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Original Article

Hyperideals and hypersystems in LA-hyperrings

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Abstract

In this paper we introduce the concept of LA-hyperrings. We explore some useful characterizations of LA-hyperrings through their hyperideals and hypersystems.

Keywords: LA-hyperrings, hyperideals, hypersystems

1. Introduction

Kazim and Naseeruddin (1972) introduced the concept of left almost semigroups (LA-semigroups). A groupoid S is called an LA-semigroup if it satisfies the left invertive law: $(ab)c = (cb)a$ for all $a, b, c \in S$. Protic and Stevanovic (1995a, 1995b) called this structure as an Abel-Grassmann's groupoid (abbreviated as an AG-groupoid). An AG-groupoid is the midway structure between a commutative semigroup and a groupoid. Later, Kamran (1993) extended the notion of LA-semigroup to left almost group (LA-group). A groupoid G is called a left almost group (LA-group), if there exists left identity $e \in G$ (that is $ea = a$ for all $a \in G$), for $a \in G$ there exists $b \in G$ such that $ba = e$ and left invertive law holds in G

Left almost ring (LA-ring) is actually an off shoot of LA-semigroup and LA-group. It is a non-commutative and non-associative structure and gradually due to its peculiar characteristics it has been emerging as useful non-associative class which intuitively would have reasonable contribution to enhance non-associative ring theory. By an LA-ring, we mean a non-empty set R with at least two elements such that $(R, +)$ is an LA-group, $(R, \cdot)$ is an LA-semigroup, both left and right distributive laws hold. For example, from a commutative ring $(R, +, \cdot)$, we can always obtain an LA-ring $(R, \oplus, \cdot)$ by defining for all $a, b \in R$, $a \oplus b = b \ominus a$ and $a \cdot b$ is same as in the ring.

Shah and Rehman (2010) discussed left almost ring (LA-ring) of finitely nonzero functions which is in fact a generalization of a commutative semigroup ring. Shah *et al.* (2012) applied the concept of intuitionistic fuzzy sets to non-associative rings and established some useful results. In Rehman *et al.* (2013) some computational work through Mace4, has been done and some interesting characteristics of LA-rings have been explored. For some more study of LA-rings, we refer the readers to see (Rehman, 2011) and Shah *et al.* (2011a, 2011b).

The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty (1934) and since then many researchers have developed this theory. Nowadays, a number of different hyperstructures are widely studied from the theoretical point of view. Hyperstructures have a lot of applications to several domains of mathematics and

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computer science and they are studied in many countries of the world. In a recent book, Corsini and Leoreanu (2003) have presented some of the numerous applications of algebraic hyperstruc-tures, especially those from the last fifteen years, in the fields of geometry, hypergraphs, binary relations, lattices, fuzzy sets, rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. A lot of papers and several books have been written on hyperstructure theory; see Corsini (1993), Vougiouklis (1994). Another book, Davvaz and Fotea (2007), is devoted especially to the study of hyperring theory. The volume ends with an outline of applications in chemistry and physics, analyzing several special kinds of hyperstructures: e-hyperstructures and transposition hypergroups. Many authors studied different aspects of semihypergroups, for instance, Bonansinga and Corsini (1982), Onipchuk (1992), Hasankhani (1999), Leoreanu (2000), Davvaz (2000), Davvaz and Poursalavati (2000), Corsini and Crisrea (2005), Hila *et al.* (2011) Abdullah *et al.* (2011, 2012), Ahmed *et al.* (2012), and Aslam *et al.* (2012, 2013).

Recently, Hila and Dine (2011) introduced the notion of LA-semihypergroups as a generalization of semigroups, semihypergroups and LA-semigroups. Yaqoob *et al.* (2013) introduced intra-regular class of left almost semihypergroups. Yaqoob and Gulistan (2015) introduced the concept of partially ordered left almost semihypergroups. For more study of LA-semihypergroups we refer the readers to see Amjad *et al.* (2014a, 2014b), Gulistan *et al.* (2015), and Yousafzai *et al.* (2015).

In this paper, first we introduce the notion of LA-hyperrings and then we establish some basic related results. We characterize LA-hyperrings based on their hyperideals and Hypersystems.

2. LA-hyperrings

Before the definition of LA-hyperring, first we give the definition and examples of LA-hypergroup.

Definition 2.1 A hypergroupoid $(H,\circ)$ is said to be an LA-hypergroup if it satisfies the following axioms:

(i) for all $x,y,z\in H$, $(x\circ y)\circ z=(z\circ y)\circ x$,

(ii) for every $x\in H$, $x\circ H=H\circ x=H$.

Example 2.2 Let $H=\{a,b,c\}$ be a set with the hyperoperations $\circ_1$, $\circ_2$ and $\circ_3$ defined as follows:

$\circ_1$	a	b	c
a	a	H	H
b	H	$\{b,c\}$	$\{b,c\}$
c	H	b	b

$\circ_2$	a	b	c
a	a	H	H
b	H	$\{b,c\}$	$\{b,c\}$
c	H	$\{a,b\}$	$\{a,b\}$

$\circ_3$	a	b	c
a	a	H	H
b	H	$\{b,c\}$	$\{b,c\}$
c	H	H	H

Then $(H,\circ_1)$, $(H,\circ_2)$ and $(H,\circ_3)$ are LA-hypergroups.

Definition 2.3 An algebraic system $(R,\oplus,\otimes)$ is said to be an LA-hyperring, if

(i) $(R,\oplus)$ is an LA-hypergroup;

(ii) $(R,\otimes)$ is an LA-semihypergroup;

(iii) $\otimes$ is distributive with respect to $\oplus$.

Example 2.4 Let $R=\{a,b,c\}$ be a set with the hyperoperations $\oplus$ and $\otimes$ defined as follows:

$\oplus$	a	b	c
a	a	R	R
b	R	$\{b,c\}$	$\{b,c\}$
c	R	R	R

$\otimes$	a	b	c
a	a	a	a
b	a	R	c
c	a	R	R

Then $(R,\oplus,\otimes)$ is an LA-hyperring. One can see that $(R,\oplus)$ and $(R,\otimes)$ both satisfy left invertive law and also both are non-associative, see $\{b,c\}=b\oplus(b\oplus c)\neq(b\oplus b)\oplus c=R$ and $\{c\}=b\otimes(b\otimes c)\neq(b\otimes b)\otimes c=R$.

3. Hyperideals in LA-hyperrings

In this section, we discuss some elementary properties of hyperideals in an LA-hyperring with pure left identity e and specifically we prove the necessary and sufficient conditions for an LA-hyperring to be a fully prime.

Definition 3.1 Let $(R,\oplus,\otimes)$ be an LA-hyperring. Then

(i) R is called with left identity (resp., pure left identity), if there exists an element, say $e\in R$, such that $x\in e\otimes x$ (resp., $x=e\otimes x$), for all $x\in R$.

(ii) an element r of an LA-hyperring R is called an idempotent (resp., weak idempotent) if $r=r\otimes r$ (resp., $r\in r\otimes r$).

(iii) a non-empty subset A of R is said to be an LA-subhyperring of R if $(A,\oplus,\otimes)$ is itself an LA-hyperring.

Definition 3.2 If A is an LA-subhyperring of an LA-hyperring $(R,\oplus,\otimes)$, then A is called a left hyperideal if $R\otimes A\subseteq A$. Right hyperideal and two sided hyperideals are defined in the usual manner.

The following proposition identify that every right hyperideal in an LA-hyperring is a two sided hyperideal.

Proposition 3.3 If $(R,\oplus,\otimes)$ is an LA-hyperring with left identity (or with pure left identity) e, then every right hyperideal is a left hyperideal.

Proof. Let I be a right hyperideal of LA-hyperring R. This implies that I is an LA-subhyperring of R. If $r\in R$ and $i\in I$, then

$$r\otimes i\subseteq\left(e\otimes r\right)\otimes i=\left(i\otimes r\right)\otimes e\subseteq(I\otimes R)\otimes R\subseteq I.$$

Thus I is also a left hyperideal of $(R,\oplus,\otimes)$. The case for pure left identity can be seen in a similar way. Now onward by hyperideal in an LA-hyperring R with pure left identity e, we mean a right hyperideal.

Definition 3.4 A hyperideal I of an LA-hyperring R is called minimal if it does not contain any hyperideal of R other than itself.

Remark 3.5 Every LA-hyperring satisfies the law $(a \otimes b) \otimes (c \otimes d) = (a \otimes c) \otimes (b \otimes d)$ for all $a, b, c, d \in R$. This law is known as medial law.

Theorem 3.6 Let R be an LA-hyperring with pure left identity e. If I is a minimal left hyperideal of R, then $a \otimes I$ is a minimal left hyperideal of R for every idempotent a.

Proof. Let I be a minimal left hyperideal of an LA-hyperring R and a is an idempotent element. Consider $a \otimes I = \{a \otimes i : i \in I\}$. Let $a \otimes i_1, a \otimes i_2 \subseteq a \otimes I$. Then

$$a \otimes i_1 - a \otimes i_2 = a \otimes (i_1 - i_2) \subseteq a \otimes I,$$

and

$$\begin{aligned}(a \otimes i_1) \otimes (a \otimes i_2) &= (a \otimes a) \otimes (i_1 \otimes i_2) \\ &= a \otimes (i_1 \otimes i_2) \subseteq a \otimes I.\end{aligned}$$

Thus $a \otimes I$ is an LA-subhyperring of R. For $r \in R$, $a \otimes i \in a \otimes I$, we have

$$r \otimes (a \otimes i) = a \otimes (r \otimes i) \subseteq a \otimes I.$$

Thus $a \otimes I$ is a left hyperideal of R. Next, let H be a non-empty left hyperideal of R which is properly contained in $a \otimes I$. Define $K = \{i \in I : a \otimes i \subseteq H\}$ and let $y \in K$. Then $a \otimes y \subseteq H$, and so we get

$$a \otimes (r \otimes y) = r \otimes (a \otimes y) \subseteq R \otimes H \subseteq H.$$

This implies that $r \otimes y \subseteq K$. Hence K is a left hyperideal properly contained in I. But this is a contradiction to the minimality of I. Thus $a \otimes I$ is a minimal left hyperideal of LA-hyperring R.

Lemma 3.7 If I is a right hyperideal of an LA-hyperring R with left identity (or with pure left identity) e, then I^2 is a hyperideal of R.

Proof. Let $i \in I^2$. Then we can write $i \in x \otimes y$, where $x, y \in I$. Consider

$$\begin{aligned}i \otimes r &\subseteq (x \otimes y) \otimes r \subseteq (x \otimes y) \otimes (e \otimes r) \\ &= (x \otimes e) \otimes (y \otimes r) \quad \text{(by medial law)} \\ &\subseteq I \otimes I = I^2.\end{aligned}$$

This implies that I^2 is a right hyperideal and hence by Proposition 3.3, I^2 is a left hyperideal. The case for pure left identity can be seen in a similar way.

Lemma 3.8 Intersection of two left (right) hyperideals of an LA-hyperring is again a left (right) hyperideal.

Proof. Let I, J be two left hyperideals of an LA-hyperring R. Let $a, b \in I \cap J$ this implies that $a, b \in I$ and $a, b \in J$. So, $a \Theta b \subseteq I$, $a \Theta b \subseteq J$ and $a \otimes b \subseteq I$ and $a \otimes b \subseteq J$. This implies that $a \Theta b \subseteq I \cap J$ and $a \otimes b \subseteq I \cap J$. Now let $r \in R$ and $a \in I \cap J$, so $a \in I$ and $a \in J$. This implies $r \otimes a \subseteq I$ and $r \otimes a \subseteq J$ and hence $r \otimes a \subseteq I \cap J$. Thus $I \cap J$ is a left hyperideal.

Addition of hyperideals I and J of an LA-hyperring R is defined as

$$I \oplus J = \{x \oplus y \ : \ x \in I, \ y \in J\} \subseteq R.$$

If for every element $x \in I \oplus J$, there exist unique elements a, b such that $a \in I$, $b \in J$ and $x \in a \oplus b$.

Theorem 3.9 The sum of two left (right) hyperideals of an LA-hyperring is again a left (right) hyperideal.

Proof. Let R be an LA-hyperring. Let I, J be left hyperideals of R. Suppose $z_1, z_2 \in I \oplus J$. This implies $z_1 \in (x_1 \oplus y_1)$ and $z_2 \in (x_2 \oplus y_2)$. Now consider

$$z_1 \Theta \subseteq I \oplus J, \ y_2) \Theta (y_1 \ \oplus \ x_2) \Theta (x_1 = (x_2 \oplus y_2) \ \Theta (x_1 \oplus y_1) \ \subseteq \ z_2$$

and

$$\begin{aligned}z_1 \otimes z_2 &= (x_1 \oplus y_1) \otimes (x_2 \oplus y_2) \\ &= (x_1 \otimes (x_2 \oplus y_2)) \oplus (y_1 \otimes (x_2 \oplus y_2)) \\ &= ((x_1 \otimes x_2) \oplus (x_1 \otimes y_2)) \oplus ((y_1 \otimes x_2) \oplus (y_1 \otimes y_2)) \\ &= ((x_1 \otimes x_2) \oplus (y_1 \otimes x_2)) \oplus ((x_1 \otimes y_2) \oplus (y_1 \otimes y_2)) \subseteq I \oplus J.\end{aligned}$$

Again suppose $z \in I \oplus J$ and $r \in R$, then $z \in x \oplus y$, for some $x \in I, y \in J$. Consider

$$r \otimes z \subseteq r \otimes (x \oplus y) = (r \otimes x) \oplus (r \otimes y) \subseteq I \oplus J.$$

Thus $I \oplus J$ is a left hyperideal.

Corollary 3.10 If R is an LA-hyperring with pure left identity e, then

(i) the sum of one left hyperideal and one right hyperideal of an LA-hyperring R is a left hyperideal of R.

(ii) addition of left hyperideals is not commutative as well as not associative.

Let I, J be hyperideals of an LA-hyperring R. Then the product of I and J is defined by

$$I \otimes J = \{\sum_{finite} x_i \otimes y_i = (...((x_1 \otimes y_1) \oplus (x_2 \otimes y_2)) \oplus ... \oplus (x_n \otimes y_n)) : \ x_i \in I, y_i \in J\}.$$

Theorem 3.11 Let R be an LA-hyperring with pure left identity e. Product of two left hyperideals of R is again a left hyperideal.

Proof. Straightforward.

Lemma 3.12 Let R be an LA-hyperring with left identity (or with pure left identity) e. If I is a proper hyperideal of R, then $e \notin I$.

Proof. Assume on contrary that $e \in I$ and let $r \in R$, then consider $r \subseteq e \otimes r \in I \otimes R \subseteq I$. This implies that $R \subseteq I$, but $I \subseteq R$. So, $I = R$. A contradiction. Hence $e \notin I$. The case for pure left identity can be seen in a similar way.

Definition 3.13 An LA-hyperring R is said to be fully idempotent if all hyperideals of R are idempotent. If R is an LA-hyperring with pure left identity e, then the principal left hyperideal generated by an element a is defined as $\langle a\rangle = R \otimes a = \{r \otimes a \ : \ r \in R\}$.

Remark 3.14 It is important to note that if I is a hyperideal of R then $I = \langle I\rangle$, and also I^2 is a hyperideal of LA-hyperring R, hence $I^2 = \langle I^2\rangle$.

Proposition 3.15 If R is an LA-hyperring with pure left identity e and I, J are hyperideals of R, then the following assertions are equivalent:

(i) R is fully idempotent,

(ii) $I \cap J = \langle I \otimes J\rangle$,

(iii) the hyperideals of R form a semilattice $(L_S, \wedge)$, where $I \wedge J = \langle I \otimes J\rangle$.

Proof. (i) $\Rightarrow$ (ii). Since $I \otimes J \subseteq I \cap J$, $\langle I \otimes J\rangle \subseteq I \cap J$. Now let $a \in I \cap J$. As $\langle a\rangle$ is principal left hyperideal generated by a fixed element a, so $a \in \langle a\rangle = \langle a\rangle \otimes \langle a\rangle \subseteq \langle I \otimes J\rangle$. Hence $I \cap J = \langle I \otimes J\rangle$.

(ii) $\Rightarrow$ (iii). $I \wedge J = \langle I \otimes J\rangle = I \cap J = J \cap I = J \wedge I$ and also $I \wedge I = \langle I \otimes I\rangle = I \cap I = I$. Hence $(L_S, \wedge)$ is a semilattice.

(iii) $\Rightarrow$ (i). Now $I = I \wedge I = \langle I \otimes I\rangle = I \otimes I$.

Definition 3.16 A hyperideal P of an LA-hperring R is said to be prime hyperideal if and only if $A \otimes B \subseteq P$ implies either $A \subseteq P$ or $B \subseteq P$, where A and B are hyperideals in R.

Definition 3.17 A hyperideal P of an LA-hyperring R is called semi-prime if for any hyperideal I of R, $I^2 \subseteq P$ implies that $I \subseteq P$.

Definition 3.18 An LA-hyperring R is said to be fully prime if every hyperideal of R is prime and it is fully semiprime if every hyperideal is semiprime.

Definition 3.19 The set of hyperideals of an LA-hyperring R is said to be a totally ordered under inclusion if for all hyperideals I, J of R, either $I \subseteq J$ or $J \subseteq I$ and is denoted by hyperideal (R).

Theorem 3.20 If R is an LA-hyperring with pure left identity e, then R is fully prime if and only if every hyperideal is idempotent and the set hyperideal (R) is totally ordered under inclusion.

Proof. Let R is fully prime and I be any hyperideal of R. By Lemma 3.7, I^2 is a hyperideal of R, and so $I^2 \subseteq I$. Also $I \otimes I \subseteq I$ which implies that $I \subseteq I^2$. So, $I^2 = I$ and hence I is idempotent. Now let A, B be hyperideals of R and $A \otimes B \subseteq A$, $A \otimes B \subseteq B$ which implies that $A \otimes B \subseteq A \cap B$. As A and B are prime hyperideals, so $A \cap B$ is also a prime hyperideal of R. Then $A \subseteq A \cap B$ or $B \subseteq A \cap B$ which implies that either $A \subseteq B$ or $B \subseteq A$. Hence the set hyperideal (R) is totally ordered under inclusion. Conversely let every hyperideal of R is idempotent and hyperideal (R) is totally ordered under inclusion. Let L, M and N be any hyperideals of R with $L \otimes M \subseteq N$ such that $L \subseteq M$. Now since L is idempotent, $L = L^2 = L \otimes L \subseteq L \otimes M \subseteq N$. This implies that $L \subseteq N$ and hence R is fully prime.

4. Hypersystems in LA-hyperrings

In this section, we discuss M-hypersystem, P-hypersystem, I-hypersystem and subtractive sets in an LA-hyperring with pure left identity e. We prove the equivalent conditions for a left hyperideal to be an M-hypersystem, P-hypersystem, I-hypersystem and establish that every M-hypersystem of elements of an LA-hyperring with pure left identity e is P-hypersystem.

Definition 4.1 A non-empty subset S of an LA-hyperring R is called an M-hypersystem if for $a, b \in S$ there exists r in R such that $a \otimes (r \otimes b) \subseteq S$.

Example 4.2 Since we assume that any LA-hyperring R has pure left identity e, so any LA-semihypergroup of $(R, \otimes)$ is an M-hypersystem.

Definition 4.3 Let I be a left hyperideal of an LA-hyperring R. Then I is said to be a quasi-prime if $H \otimes K \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are left hyperideals of R. For any left hyperideal H of R such that $H^2 \subseteq I$, we have $H \subseteq I$, then I is called quasi-semiprime hyperideal.

Proposition 4.4 Let I be a left hyperideal of R with pure left identity e, then the following are equivalent:

(i) I is quasi-prime hyperideal.

(ii) $H \otimes K = \langle H \otimes K\rangle \subseteq I$ implies that either $H \subseteq I$ or $K \subseteq I$, where H and K are any left hyperideals of R.

(iii) If $H \not\subseteq I$ and $K \not\subseteq I$ then $H \otimes K \not\subseteq I$, where H and K are any left hyperideals of R.

(iv) If h, k are elements of R such that $h \notin I$ and $k \notin I$ then $\langle h\rangle \otimes \langle k\rangle \not\subseteq I$, then either $h \in I$ or $k \in I$.

(v) If h, k are elements of R satisfying $h \otimes (R \otimes k) \subseteq I$, then either $h \in I$ or $k \in I$.

Proof. (i) $\Leftrightarrow$ (ii) Let I is quasi-prime. Now by definition if $H \otimes K = \langle H \otimes K \rangle \subseteq I$, then obviously it implies that either $H \subseteq I$ or $K \subseteq I$ for all left hyperideals H and K of R. Converse is trivial.

(ii) $\Leftrightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (iv) Let $\langle h \rangle \otimes \langle k \rangle \subseteq I$, then either $\langle h \rangle \subseteq I$ or $\langle k \rangle \subseteq I$, which implies that either $h \in I$ or $k \in I$.

(iv) $\Rightarrow$ (ii) Let $H \otimes K \subseteq I$. If $h \in H$ and $k \in K$, then $\langle h \rangle \otimes \langle k \rangle \subseteq I$ and hence by hypothesis either $h \in I$ or $k \in I$. This implies that either $H \subseteq I$ or $K \subseteq I$.

(i) $\Leftrightarrow$ (iv) Let $h \otimes (R \otimes k) \subseteq I$, then $R \otimes (h \otimes (R \otimes k)) \subseteq R \otimes I \subseteq I$. Now using medial law and paramedial law, we have

$$\begin{aligned} R \otimes (h \otimes (R \otimes k)) &= (R \otimes R) \otimes (h \otimes (R \otimes k)) \\ &= (R \otimes h) \otimes (R \otimes (R \otimes k)) \\ &= (R \otimes h) \otimes ((R \otimes R) \otimes (R \otimes k)) \\ &= (R \otimes h) \otimes ((k \otimes R) \otimes (R \otimes R)) \\ &= (R \otimes h) \otimes ((R \otimes R) \otimes k) \\ &= (R \otimes h) \otimes (R \otimes k) \subseteq I. \end{aligned}$$

Since $R \otimes h$ and $R \otimes k$ are left hyperideals for all $h \in H$ and $k \in K$, hence either $h \in I$ or $k \in I$. Conversely, let $H \otimes K \subseteq I$ where H and K are any left hyperideals of R. Let $H \not\subseteq I$ then there exists $l \in H$ such that $l \notin I$. For all $m \in K$, we have

$$l \otimes (R \otimes m) \subseteq H \otimes (R \otimes K) \subseteq H \otimes K \subseteq I.$$

This implies that $K \subseteq I$ and hence I is quasi-prime hyperideal of R.

Proposition 4.5 A left hyperideal I of an LA-hyperring R with pure left identity e is quasi-prime if and only if $R \setminus I$ is an M-hypersystem.

Proof. Suppose I is a quasi-prime hyperideal. Let $a, b \in R \setminus I$ which implies that $a \notin I$ and $b \notin I$. So by Proposition 4.4, $a \otimes (R \otimes b) \not\subseteq I$. This implies that there exists some $r \in R$ such that $a \otimes (r \otimes b) \not\subseteq I$ which further implies that $a \otimes (r \otimes b) \subseteq R \setminus I$. Hence $R \setminus I$ is an M-hypersystem. Conversely, let $R \setminus I$ is an M-hypersystem. Suppose $a \otimes (R \otimes b) \subseteq I$ and let $a \notin I$ and $b \notin I$. This implies that $a, b \in R \setminus I$. Since $R \setminus I$ is an M-hypersystem, so there exists $r \in R$ such that $a \otimes (r \otimes b) \subseteq R \setminus I$ which implies that $a \otimes (R \otimes b) \not\subseteq I$. A contradiction. Hence either $a \in I$ or $b \in I$. This shows that I is a quasi-prime hyperideal.

Definition 4.6 A non-empty subset Q of an LA-hyperring R is called P-hypersystem if for all $a \in Q$, there exists $r \in R$ such that $a \otimes (r \otimes a) \subseteq Q$.

Proposition 4.7 If I is a left hyperideal of an LA-hyperring R with pure left identity e, then the following are equivalent:

(i) I is quasi-semiprime.

(ii) $H^2 = \langle H^2 \rangle \subseteq I \Rightarrow H \subseteq I$, where H is any left hyperideal of R.

(iii) For any left hyperideal H of R, $H \not\subseteq I \Rightarrow H^2 \not\subseteq I$.

(iv) If a is any element of R such that $\langle a \rangle^2 \subseteq I$, then it implies that $a \in I$.

(v) For all $a \in R$, $a \otimes (R \otimes a) \subseteq I \Rightarrow a \in I$.

Proof. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) is trivial.

(i) $\Rightarrow$ (iv). Let $\langle a \rangle^2 \subseteq I$. But by hypothesis I is quasi-semiprime, so it implies that $\langle a \rangle \subseteq I$ which further implies that $a \in I$.

(iv) $\Rightarrow$ (ii). For all left hyperideals H of R, let $H^2 = \langle H^2 \rangle \subseteq I$. If $a \in H$, then by hypothesis $\langle a \rangle^2 \subseteq I$ and this implies that $a \in I$. Hence it shows that $H \subseteq I$.

(i) $\Leftrightarrow$ (v) is straightforward.

Proposition 4.8 A left hyperideal I of an LA-hyperring R with pure left identity e is quasi-semiprime if and only if $R \setminus I$ is a P-hypersystem.

Proof. Let I is quasi-semiprime hyperideal of R and let $a \in R \setminus I$. On contrary suppose that there does not exist an element $x \in R$ such that $a \otimes (x \otimes a) \subseteq R \setminus I$. This implies that $a \otimes (x \otimes a) \subseteq I$. Since I is quasi-semiprime, so by Proposition 4.7, $a \in I$ which is a contradiction. Thus there exists $x \in R$ such that $a \otimes (x \otimes a) \subseteq R \setminus I$. Hence, $R \setminus I$ is a P-hypersystem. Conversely, suppose for all $a \in R \setminus I$ there exists $x \in R$ such that $a \otimes (x \otimes a) \subseteq R \setminus I$. Let $a \otimes (R \otimes a) \subseteq I$. This implies that there does not exist $x \in R$ such that $a \otimes (x \otimes a) \subseteq R \setminus I$ which implies that $a \in I$. Hence by Proposition 4.7, I is quasi-semiprime.

Lemma 4.9 An M-hypersystem of elements of an LA-hyperring R is a P-hypersystem.

Proof. The proof is obvious.

Definition 4.10 A hyperideal I of an LA-hyperring R is strongly irreducible if and only if for hyperideals H and K of R, $H \cap K \subseteq I$ implies that $H \subseteq I$ or $K \subseteq I$ and I is said to be irreducible if for hyperideals H and K, $I = H \cap K$ implies that $I = H$ or $I = K$.

Lemma 4.11 Every strongly irreducible hyperideal of an LA-hyperring R with pure left identity e is irreducible.

Proof. The proof is obvious.

Proposition 4.12 A hyperideal I of an LA-hyperring R with pure left identity e is prime if and only if it is semiprime and strongly irreducible.

Proof. The proof is obvious.

Definition 4.13 A non-empty subset S of an LA-hyperring R is called an I-hypersystem if for all $a,b \in S$, $(\langle a\rangle \cap \langle b\rangle) \cap S \neq \varnothing$.

Proposition 4.14 The following conditions on hyperideal I of an LA-hyperring R are equivalent:

(i) I is strongly irreducible.

(ii) For all $a,b \in R : \langle a\rangle \cap \langle b\rangle \subseteq I$ implies that either $a \in I$ or $b \in I$.

(iii) $R \setminus I$ is an I-hypersystem.

Proof. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii). Let $a,b \in R \setminus I$. Let $(\langle a\rangle \cap \langle b\rangle) \cap R \setminus I = \varnothing$. This implies that $\langle a\rangle \cap \langle b\rangle \subseteq I$ and so by hypothesis either $a \in I$ or $b \in I$ which is a contradiction. Hence $(\langle a\rangle \cap \langle b\rangle) \cap R \setminus I \neq \varnothing$.

(iii) $\Rightarrow$ (i). Let H and K be hyperideals of R such that $H \cap K \subseteq I$. Suppose H and K are not contained in I, then there exist elements a,b such that $a \in H \setminus I$ and $b \in K \setminus I$. This implies that $a,b \in R \setminus I$. So by hypothesis $(\langle a\rangle \cap \langle b\rangle) \cap R \setminus I \neq \varnothing$ which implies that there exists an e lement $c \in \langle a\rangle \cap \langle b\rangle$ such that $c \in R \setminus I$. It shows that $c \in \langle a\rangle \cap \langle b\rangle \subseteq H \cap K \subseteq I$ which further implies that $H \cap K \not\subseteq I$. A contradiction. Hence either $H \subseteq I$ or $K \subseteq I$ and so I is strongly irreducible.

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