, $\overline{f}(x \cdot y, q) \geq \min\{\overline{f}(x,q), \overline{f}(y,q)\}$ for all $x, y \in A$ and $q \in Q$. Therefore, $\overline{f}$ is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Consequently, $\overline{f}$ is a Q-fuzzy UP-subalgebra of A.

(3) Assume that f is a Q-fuzzy UP-subalgebra of A. Then f is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Let $q \in Q$ and $t \in [0,1]$ be such that $U(f;t,q) \neq \emptyset$ and let $x, y \in U(f;t,q)$. Then $f(x,q) \geq t$ and $f(y,q) \geq t$, we have $f(x \cdot y, q) \geq \min\{f(x,q), f(y,q)\} \geq t$. Thus $x \cdot y \in U(f;t,q)$. Hence, $U(f;t,q)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and $q \in Q$ and let $t = \min\{f(x,q), f(y,q)\}$. Thus $f(x,q) \geq t$ and $f(y,q) \geq t$, so $x, y \in U(f;t,q) \neq \emptyset$. By assumption, we have $U(f;t,q)$ is a UP-subalgebra of A. It follows that $x \cdot y \in U(f;t,q)$. Thus $f(x \cdot y, q) \geq t = \min\{f(x,q), f(y,q)\}$. Hence, f is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Consequently, f is a Q-fuzzy UP-subalgebra of A.

(4) Similarly to as in the proof of the necessity of (3).

Conversely, assume that the condition $(\star)$ holds. Assume that there exist $x, y \in A$ and $q \in Q$ such that $f(x \cdot y, q) < \min\{f(x,q), f(y,q)\}$. Then $f(x \cdot y, q) \in [0,1]$. Choose $t = f(x \cdot y, q)$. Thus $f(x,q) > t$ and $f(y,q) > t$, so $x, y \in U^+(f;t,q) \neq \emptyset$. By assumption, we have $U^+(f;t,q)$ is a UP-subalgebra of A and so $x \cdot y \in U^+(f;t,q)$. Thus $f(x \cdot y, q) > t = f(x \cdot y, q)$ which is a contradiction. Hence, $f(x \cdot y, q) \geq \min\{f(x,q), f(y,q)\}$ for all $x, y \in A$ and $q \in Q$. Therefore, f is a q-fuzzy UP-subalgebra of A for all $q \in Q$. Consequently, f is a Q-fuzzy UP-subalgebra of A.

Corollary 2.12. *Let f be a Q-fuzzy set in A. Then the following statements hold:*

(1) if $\overline{f}$ is a Q-fuzzy UP-subalgebra of A, then for any $t \in [0,1]$, $L(f;t)$ is either empty or a UP-subalgebra of A,

(2) if $\overline{f}$ is a Q-fuzzy UP-subalgebra of A, then for any $t \in [0,1]$, $L^-(f;t)$ is either empty or a UP-subalgebra of A,

(3) if f is a Q-fuzzy UP-subalgebra of A, then for any $t \in [0,1]$, $U(f;t)$ is either empty or a UP-subalgebra of A, and

(4) if f is a Q-fuzzy UP-subalgebra of A, then for any $t \in [0,1]$, $U^+(f;t)$ is either empty or a UP-subalgebra of A.

Proof. (1) Assume that $\overline{f}$ is a Q-fuzzy UP-subalgebra of A. By Theorem 2.11 (1), we have for any $t \in [0,1]$ and $q \in Q$, $L(f;t,q)$ is either empty or a UP-subalgebra of A. Let $t \in [0,1]$. If $L(f;t,q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (1) that $L(f;t) = \bigcap_{q\in Q} L(f;t,q) = \emptyset$. If $L(f;t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.11 (1) that $L(f;t,q)$ is a UP-subalgebra of A for all $q \in Q$. By Lemma 2.7 (1) and Theorem1.7, we have $L(f;t) = \bigcap_{q\in Q} L(f;t,q)$ is a UP-subalgebra of A.

(2) Similarly to as in the proof of (1).

(3) Assume that f is a Q-fuzzy UP-subalgebra of A. By Theorem 2.11 (3), we have for any $t \in [0,1]$ and $q \in Q$, $U(f;t,q)$ is either empty or a UP-subalgebra of A. Let $t \in [0,1]$. If $U(f;t,q) = \emptyset$ for some $q \in Q$, it follows from Lemma 2.7 (3) that $U(f;t) = \bigcap_{q\in Q} U(f;t,q) = \emptyset$. If $U(f;t,q) \neq \emptyset$ for all $q \in Q$, it follows from Theorem 2.11 (3) that $U(f;t,q)$ is a UP-subalgebra of A for all $q \in Q$. By Lemma 2.7 (3) and Theorem 1.7, we have $U(f;t) = \bigcap_{q\in Q} U(f;t,q)$ is a UP-subalgebra of A.

(4) Similarly to as in the proof of (3).

Corollary 2.13. *Let I be a UP-ideal of A. Then the following statements hold:*

(1) for any $k \in (0,1]$, then there exists a Q-fuzzy UP-ideal g of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$, and

(2) for any $k \in [0,1)$, then there exists a Q-fuzzy UP-ideal f of A such that $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

Proof. (1) Let f be a Q-fuzzy set in A defined by

$$f(x,q) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$.

Case 1: To show that $L(f;t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f;t)$. Then $f(x,q) \leq t < k$ for all $q \in Q$. Thus $f(x,q) \neq k$ for all $q \in Q$, so $f(x,q) = 0$ for all $q \in Q$. Thus $x \in I$, so $L(f;t) \subseteq I$. Now, let $x \in I$. Then $f(x,q) = 0 \leq t$ for all $q \in Q$. Thus $x \in L(f;t)$, so $I \subseteq L(f;t)$. Hence, $L(f;t) = I$ for all $t < k$.

Case 2: To show that $L(f;t) = A$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x,q) = \begin{cases} 0 < t & \text{if } x \in I, \\ k \leq t & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$. Thus $x \in L(f;t)$, so $A \subseteq L(f;t)$. Hence, $L(f;t) = A$ for all $t \geq k$. We claim that $L(f;t,q) = L(f;t,q')$ for all $q,q' \in Q$. For $q,q' \in Q$, we obtain

$$\begin{aligned} x \in L(f;t,q) &\Leftrightarrow f(x,q) \leq t \\ &\Leftrightarrow f(x,q') \leq t && (f(x,q) = f(x,q')) \\ &\Leftrightarrow x \in L(f;t,q'). \end{aligned}$$

Hence, $L(f;t,q) = L(f;t,q')$ for all $q, q' \in Q$. By Lemma 2.7 (1), we have $L(f;t) = \bigcap_{q\in Q} L(f;t,q)$. By the claim, we have $L(f;t) = L(f;t,q)$ for all $q \in Q$. Since $L(f;t,q) = L(f;t) = I$ for all $t < k$ and $L(f;t,q) = L(f;t) = A$ for all $t \geq k$, it follows from Theorem 2.9 (1) that $\overline{f}$ is a Q-fuzzy UP-ideal of A. By Remark 1.17, we have $L(\overline{\overline{f}};t) = L(f;t) = I$ for all $t < k$ and $L(\overline{\overline{f}};t) = L(f;t) = A$ for all $t \geq k$. Let $\overline{f} = g$. Then g is a Q-fuzzy UP-ideal of A such that $L(\overline{g};t) = I$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$.

(2) Let f be a Q-fuzzy set in A defined by

$$f(x,q) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I, \end{cases}$$

for all $q \in Q$.

Case 1: To show that $U(f;t) = I$ for all $t > k$, let $t \in [0,1]$ be such that $t > k$. Let $x \in U(f;t)$. Then $f(x,q) \geq t > k$ for all $q \in Q$. Thus $f(x,q) \neq k$ for all $q \in Q$, so $f(x,q) = 1$ for all $q \in Q$. Thus $x \in I$, so $U(f;t) \subseteq I$. Now, let $x \in I$. Then $f(x,q) = 1 \geq t$ for all $q \in Q$. Thus $x \in U(f;t)$, so $I \subseteq U(f;t)$. Hence, $U(f;t) = I$ for all $t > k$.

Case 2: To show that $U(f;t) = A$ for all $t \leq k$, let $t \in [0,1]$ be such that $t \leq k$. Clearly, $U(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x,q) = \begin{cases} k \geq t & \text{if } x \notin I, \\ 1 > t & \text{if } x \in I, \end{cases}$$

for all $q \in Q$. Thus $x \in U(f;t)$, so $A \subseteq U(f;t)$. Hence, $U(f;t) = A$ for all $t \leq k$. We claim that $U(f;t,q) = U(f;t,q')$ for all $q, q' \in Q$. For $q, q' \in Q$, we obtain

$$\begin{aligned} x \in U(f;t,q) &\Leftrightarrow f(x,q) \geq t \\ &\Leftrightarrow f(x,q') \geq t && (f(x,q) = f(x,q')) \\ &\Leftrightarrow x \in U(f;t,q'). \end{aligned}$$

Hence, $U(f;t,q) = U(f;t,q')$ for all $q, q' \in Q$. By Lemma 2.7 (3), we have $U(f;t) = \bigcap_{q\in Q} U(f;t,q)$. By the claim, we have $U(f;t) = U(f;t,q)$ for all $q \in Q$. Since $U(f;t,q) = U(f;t) = I$ for all $t > k$ and $U(f;t,q) = U(f;t) = A$ for all $t \leq k$, it follows from Theorem 2.9 (3) that f is a Q-fuzzy UP-ideal of A.

Corollary 2.14. *Let S be a UP-subalgebra of A. Then the following statements hold:*

(1) for any $k \in (0,1]$, then there exists a Q-fuzzy UP-subalgebra g of A such that $L(\overline{g};t) = S$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$, and

(2) for any $k \in [0,1)$, then there exists a Q-fuzzy UP-subalgebra f of A such that $U(f;t) = S$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

Proof. (1) Let f be a Q-fuzzy set in A defined by

$$f(x,q) = \begin{cases} 0 & \text{if } x \in S, \\ k & \text{if } x \notin S, \end{cases}$$

for all $q \in Q$.

In the proof of Corollary 2.13 (1), we have $L(f;t) = S$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$, and $L(f;t,q) = L(f;t,q')$ for all $q, q' \in Q$. By Lemma 2.7 (1), we have $L(f;t) = \bigcap_{q\in Q} L(f;t,q)$. By the claim, we have $L(f;t) = L(f;t,q)$ for all $q \in Q$. Since $L(f;t,q) = L(f;t) = S$ for all $t < k$ and $L(f;t,q) = L(f;t) = A$ for all $t \geq k$, it follows from Theorem 2.11 (1) that $\overline{f}$ is a Q-fuzzy UP-subalgebra of A. By

Remark 1.17, we have $L(\overline{\overline{f}};t) = L(f;t) = S$ for all $t < k$ and $L(\overline{\overline{f}};t) = L(f;t) = A$ for all $t \geq k$. Let $\overline{f} = g$. Then g is a Q-fuzzy UP-subalgebra of A such that $L(\overline{g};t) = S$ for all $t < k$ and $L(\overline{g};t) = A$ for all $t \geq k$.

(2) Let f be a Q-fuzzy set in A defined by

$$f(x,q) = \begin{cases} 1 & \text{if } x \in S, \\ k & \text{if } x \notin S, \end{cases}$$

for all $q \in Q$.

In the proof of Corollary 2.13 (2), we have $U(f;t) = S$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$, and $U(f;t,q) = U(f;t,q')$ for all $q, q' \in Q$. By Lemma 2.7 (3), we have $U(f;t) = \bigcap_{q \in Q} U(f;t,q)$. By the claim, we have $U(f;t) = U(f;t,q)$ for all $q \in Q$. Since $U(f;t,q) = U(f;t) = S$ for all $t > k$ and $U(f;t,q) = U(f;t) = A$ for all $t \leq k$, it follows from Theorem 2.11 (3) that f is a Q-fuzzy UP-subalgebra of A.

Theorem 2.15. *Let f be a Q-fuzzy set in A and $s < t$ for $s, t \in [0,1]$. Then the following statements hold:*

(1) $L(f;s,q) = L(f;t,q)$ if and only if there is no $x \in A$ such that $s < f(x,q) \leq t$,

(2) $L^-(f;s,q) = L^-(f;t,q)$ if and only if there is no $x \in A$ such that $s \leq f(x,q) < t$,

(3) $U(f;s,q) = U(f;t,q)$ if and only if there is no $x \in A$ such that $s \leq f(x,q) < t$, and

(4) $U^+(f;s,q) = U^+(f;t,q)$ if and only if there is no $x \in A$ such that $s < f(x,q) \leq t$.

Proof. (1) Assume that $L(f;s,q) = L(f;t,q)$. Suppose that there is $x \in A$ such that $s < f(x,q) \leq t$. Then $x \in L(f;t,q)$ but $x \notin L(f;s,q)$, so $L(f;t,q) \neq L(f;s,q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s < f(x,q) \leq t$.

Conversely, assume that there is no $x \in A$ such that $s < f(x,q) \leq t$. Let $x \in L(f;s,q)$. Then $f(x,q) \leq s < t$, so $x \in L(f;t,q)$. Thus $L(f;s,q) \subseteq L(f;t,q)$. Suppose that $L(f;t,q) \nsubseteq L(f;s,q)$. Then there exists $x \in L(f;t,q)$ but $x \notin L(f;s,q)$. Thus $f(x,q) \leq t$ and $f(x,q) > s$, so $s < f(x,q) \leq t$ which is a contradiction. Thus $L(f;t,q) \subseteq L(f;s,q)$. Hence, $L(f;s,q) = L(f;t,q)$.

(2) Similarly to as in the proof of (1).

(3) Assume that $U(f;s,q) = U(f;t,q)$. Suppose that there is $x \in A$ such that $s \leq f(x,q) < t$. Then $x \in U(f;s,q)$ but $x \notin U(f;t,q)$, so $U(f;s,q) \neq U(f;t,q)$ which is a contradiction. Hence, there is no $x \in A$ such that $s \leq f(x,q) < t$.

Conversely, assume that there is no $x \in A$ such that $s \leq f(x,q) < t$. Let $x \in U(f;t,q)$. Then $f(x,q) \geq t > s$, so $x \in U(f;s,q)$. Thus $U(f;t,q) \subseteq U(f;s,q)$. Suppose that $U(f;s,q) \nsubseteq U(f;t,q)$. Then there exists $x \in U(f;s,q)$ but $x \notin U(f;t,q)$. Thus $f(x,q) \geq s$ and $f(x,q) < t$, so $s \leq f(x,q) < t$ which is a contradiction. Thus $U(f;s,q) \subseteq U(f;t,q)$. Hence, $U(f;s,q) = U(f;t,q)$.

(4) Similarly to as in the proof of (3).

Corollary 2.16. *Let f be a Q-fuzzy set in A and $s < t$ for $s, t \in [0,1]$. Then the following statements hold:*

(1) $L(f;s,q) = L(f;t,q)$ if and only if $U^+(f;s,q) = U^+(f;t,q)$, and

(2) $U(f;s,q) = U(f;t,q)$ if and only if $L^-(f;s,q) = L^-(f;t,q)$.

Proof. (1) It follows from Theorem 2.15 (1) and Theorem 2.15 (4).

(2) It follows from Theorem 2.15 (2) and Theorem 2.15 (3).

Theorem 2.17. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if μ is a q-fuzzy UP-ideal of B, then μ_f is also a q-fuzzy UP-ideal of A, and

(2) if μ is a q-fuzzy UP-subalgebra of B, then μ_f is also a q-fuzzy UP-subalgebra of A.

Proof. (1) Assume that μ is a q-fuzzy UP-ideal of B. Let $x \in A$. Then

$$\begin{aligned} \mu_f(0_A,q) &= \mu(f(0_A),q) \\ &= \mu(0_B,q) && \text{(Proposition 1.22)} \\ &\geq \mu(f(x),q) && \text{(Definition 1.10 (1))} \\ &= \mu_f(x,q). \end{aligned}$$

Let $x,y,z \in A$. Then

$$\begin{aligned} \mu_f(x\cdot z,q) &= \mu(f(x\cdot z),q) \\ &= \mu(f(x)*f(z),q) \\ &\geq \min\{\mu(f(x)*(f(y)*f(z)),q),\mu(f(y),q)\} && \text{(Definition 1.10 (2))} \\ &= \min\{\mu(f(x)*f(y\cdot z),q),\mu(f(y),q)\} \\ &= \min\{\mu(f(x\cdot(y\cdot z)),q),\mu(f(y),q)\} \\ &= \min\{\mu_f(x\cdot(y\cdot z),q),\mu_f(y,q)\}. \end{aligned}$$

Hence, μ_f is a q-fuzzy UP-ideal of A.

(2) Assume that μ is a q-fuzzy UP-subalgebra of B. Let $x,y \in A$. Then

$$\begin{aligned} \mu_f(x\cdot y,q) &= \mu(f(x\cdot y),q) \\ &= \mu(f(x)*f(y),q) \\ &\geq \min\{\mu(f(x),q),\mu(f(y),q)\} && \text{(Definition 1.13)} \\ &= \min\{\mu_f(x,q),\mu_f(y,q)\}. \end{aligned}$$

Hence, μ_f is a q-fuzzy UP-subalgebra of A.

With Definition 1.10 and 1.13 and Theorem 2.17, we obtain the corollary.

Corollary 2.18. *Let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) if μ is a Q-fuzzy UP-ideal of B, then μ_f is also a Q-fuzzy UP-ideal of A, and

(2) if μ is a Q-fuzzy UP-subalgebra of B, then μ_f is also a Q-fuzzy UP-subalgebra of A.

Theorem 2.19. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-isomorphism. Then the following statements hold:*

(1) if μ_f is a q-fuzzy UP-ideal of A, then μ is also a q-fuzzy UP-ideal of B, and

(2) if μ_f is a q-fuzzy UP-subalgebra of A, then μ is also a q-fuzzy UP-subalgebra of B.

Proof. (1) Assume that μ_f is a q-fuzzy UP-ideal of A. Let $y \in B$. Then there exists $x \in A$ such that $f(x) = y$, we have

$$\begin{aligned}
\mu(0_B, q) &= \mu(y * 0_B, q) && \text{(UP-3)}\\
&= \mu(f(x) * f(0_A), q) && \text{(Proposition 1.22)}\\
&= \mu(f(x \cdot 0_A), q)\\
&= \mu_f(x \cdot 0_A, q)\\
&= \mu_f(0_A, q) && \text{(UP-3)}\\
&\geq \mu_f(x, q) && \text{(Definition 1.10 (1))}\\
&= \mu(f(x), q)\\
&= \mu(y, q).
\end{aligned}$$

Let $a, b, c \in B$. Then there exist $x, y, z \in A$ such that $f(x) = a$, $f(y) = b$ and $f(z) = c$, we have

$$\begin{aligned}
\mu(a * c, q) &= \mu(f(x) * f(z), q)\\
&= \mu(f(x \cdot z), q)\\
&= \mu_f(x \cdot z, q)\\
&\geq \min\{\mu_f(x \cdot (y \cdot z), q), \mu_f(y, q)\} && \text{(Definition 1.10 (2))}\\
&= \min\{\mu(f(x \cdot (y \cdot z)), q), \mu(f(y), q)\}\\
&= \min\{\mu(f(x) * (f(y) * f(z)), q), \mu(f(y), q)\}\\
&= \min\{\mu(a * (b * c), q), \mu(b, q)\}.
\end{aligned}$$

Hence, μ is a q-fuzzy UP-ideal of B.

(2) Assume that μ_f is a q-fuzzy UP-subalgebra of A. Let $a, b \in B$. Then there exist $x, y \in A$ such that $f(x) = a$ and $f(y) = b$, we have

$$\begin{aligned}
\mu(a * b, q) &= \mu(f(x) * f(y), q)\\
&= \mu(f(x \cdot y), q)\\
&= \mu_f(x \cdot y, q)\\
&\geq \min\{\mu_f(x, q), \mu_f(y, q)\} && \text{(Definition 1.13)}\\
&= \min\{\mu(f(x), q), \mu(f(y), q)\}\\
&= \min\{\mu(a, q), \mu(b, q)\}.
\end{aligned}$$

Hence, μ is a q-fuzzy UP-subalgebra of B.

With Definition 1.10 and 1.13 and Theorem 2.19, we obtain the corollary.

Corollary 2.20. *Let $f: A \to B$ be a UP-isomorphism. Then the following statements hold:*

(1) if μ_f is a Q-fuzzy UP-ideal of A, then μ is also a Q-fuzzy UP-ideal of B, and

(2) if μ_f is a Q-fuzzy UP-subalgebra of A, then μ is also a Q-fuzzy UP-subalgebra of B.

Lemma 2.21. *(Bali, 2005) For any $a, b, c, d \in \mathbb{R}$, the following properties hold:*

(1) $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$, *and*

(2) $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. We can easily prove that $A \times B$ is a UP-algebra defined by

$$(x_1, x_2) \diamond (y_1, y_2) = (x_1 \cdot y_1, x_2 * y_2)$$

for all $x_1, y_1 \in A$ and $x_2, y_2 \in B$.

Theorem 2.22. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras. Then the following statements hold:*

(1) if μ is a q-fuzzy UP-ideal of A and δ is a q-fuzzy UP-ideal of B, then $\mu\cdot\delta$ is a q-fuzzy UP-ideal of $A\times B$, and

(2) if μ is a q-fuzzy UP-subalgebra of A and δ is a q-fuzzy UP-subalgebra of B, then $\mu\cdot\delta$ is a q-fuzzy UP-subalgebra of $A\times B$.

Proof. (1) Assume that μ is a q-fuzzy UP-ideal of A and δ is a q-fuzzy UP-ideal of B. Let $(x_1,x_2)\in A\times B$. Then

$$\begin{aligned}(\mu\cdot\delta)((0_A,0_B),q)&=\min\{\mu(0_A,q),\delta(0_B,q)\}\\&\geq\min\{\mu(x_1,q),\delta(x_2,q)\} &&\text{(Definition 1.10 (1))}\\&=(\mu\cdot\delta)((x_1,x_2),q).\end{aligned}$$

Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in A\times B$. Then

$$\begin{aligned}&(\mu\cdot\delta)((x_1,x_2)\diamond(z_1,z_2),q)\\&=(\mu\cdot\delta)((x_1\cdot z_1,x_2*z_2),q)\\&=\min\{\mu(x_1\cdot z_1,q),\delta(x_2*z_2,q)\}\\&\geq\min\{\min\{\mu(x_1\cdot(y_1\cdot z_1),q),\mu(y_1,q)\},\\&\quad\min\{\delta(x_2*(y_2*z_2),q),\delta(y_2,q)\}\} &&\text{(Definition 1.10 (2))}\\&=\min\{\min\{\mu(x_1\cdot(y_1\cdot z_1),q),\delta(x_2*(y_2*z_2),q)\},\\&\quad\min\{\mu(y_1,q),\delta(y_2,q)\}\} &&\text{(Lemma 2.21 (2))}\\&=\min\{(\mu\cdot\delta)((x_1\cdot(y_1\cdot z_1),x_2*(y_2*z_2)),q),(\mu\cdot\delta)((y_1,y_2),q)\}\\&=\min\{(\mu\cdot\delta)((x_1,x_2)\diamond(y_1\cdot z_1,y_2*z_2),q),(\mu\cdot\delta)((y_1,y_2),q)\}\\&=\min\{(\mu\cdot\delta)((x_1,x_2)\diamond((y_1,y_2)\diamond(z_1,z_2)),q),(\mu\cdot\delta)((y_1,y_2),q)\}.\end{aligned}$$

Hence, $\mu\cdot\delta$ is a q-fuzzy UP-ideal of $A\times B$.

(2) Assume that μ is a q-fuzzy UP-subalgebra of A and δ is a q-fuzzy UP-subalgebra of B. Let $(x_1,x_2),(y_1,y_2)\in A\times B$. Then

$$\begin{aligned}&(\mu\cdot\delta)((x_1,x_2)\diamond(y_1,y_2),q)\\&=(\mu\cdot\delta)((x_1\cdot y_1,x_2*y_2),q)\\&=\min\{\mu(x_1\cdot y_1,q),\delta(x_2*y_2,q)\}\\&\geq\min\{\min\{\mu(x_1,q),\mu(y_1,q)\},\min\{\delta(x_2,q),\delta(y_2,q)\}\} &&\text{(Definition 1.13)}\\&=\min\{\min\{\mu(x_1,q),\delta(x_2,q)\},\min\{\mu(y_1,q),\delta(y_2,q)\}\} &&\text{(Lemma 2.21 (2))}\\&=\min\{(\mu\cdot\delta)((x_1,x_2),q),(\mu\cdot\delta)((y_1,y_2),q)\}.\end{aligned}$$

Hence, $\mu\cdot\delta$ is a q-fuzzy UP-subalgebra of $A\times B$.

Give examples of conflict that μ and δ are q-fuzzy UP-ideals (resp. q-fuzzy UP-subalgebras) of A but $\mu\times\delta$ is not a q-fuzzy UP-ideal (resp. q-fuzzy UP-subalgebra) of $A\times A$.

Example 2.23. Let $A=\{0,1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A;\cdot,0)$ is a UP-algebra. Let $Q=\{q\}$. We define Q-fuzzy sets μ and δ in A as follows: $\mu(0,q)=0.2,\delta(0,q)=0.3,\mu(1,q)=0.1$ and $\delta(1,q)=0.1$. Using this data, we can show that μ and δ are q-fuzzy UP-ideals of A. Let $(x_1,x_2)=(0,0),(y_1,y_2)=(1,0),(z_1,z_2)=(1,1)\in A\times A$. Then

$$(\mu \times \delta)((x_1, x_2) \diamond (z_1, z_2), q) = 0.1$$

and

$$\min\{(\mu \times \delta)((x_1, x_2) \diamond [(y_1, y_2) \diamond (z_1, z_2)], q), (\mu \times \delta)((y_1, y_2), q)\} = 0.2.$$

Hence, $(\mu \times \delta)((x_1, x_2) \diamond (z_1, z_2), q) \ngeq \min\{(\mu \times \delta)((x_1, x_2) \diamond [(y_1, y_2) \diamond (z_1, z_2)], q), (\mu \times \delta)((y_1, y_2), q)\}$. Therefore, $\mu \times \delta$ is not a q-fuzzy UP-ideal of $A \times A$.

Example 2.24. Let $A = \{0, 1, 2\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1	2
0	0	1	2
1	0	0	1
2	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{q\}$. We defined a Q-fuzzy set μ and δ in A as follows: $\mu(0, q) = 0.4, \delta(0, q) = 0.7, \mu(1, q) = 0.1, \delta(1, q) = 0.1, \mu(2, q) = 0.3$ and $\delta(2, q) = 0.3$. Using this data, we can show that μ and δ are q-fuzzy UP-subalgebras of A. Let $(x_1, x_2) = (0, 1), (y_1, y_2) = (1, 2) \in A \times A$. Then

$$(\mu \times \delta)((x_1, x_2) \diamond (y_1, y_2), q) = 0.1$$

and

$$\min\{(\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q)\} = 0.3.$$

Hence, $(\mu \times \delta)((x_1, x_2) \diamond (y_1, y_2), q) \ngeq \min\{(\mu \times \delta)((x_1, x_2), q), (\mu \times \delta)((y_1, y_2), q)\}$. Therefore, $\mu \times \delta$ is not a q-fuzzy UP-subalgebra of $A \times A$.

With Definition 1.10 and 1.13 and Theorem 2.22, we obtain the corollary.

Corollary 2.25. *The following statements hold:*

(1) if μ is a Q-fuzzy UP-ideal of A and δ is a Q-fuzzy UP-ideal of B, then $\mu \cdot \delta$ is a Q-fuzzy UP-ideal of $A \times B$, and

(2) if μ is a Q-fuzzy UP-subalgebra of A and δ is a Q-fuzzy UP-subalgebra of B, then $\mu \cdot \delta$ is a Q-fuzzy UP-subalgebra of $A \times B$.

Theorem 2.26. *If μ is a Q-fuzzy set in A and δ is a Q-fuzzy set in B such that $\mu \cdot \delta$ is a q-fuzzy UP-ideal of $A \times B$, then the following statements hold:*

(1) either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$,

(2) if $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$, then either $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, and

(3) if $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, then either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

Proof. (1) Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(0_A, q) < \mu(x, q)$ and $\delta(0_B, q) < \delta(y, q)$. Then

$$\begin{aligned}(\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\} \\ &> \min\{\mu(0_A, q), \delta(0_B, q)\} \\ &= (\mu \cdot \delta)((0_A, 0_B), q)\end{aligned}$$

which is a contradiction. Hence, $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$.

(2) Assume that $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$. Suppose that there exist $x \in A$ and $y \in B$ such that $\delta(0_B, q) < \mu(x, q)$ and $\delta(0_B, q) < \delta(y, q)$. Then $\mu(0_A, q) \geq \mu(x, q) > \delta(0_B, q)$. Thus

$$\begin{aligned}(\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\}\\ &> \min\{\delta(0_B, q), \delta(0_B, q)\}\\ &= \delta(0_B, q)\\ &= \min\{\mu(0_A, q), \delta(0_B, q)\}\\ &= (\mu \cdot \delta)((0_A, 0_B), q)\end{aligned}$$

which is a contradiction. Hence, $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$.

(3) Assume that $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that there exist $x \in A$ and $y \in B$ such that $\mu(0_A, q) < \mu(x, q)$ and $\mu(0_A, q) < \delta(y, q)$. Then $\delta(0_B, q) \geq \delta(x, q) > \mu(0_A, q)$. Thus

$$\begin{aligned}(\mu \cdot \delta)((x, y), q) &= \min\{\mu(x, q), \delta(y, q)\}\\ &> \min\{\mu(0_A, q), \mu(0_A, q)\}\\ &= \mu(0_A, q)\\ &= \min\{\mu(0_A, q), \delta(0_B, q)\}\\ &= (\mu \cdot \delta)((0_A, 0_B), q)\end{aligned}$$

which is a contradiction. Hence, $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

With Definition 1.10 and 1.13 and Theorem 2.26, we obtain the corollary.

Corollary 2.27. *If μ is a Q-fuzzy set in A and δ is a Q-fuzzy set in B such that $\mu \cdot \delta$ is a Q-fuzzy UP-ideal of $A \times B$, then the following statements hold:*

(1) for all $q \in Q$, either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$,

(2) for all $q \in Q$, if $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$, then either $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, and

(3) for all $q \in Q$, if $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$, then either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$.

Theorem 2.28. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let μ be a Q-fuzzy set in A and δ be a Q-fuzzy set in B. Then the following statements hold:*

(1) if $\mu \cdot \delta$ is a q-fuzzy UP-ideal of $A \times B$, then either μ is a q-fuzzy UP-ideal of A or δ is a q-fuzzy UP-ideal of B, and

(2) if $\mu \cdot \delta$ is a q-fuzzy UP-subalgebra of $A \times B$, then either μ is a q-fuzzy UP-subalgebra of A or δ is a q-fuzzy UP-subalgebra of B.

Proof. (1) Assume that $\mu \cdot \delta$ is a q-fuzzy UP-ideal of $A \times B$. Suppose that μ is not a q-fuzzy UP-ideal of A and δ is not a q-fuzzy UP-ideal of B. By Theorem 2.26 (1), we have $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Suppose that $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$. By Theorem 2.26 (2), either $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$ or $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. If $\delta(0_B, q) \geq \mu(x, q)$ for all $x \in A$, then $(\mu \cdot \delta)((x, 0_B), q) = \min\{\mu(x, q), \delta(0_B, q)\} = \mu(x, q)$. We consider, for all $x, y, z \in A$,

$$
\begin{aligned}
\mu(x \cdot z, q) &= \min\{\mu(x \cdot z, q), \delta(0_B, q)\} \\
&= (\mu \cdot \delta)((x \cdot z, 0_B), q) && \text{(Definition 1.25)} \\
&= (\mu \cdot \delta)((x \cdot z, 0_B * 0_B), q) && \text{(Proposition 1.2 (1))} \\
&= (\mu \cdot \delta)((x, 0_B) \diamond (z, 0_B), q) \\
&\geq \min\{(\mu \cdot \delta)((x, 0_B) \diamond [(y, 0_B) \diamond (z, 0_B)], q), \\
&\quad (\mu \cdot \delta)((y, 0_B), q)\} && \text{(Definition 1.10 (2))} \\
&= \min\{(\mu \cdot \delta)((x \cdot (y \cdot z), 0_B * (0_B * 0_B)), q), (\mu \cdot \delta)((y, 0_B), q)\} \\
&= \min\{(\mu \cdot \delta)((x \cdot (y \cdot z), 0_B), q), (\mu \cdot \delta)((y, 0_B), q)\} && \text{(Proposition 1.2 (1))} \\
&= \min\{\min\{\mu(x \cdot (y \cdot z), q), \delta(0_B, q)\}, \\
&\quad \min\{\mu(y, q), \delta(0_B, q)\}\} && \text{(Definition 1.25)} \\
&= \min\{\mu(x \cdot (y \cdot z), q), \mu(y, q)\}.
\end{aligned}
$$

Hence, μ is a q-fuzzy UP-ideal of A which is a contradiction. Suppose that $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. By Theorem 2.26 (3), either $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ or $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$. If $\mu(0_A, q) \geq \delta(x, q)$ for all $x \in B$, then $(\mu \cdot \delta)((0_A, x), q) = \min\{\mu(0_A, q), \delta(x, q)\} = \delta(x, q)$. We consider, for all $x, y, z \in B$,

$$
\begin{aligned}
\delta(x * z, q) &= \min\{\mu(0_A, q), \delta(x * z, q)\} \\
&= (\mu \cdot \delta)((0_A, x * z), q) && \text{(Definition 1.25)} \\
&= (\mu \cdot \delta)((0_A \cdot 0_A, x * z), q) && \text{(Proposition 1.2 (1))} \\
&= (\mu \cdot \delta)((0_A, x) \diamond (0_A, z), q) \\
&\geq \min\{(\mu \cdot \delta)((0_A, x) \diamond [(0_A, y) \diamond (0_A, z)], q), \\
&\quad (\mu \cdot \delta)((0_A, y), q)\} && \text{(Definition 1.10 (2))} \\
&= \min\{(\mu \cdot \delta)((0_A \cdot (0_A \cdot 0_A), x * (y * z)), q), (\mu \cdot \delta)((0_A, y), q)\} \\
&= \min\{(\mu \cdot \delta)((0_A, x * (y * z)), q), \\
&\quad (\mu \cdot \delta)((0_A, y), q)\} && \text{(Proposition 1.2 (1))} \\
&= \min\{\min\{\mu(0_A, q), \delta(x * (y * z), q)\}, \\
&\quad \min\{\mu(0_A, q), \delta(y, q)\}\} && \text{(Definition 1.25)} \\
&= \min\{\delta(x * (y * z), q), \delta(y, q)\}.
\end{aligned}
$$

Hence, δ is a q-fuzzy UP-ideal of B which is a contradiction. Since μ is not a q-fuzzy UP-ideal of A and δ is not a q-fuzzy UP-ideal of B, we have $\mu(0_A, q) \geq \mu(x, q)$ for all $x \in A$ and $\delta(0_B, q) \geq \delta(x, q)$ for all $x \in B$. Let $x_1, x_2, x_3 \in A$ and $y_1, y_2, y_3 \in B$ be such that $\mu(x_1 \cdot x_3, q) < \min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}$ and $\delta(y_1 * y_3, q) < \min\{\delta(y_1 * (y_2 * y_3), q), \delta(y_2, q)\}$, so $\min\{\mu(x_1 \cdot x_3, q), \delta(y_1 * y_3, q)\} <$

$\min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}, \min\{\delta(y_1 * (y_2 * y_3), q), \delta(y_2, q)\}\}$. Thus

$$\begin{aligned}
&\min\{\mu(x_1 \cdot x_3, q), \delta(y_1 * y_3, q)\} \\
&\quad = (\mu \cdot \delta)((x_1 \cdot x_3, y_1 * y_3), q) && \text{(Definition 1.25)} \\
&\quad = (\mu \cdot \delta)((x_1, y_1) \diamond (x_3, y_3), q) \\
&\quad \geq \min\{(\mu \cdot \delta)((x_1, y_1) \diamond [(x_2, y_2) \diamond (x_3, y_3)], q), \\
&\qquad (\mu \cdot \delta)((x_2, y_2), q)\} && \text{(Definition 1.10 (2))} \\
&\quad = \min\{(\mu \cdot \delta)((x_1 \cdot (x_2 \cdot x_3), y_1 * (y_2 * y_3)), q), (\mu \cdot \delta)((x_2, y_2), q)\} \\
&\quad = \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \delta(y_1 * (y_2 * y_3), q)\}, \\
&\qquad \min\{\mu(x_2, q), \delta(y_2, q)\}\} && \text{(Definition 1.25)} \\
&\quad = \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}, \\
&\qquad \min\{\delta(y_1 * (y_2 * y_3), q), \delta(y_2, q)\}\}. && \text{(Lemma 2.21 (2))}
\end{aligned}$$

It follows that $\min\{\mu(x_1 \cdot x_3, q), \delta(y_1 * y_3, q)\} \nless \min\{\min\{\mu(x_1 \cdot (x_2 \cdot x_3), q), \mu(x_2, q)\}, \min\{\delta($ $(y_2 * y_3), q), \delta(y_2, q)\}\}$ which is a contradiction. Hence, μ is a q-fuzzy UP-ideal of A or δ is a q-fuzzy UP-ideal of B.

(2) Assume that $\mu \cdot \delta$ is a q-fuzzy UP-subalgebra of $A \times B$. Suppose that μ is not a q-fuzzy UP-subalgebra of A and δ is not a q-fuzzy UP-subalgebra of B. Then there exist $x, y \in A$ and $a, b \in B$ such that

$$\mu(x \cdot y, q) < \min\{\mu(x, q), \mu(y, q)\} \text{ and } \delta(a * b, q) < \min\{\delta(a, q), \delta(b, q)\}.$$

Then $\min\{\mu(x \cdot y, q), \delta(a * b, q)\} < \min\{\min\{\mu(x, q), \mu(y, q)\}, \min\{\delta(a, q), \delta(b, q)\}\}$. Consider,

$$\begin{aligned}
\min\{\mu(x \cdot y, q), \delta(a * b, q)\} &= (\mu \cdot \delta)((x \cdot y, a * b), q) && \text{(Definition 1.25)} \\
&= (\mu \cdot \delta)((x, a) \diamond (y, b), q) \\
&\geq \min\{(\mu \cdot \delta)((x, a), q), \\
&\quad (\mu \cdot \delta)((y, b), q)\} && \text{(Definition 1.13)} \\
&= \min\{\min\{\mu(x, q), \delta(a, q)\}, \\
&\quad \min\{\mu(y, q), \delta(b, q)\}\} && \text{(Definition 1.25)} \\
&= \min\{\min\{\mu(x, q), \mu(y, q)\}, \\
&\quad \min\{\delta(a, q), \delta(b, q)\}\}. && \text{(Lemma 2.21 (2))}
\end{aligned}$$

Thus $\min\{\mu(x \cdot y, q), \delta(a * b, q)\} \nless \min\{\min\{\mu(x, q), \mu(y, q)\}, \min\{\delta(a, q), \delta(b, q)\}\}$ which is a contradiction. Hence, μ is a q-fuzzy UP-subalgebra of A or δ is a q-fuzzy UP-subalgebra of B.

Give examples of conflict that μ and δ are not Q-fuzzy UP-ideals (resp. Q-fuzzy UP-subalgebras) of A but $\mu \cdot \delta$ is a Q-fuzzy UP-ideal (resp. Q-fuzzy UP-subalgebra) of $A \times A$.

Example 2.29. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We define two Q-fuzzy sets μ and δ in A as follows:

μ	a	b
0	0.1	0.3
1	0.3	0.3

and

δ	a	b
0	0.3	0.1
1	0.3	0.3

Since $\mu(0,a) = 0.1 < 0.3 = \mu(1,a)$, we have $\mu(0,a) \ngeq \mu(1,a)$. Thus μ is not an a-fuzzy UP-ideal of A. Since $\delta(0,b) = 0.1 < 0.3 = \delta(1,b)$, we have $\delta(0,b) \ngeq \delta(1,b)$. Thus δ is not a b-fuzzy UP-ideal of A. Therefore, μ and δ are not Q-fuzzy UP-ideals of A. Using the above data, we can show that $\mu \cdot \delta$ is a Q-fuzzy UP-ideal of $A \times A$.

Example 2.30. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. Let $Q = \{a, b\}$. We defined two Q-fuzzy sets μ and δ in A as follows:

μ	a	b
0	0.1	0.3
1	0.3	0.3

and

δ	a	b
0	0.3	0.1
1	0.3	0.3

Since $\mu(1 \cdot 1, a) = \mu(0, a) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\mu(1,a), \mu(1,a)\}$, we have $\mu(1 \cdot 1, a) \ngeq \min\{\mu(1,a), \mu(1,a)\}$. Thus μ is not an a-fuzzy UP-subalgebra of A. Since $\delta(1 \cdot 1, b) = \delta(0, b) = 0.1 < 0.3 = \min\{0.3, 0.3\} = \min\{\delta(1,b), \delta(1,b)\}$, we have $\delta(1 \cdot 1, b) \ngeq \min\{\delta(1,b), \delta(1,b)\}$. Thus δ is not a b-fuzzy UP-subalgebra of A. Therefore, μ and δ are not Q-fuzzy UP-subalgebras of A. By Example 2.29, we have $\mu \cdot \delta$ is a Q-fuzzy UP-ideal of $A \times A$. By Corollary 2.2, we have $\mu \cdot \delta$ is a Q-fuzzy UP-subalgebra of $A \times A$.

Acknowledgements

The authors wish to express their sincere thanks to the referees for the valuable suggestions which lead to an improvement of this paper.

References

Bali, N. P. (2005). *Golden real analysis*. Chennai, India: Laxmi Publications.

Iampan, A. (2017). A new branch of the logical algebra: UP-algebras. *Journal of Algebra and Related Topics*, *5*(1), 35-54.

Jun, Y. B. (2001). Q-fuzzy subalgebras of BCK/BCI-algebras. *Scientiae Mathematicae Japonicae Online*, *4*, 197-202.

Kim, K. H. (2006). On intuitionistic Q-fuzzy semiprime ideals in semigroups. *Advance Fuzzy Math*, *1*(1), 15-21.

Malik, S. C., & Arora, S. (2014). *Mathematical analysis (4th Ed.)*. New Delhi, India: New Age International.

Mostafa, S. M., Abdel-Naby, M. A., & Elgendy, O. R. (2012). Fuzzy Q-ideals in Q-algebras. *World Applied Programming*, *2*(2), 69-80.

Muthuraj, R., Sridharan, M., Muthuraman, M. S., & SitharSelvam, P. M. (2010). Anti Q-fuzzy BG-ideals in BG-algebra. *International Journal of Computer Applications*, *4*(11), 27-31.

Priya, T., &Ramachandran, T. (2014). A note on anti Q-fuzzy R-closed PS-ideals in PS-algebras. *Advances in Pure and Applied Mathematics*, *6*(2), 150-159.

Roh, E. H., Kim, K. H., & Lee, J. G. (2006). Intuitionistic Q-fuzzy subalgebras of BCK/BCI-algebras. *International Mathematical Forum*, *1*(24), 1167-1174.

Sithar Selvam, P. M., Priya, T., Nagalakshmi, K. T., & Ramachandran, T. (2013). A note on anti Q-fuzzy KU-subalgebras and homomorphism of KU-algebras *Bulletin of Mathematics and Statistics Research*, *1*(1), 42-49.

Sithar Selvam, P. M., Priya, T., & Ramachandran, T. (2012). Anti Q-fuzzy KU-ideals in KU-algebras and its lower level cuts. *International Journal of Engineering Research and Technology*, *2*(4), 1286-1289.

Sithar Selvam, P. M., Priya, T., Ramachandran, T., & Nagalakshmi, K. T. (2014). Anti Q-fuzzy R-closed KU-ideals in KU-algebras and its lower level cuts. *International Journal of Fuzzy Mathematical Archive*, *4*(2), 61-71.

Somjanta, J., Thuekaew, N., Kumpeangkeaw, P., & Iampan, A. (2016). Fuzzy sets in UP-algebras. *Annals of Fuzzy Mathematics and Informatics*, *12*(6), 739-756.

Zadeh, L. A. (1965). Fuzzy sets. *Information and Control*, *8*, 338-353.