

Ideas for using GeoGebra and Origami in Teaching Regular Polyhedrons Lessons

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Abstract

The approach of combining GeoGebra and origami is well accepted among students in the school "Petro Kuzmjak" where it is used to teach geometry lessons. This article elaborates on how to introduce students (upper elementary and high school students, age 14-18) to Platonic solids and their properties through combination of GeoGebra and origami activities. Some of the important mathematical concepts related to these well-known geometrical solids can be explained to students applying hands-on activities along with educational software.

Keywords: Geogebra, Origami, Platonic solids, hands-on activities, geometry lessons

Introduction

In our educational approach, we combine origami and GeoGebra and offer examples of how to use them in teaching mathematics, especially for geometry lessons. In conjunction with traditional origami, we propose the use of dynamic mathematics software known as GeoGebra, which is widely used by millions of students and teachers around the world (International GeoGebra Institute, 2017). In our approach, we use a problem-solving strategy with hands-on models and software that supplement each other. Exploring different strategies of problem solving in hands-on and digital environments develops reasoning skills and could support deeper understanding of problems (Boakes, 2009). Origami requires following a certain procedure, while GeoGebra allows crafting other procedures leading to a solution.

There has already been some implementation of this principle in teaching. In the literature there is an example of teaching the Delian problem combining origami and GeoGebra to students (Fenyvesi et al., 2014) where efficient practice of joint origami and GeoGebra is highlighted. The Delian problem is based on the mathematical knowledge of calculating the volume of geometric solids that is impossible to solve within the constraints of Euclidean geometry.

In this paper, we describe examples of how to combine origami and GeoGebra in teaching properties of regular polyhedrons known as Platonic solids. These trial problems were done in the classrooms in "Petro Kuzmjak" school in Serbia as part of a research project about the benefits of engagement in folding and digital problems.

Origami in mathematical lessons

In the past decades, origami has become an interesting and dynamically developing part of mathematical sciences. The Huzita axioms provided the first formal description of the types of geometric constructions that were possible to carry out with paper folding (Meyer &

Meyer, 1999). Mathematical axiomatization of origami put many geometrical constructions that were previously impossible to solve, to the straight ruler and compass through the solution (Auckly & Cleveland, 1995).

The definition of origami implies that paper should be folded without using scissors and glue, and it should result in a variety of forms that we use in everyday life. It is shown that origami could be beneficial for mathematics teaching as, among other reasons: 1) students feel satisfaction from creation; 2) it blends mathematical vocabulary and content during the steps of folding; 3) it supports community building; 4) it encourages cooperative learning; 5) it develops planar and spatial reasoning; 6) it allows students to create and manipulate basic geometric shapes with ease; 7) it contributes to the students' cognitive development; 8) and it offers affordable uses in classrooms.

It is also known that origami contributes to students' understanding of mathematical concepts, especially concepts related to geometry. Educational researchers perceived its beneficial influence on teaching and learning geometry (Boakes, 2009; Cipoletti & Wilson, 2004; Fenyvesi et al., 2014a; Fenyvesi et al., 2014b; Robichaux & Rodrigue, 2003) and it is getting significant attention in the educational circles promoting principles of STEAM education (Science, Technology, Engineering, Art, and Mathematics). For example, it can be used in teaching topics such as polyhedrons or symmetry, where models can be made by folding paper and thereby contribute to students' understanding of the properties of objects.

Geogebra and the 3D properties in mathematical lessons

Dynamic Geometry Software (DGS) such as GeoGebra can be used to extend investigations and foster deeper understanding of a proof (www.geogebra.org). GeoGebra is accessible and engaging, and helps to encourage students to further explore the geometrical situation. It also provides possibilities for making and evaluating conjectures and results. GeoGebra, as its name implies, connects algebra and geometry. This software is useful in high school education, because many curriculum topics in mathematics are algebraic and geometrical. In addition, it is easy to use and has many powerful features, as the software is dynamic. There are numerous materials available on the internet that can be easily used in classrooms and the software is available in many languages, has free access and the open-source nature also contributes to its popularity. More and more accepted in educational circles, GeoGebra is developing and adding new features, such as a 3D grapher featuring surfaces, geometric objects, quadrics and nets.

Platonic solids lesson with GeoGebra and Origami

In this section, we provide guidelines on how to introduce students to Platonic solids and how to use origami and GeoGebra during that process. The lesson plan is created with the idea that the aid of visual imagination can be helpful to students while discovering facts without acquiring formal definitions or concepts (Hilbert & Cohn-Voss, 1999). Also, this type of lesson should guide students through geometry and spatial relations. Manipulation of paper and assembling physical models, together with GeoGebra and its dynamic and 3D properties should open a mathematical discourse and encourage students to explore and learn.

The lessons should start with a short introduction about Platonic solids, which have an interesting and intriguing history, not only mathematical but also philosophical and artistic. Also, students should know that there are only five Platonic solids: tetrahedron, cube, octahedron, dodecahedron and icosahedron. The simplest is the tetrahedron, consisting of four equilateral triangle faces. The six square face solid is well-known as a cube or hexahedron. Eight equilateral triangular faces form the octahedron, while an icosahedron consists of twenty. The dodecahedron has twelve regular pentagon faces. These solids, labeled Platonic, were named after the ancient philosopher Plato, who

associated them with classical elements such as earth, air, water and fire. Figure 1 presents five Platonic solids made by origami techniques.

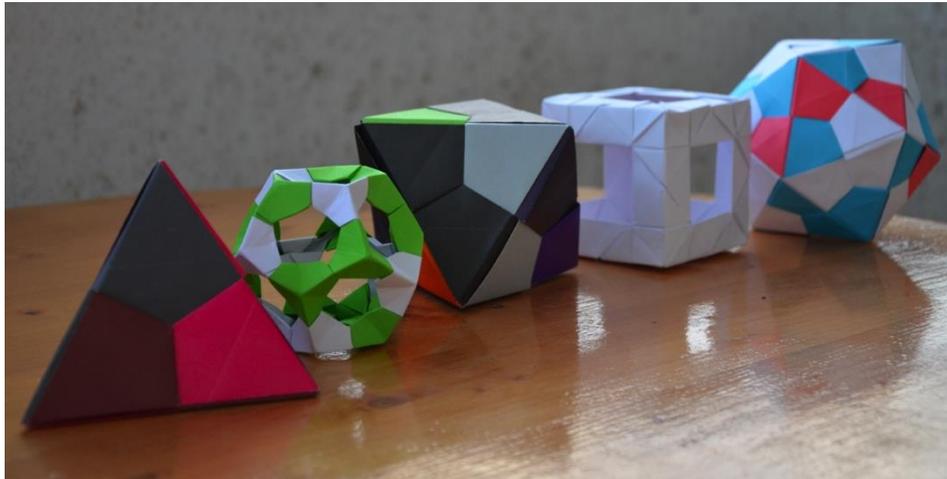


Figure 1. Five Platonic solids made by origami techniques: (L-R) tetrahedron, dodecahedron, octahedron, cube and icosahedron

In the main part of the lessons students should learn about properties of Platonic solids and polyhedrons in general. GeoGebra may be used in the introductory part where students are introduced to basic notions connected to polyhedrons such as faces (polygons of which the polyhedron is consisted), vertices (the segments where faces meet) and edges (the points where vertices meet). The prior made applets can be used for that purpose. An example of one of the applets found at GeoGebraTube is shown in Figure 2.

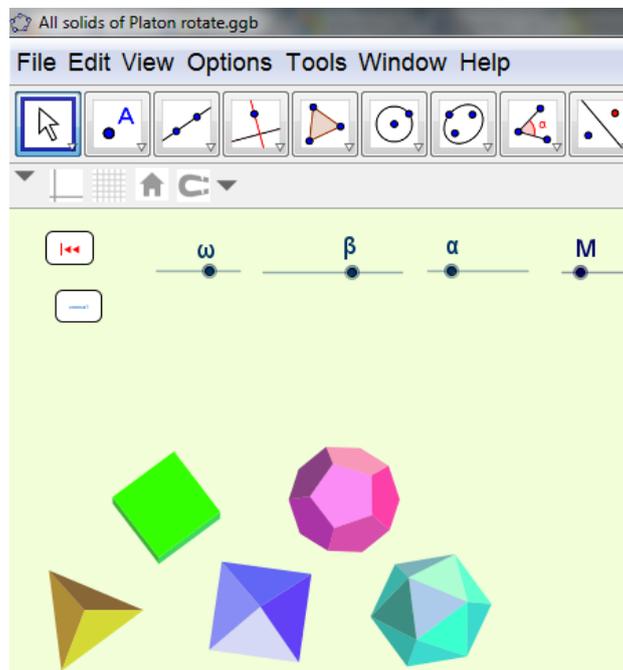


Figure 2. Geogebra applet about Platonic solids

Also, it can be visualized with help of GeoGebra and 3D options as can be seen in Figure 3. Geogebra can also be useful for reminding students of the regular polyhedrons' definitions.

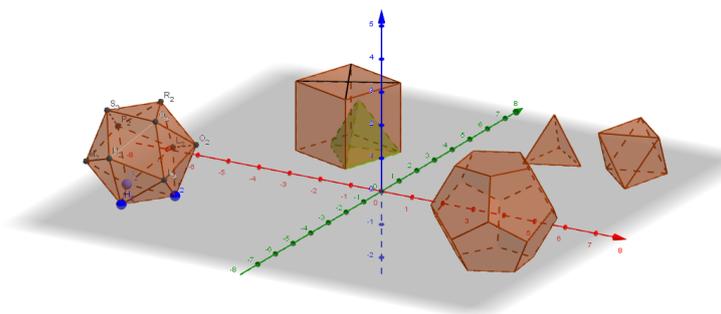


Figure 3. GeoGebra representation of Platonic solids made by students

Students can receive instructions on paper folding before the lessons. An example of one of the instructions, folding a tetrahedron unit, is shown in Figure 4. Also, students can follow YouTube tutorials about Platonic solids paper folding.

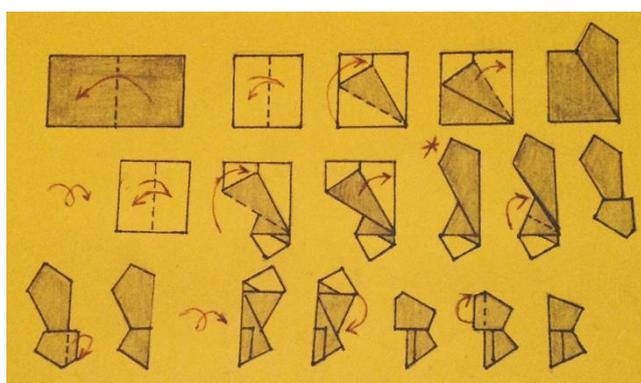


Figure 4. Instructions for folding tetrahedron units

As certain skills in paper folding are required, parts or models made beforehand could be provided during the lesson to make the process more efficient. As homework, students can assemble some sophisticated models prior to the lesson. There are many ways of folding paper origami models, but we propose techniques recommended by Glassner (1996). The property of convexity can be conveyed through the observation of origami models. By observing models, students can conclude that regular polyhedrons are convex. Also, origami models can be used to check that each vertex of the Platonic solid has the same number of meeting faces (vertices).

The fact that there are only five Platonic solids would be visualized by the constructed origami model. The proof can be explained by the fact that there are at least three faces at each vertex of a solid. That implies that the sum of internal angles at the vertex must be less than 360° . At 360° the shape is flattened. The fact that hexagons cannot be Platonic solid faces should be mentioned to students. In addition, students should be introduced to the formulae regarding Platonic solids. Finally, the participants should be introduced to numerous mathematical facts and formulae regarding Platonic solids. Some of them are shown in Table 1. Derivation of those formulae and their applications can be explained with help of GeoGebra.

Table 1

Formulae related to Platonic solids

Polyhedron	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Faces	Equilateral triangles	Squares	Equilateral triangles	Regular pentagons	Equilateral triangles
Number of faces	4	6	8	12	20
Dihedral angles	$\arccos\left(\frac{1}{3}\right)$	$\frac{\pi}{2}$	$\pi - \arccos\left(\frac{1}{3}\right)$	$\pi - \arccos(2)$	$\pi - \arccos\left(\frac{\sqrt{5}}{3}\right)$
Area	$\sqrt{3}a^2$	$6a^2$	$2\sqrt{3}a^2$	$3\sqrt{25+10\sqrt{5}}a^2$	$5\sqrt{3}a^2$
Volume	$\frac{a^3}{6\sqrt{2}}$	a^3	$\frac{\sqrt{2}}{3}a^3$	$\frac{15+7\sqrt{5}}{4}a^3$	$\frac{5(3+\sqrt{5})}{12}a^3$
Circumradius	$\frac{\sqrt{6}}{4}a$	$\frac{\sqrt{3}}{2}a$	$\frac{a}{2}\sqrt{2}$	$\frac{a\sqrt{3}}{4}(1+\sqrt{5})$	$\frac{a}{4}\sqrt{10+2\sqrt{5}}$
Inradius	$\frac{a}{\sqrt{24}}$	$\frac{a}{2}$	$\frac{a}{6}\sqrt{6}$	$\frac{a}{2}\sqrt{\frac{15}{2}+\frac{11}{10}\sqrt{5}}$	$\frac{\sqrt{3}(3+\sqrt{5})}{12}a$

Conclusions

The use of GeoGebra in the teaching process is not a new one. The idea of using origami in mathematical lessons is recognized as beneficial to the teaching process. The combination of these two is getting significant attention, because it is not a very common educational practice.



Natalija Budinski holds a master's degree in mathematics and works as a teacher in the primary and secondary school Petro Kuzmjak in Serbia. The field of her interest is mathematics education and teaching mathematical content through origami activities. She is promoting the approach by giving workshops and lectures to students and teachers in her home country, and also at international conferences. In May 2016 her origami approach to mathematical lessons was recognized as one of the ten best mathematical education practices in Serbia by the Centre for Science Promotion. She received a Bridges organization scholarship for achievement in connecting mathematics with art in education in 2016 in Finland.



Zsolt Lavicza received his degrees in mathematics and physics in Hungary, then began his postgraduate studies in applied mathematics at the University of Cincinnati. While teaching mathematics at the University of Cincinnati he became interested in researching issues in teaching and learning mathematics. In particular, he focused on investigating issues in relation to the use of technology in undergraduate mathematics education. Since then, both at the University of Michigan and Cambridge, he has worked on several research projects examining technology and mathematics teaching in a variety of classroom environments. Currently, Dr. Lavicza is working on numerous research projects worldwide related to technology integration into schools, offering educational research training courses at a number of universities, leading a doctoral program in STEM education at Johannes Kepler University, and coordinating research projects within the International GeoGebra Institute.



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