DEVELOPING CONTEXTUAL-AND-GRAPHING-ACTIVITY-BASED LEARNING CYCLE UNIT TO ENHANCE STUDENTS' UNDERSTANDING OF THE FUNDAMENTALS OF CALCULUS AND THE RELATIONSHIP BETWEEN DIFFERENTIATION AND INTEGRATION

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…………………………………………
Mr. Kinley
Candidate

…………………………………………
Lect. Parames Laosinchai, Ph.D.
Major advisor

…………………………………………
Lect. Wararat Wongkia, Ph.D.
Co-advisor

…………………………………………
Prof. Banchong Mahaisavariya,
M.D., Dip Thai Board of Orthopedics
Dean
Faculty of Graduate Studies
Mahidol University

…………………………………………
Lect. Namkang Sriwattanarothai, Ph.D.
Program Director
Master of Science Program in
Science and Technology Education
Institute for Innovative Learning,
Mahidol University

Thesis
entitled

DEVELOPING CONTEXTUAL-AND-GRAPHING-ACTIVITY-BASED LEARNING CYCLE UNIT TO ENHANCE STUDENTS' UNDERSTANDING OF THE FUNDAMENTALS OF CALCULUS AND THE RELATIONSHIP BETWEEN DIFFERENTIATION AND INTEGRATION

was submitted to the Faculty of Graduate Studies, Mahidol University
for the degree of Master of Science (Science and Technology Education)
on
May 29, 2014

…………………………………………
Mr. Kinley
Candidate

…………………………………………
Lect. Dusadee Sukawat, Ph.D.
Chair

…………………………………………
Lect. Wararat Wongkia, Ph.D.
Member

…………………………………………
Lect. Parames Laosinchai, Ph.D.
Member

…………………………………………
Prof. Banchong Mahaisavariya,
M.D., Dip Thai Board of Orthopedics
Dean
Faculty of Graduate Studies
Mahidol University

…………………………………………
Assoc. Prof. Wannapong Triampo, Ph.D.
Director
Institute for Innovative Learning,
Mahidol University

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KINLEY 5536885 ILSE/M

M.Sc. (SCIENCE AND TECHNOLOGY EDUCATION)

THESIS ADVISORY COMMITTEE: PARAMES LAOSINCHAI, Ph.D., WARARAT WONGKIA, Ph.D.

ABSTRACT

Calculus is one of the greatest achievements of human intellect and demonstrates the power to illuminate the most fundamental problems in mathematics, physical sciences, biological sciences, and engineering. Calculus can reduce complicated problems to simple rules and procedures by using symbols and notations. However, use of symbols and notations might lead to losing the original pictures of the problems. Despite its importance, the teaching of introductory calculus always emphasizes manipulation of algebraic notations and rote learning. Students memorize algebraic procedural steps rather develop conceptual understanding. Most students learn the how instead of the why of calculus due to extensive use of algebraic symbols and notations. The real meanings of symbols and notations learned in the classroom are not interpreted explicitly in the context of real world situations.

To address this issue, contextual and graphing activities based on the learning cycle approach were developed to enhance students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration. Experimentally real activities for students were developed to convey the concepts of the fundamentals of calculus realistically and then represented in the form of graphs. The study was conducted with eleventh grade students in the south west of Bhutan.

The experimental group results showed that the developed learning units significantly improved the students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration and they also showed positive attitude towards the developed learning units.

KEY WORDS: CALCULUS/ DIFFERENTIATON/ INTEGRATION/ LEARNING CYCLE/ CONTEXTUAL AND GRAPHING ACTIVITIES

162 pages

CONTENTS

CONTENTS (cont.)

CONTENTS (cont.)

CONTENTS (cont.)

LIST OF TABLES

LIST OF TABLES (cont.)

Tables **Page**

LIST OF FIGURES

LIST OF FIGURES (cont.)

LIST OF FIGURES (cont.)

Figures **Page**

LIST OF FIGURES (cont.)

LIST OF ABBREVIATIONS

Abbreviation	Meaning
CAI	Computer-assisted instruction
FC	Fundamentals of calculus
LAMLC	Lawson-Abraham's model of learning cycle
MU-IRB	Mahidol University Institutional Board
NAPE	New Approach to Primary Education
NCTM	National Council of Teachers of Mathematics
RCSC	Royal Civil Service Commission
R-DI	Relationship between differentiation and integration
RME	Realistic Mathematics Education
SCIS	Science Curriculum Improvement Study
m/s	Meter per second
m/s^2	Meter per second squared

CHAPTER I
INTRODUCTION

Overview

This chapter introduces the background information and the rationale for this study. This part includes the objectives, research questions, and scope of the study. Along with the above, the definition of terms and an outline of the thesis are also provided.

1.1 Background and Rationale of the Study

The ideas of calculus are one of the greatest achievements of the human intellect (Hughes-Hallett et al., 2003) since calculus has demonstrated the power to illuminate the most fundamental problems in mathematics, physical sciences, biological sciences, and engineering. Calculus has reduced complicated problems to simple rules and procedures by using symbols and notations not only to represent a shorter way of writing but also to find the solutions easier. However, using those symbols and notations might lose the original pictures of the problems.

By contrast, teaching and learning of calculus begin with the symbols and notations and follow by several examples of its applications. Lecture-based teaching methods are the top choices in calculus instruction for decades. Students develop a skill at memorizing formulas and algebraic procedural steps instead of conceptual understanding. Besides, a vast number of calculus textbooks are available, covering every conceivable approach; these calculus textbooks contain a lot of abstract symbols and notations. A wide range of problems in the textbooks are solved using memorized formulas and procedural steps. Most students have learned calculus 'how' rather than 'why' due to extensive use of notations and symbols in teaching and learning. The real meanings of symbols and notations that students learned in the classrooms are not interpreted explicitly in the context of real world situations.

With these reasons, mathematics educators felt that calculus education needed reform focusing on conceptual understanding rather than the acquisition of procedural skills (Peterson, 1986; Steen, 1988). Since the calculus reform movement began, many researches were conducted on concept-based approaches to teach the fundamentals of calculus, e.g. a realistic approach (Kaput, 1994), a guided reinvention (Gravemeijer & Doorman, 1999), a computer-assisted approach (Lang, 1999), and a graphical approach (Tall, 1986). Most of these approaches employed contextual examples (e.g., distance and velocity of a moving car) without the real activities. As a result students were required to imagine the context of the situations. If students cannot form clear mental pictures of contextual examples, they memorize the steps and follow rote learning. Graphing is a powerful tool in teaching as well as learning calculus. Graphical representations in calculus can help students to visualize underlying concepts. Graphs can translate and interpret algebraic formulas and data. Moreover, graphs often reveal mathematical results simply and clearly.

Students have a vague concept of algebraic notations in relation to geometric interpretations. Mundy and Graham (1994) also reported that the difference between the performances on the procedural items and the conceptual ones was due to the separate understanding of geometrical and algebraic context in calculus. From our pilot study which was carried out in undergraduate students, they had difficulty in explaining the relationship between differentiation and integration. They recognized that “integration is the inverse process of differentiation” whereas they remained silent when asked to explain “how”. Since the concepts of calculus are originated from contextual applications, using contextual activities and interpreting them in the form of graphs would ultimately help students to relate the concepts of calculus to algebraic symbols and notations. In addition, the students will get the physical feel and visualization of the concepts.

In this study, contextual and graphing activities were embedded in the learning cycle approach to develop the concepts of limits, continuity of a function, differentiation, integration, and the relationship between differentiation and integration in calculus. These activities were introduced to extend students’ understanding of calculus concepts from real activities to graphical, numerical, and symbolic representations. The learning cycle approach, described by Anton Lawson, Michael

Abraham, and John Renner (1989) and Michael Abraham (1997), known as "The Lawson-Abraham's model" of learning cycle, was used as teaching pedagogy for each learning unit. It consists of three phases, namely Exploration, Concept Introduction, and Concept Application. It is also based on the theory of constructivism.

1.2 Research Objectives

This research has four main objectives:

1) To develop contextual and graphing activity–based learning cycle units for limits, continuity, differentiation, integration, and the relationship between differentiation and integration in calculus.
2) To find out the effectiveness of the developed learning units on the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration.
3) To find out whether the developed learning units are more effective than the conventional teacher-centered and textbook-oriented instruction.
4) To measure the students' attitude towards the developed learning units.

1.3 Research Questions

This research study addresses the following questions:

1) To what extent can the developed learning units enhance the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration?
2) To what extent are the developed learning units more effective than the conventional instruction in improving students' understanding of the fundamentals of calculus and the relationship between differentiation and integration?
3) What are students' attitudes towards the developed learning units?

1.4 Scope of the Study

This study is to determine the effectiveness of the developed learning units on grade-11 students in Bhutanese classrooms. It could provide guidelines for Bhutanese teachers on the implementation of the learning cycle approach in teaching mathematics.

1.5 Definition of Terms

Conceptual understanding: is the ability to describe the relationship among the concepts, to elaborate knowledge and recognize the applications of the learned scientific concepts and phenomena in everyday life situations (Godino, 1996).

Constructivist teaching and learning: is a teaching and learning approach that considers the learners as actively making meaning and constructing new knowledge based on existing knowledge within an individual, either individually or socially (Gray, 1997).

Fundamentals of calculus: includes limits, continuity, differentiation, and integration.

Learning cycle approach: is an inquiry-based teaching model which can be useful to teachers in designing curriculum materials and instructional strategies in science which is derived from constructivist ideas of the nature of science and the developmental theory of Jean Piaget (Abraham, 1997). In this study, the learning cycle approach is based on Lawson-Abraham's model of learning cycle which consists of Exploration Phase, Concept Introduction Phase, and Concept Application Phase.

Constructivism: is Piaget's theory of learning and development.

Social constructivism: is Vygotsky's theory of learning and development.

1.6 An Outline of the Thesis

This research is organized into five chapters as follows:

Chapter one provides the background information and the rationale for this study. This part also includes research objectives, research questions, and scope of the study, followed by the definition of terms relevant to this study and an outline of the thesis.

Chapter two provides literature review related to the scope of this study. The review involves the importance of calculus, calculus reform movement, and calculus teaching approaches. This chapter also describes the learning cycle approach based on the idea of constructivism and mathematics education in Bhutanese context.

Chapter three provides the methodology and methods employed in the studies. The pilot study, the participants' information, the development of the learning units on limits, continuity, differentiation, and integration based on the learning cycle approach using contextual and graphing activities, the implementation of the developed learning units, and the procedures used for data collection and data analysis were also described.

Chapter four describes the analysis of the data and the interpretation of the results.

Chapter five presents the discussion and conclusions of the overall findings of the research study. This chapter also provides the contributions of the research study and recommendations for future study.

CHAPTER II
LITERATURE REVIEW

Overview

This chapter presents the importance of calculus, calculus reform movement, calculus teaching approaches, the description of the learning cycle approach based on the idea of constructivism, and mathematics education in Bhutan.

2.1 Importance of Calculus

Calculus, the mathematical study of change, is one of the great endeavors of the human mind (Goodman, 1991). Calculus was developed from algebra and geometry in the 17th century by Sir Isaac Newton and Gottfried Leibniz. It is made up of two interconnected topics: differential calculus concerning rates of change and the slopes of curves, and integral calculus concerning accumulation of quantities and the area under the curves.

Calculus occupies an important role in the 21st-century education as it opens a door into a higher area of mathematics concerned with analysis and abstract operations. It is deeply integrated into every branch of science, such as physics, chemistry, and biology. It is also found in computer science, statistics, engineering, economics, business, and medicine. All the modern developments such as architecture, aviation, and other technologies all make use of what calculus has to offer and its ability to solve many problems that algebra alone cannot.

Calculus and science have a rich history of mutually enriching each other's growth, particularly physics when Newton needed a way of mathematically describing the laws of motion in the 17th century. Today, the use of calculus continues to grow with new research. The advent of computers and graphing calculators reduces the human part to providing input of a problem; e.g. if we ask students a direct question on a definite integral or a system of linear equations, they will answer immediately using a graphing calculator or a computer. The contents of calculus courses have changed little over the last century (Strang, 1988) even though major efforts have been put to change the teaching and learning pedagogy to enhance the conceptual understanding of the fundamentals of calculus as parts of reforming calculus.

2.2 Calculus Reform Movement

Calculus teaching has been revolutionized for the last three decades due to the contributions of a number of mathematics educators. The technology-driven changes inspired a major university-centered “Calculus Reform Movement” in the late 1980s and early 1990s (Tucker, 1990). The three events, “Towards a Lean and Lively Calculus” conference in 1986, “Calculus for a New Century” symposium in 1987, and “Priming the Calculus Pump” in 1990, signaled the beginning of calculus reform (Park & Travers, 1996) to make calculus lean and lively, relevant to applications by including contemporary mathematics that reflected new technology. The themes of calculus reform movement include: involving students in doing mathematics instead of lecturing at them; stressing conceptual understanding rather than only computation; developing meaningful problem-solving abilities; exploring patterns and relationships instead of just memorizing formulas; becoming engaged in open-ended, discovery-type problems rather than doing routine, closed-ended exercises; and approaching mathematics as a live exploratory subject, not merely as a description of past works.

Calculus reform movement has questioned the role of traditional structured calculus courses in the curriculum. There was much concern about a large number of students taking calculus courses who were mostly taught using rote learning and manipulative techniques (Cipra, 1988; Steen, 1988; White & Mitchelmore, 1996). The structured calculus courses have also come under scrutiny as computers and calculators have performed most of the manipulative procedures taught in the calculus courses (Korey, Rheinlander, & Wallace, 2011; Lang, 1999; Steen, 1988; Tall, 1987). Several studies (Amoah, 2003; Awang & Zakaria, 2012; Heid, 1988; Herceg & Herceg, 2009; Palmiter, 1991; Slavickova, 2009) indicated that using computer as a tool for performing procedures of calculus freed students from algebraic manipulations to explore applications. Some projects undertaken to improve the teaching of calculus were:

redesigning curriculum and pedagogy using the power of the new computer tools, e.g. Mathematica (Alsawaie, Alghazo, & Travers, 2000); Excel (Kadry & Shalkamy, 2012; Lim, 2008), and graphing calculators (Leng, 2011), and using constructivist approaches, e.g. group work (Mokhtar, Tarmizi, Tarmizi, & Ayub, 2010; Mokhtar, Tarmizi, Ayub, & Tarmizi, 2010) and collaborative learning (Asiala, Cottrill, Dubinsky, & Schwingendorf, 1997; Engelbrecht & Harding, 2002).

Orton (1983a, 1983b) used graphs to explore equations of the curves to build sound knowledge in differentiation and integration and also to find out the misconceptions the students possessed. Several other studies revealed the use of graphs to teach calculus. For example; Tall (1986) presented a process that involved graduating from paper to a calculator and then to the graphical interface of a computer in order to obtain insight into the concepts of integration. Asiala et al. (1997) explored the students' graphical understanding of the function and its derivative using computers and graphs. Nevill and John (1998) used a contextual example of a moving vehicle and graphs in a computer to develop the concepts of differentiation. Ubuz (2007) investigated 147 first-year engineering students' ability to interpret the graph of a function and then construct its derivative graph. Berry and Nyman (2003) and Orhun (2012) studied the difficulty faced by the students when they made connection between the graph of the derivative of function and the original function.

Besides using computers and graphing methods, there were various studies conducted to enhance the conceptual understanding of calculus using contextual examples. Schwalbach and Dosemagen (2000) used concrete examples of pendulum movement in a physics class where students discovered the connection between the physics concepts of position, velocity, and acceleration, and the calculus concepts of function, derivative, and anti-derivative. The connection between calculus and physics can yield deep understanding of semantic as well as procedural knowledge. Yoon, Dreyfus, and Thomas (2009) presented a mathematical modeling activity that was set within the context of tramping, and asked students to find the graphical anti-derivative of a function presented graphically. The students mathematized the real world context, and applied the knowledge of integration to solve the problems.

2.3 Teaching Approaches in Calculus

There are certainly several approaches, but only a few approaches used in teaching calculus that focus specifically on differentiation and integration are reviewed: algebraic, graphical, contextual or realistic, and computer-assisted approaches.

2.3.1 Algebraic approach

An algebraic approach is most closely related to the traditional approach of teaching calculus. Elk (1998) utilized a lecture-based format representing a logical extension of algebra to transition into calculus courses. He proposed the basic algebraic ideas imbedded in the concepts of derivative and integral. He cited an example of three situations that involve division by zero: 0/*a*, *a*/0, and 0/0 where '*a*' is any number not equal to zero. He emphasized that differentiation gives meaning to the third case (0/0) under the appropriate conditions. He introduced the idea of the changes in both the *y*-coordinate and the *x*-coordinate approaching zero. As the distance between the points decreases, the two respective components of the distance decrease and become very small. Thus, the change in the *y*-coordinate over the change in the *x*-coordinate would cause both the numerator and the denominator to be zero, resulting in the 0/0 situation.

Consider the function $f(x)=\frac{(x^2-4)}{(x-2)}$. At $x=2$, the function reduced to 0/0, and at this point, the function is not defined. In order for the function to be defined at all points, an additional criteria needs to be added; for example, if $f(2)=4$ were added, it would produce a continuous function for all values of *x*. Furthermore, any function $f(2)=b$ where *b* is any number not equal to 4 would produce a discontinuous function. Using this information, Elk presented an algorithm to compute the derivative of the given function:

i). From the given function $y=f(x)$, form $f(x+\Delta x)$.

ii). Subtract $f(x)$ from $f(x+\Delta x)$, a difference we call Δy.

iii). Divide Δy by Δx and simplify the algebra.

iv). Take the limit as $\Delta x \to 0$.

The algorithm results in the equation $y' = \lim_{x \to 0} \frac{\Delta y}{\Delta x}$, or $y' = \lim_{x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$, the definition of the derivative. The key to giving meaning to 0/0 is taking the limit at the end of the algorithm rather than during one of the earlier steps. By doing this, an initially undefined computation can bring about information.

Similarly, Elk proposed that integration gives meaning to multiplying infinity by zero. The process of dividing a figure into small pieces to calculate the sum of the areas and repeating the calculation by sub-dividing further until the pieces are infinitesimal yields the desired area as the limit. This is equivalent to multiplying infinity by 0. He stressed the importance of noting that this multiplication oftentimes yields a finite product, contrasting the common notions that zero multiplied by anything is zero and that infinity multiplied by anything is infinity. Thus, not only does integration give meaning to infinity times zero, but that meaning is oftentimes finite. This can be seen given the standard definition of an integral: $\int_a^b f(x) = \lim_{\Delta x \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$; the limit is the sum of an infinite number of zero terms since the limit of each Δx_i is zero.

Although the algebraic approach encourages a better conceptual understanding of the ideas of differentiation and integration, utilizing the algebraic approach may limit the capability of students to extend their knowledge beyond the definitions of derivative and integral to problems concerning real world applications. Elk also mentioned that integration often seems to give a finite value to the meaning of infinity times zero, it does not explain how that value is reached other than the inclusion of the limit in the standard definition. Furthermore, the definition of the derivative is still structured procedurally, and a concern may be that the step-by-step instructions do not adequately deviate away from lecture-based instruction.

2.3.2. Graphical approach

The advantages of using computer technology were beginning to surface. Tall (1986) saw the potential of computers and presented a graphical approach to teach integration and the fundamental theorem. He mentioned the difficulty of going beyond simple examples when using the algebraic method of summation, and the tediousness and obscurity of interpretation that occurred when using a calculator. For these reasons, he presented a process that involves progressing from paper to calculator and then to the graphical interface of a computer in order to obtain insight into the concepts of integration.

His main approach to teaching integration was to have students infer possible functions that described the behavior observed through the approach taken to solve the integral, be it through the calculator or computer software. He illustrated the difficulty in conjecturing the formula for the area under a graph over an interval using the averages of upper and lower sums (the trapezium rule) since the numerical work begins to get oppressive unless a computer is available. As a solution to this, he mentioned the study of Neill and Shuard (1982) where they cunningly involved the entire class to produce a table of areas under the graphs of x^n from 0 to 1 using the trapezium rule with 10 strips with the intention of leading students to conjecture that the area under $f(x) = x^n$ from 0 to 1 is $\frac{1}{n+1}$. From the table, students could see the common fractions $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},$ etc.

Tall proposed using the information obtained from partial sums to conjecture the areas of functions more directly than the results of area calculations. By plotting the successive accumulations of areas of the function $f(x) = x^2$ (see Figure 2.1) as separate points that represented the areas under the graph from $x = 0$ to the current x-coordinate using the mid-ordinate rule, the area graph looked like a higher power of x. Students then conjectured that it looked about like kx^3 for some constant k. Since the area graph and the graph of x^2 crossed at $x = 3$, substituting the value of x yielded the value of $k = \frac{1}{3}$.

Tall's idea behind the graphical approach was to concentrate on the ideas rather than the technicalities. For example, as explained above, students were meant to become aware of two ideas: approximations of the area grew closer to the true area when smaller strips were taken, and the results of the graphs displayed the patterns. However, it should be noted that he expected students to be able to calculate approximations numerically before graduating to software that eliminated the need for calculations by hand. With that knowledge base, students could use the software to explore cases such as negative areas and develop concepts such as the fundamental theorem graphically. There are two kinds of negative areas: one resulting from areas under the *x*-axis and another resulting from integrating toward the left of the lower limit of the integral. The latter involves taking steps in the negative direction which was rarely done in traditional courses and teachers often said that it was too difficult for their students. Graphically, students could understand that a negative step and a positive ordinate resulted in a negative product, and that a negative step and a negative ordinate resulted in a positive product. Tall presented a computer simulation that went beyond a static picture to see the negative steps and sense the growth of the area as the picture developed. By investigating the negative steps, students could uncover all four sign combinations.

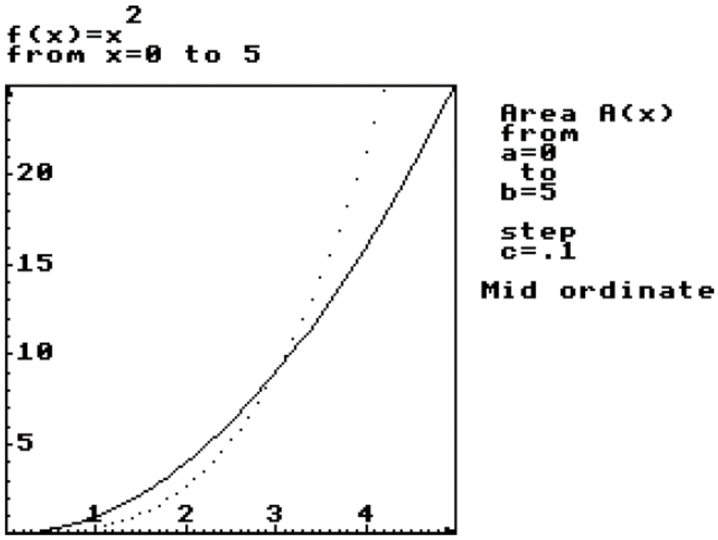


Figure 2.1 The cumulative area under $f(x) = x^2$ from 0 to 5.

After observing that smaller strips resulted in area approximations that came closer to the true area and understanding the signs, Tall believed that the students now had the background knowledge necessary to find the area between a graph and the *x*-axis from a fixed point *a* to a variable *x* as a function of *x*. By plotting the cumulative area function for the graph $f(x) = x$, starting at the origin and moving right, and then restarting at the origin and moving left, the graph took on the recognizable shape of a quadratic function kx^2 (Figure 2.2). Then by noting that when $x = 2$, $y = 2$ the students could conjecture that the area of the function was $\frac{x^2}{2}$. The same process can be applied to $f(x) = x^2$ and the general pattern of the area under $f(x) = x^n$ from 0 to x can be thought of as $I(x) = \frac{x^{n+1}}{n+1}$.

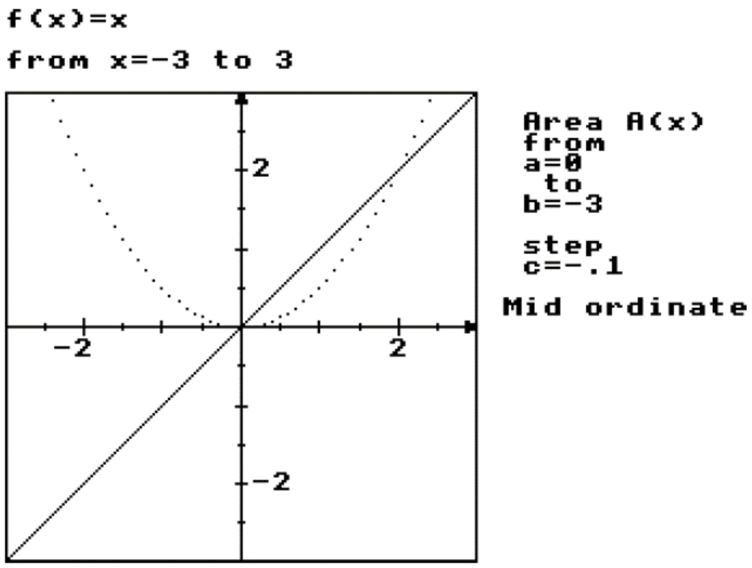


Figure 2.2 The area under $f(x) = x$ calculated from $x = 0$.

Gravemeijer and Doorman (1999) noted the characteristic dynamical aspect of Tall's (1985) graphical approach to teaching the introductory calculus concepts. When teaching derivatives, the graphs showed how the dependent variable (y or $f(x)$) changed when the independent value x changed at a constant rate. Students then observed the changes in the dependent variable and the rate of these changes and began to develop an intuitive idea of these changes in terms of increase, decrease, and gradient.

Gravemeijer and Doorman also demonstrated that focusing on the gradient of the graph leaved the idea of the derivative as a measure of the rate of change as implicit. The difficulty lay in the transition from visual imagery discussion to formal mathematical reasoning. Hence, they concluded that students interpreted a definition that was based on visual imagery as a description of the picture, instead of a mathematical definition that could be used for formal reasoning.

2.3.3 Contextual approach

James Kaput (1994) developed an approach to teaching calculus that related mathematical symbols and graphs to everyday realities. Kaput devoted his time describing an early version of the video system *MathCars* (Roschelle, Kaput, & Stroup, 2000) that simulated driving a car through interactive technology (Kaput, 1994). The idea behind the software was to map the phenomenologically rich experience of motion in a vehicle (sights and sounds) onto coordinate graphical and other mathematical notations by controlling aspects such as time, distance, and velocity which were visible as visual representations or graphs (Dubinsky, 1996).

Gravemeijer and Doorman (1999) used the metaphor of the software linking "the gap between the island of formal mathematics and the mainland of real human experience". In the case of *MathCars*, the software linked the everyday experience of motion in a vehicle with formal graphical representations while Elk's (1998) approach was built upon knowledge learned in algebra.

The Dutch approach called Realistic Mathematics Education (RME) uses contextual problems as "models of" reality which gradually progress to "models for" mathematical reasoning (Gravemeijer & Doorman, 1999). Instead of a procedural (Elk, 1998; Powell, 1985) or an entirely graphical approach (Berry & Nyman, 2003; Orhun, 2012; Tall, 1986), which Gravemeijer and Doorman believed that it may leave students unable to understand the whole picture or unable to connect concepts to formal mathematics, RME suggests that teachers employ guided re-invention, often by considering the historical origins of the concepts. The idea is to keep the gap between where the students are and what is being introduced as small as possible by designing a hypothetical learning trajectory for the students to be able to reinvent formal mathematics. While Kaput (1994) used a historically based approach as well, the perspective shift from considering notation and human experience as distinct to viewing them as a development from one to the next is not comparing a ready-made system with real world experiences but rather a reinventing process of the system.

The process of guided re-invention emphasizes the character of the learning process over the invention. The idea is for the students to develop their own private knowledge and understanding individually, with guidance, and no answers from the teacher. The idea of progressive mathematization that Gravemeijer and Doorman presented stems from a combination of Treffers's idea of horizontal and vertical mathematization (Treffers, 1987). Horizontal mathematization is the process of describing a context problem in mathematical terms. Vertical mathematization is mathematizing one's own mathematical activity, reaching higher levels of mathematics through the process.

By using contextual problems, students develop informal, context-specific solution strategies that are meant to guide them to generalizations. These generalizations become models for mathematical reasoning. In the case of RME, contextual problems are defined as problem situations that are experientially real to the students (Gravemeijer & Doorman, 1999). As an overview, in order to develop calculus concepts, contextual problems consisting of modeling problems about velocity and distance were employed. Using this method to understand integrals, students initially developed discrete approximations of a function denoting varying velocities and distances. The resulting inscribed discrete graphs later became continuous and the model for formal mathematical reasoning. The act of summing the discrete intervals made on the graph or noting the differences in the increments were the reifying processes, while the integral and derivative became the mathematical objects. As the students worked through the problems, they would likely experience a nonlinear learning process that would end with a result between a process and an object. The goal for the teacher was to emphasize that the underlying process was an integral part of the mathematical object that was developed (Gravemeijer & Doorman, 1999).

The following presented the reification process of the integral and then the derivative. Instead of simply being told that the area of the graph coincided with the total distance covered over a period of time, students developed the ability to model and conceptualize motion through representations and approximations. In order to accomplish this, the students were first presented with the story of Galileo. He presumed that a free-falling object would move with a constantly increasing velocity. Students were asked to graph the discrete approximation of that motion and then asked to discover the distance covered by the object (see Figure 2.3) (Gravemeijer & Doorman, 1999).

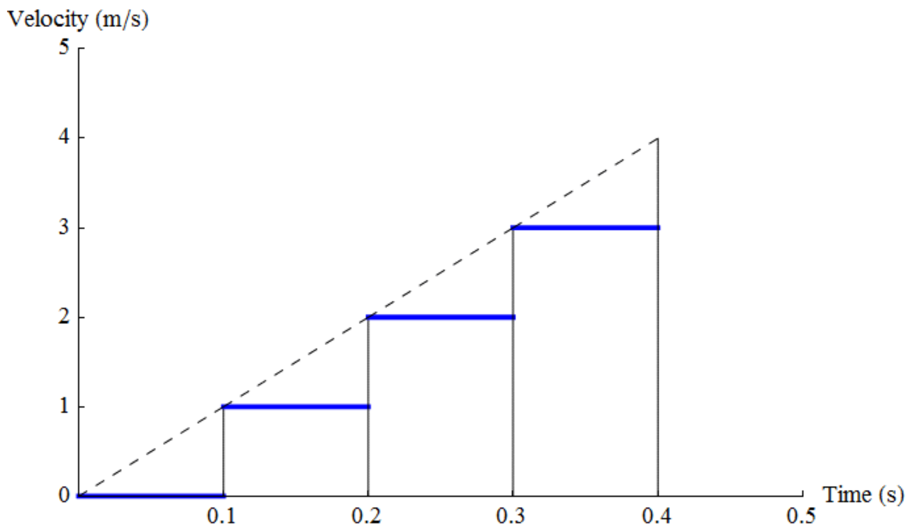


Figure 2.3 A discrete approximation of a constantly changing velocity.

Students solved these discrete approximations and were then expected to connect the area of the discrete graph with the area of a continuous graph, $s(t) = \frac{t \times 10t}{2} = 5t^2$, which is the area of a triangle (with base t and height $10t$). From this, they discovered the quadratic relationship between time and distance, which was what Galileo used to test the above-mentioned hypothesis. The next step was to progressively rediscover the differential calculus. This was done by determining velocities from the distance-time graphs and formulas. As students worked through examples, the model they had developed for reasoning through the problems would begin to function as a model for reasoning arbitrary function as well as standard algebraic functions. At this point, the teacher should shift from an everyday-life contextual problem to a focus on mathematical concepts and relations. This shift could be possible for the students when they were able to develop a framework that enabled them to view the problems mathematically.

Kaput (1994) also suggested employing a curriculum that introduced the ideas of calculus (variable rates of changing quantities, the accumulation of those quantities, the connections between rates and accumulations, and approximations) in early grades. The idea was to build up concepts gradually in the mathematics found in calculus. In order to develop curriculum, he considered the implications of history and desired to look closely at the origins of the major ideas of calculus for clues regarding how calculus might be regarded as a web of ideas that should be approached gradually, from elementary school onward, in a coherent school mathematics curriculum. Overall, Kaput mentioned that mathematicians in the past developed the ideas gradually based on real world experiences. With this in mind, Kaput suggested a three-step approach for developing the ideas of derivatives and integrals.

Dubinsky (1996) outlined the three steps mentioned by Kaput. Firstly, at an early age, students should be able to represent quantities such as temperature, velocity, acceleration, time, and distance as geometric objects (i.e. lines and rectangles). Secondly, teachers incorporated video technology such as *MathCars* in order to help students conceptualize continuous phenomena. Lastly, students discussed and moved towards firm, logical foundations of the principles of calculus.

This approach aims for students to develop a conceptual understanding of the concept smoothly as they are guided from contextual problems to a focus on mathematical concepts and relations. However, the formal definition, containing a limit and based on summing the areas of infinitely many rectangles, is not introduced; only the geometric area under a curve is discussed. Although the jump from true area to approximated area may not be a substantial one, it is to be considered if students are expected to be able to understand the formal definition of an integral.

2.3.4 Computer-assisted approach

Lang (1999) developed computer-assisted calculus instruction (CAI) in China. Although technology has developed significantly since the article was written, it presented the idea of implementing laboratory courses to foster the scientific exploratory spirit of the undergraduate calculus students at an engineering school. Lang emphasized the role of the computer as a tool, not a replacement for brainpower, and noted that good problems should encourage students to think deeply. The goal was for the laboratory to be continuously in development, and Lang encourages teacher collaboration to develop good problems.

Since it was assumed that the students should be taught the formulas for integrals and derivatives in a more traditional setting in the classroom, the laboratory was the time for students to be able to understand concepts and pursue novelty in mathematics (Lang, 1999). The idea was to shift the students away from memorizing the proof of a theorem to truly understanding the theorem. Therefore, the laboratory was the time for students to find ‘explanations’ instead of ‘proofs’ for the definitions they had learned. Lang illustrated that good explanations included important ideas in proof, and thus a student’s understanding of a mathematical theorem was not necessarily weakened by a shift towards a concentration on explanation over proof.

The following was an example of a computer experiment to develop the understanding of the relationship between functional increment and differential (Lang, 1999):

For $f(x)=\sqrt{x}$ when $x_0 = 64, \Delta x = -1$, find the true increment and its differential approximation, make a comparison, and calculate the relative error when using the approximation.

For $g(x)=x^{100}$ when $x_0 = 1$, calculate the increment and differential approximation, make a comparison and calculate the errors at $\Delta x = 0.03, \Delta x = 0.003$ and $\Delta x = 0.0003$ respectively.

By calculating and comparing increments and their differential approximations of various functions that resulted from different increments for different values of the independent variable, students could discover the accuracy of the approximation. Students noted that the differential approximation became a good approximation of the function increment when the sizes of the increment of the independent variable were smaller.

The idea of the laboratory that Lang mentioned was to shift the students away from the memorization of definitions and proofs and towards an ability to explain these definitions. Although only a couple examples of questions posed to students in the laboratory were presented in the article, the questions seemed to focus on either exploring the limits of the procedures used to calculate integrals and derivatives or identifying patterns in relationships. While this knowledge was helpful in understanding the processes, it did not suggest a change in the way the definitions of integrals and derivatives were presented to the students in the lecture portion of the classroom or hinted towards any contextual problems or applications.

The four aforementioned approaches were an effort to challenge traditional calculus instruction and to aid students in developing a better conceptual understanding of the meanings of differentiation and integration. Although each approach presented various strategies to help students develop the concepts, each strived to move beyond procedural fluency. The differing methods incorporated students' prior knowledge, historical influence, real-world contexts, graphical images, and computer-enhanced computations.

2.4 Constructivism Theory in Teaching and Learning of Mathematics

Constructivism, as a perspective in education, was founded on the belief that the learners construct their own understanding and knowledge of the world through experiencing things and reflecting on those experiences either individually or socially. It was widely accepted as a viable theory of knowledge (Thompson, 1979, 2013) and became the foundation for many instructional methods in mathematics education (Gupta, 2008) and contemporary science (Matthews, 2002, 2003). Constructivism asserts that knowledge about the outside world is a human construction. Thus, learning is not viewed as a transfer but an active construction of knowledge by an individual through interaction with physical phenomena and interpersonal exchanges (Duit & Treagust, 1998). The theory of constructivism can be used to explain the process of learning mathematics based on Jean Piaget's and Lev Vygotsky's works, namely cognitive and social constructivism respectively.

The cognitive constructivism is based on Jean Piaget (1977) who believed that the learners independently build understanding of their own world by connecting new ideas with existing ones, generates their own rules and mental models to make sense of their own experiences. All learning takes place within the individual's mind, through the lens of previous knowledge. Thus the learners are taught to use and reconstruct their pre-existing mental model to make sense of new experience, in which old and new experiences merges to form the new knowledge (Karplus, 1980). Piaget (1977) suggested that learners construct new knowledge based on their experiences through the process of assimilation and accommodation. Assimilation occurs when the learner uses existing concepts to deal with new experiences or information from the outside world. However, if there is a conflict between the new information and the existing mental structure, accommodation takes place. Accommodation occurs when learners' current concepts are inadequate to grasp the new experience successfully (Marin, Benarroch, &

Gomez, 2000). Accommodation required the learner to replace or restructure the current experience or knowledge. The physical experiences are better at inducing cognitive conflicts and encouraging learners to develop new knowledge schemes (Houses, 1995).

The social constructivism is associated with Lev Vygotsky who believed in the role of social interactions and cooperative learning in constructing both cognitive and emotional images of reality as an integral part of the learning process. The learner's thinking and meaning making is socially constructed and emerges out of their social interactions with the environment (Settlage & Southerland, 2007). The knowledge and understanding are constructed when learners are engaged socially in talks and activities about the shared problems or tasks. The dialogue exchange in conversation and the collaboration with each other help the learners to construct knowledge. In a group, a more experienced learner can help a less experienced one by reconstructing the tasks that make it possible for the less experienced to perform and to internalize the process or knowledge (Driver & Bell, 1986). The learning is considered to take place first in a group setting as thoughts are exchanged, and then it becomes incorporated into the individual's mind. A social constructivist's view of learning considers both the development of the individual construction of meaning towards the socially agreed to knowledge and the reconstruction of the culture and social knowledge (Palmer, 2005). Social interaction is essential for effective language usage and the development of efficient communication.

In the common views of constructivists, both cognitive and social constructivism share the idea that the development of understanding requires the learners' active engagement in constructing the knowledge (Narli, 2011). It also involves a process of hypothesizing, predicting, manipulating, and constructing knowledge and constructivists believe that knowledge is more than facts and information (Alemu & Schulze, 2012; MacMath, Wallace, & Chi, 2009).

In mathematics, the teaching has become more effective and interesting due to mathematics education incorporated with constructivism (Chan, 2010; Czarnocha & Maj, 2008; Gupta, 2008; Jaworski, 2006; Williams, 2007). Constructivist approaches emphasize that the learning environment is as rich as possible, dealing with students' existing ideas and concepts, and encouraging students' participation. It can be implemented with the use of several teaching and learning approaches. Some of the teaching and learning approaches based on constructivism are project-based learning, problem-based learning, and the different forms of learning cycles, i.e. 3E, 4E, 5E, and 7E learning cycles. The Lawson-Abraham model of learning cycle is closely related to 3E in terms of the sequence of phases and teaching and learning format but not in the names of the 3E phases (Campbell & Fuller, 1982).

2.5 Learning Cycle Approach

The learning cycle approach is an inquiry-based teaching model, which first emerged in the 1960s when Robert Karplus and his colleagues implemented in the Science Curriculum Improvement Study (SCIS) program (Lawson et al., 1989). It was developed by Robert Karplus based on the constructivist theory of intellectual development proposed by Jean Piaget (Karplus, 1980). In Piaget's intellectual development, the learning is not view as a transfer, but as an active construction of knowledge by the individual based on the knowledge already held (Piaget, 1952).

The learning cycle approach has an advantage in learning by ordering the instructional activities compatible with Paiget's cognitive development. In order to facilitate accommodation, the activities in the exploration phase expose the learner to a segment of the environment that demonstrates the information to be accommodated. In the second phase, the activities help the learner to accommodate the information. Finally, to organize the accommodated information, the activities are developed to help the learner to see the relationship between new information and other previously learned information (Abraham, 1997).

The learning cycle has been implemented in various studies even though the names of the phases have changed. Originally, Exploration, Invention, and Discovery phases were named by Karplus and his colleagues. Many authors have modified the names of these phases, eg. Barnes, Driver, Karplus, Erickson, Nussbaum and Novic, Renner, and Rowell and Dawson, but the learning format and sequence of the phases remain the same (Lawson et al., 1989; Sunal, 2007). However the phases of exploration, concept introduction, and concept application described by Anton Lawson (1988) and Michael Abraham (1989) are the foundation phases most closely related to the pioneer in the learning cycle, Robert Karplus, and called Lawson-Abraham model of learning cycle.

The Lawson-Abraham's model of learning cycle consists of the Exploration, the Concept Introduction, and the Concept Application phases.

Exploration phase

This is the most active phase for the students. They learn through their own actions and reactions with minimum guidance in an activity to expose them to the concepts. The students try out their knowledge by observation and investigation through the activity. The students are expected to encounter situations that they cannot explain with their present ideas or reasoning patterns. The teacher acts a facilitator by probing guiding questions and serving as a resource for the students.

Concept Introduction phase

In this phase, the concept is introduced and explained with help from the teacher. The concept is usually derived from the data or classroom discussions. This step should always follow exploration and relate directly to the pattern discovered during the exploration activity. The students should be encouraged to identify as many new patterns as possible before the concept is revealed to the class.

Concept Application phase

In this phase, the students explore the usefulness of the concept they have learned and apply it to new situations. This phase is necessary to extend the range of applicability of the new concept. Without a number and variety of applications, the concept's meaning may remain restricted to the examples used at the time it was initially defined and discussed. In addition, application activities aid students whose conceptual reorganization takes place more slowly than average or who do not adequately relate the teacher's original explanation to their experiences.

Many science educators considered the learning cycle as a useful model for instruction and curriculum development (Abraham, 1997). For the instruction developed based on the learning cycle approach, the students' improvements were found in terms of better understanding of scientific concepts, better thinking skills, and better attitudes towards learning science (Bevevino, Dengel, & Adams, 1999; Qarareh, 2012). The learning cycle approach was seen as an effective hands-on, minds-on, inquiry-based scientific pedagogy, especially for enhancing students' understanding by the ways in which they learned the nature of the world (Bybee et al., 2006). The learning cycle approach can result in greater achievement in learning, better retention of concepts, improved attitudes toward learning subjects, improved reasoning ability, and superior process skills (Hanuscin & Lee, 2007; Marek, 2008; Zollman, 1990). Karplus (1980) believed that text book teaching alone did not give students at any age the integration of conceptual understanding and the process skills. Therefore, the guided-discovery system would make the complex concepts easier to learn by involving them in the activities (Fuller, 2002). Later, Lawson (1988) recognized that teachers can give students new knowledge but students must actively invent or generate the concepts. The learning cycle approach appears to be a potential means of promoting students' understanding of difficult mathematical concepts besides difficult scientific ones.

2.6 Mathematics Education in Bhutanese Context

Mathematics education in Bhutan started with a curriculum borrowed from neighboring India since the modern education started in Bhutan in the 1960s (Denman & Namgyel, 2008). Mathematics is a compulsory subject from pre-primary to middle secondary education in Bhutan. However, it's an optional subject in the higher secondary level depending on the students' choices of the subjects. After grade 10, the students have three choices to choose the subjects; science, commerce, and arts. For the students opting for the science stream, mathematics is a major subject for those who study physical sciences and optional for those studying biological sciences. Similarly, it was optional for arts students and compulsory for commerce students.

From the middle of the 1980s, the Education Department started to change the education system to the Bhutanese context, so that teaching and learning could be based on the national needs and aspirations (Gyamtso & Dukpa, 1998). The government had adopted initiatives to make education more meaningful and improve students' learning environment. One of them was the New Approach to Primary Education (NAPE), which employed a new curriculum based on activities and rooted in the local environment and culture. This method was universalized during the past decade in every discipline (Bray, 1996; Dorji, 2003). However, NAPE did not fully address the objectives related to values, skills, and knowledge required by the future generation especially in mathematics education due to non-availability of mathematics textbook based on the Bhutanese context, very rigid Anglo-Indian mathematics syllabus oriented towards results, and strong influence of monastic traditional teaching and learning. The trend continued for more than a decade from pre-primary education to middle and then to higher secondary education. In fact, NAPE emphasized activity-based learning and changed the focus from teacher-centered to child centeredness (Dolkar, 1995) but nothing has changed in mathematics classrooms. The rote-learning with tedious problem solving from the textbook became more rigorous as the board examination approached and students started avoiding mathematics, and the fail rate increased drastically (Bhutan Board of Examinations, 2005).

In 2007, the mathematics curriculum was reformed, funded by the World Bank in collaboration with the University of New Brunswick, Canada, to produce the new and realistic version of the textbooks (Understanding Mathematics) developed uniquely to address the Bhutanese context and aligned with international foci. The new mathematics curriculum closely resembles New Brunswick's curriculum, which explicitly follows principles and standards established by the National Council of Teachers of Mathematics (NCTM) (Wagner, 2010). The "Understanding Mathematics" textbooks are available from pre-primary to grade 10. However, for grade–11 and 12 mathematics, only learning outcomes and syllabus follow the current reform, but old mathematics textbooks written by Indian authors are still in use.

The mathematics curriculum and textbooks have changed several of times in Bhutan, but current textbooks need revision and improvement in line with professionalism. Acquisition of the content knowledge alone or just a good mark in an examination may not be adequate anymore. For a successful sail beyond grade 12, students would be required to learn mathematics with in-depth understanding. The Bhutanese mathematics curriculum and instructions in the classrooms should play proactive roles (Department for Curriculum and Research Development, 2007).

In this study, fundamentals-of-calculus learning units based on the learning cycle approach using contextual and graphing activities were developed after a careful study of the grade-11 mathematics syllabus of Bhutan. The developed learning units do not cover all the components prescribed in the syllabus. Particularly, they were developed to enhance students' conceptual understanding of calculus topics and the relationship between differentiation and integration. The prescribed syllabus for calculus unit in grade-11 mathematics according to the Department for Curriculum and Research Development (2014) is given below.

1. Limits (5 hours)

Notion and meaning of limits; Fundamental theorems on limits; Limits of algebraic and trigonometric functions.

2. Continuity (5 hours)

Continuity of a function at a point $x = a$; Continuity of a function in a range.

3. Differentiation (15 hours)

Meaning and geometrical interpretation of derivatives; Differentiation from first principle; Derivative of simple algebraic and trigonometric functions and their formulae; Derivative of sums, differences, products and quotients of functions; Application of derivatives: Equation of tangent and normal; Approximation; Rate measure.

4. Integration (5 hours)

Indefinite integral: Integration as the inverse of differentiation; Anti-derivatives of polynomials and functions like $(ax+b)^n$, $\sin x$, $\cos x$, $\sec^2 x$, and $\operatorname{cosec}^2 x$; Integration by simple substitution for simple polynomial functions and simple trigonometric functions.

CHAPTER III
RESEARCH METHODOLOGY

Overview

This chapter presents the methodology of the pilot and the main research studies. The framework of the research study, participants, instruments, development of the fundamentals of calculus learning units based on the Lawson-Abraham's model of learning cycle (LAMLC), implementation of learning units on the fundamentals of calculus, and procedures used for data collection and analysis of the research are described.

To address the research questions, the framework of the research was developed as shown in Figure 3.1. The framework of the research was divided into three parts: (i) the pilot study, (ii) the improvement and extension of the learning unit on the relationship between differentiation and integration in calculus to the learning units on fundamentals of calculus, and (iii) the implementation of the learning units on the fundamentals of calculus.

In the pilot study, the learning unit on the relationship between differentiation and integration in calculus, called the R-DI learning unit, was developed based on the learning cycle approach with the contextual and graphing activities and implemented with undergraduate students to determine the effectiveness of the R-DI learning unit. To further enhance the relationship between differentiation and integration, the results of the pilot study were used to extend the R-DI learning unit into the learning units on the fundamentals of calculus, called the FC learning units. The FC learning units included limits, continuity, differentiation, and integration. The limits and continuity learning units were designed to provide basic backgrounds of calculus while the relationship between differentiation and integration were embedded into the last two learning units. Then, the FC learning units were implemented to grade-11 students in one of the higher secondary schools in Bhutan.

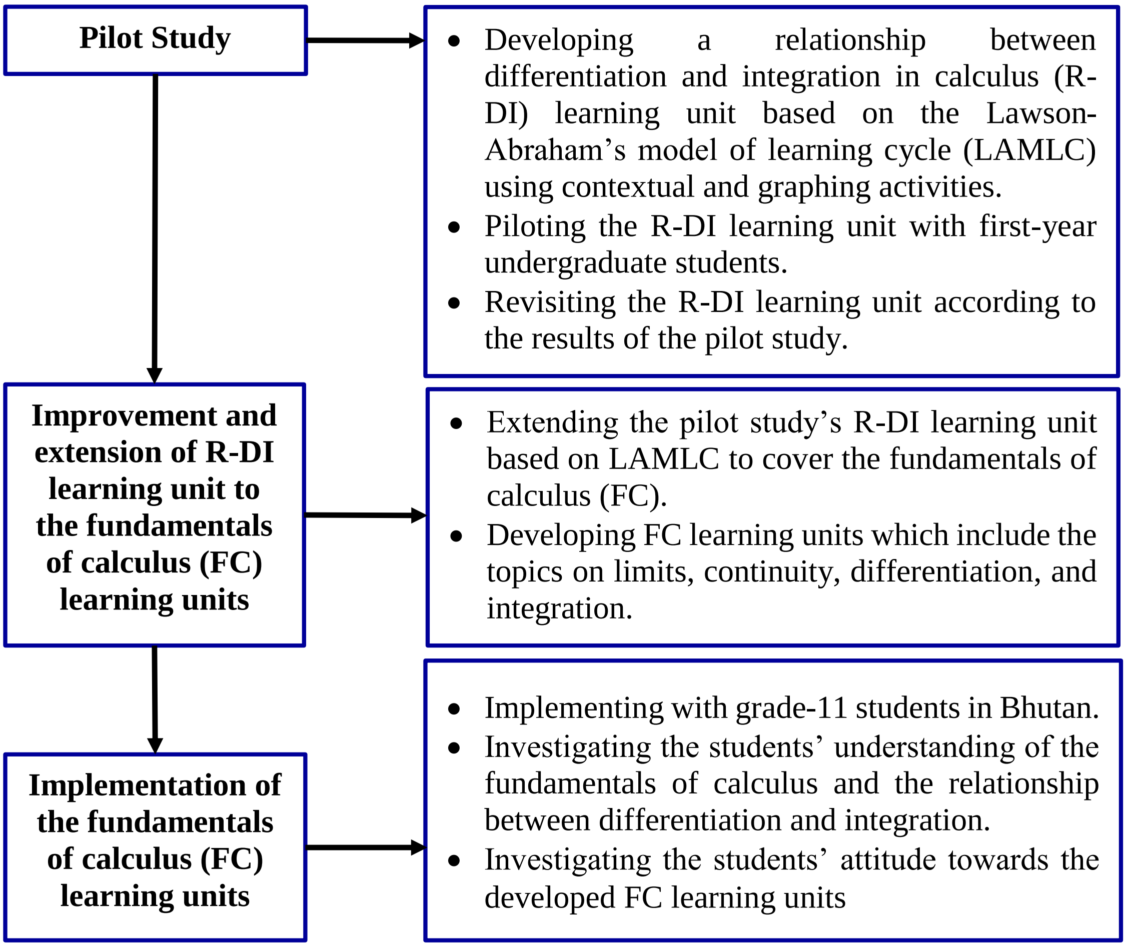


Figure 3.1 The framework of the development and implementation of FC learning units.

3.1 Pilot Study

From the review of a textbook used in introductory calculus (Malhotra, Gupta, & Gangal, 2011), it was found that the textbook is full of algebraic symbols and notations. Students learned only algebraic manipulation in calculus following the traditional mathematics teaching and learning where a teacher showed different examples of solving the algebraic problems. Then, students copied the procedure of solving the problems and practiced the exercises. An over-emphasis on proficiency in problem solving drove the students to focus on calculating correctly while ignoring the true meaning of the concepts behind the calculation. On the other hand, the relationship between differentiation and integration in the textbook was mentioned in words as

"Integration is the inverse process of differentiation" without further explanation and left it to the students to figure out the relationship.

Seeing only algebraic symbols and notations would make it difficult for the students to visualize the relationship between differentiation and integration. After reviewing the literatures on teaching approaches used in learning calculus, the learning unit on the relationship between differentiation and integration (R-DI) was designed for students who have already taken elementary calculus using contextual examples and graphing activities based on the Lawson-Abraham's model of learning cycle (LAMLC) as shown in Figure 3.2.

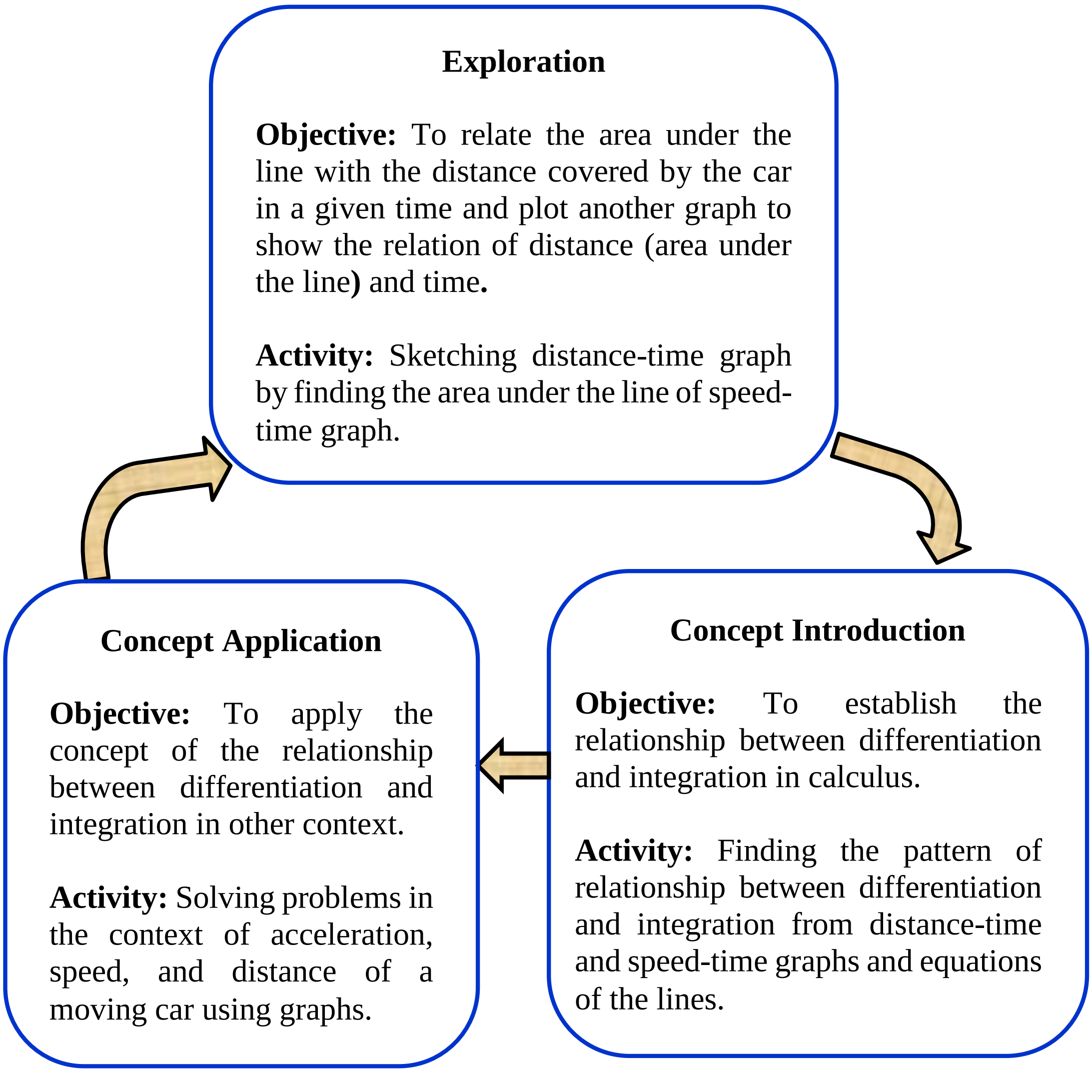


Figure 3.2 Diagram representing the overall activities and objectives of R-DI learning unit.

The R-DI learning unit based on LAMLC consisted of three phases: exploration, concept introduction, and concept application.

3.1.1 Exploration phase

In this phase, students were asked to sketch the graph of the constant speed of a moving car for five hours, and to divide the area under the line into five equal parts as shown in Figure 3.3. Students were asked to find the area of each part under the line and also to find the unit and the meaning of those areas, which should help them realize the graphical relationship between speed and distance.

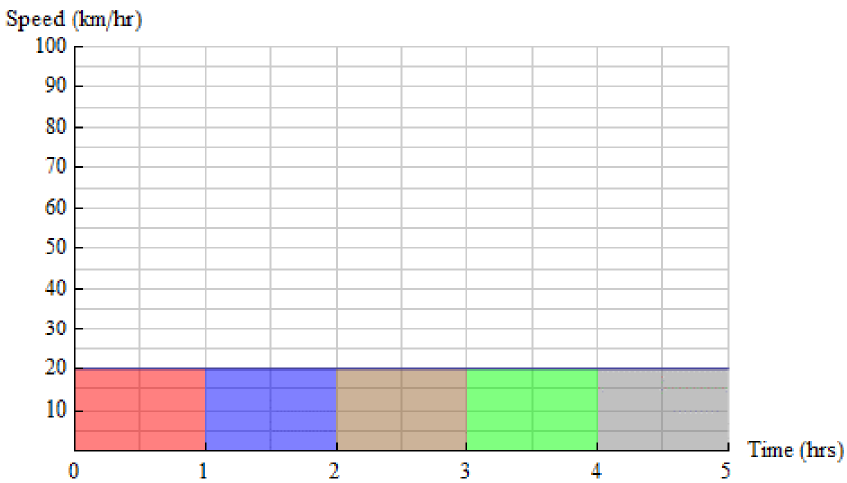
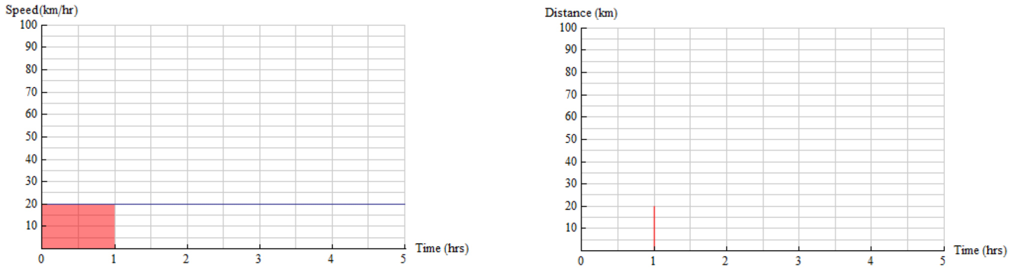


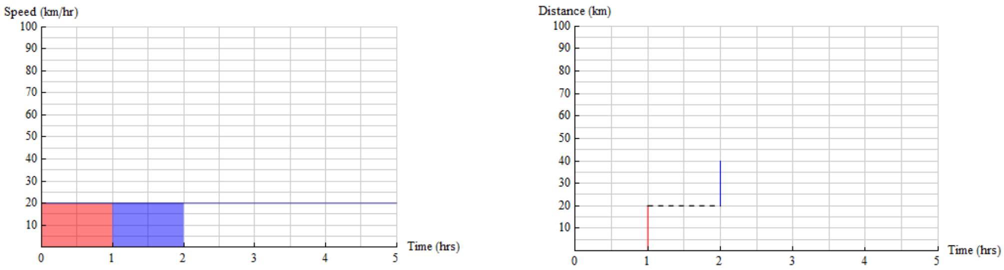
Figure 3.3 Graph showing constant speed of a car and area under the line.

Students were asked to plot each area (distance) on another graph paper and compare the two graphs, which were actually the graphs of a derivative and its anti-derivative as shown in Figures 3.4–3.8. Students found the area under the line from $t = 0$ to 1 hour, 1 to 2 hours, 2 to 3 hours, 3 to 4 hours and 4 to 5 hours (see Figures 3.4(a)–3.8(a)) and sketched the area on another graph as shown in Figures 3.4(b)–3.8(b) which eventually yielded the area under the speed-time graph (see Figure 3.9(a)) and the distance-time graph (see Figure 3.9(b)). They were further asked to find the equations of the two graphs in Figure 3.9. Being more familiar with algebraic notation, the equations should help them in confirming the relationship between the two graphs.



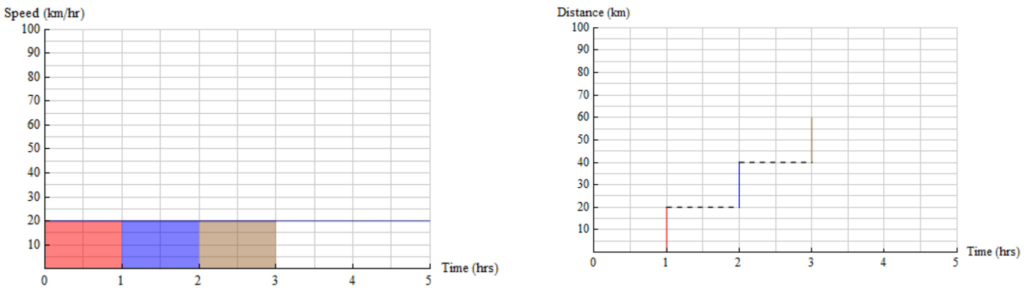
(a) (b)

Figure 3.4 Area under the line and distance during the first hour.



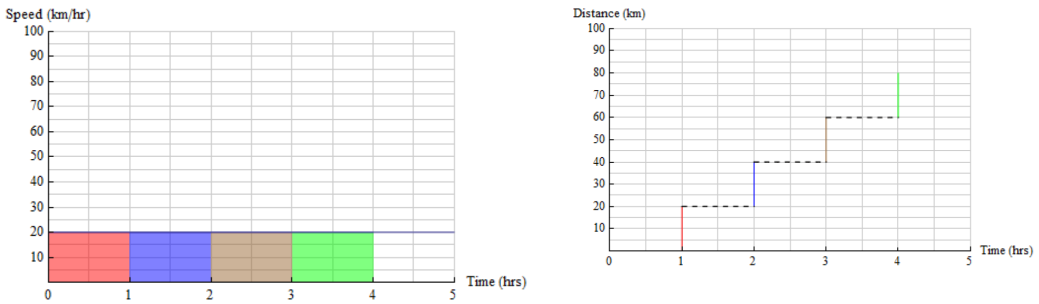
(a) (b)

Figure 3.5 Area under the line and distance during the second hour.



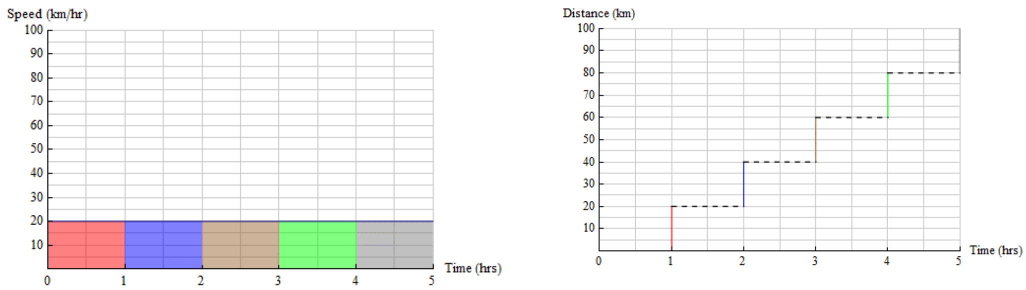
(a) (b)

Figure 3.6 Area under the line and distance during third hour.



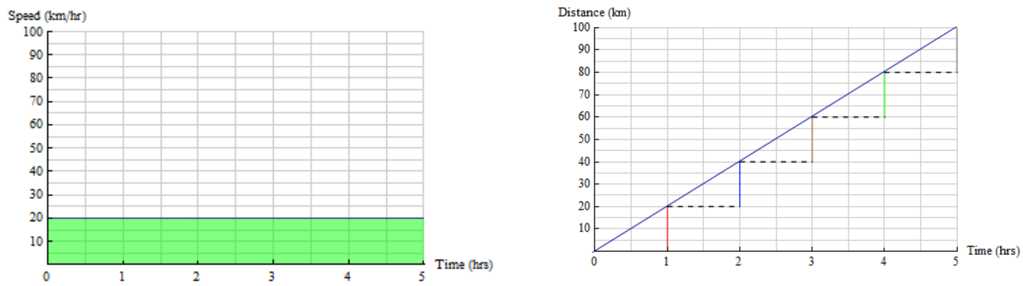
(a) (b)

Figure 3.7 Area under the line and distance during fourth hour.



(a) (b)

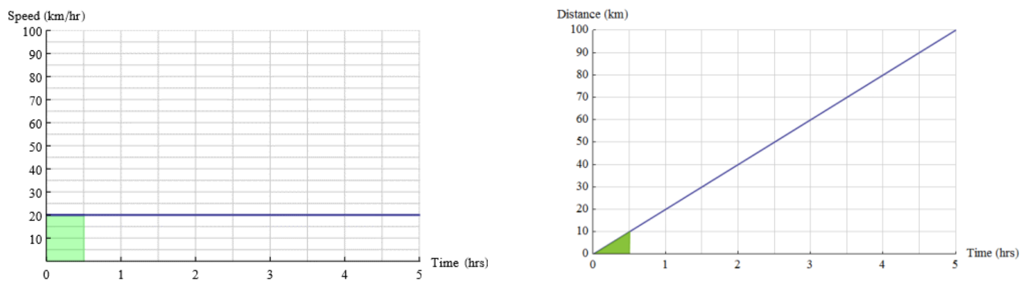
Figure 3.8 Area under the line and distance during fifth hour.



(a) (b)

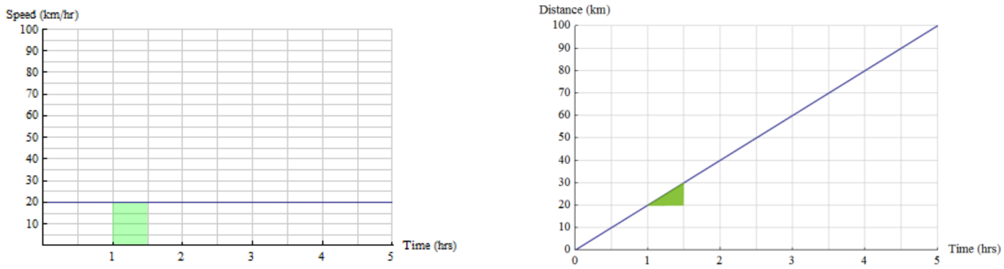
Figure 3.9 Area under the speed-time graph and slope of the distance-time graph.

To help students see that the graphical relationship work for non-integers as well, they were asked to find the areas and the distances during the last half hour prior to t = 0.5, 1.5, 2.5, 3.5 and 4.5 hours as shown in Figure 3.10–3.14.



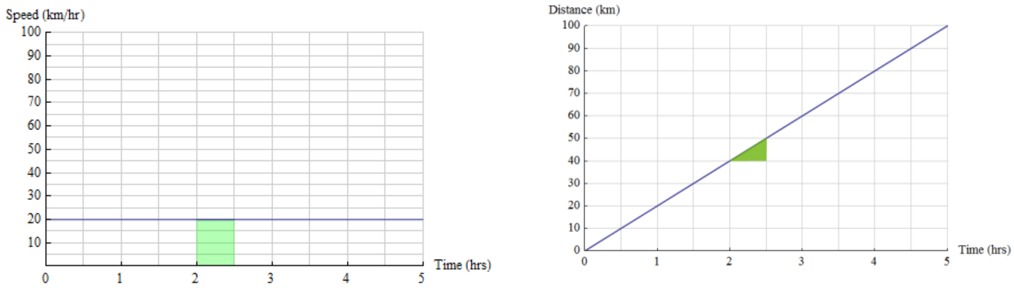
(a) (b)

Figure 3.10 Area under the line for $t = 0.5$ hours and slope of the line at $t = 0.5$ hours.



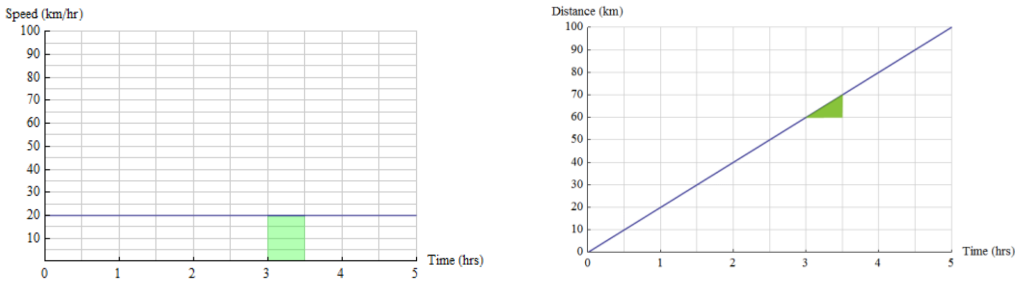
(a) (b)

Figure 3.11 Area under the line for $t = 1.5$ hours and slope of the line at $t = 1.5$ hours.



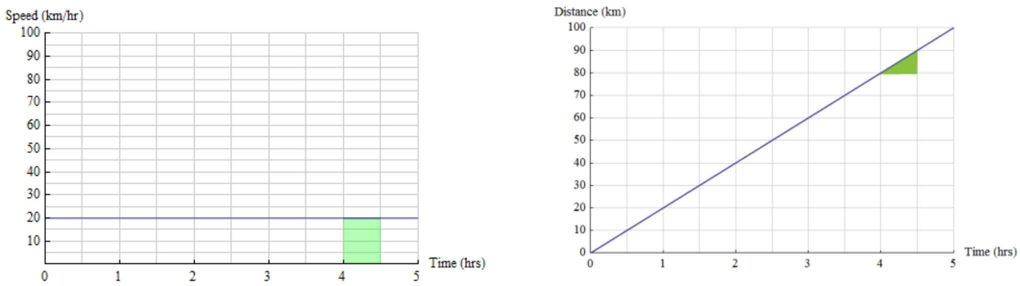
(a) (b)

Figure 3.12 Area under the line for $t = 2.5$ hours and slope of the line at $t = 2.5$ hours.



(a) (b)

Figure 3.13 Area under the line for $t = 3.5$ hours and slope of the line at $t = 3.5$ hours.



(a) (b)

Figure 3.14 Area under the line for $t = 4.5$ hours and slope of the line at $t = 4.5$ hours.

Finally, they were asked to directly calculate the area under the line in Figure 3.9(a) from $t = 0$ to $t = 5$ hours by integration and to confirm that the result agreed with the area and the distance in the two graphs. They were also asked to find the slope of the line in Figure 3.9(b) at $t = 0.5$ hours which gave the point on the line at $t = 0.5$ hours in Figure 3.9(a).

3.1.2 Concept introduction phase

From the exploration phase, students should begin to have an idea about the relationship between differentiation and integration. The concept introduction phase should help them formulate the idea more completely. The students were asked the following questions.

i). What do you get if you find the area under the graph in Figure 3.9(a) by integrating the equation of the line from $t = 0$ to $t = 5$ hours algebraically?

ii). What is the unit of the area? And what does the unit of the area tell you?

The students should be able to see that finding the area under the line and integrating the line of the equation would give 100 km, which would indicate the distance travelled by the car in 5 hours as shown in Figure 3.9(b). Then, the students were also asked the following questions.

i). What do you get if you find the slope of the line graphically and differentiate the equation of the line algebraically of the graph in Figure 3.9(b)?

ii). What is the unit of the slope?

The students should be able to see that the finding the slope of the line graphically and differentiating the equation of line algebraically would give 20 km/hr which would indicate the speed of the car on the graph as shown in Figure 3.9(a). Then, the students were asked; do you see any relationship between the graphs in Figures 3.9(a) and 3.9(b) in terms of differentiation and integration in calculus? Explain?

Now, the students should be able to see the relationship between differentiation and integration graphically, algebraically, and contextually from the activity and to conceptually understand that integration is the inverse process of differentiation.

3.1.3 Concept application phase

In this phase, the context was still a moving car but accelerating at 2 m/s^2 for 10 seconds instead of travelling at a constant velocity. From the exploration and concept introduction phases, students should have an idea how to figure out the equation of the area under the line from a given graph in general. The students should be able to figure out the equations of the lines from the graphs as shown in Figure 3.15.

Finding the area of the shaded region under the graph in Figure 3.15(a) would yield $2t_0$ which could be generalized to the equation of the line ($v(t) = 2t$) in Figure 3.15(b), and finding the area under the shaded region under the graph in Figure 3.15(b) would give $\frac{1}{2} \times 2t_0 \times t_0 = t_0^{\ 2}$ which could be generalized to the equation of the line in Figure 3.15(c).

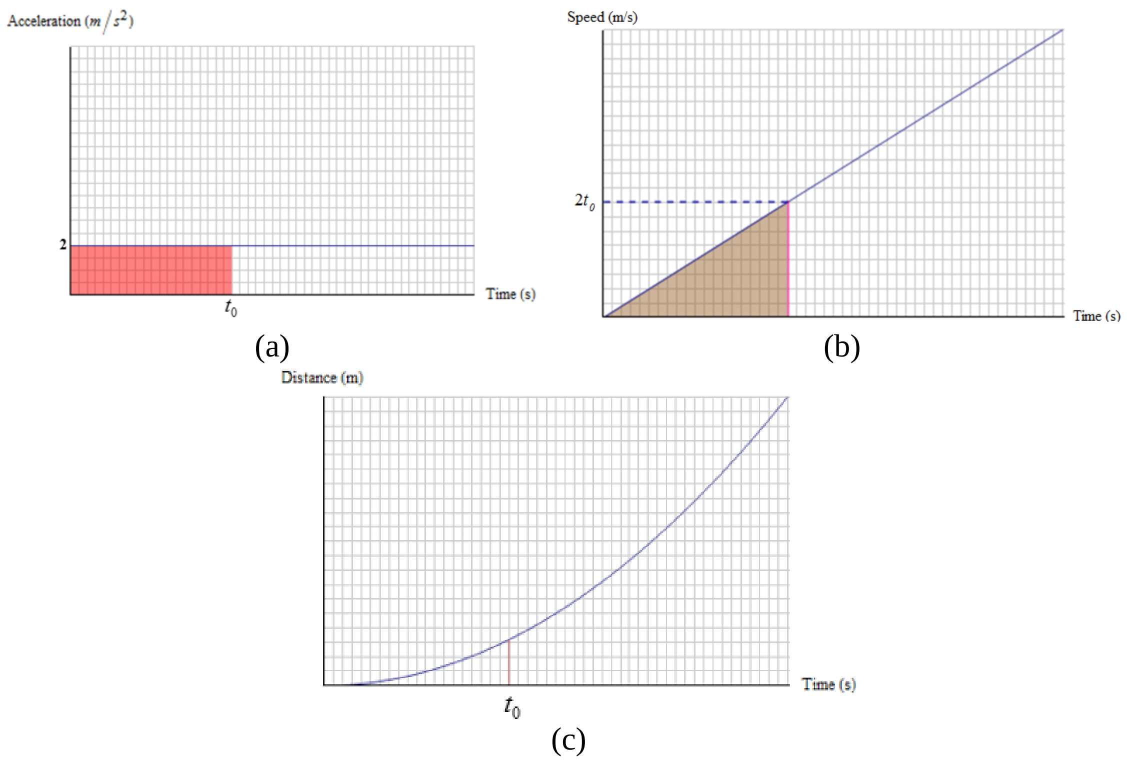


Figure 3.15 Graph showing how to derive the general equations of the lines.

The students were asked to sketch the acceleration-time graph, to find the area under the graph as shown in Figure 3.16, and to plot the area, which was in fact the velocity, on another graph paper as shown in Figure 3.17. Then, finding the area under the graph in Figure 3.18 would give the distance, whose graph is shown in Figure 3.19.

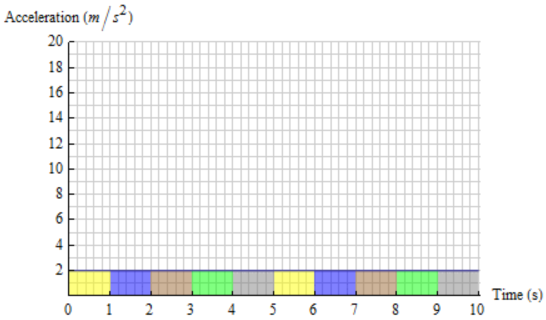


Figure 3.16 Acceleration-time graph.

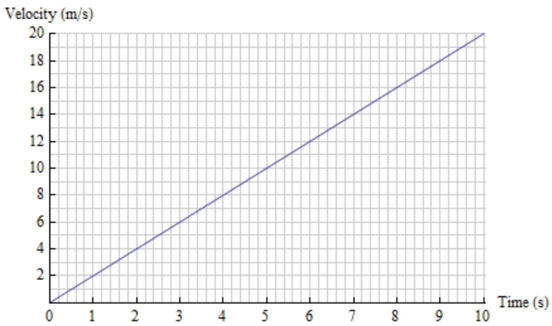


Figure 3.17 Velocity-time graph.

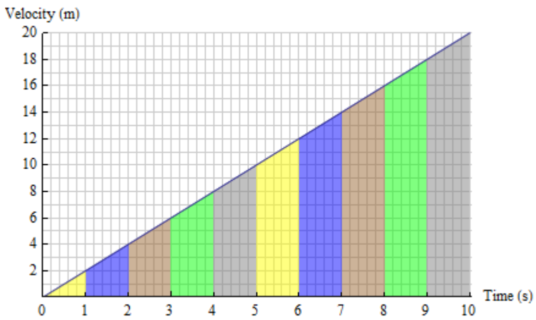


Figure 3.18 Velocity-time graph showing area under the line.

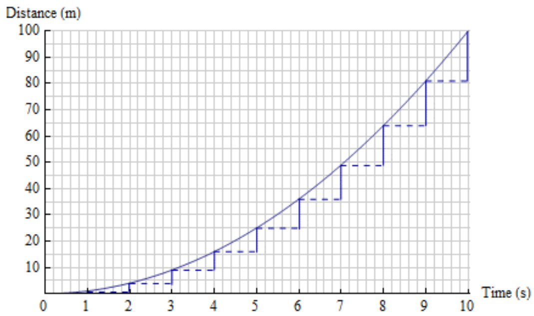


Figure 3.19 Distance-time graph.

Finding the derivatives of the distance-time graph in Figure 3.19 would give back the velocity-time graph in Figure 3.17 and finding the derivative of the velocity-time graph would give the acceleration-time graph as shown in Figure 3.16. Then, students could use the algebraic method of integration to confirm the findings of the velocity-time and distance-time graphs from the graphical method and the same was true for the reverse process of differentiation. It should be noted that the non-linearity of the distance-time graph could also be used to emphasize the instantaneous nature of a derivative, both graphically and algebraically.

3.1.4 Results and conclusions of the pilot study

The developed R-DI learning unit was piloted with eight preparatory-college students and six first-year undergraduate mathematics-major students. The participants were chosen as they had already studied the introductory calculus course during the higher secondary level. The study was piloted to find out whether the students were able to relate differentiation to integration before the intervention of the R-DI learning unit, the effectiveness of the developed learning unit on the students' understanding of the relationship between differentiation and integration, and to measure the students' attitude towards the learning unit.

The conceptual understanding test consists of four open-ended questions. The same set of questions was used for both the pre-test and the post-test. The pre-test was conducted prior to the intervention of the learning unit. At the end of the instruction, the post-test and an attitude test were conducted. The collected data from the conceptual understanding test were analyzed using a paired-sample *t* test as shown in Table 3.1. The total score of the conceptual understanding test was 17.

Table 3.1 Paired-sample *t* test of the pre-test and post-test results.

Test	N	Mean	SD	*t*	p-value
Pre-test	14	1.75	0.83	5.46	0.00
Post-test	14	4.54	1.89		

A paired sample *t* test showed that the average post-test score was significantly greater than the average pre-test score (see Table 3.1). After the intervention, there was significant improvement in students' performance. The mean score in the pre-test was extremely low as many students found the questions difficult which indicated that students had no conceptual understanding in regard to differentiation and integration in general and to the relationship between differentiation and integration in particular. Some of the students scored much higher in the post-test despite the fact that the intervention lasted for only two hours.

The attitude questionnaire consisted of thirteen Likert-type items and three open-ended questions. The Likert scale in the questionnaire included "1 = Strongly disagree", "2 = Disagree", "3 = Neutral", "4 = Agree", and "5 = Strongly agree". The frequencies of the responses to each questionnaire item were separately tabulated and interpreted as shown in Table 3.2.

Table 3.2 Students' attitude questionnaire responses.

	Items	1	2	3	4	5	Mean	SD
1	I have learned about differentiation and integration in calculus before	0	0	5	8	1	3.71	0.61
2	My Mathematics teacher never taught me the relationship between differentiation and integration in calculus	3	3	7	1	0	2.43	0.94
3	I understood the relationship between differentiation and integration in calculus from the activities in the lesson	0	0	6	8	0	3.57	0.51
4	I found it difficult to see the relationship between the area under the line in integration and the slope of the line in differentiation	0	8	2	4	0	3.14	0.91
5	I found it difficult to see the relationship using mathematical notations used in differentiation and integration in calculus	0	7	2	5	0	3.21	0.95
6	Did the lesson improve your understanding of the topic?	0	1	4	6	3	3.79	0.89
7	The lesson was well organized in a way that helps me understand the relationship between differentiation and integration in calculus	0	1	4	6	3	3.64	0.89
8	The lesson was useful to understand the differentiation and integration better	0	2	4	3	5	3.79	1.12
9	The instructor has been well-prepared for the class	0	0	1	8	5	4.29	0.61
10	The instructor has encouraged the students to participate actively in class	0	0	4	7	3	3.93	0.73
11	The instructor has made an effort to enhance learning	0	0	2	9	3	4.07	0.62
12	The instructor has been open to students' opinion	0	0	2	9	3	4.07	0.62
13	I found the class very interesting and enriching	0	0	1	8	5	4.29	0.61

Nine students indicated that they had learned differentiation and integration in an introductory calculus course before but five students were not sure about it. Thirteen students found that the activities were interesting and enriching and helped the majority of them understand the relationship between differentiation and integration better. The majority of the students found that the lesson was well organized and the instructor encouraged the students in the learning process. The students found the activities interesting and enriching probably because they took active roles in the lesson and felt motivated to learn calculus, hence leading to better performance.

To really understand the relationship between differentiation and integration, students obviously need to understand both differentiation and integration which are themselves based on more fundamental concepts like limits, and continuity and discontinuity of a function. The coverage of the fundamentals of calculus is a must before the learning of the relationship between differentiation and integration can take place. Such a coverage will require a much longer intervention duration. Therefore, coherent lessons on the fundamentals of calculus based on the learning cycle approach using contextual and graphing activities were developed.

3.2 Development of Learning Units on the Fundamentals of Calculus

The fundamentals of calculus (FC) learning units were extended from the R-DI learning unit based on the Lawson-Abraham's model of learning cycle (LAMLC). The FC learning units were developed strictly following the grade-11 mathematics syllabus of the Department of Curriculum Research and Development (DCRD), Ministry of Education, Bhutan. Each learning unit consisted of contextual and graphing activities.

The first learning unit began with an activity involving using a rope to find the circumference of a circle. The activity was designed to help students to develop the concept of limits intuitively. Graphs were used to introduce the concept of continuity of functions. An inclined plane experiment was designed to find the speed of a rolling ball and the data collected from the experiment were used to introduce the concept of derivative. The experiment should also help students to develop the concepts of integration and establish the relationship between differentiation and integration. The details of learning units are explained below.

3.2.1 Limits (240 minutes)

The learning unit on limits started with Exploration I, measuring the circumference of a circle using a non-elastic rope, to help students develop the concept of limits intuitively. In Concept Introduction I, particular questions were used to introduce the concept of limits to students. Students should be able to link the finding of the circumference of the circle from the experiment to the formula for the circumference of a circle ($C = 2\pi r$). Next, students again explored the graphs and values of functions (Exploration II). Then, the concepts of left hand limits and right hand limits were introduced in Concept Introduction II with guiding questions. Finally, students solved various limit problems by applying the concepts they had learned in the Exploration and Concept Introduction.

Exploration I (90 minutes)

Activity: Measuring the circumference of a circle using a rope

Students formed groups consisting of 4–5 students. Then students were taken outside the classroom on a parking lot.

1) Students in a group were given following materials: a 100-cm non-elastic rope, a ruler, chalks and a worksheet

2) Students drew a circle using the 100 cm non-elastic rope and chalks on the parking lot as shown in Figure 3.20.

Figure 3.20 Students drew a circle on a parking lot.

3) A fixed point is a point on the rope, except the starting point, that lies on the circumference of the circle. In Figure 3.21(a), point A is the starting point of the rope, and point B on the circumference of the circle, which is also the end point of the rope, is a fixed point. In this case, there is 1 fixed point per rope.

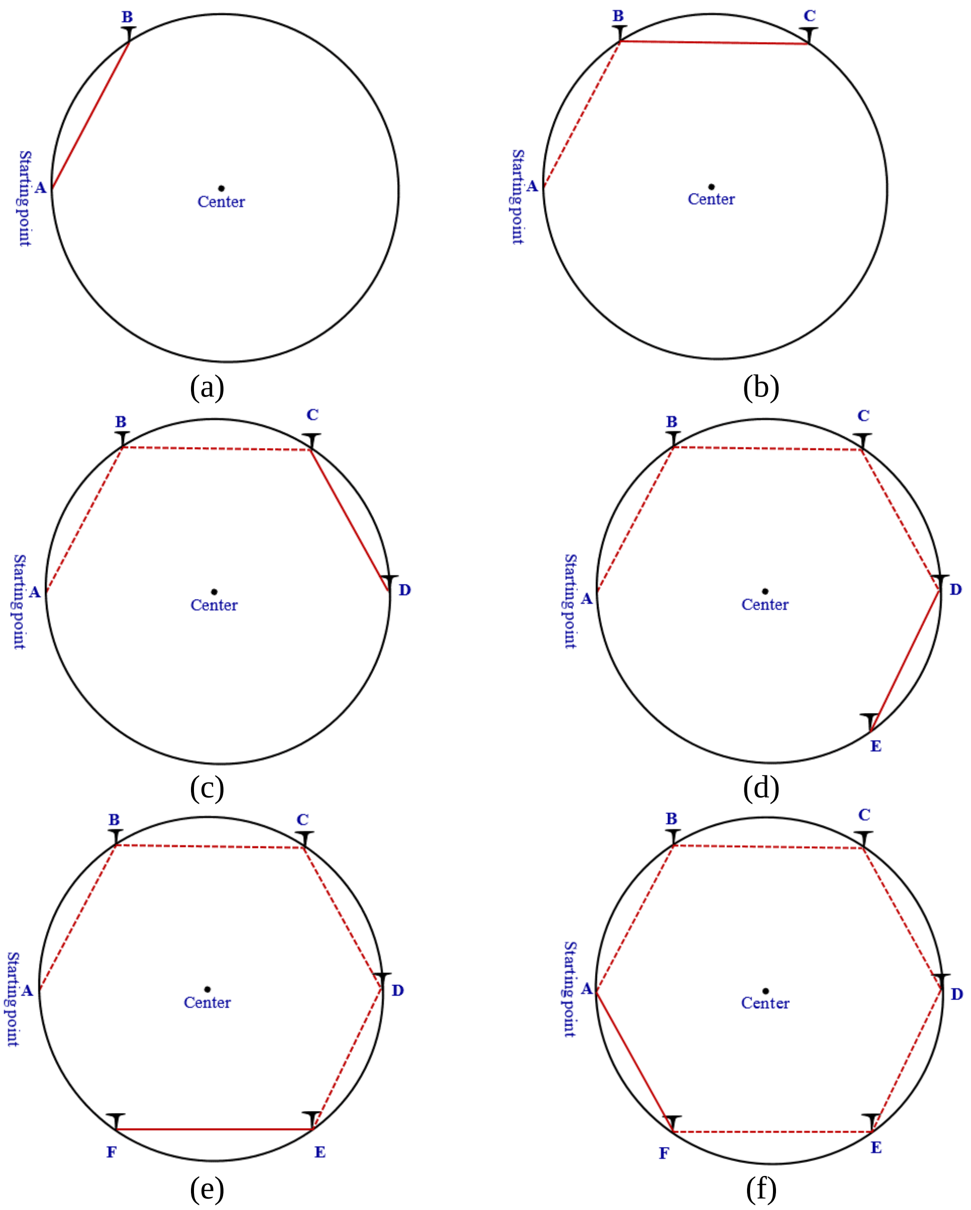


Figure 3.21 Measuring the circumference of the circle by marking one fixed point per rope.

4) Students measured the circumference of the circle using the non-elastic rope.

a) Students marked one fixed point per rope on the circumference of the circle. Point A was the starting point of the rope and point B was the fixed point as shown in Figure 3.21(a).

b) With still one fixed point per rope, point B became the starting point of the rope and point C became the fixed point as shown in Figure 3.21(b).

c) Students continued with one fixed point per rope. The previous fixed point became the new starting point of the rope and the end of the rope became the new fixed point as shown in Figures 3.21(c)–(f) until point A became the last fixed point.

d) Students counted the number of ropes taken to measure the circumference of the circle, and recorded on the worksheet.

e) Students find the perimeter of the polygon ABCEDF (Figure 3.21(f)).

5) Students measured the circumference of the circle again using the same procedure but increasing the number of fixed points per rope to 2 as shown in Figure 3.22.

a) Students marked two fixed points per rope on the circumference of the circle (Figure 3.22(a)). Point A was the starting point, and points A' and B were the two fixed points.

b) With still two fixed points per rope, point B became the starting point of the rope, and points B' and C became the two fixed points (Figure 3.22(b)).

c) Students continued with two fixed points per rope until the remaining length is shorter than the length of the rope.

d) Students counted the number of ropes, measured the remaining length between the end of the last full rope or fixed point G and the starting point A (Figure 3.22(c)), and recorded the length of the perimeter of the polygon on the worksheet.

6) Students continued increasing the number of fixed points per rope (3 fixed points, 4 fixed points, …) on the circumference of the circle, and recorded the numbers of rope and the remaining lengths on the worksheet until the change in the length of the perimeter was not noticeable.

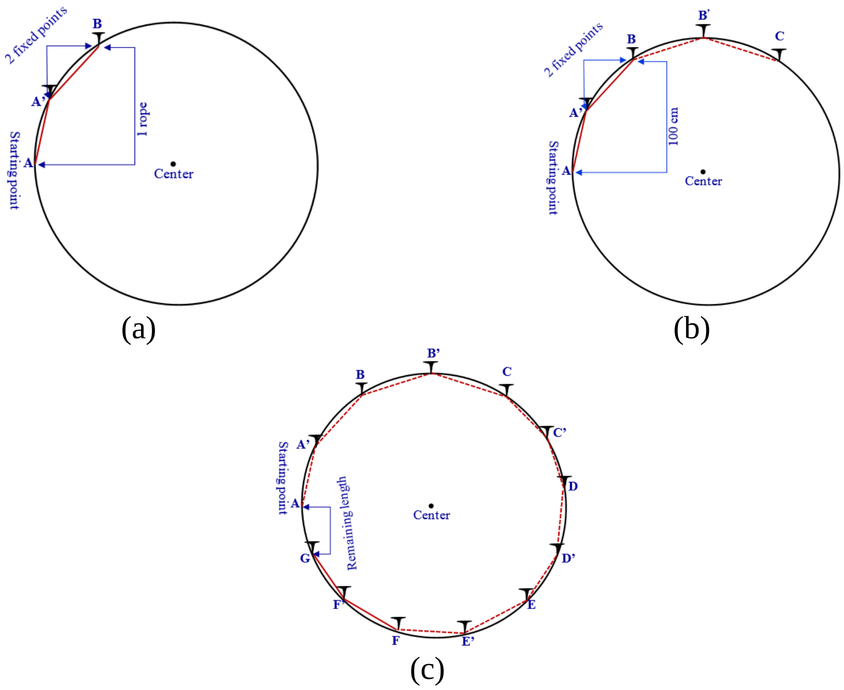


Figure 3.22 Measuring the circumference of the circle by marking two fixed points per rope.

The data in Table 3.3 shows the theoretical measurement of the circumference of the circle using a 100-cm non-elastic rope. Table 3.3 should serve as a guide for a teacher to guide students during the activity.

Table 3.3 The theoretical measurement of the circumference of the circle using a 100-cm rope.

No. of fixed points per rope	Number of ropes on the circumference of a circle	Perimeter of the polygon
1	6 ropes	600 cm
2	6 ropes and 21.8 cm	621.8 cm
3	6 ropes and 25.4 cm	625.4 cm
4	6 ropes and 26.6 cm	626.6 cm
5	6 ropes and 27.2 cm	627.2 cm
6	6 ropes and 27.5 cm	627.5 cm
7	6 ropes and 27.7cm	627.7 cm
8	6 ropes and 27.8 cm	627.8 cm
9	6 ropes and 27.9 cm	627.9 cm

Concept Introduction I (30 minutes)

Students were asked to answer the following questions to introduce the concept of limits in relation to the activity in Exploration I:

i). What did you observe from the data collected?

ii). What happened when you increased the number of fixed points per rope?

iii). What will happen if you further increase the number of fixed points per rope?

iv). What would be the circumference of the circle?

From the above questions, students were introduced to the idea of limits. As the number of fixed points on the circumference of the circle per rope increased, the perimeter of the formed polygon approached the circumference of the circle. Students were asked to compare the collected data with the calculated circumference of the circle when r = 100 cm: $C = 2 \times \pi \times 100 = 628.32$ cm.

The teacher and students should discuss together that the circumference of a circle, by increasing the number of fixed points per rope, would approach the actual circumference of a circle. This activity should help students to form the concept of limits intuitively.

Exploration II (60 minutes)

Students were provided with graph papers and asked to find the limits by completing the tables and sketching the graphs of the given functions. This activity aimed to help students to discover intuitive ideas of limits from tables and graphs.

1) Students were asked to find the value of $f(x) = 2$ when x was given, and complete the table below. Then, students were also asked to sketch the graph of the function $f(x) = 2$, and find the limit as $x \to 1$.

x	0.9	0.99	0.999	1	1.001	1.01	1.1
$f(x)$							

2) Students were asked to find the values of $g(x) = x$ when x was given, and complete the table below. Students were also asked to sketch the graph of $g(x)$, and find the limit as $x \to 4$.

x	3.9	3.99	3.999	4	4.001	4.01	4.1
$g(x)$							

3) Similarly, students were asked to find the values of $h(x)=\begin{cases}2, & \text{if } x>0\\ -2, & \text{if } x<0\end{cases}$ when x was given, and complete the table below. Then students were also asked to sketch the graph of $h(x)$, and find the limit as $x\to 0$.

x	-0.1	-0.01	-0.001	-0.0001	0	0.0001	0.001	0.01	0.1
$h(x)$									

4) Students were asked to find the values of $k(x)=\dfrac{x^2-1}{x-1}$ when x was given, and complete the table below. Then, students were also asked to sketch the graph of $k(x)$, and find the limit as $x\to 1$.

x	0.5	0.9	0.99	0.999	0.999	1	1.0001	1.001	1.01	1.1	1.5
$k(x)$											

5) Students were also asked to modify the function $k(x)$ to $k'(x)=\begin{cases}\dfrac{x^2-1}{x-1}, & \text{if } x\neq 1\\ 2, & \text{if } x=1\end{cases}$ and compare the graphs of the two functions.

The functions $f(x)$ and $g(x)$ were there to let students discover that the limits existed and were equal to the values of the functions at $x=1$ and $x=4$ respectively. The purpose of the function $h(x)$ was to let students discover that the limit did not exist when $x\to 0$. The function $k(x)$ helped students discover what would happen to the function at $x=1$ and when *x g*et closer and closer to 1. Students should see that as *x* got close to 1, the function got close to 2 from the table. Students also needed to know what the graph of the function looked like around $x=1$ even though we could not say what would be the value of the function at $x=1$. However the graph of $k'(x)$ became continuous without a gap at $x=1$, unlike the graph of $k(x)$.

Concept Introduction II (60 minutes)

Based on questions 1, 2 and 3 in Exploration II, students were asked to answer the following questions to let them discover the conditions for the limits of functions to exist.

i). What happened to $f(x)=2$ when x approached 1 from the left (less than 1)?

ii). What happened to $f(x)=2$ when x approached 1 from the right (greater than 1)?

iii). What happened to $g(x)=x$ when x approached 4 from the right?

iv). What happen to $g(x)=x$ when x approached 4 from the left?

v). What happened to $h(x)=\begin{cases} 2, & \text{if } x>0 \\ -2, & \text{if } x<0 \end{cases}$ when x approached 0 from the right?

vi). What happen to $h(x)=\begin{cases} 2, & \text{if } x>0 \\ -2, & \text{if } x<0 \end{cases}$ when x approached 0 from the left?

From the above activity, students should notice that the left hand limit was equal to the right hand limit of the function $f(x)$ as well as $g(x)$. However $h(x)$ had different left hand and right hand limits. Then, students were introduced to the left hand limit and the right hand limit of the function $f(x)$ as follows:

- When x became bigger and approached 1 from the left, $f(x)$ approached 2. So $f(x)$ possessed a left hand limit at 2 and can be expressed as $\lim_{x\to 1^-} f(x)=2$ or $\lim_{x\to 1^-} 2=2$.
- When x became smaller and approached 1 from the right, $f(x)$ approached 2. So $f(x)$ possessed a right hand limit at 2 and can be expressed as $\lim_{x\to 1^+} f(x)=2$ or $\lim_{x\to 1^+} 2=2$.

Students were also introduced to the limit of a function. The limit of a function exists only if the right hand limit equals to the left hand limit. Otherwise, the limit of the function does not exist. This was elicited by the functions $f(x)$ and $g(x)$ in Exploration II. The functions $f(x)$ and $g(x)$ showed that limits of the functions existed, and $h(x)$ showed that limit of the function did not exist.

The function $k(x)$ aimed to familiarize students with the concept that even though the function was not defined at $x=1$, as x got closer and closer to 1, the function got closer and closer to 2. To induce the idea of limits to students, they were asked to answer the following questions:

i). How did you sketch the graph of the function $k(x) = \frac{x^2 - 1}{x - 1}$?

ii). Do you think that the limit of the function exists at $x = 1$? Explain.

iii). What should be the limit of the function as *x* gets closer and closer to 1?

Students must have seen that as *x* got close to 1, $\frac{x^2 - 1}{x - 1}$ got close to 2. Here there were two interesting situations: (i) when $x = 1$, we did not know the value of the function (it was indeterminate); (ii) but we could see that it was going to be 2. So students should understand that ignoring what happened when $x = 1$, as *x* got closer and closer to 1, the value of the function got closer and closer to 2. They should be able to express mathematically that $\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2$; i.e. the limit of $\frac{x^2 - 1}{x - 1}$ as *x* approaches 1 is 2.

Students were also asked to find the limit of *k*(*x*).by applying an algebraic approach: $k(x) = \frac{x^2 - 1}{x - 1} \Rightarrow \lim_{x \to 1} k(x) = \lim_{x \to 1} \frac{x^2 - 1^2}{x - 1} \Rightarrow \lim_{x \to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x \to 1}(x+1) = 2$.

They should able to see that the limit of *k*(*x*) was 2 when they simplified the equation even though the value at $x = 1$ is indeterminate. Here, the simplification was justifiable because even though *x* approached 1, it never took the value of 1 exactly. Thus (*x* – 1) could cancel each other. Therefore, we could not say that the limit equals the value of the function at that point. We needed a more formal definition. Students were introduce to the formal definition of the limit of a function at a certain point. The teacher should begin with the precise ($\varepsilon - \delta$)-definition.

The epsilon-delta definition of the limit of a function

Let $f(x)$ be a function of *x*. If for every positive number ε, however small it may be, there exists a positive number δ such that whenever $0 < |x - a| < \delta$, we have $|f(x) - L| < \varepsilon$. Then we say $f(x)$ tends to the limit *L* as *x* tends to *a* and write $\boldsymbol{\lim_{x \to a} f(x) = L}$.

In general, the value of δ will depend on the value of ε and one always begins with $\varepsilon > 0$ before determining an appropriate corresponding value for $\delta > 0$. There are many values of δ which work. Once an appropriate value of δ is found, all smaller values of δ also work.

The meaning behind the abstract idea of the $(\varepsilon - \delta)$-definition of a limit is shown in Figure 3.23.

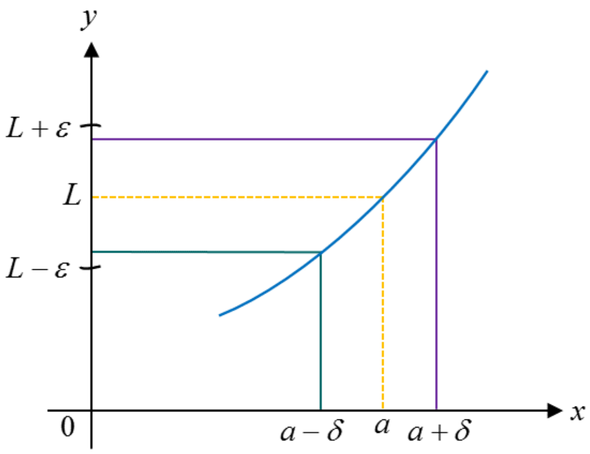


Figure 3.23 Representation of the epsilon-delta definition of a limit.

Basically, the $(\varepsilon - \delta)$-definition of a limit states the following:

- Suppose someone gives us a small value $\varepsilon > 0$.
- We can form horizontal band around your limit L of width 2ε centered about $y = L$ (all the points are within the distance ε of the line $y = L$).
- Given such an epsilon band, the limit will exist if we can find some number $\delta > 0$ such that when we create a vertical band of width 2δ centered at $x = a$, all of the function values in this vertical band will be contained within the horizontal epsilon band. This vertical delta band contains all points which are within the distance δ of the line $x = a$. The key is that for every ε we are able to choose δ. If we can always do this no matter what ε is given, then our limit exists.

Then students were asked to prove the limit of the functions given in questions 1 and 2 in Exploration II using the $(\varepsilon - \delta)$-definition of a limit. For example to prove that $\lim_{x\to 1} 2 = 2$, begin by letting $\varepsilon > 0$ be given. Find $\delta > 0$ so that if $0 < |x-1| < \delta$, then $|f(x)-2| < \varepsilon$. Since $|2-2| = 0 < \varepsilon$ no matter what value is chosen for δ, $\delta = \frac{1}{2}$, for example, will work. Thus, if $0 < |x-1| < \delta$, then it follows that $|f(x)-2| < \varepsilon$. This completes the proof. Similarly, students were asked to prove that $\lim_{x\to 4} x = 4$ using $(\varepsilon-\delta)$ -definition of a limit. This is to familiarize students with the formal definition of limits.

Concept Application (60 minutes)

Students were asked to solve the following problems by applying the concepts that they had learned in the concept introduction phases.

i). Find the limit as $x \to 1$ for the function $f(x) = 2x+4$. Does the limit exist?

ii). Find the limit as $x \to 0$ for the function $g(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$. Does the limit exist?

iii). Does $\lim_{x\to 3}\left(\frac{x^2-9}{x-3}\right)$ exist? Explain.

Students should figure out that the limit of the function $f(x) = 2x+4$ exists when $x \to 1$ as the left hand limit is equal to the right hand limit. The limit of the function $g(x) = \begin{cases} 1, & \text{if } x > 0 \\ -1, & \text{if } x < 0 \end{cases}$ does not exist when $x \to 0$ as the left hand limit is not equal to the right hand limit, and the function $\left(\frac{x^2-9}{x-3}\right)$ approaches 6 when *x* gets closer and closer to 3. So, the limit of the function as x tends to 3 is 6.

The teacher moved around to ensure that students were able to solve the problems. The teacher can guide students by asking probing questions instead of giving direct answers.

3.2.2 Continuity of functions (120 minutes)

The learning unit on continuity of functions started with Exploration where students discussed in pairs to sketch graphs and complete tasks provided in a worksheet. In Concept Introduction, students developed the concepts of continuity of functions by answering probing questions in relation to Exploration's activities. Finally, in Concept Application, students used the definition of the continuity of functions to solve problems.

Exploration (60 minutes)

Each pair of students was provided with a worksheet. Students discussed in pairs, and tried to solve the following tasks provided in the worksheet.

Task 1: Sketch the graph of the function $f(x) = 2x$ and find the limit of the function $f(x) = 2x$ at your chosen point.

Task 2: Let the function $g(x) = \left(\dfrac{x^2 - 4}{x - 2}\right)$.

i). Is the function defined at $x = 2$?

ii). Find the value of $\lim_{x \to 2}\left(\dfrac{x^2 - 4}{x - 2}\right)$.

iii). Sketch the graph of the function $g(x) = \left(\dfrac{x^2 - 4}{x - 2}\right)$ from $x = 0$ to $x = 4$.

iv). Sketch the graph of $g'(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if } x \neq 2 \\ 4, & \text{if } x = 2 \end{cases}$. What is the difference between the graph of (iii) and (iv)?

Task 3: Sketch the graph of function $h(x) = \begin{cases} 2, & \text{if } x \leq 1 \\ x, & \text{if } x > 1 \end{cases}$. Find the limit of the function at $x = 1$.

Task 4: Sketch the graph of the function $k(x) = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$. Find the limit of function $k(x)$ at $x = 0$.

Students should figure out the difference between tasks 1, 2, and 3 from the sketched graphs. Students should get a continuous line on the graph and also the limit exists at the chosen point from task 1. The function in task 2 is undefined when $x = 2$, so it is not a continuous function. However, the function can be modified to be a continuous function. The function in task 3 is also not a continuous function because the limit of the function does not exist at $x=1$ even though the function is defined there because there are two competing answers: 2 from the left and 1 from the right. Moreover, there is a jump in the line when the function is sketched on the graph. The function in task 4 is a continuous function because it is defined at $x=0$ (no "hole") and the limit exists at $x=0$ because the limits from both sides are 0 (no "jump").

Concept Introduction (30 minutes)

From tasks 1, 2, and 3 in Exploration, probing questions were asked in relation to the tasks. Students should able to figure out the difference in each task and develop the concept of continuity and discontinuity of a function.

i). Is the function $f(x)=2x$ defined at your chosen value of x? What does the graph of the function look like? Does the limit of the function exist at your chosen point?

ii). What happens to the function $g(x)=\left(\dfrac{x^2-4}{x-2}\right)$ at $x=2$? Does the limit exist at $x=2$? What does the graph of the function look like for all values of real numbers except at $x=2$? Answer the first two questions for $g'(x)=\begin{cases}\dfrac{x^2-4}{x-2}, & \text{if } x\neq 2\\ 4, & \text{if } x=2\end{cases}$.

iii). Does the limit of the function $h(x)=\begin{cases}2, & \text{if } x\leq 1\\ x, & \text{if } x>1\end{cases}$ exist at $x=1$? What do you observe from the sketched graph?

iv). Does the limit of the function $k(x)=\begin{cases}x, & \text{if } x\geq 0\\ -x, & \text{if } x<0\end{cases}$ exist at $x=0$?

Students should find differences in tasks 1, 2, 3, and 4 from the graphs as well as from the limits of functions. Then the teacher introduced the term "continuous function" for the functions in tasks 1 and 4, and the term "discontinuous functions" for the functions in tasks 2 and 3. Students were also introduced to the definition and conditions for continuity of a function as follows.

Definition of continuity of a function at a point

A function $f(x)$ defined in an open interval centered at a point a (and also at a) is said to be continuous at $x=a$ if $\boldsymbol{\lim_{x \to a} f(x) = f(a)}$.

According to the above definition, the following three conditions are necessary for the function $f(x)$ to be continuous at $x=a$.

i). $f(x)$ is defined at $x=a$, i.e. $f(a)$ exists.

ii). $\lim_{x \to a} f(x)$ exists.

iii). $\lim_{x \to a} f(x) = f(a)$.

Since $\lim_{x \to a} f(x)$ exists only if both the left hand and right hand limits of $f(x)$ as $x \to a$ exist and are equal, we may also say that a function $f(x)$ is continuous at the point $x=a$ if $\boldsymbol{\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = f(a)}$.

Then, let students think what could be the conditions for discontinuity of a function? The teacher can provoke students by using probing questions regarding the negation of the conditions required for the continuity of a function. If not the teacher can introduce the conditions for discontinuity of a function at $x=a$ as follows.

i). The function is not defined at $x=a$;

ii). One or both of the left hand and right hand limits does not exist as $x \to a$;

iii). Both the left hand and right hand limits exist but not equal; or

iv). Both the left hand and right hand limits are equal but not equal to the value of the function at $x=a$.

Students were asked to find the continuity of functions to familiarize themselves with the definition.

i). Determine whether the function $f(x) = \begin{cases} 3x-5, & \text{if } x \neq 1 \\ 2, & \text{if } x = 1 \end{cases}$ is continuous at $x=1$.

ii). Determine whether the function $g(x) = x^2 - 2x + 1$ is continuous everywhere.

iii). Determine whether the function $h(x) = \begin{cases} x^3 + 2, & \text{if } x < 2 \\ x^2 + 6, & \text{if } x \geq 2 \end{cases}$ is continuous at $x = 2$.

Students should apply the conditions for continuity to find out whether the functions are continuous at $x = 1$, everywhere, and at $x = 2$ respectively. The first function is discontinuous at $x = 1$ while the others are continuous everywhere.

Concept Application (30 minutes)

Students were asked to solve the following problems using the concepts they have learned from Concept Introduction. The teacher moved around the class to ensure that students were able to solve the following questions. The teacher should guide students by asking probing questions instead of giving direct solutions.

i). Let $f(x) = \begin{cases} 2x + 1, & x \leq 1 \\ x^2 - 1, & x > 1 \end{cases}$. Is $f(x)$ continuous at $x = 1$? If not, what type of discontinuity does $f(x)$ have at $x = 1$?

ii). Let $g(x) = \begin{cases} x + 1, & x \neq 0 \\ 2, & x = 0 \end{cases}$. Sketch the graph of $g(x)$ and at which point is $g(x)$ discontinuous?

iii). Let $h(x) = \begin{cases} x^3 + 2, & \text{if } x < 2 \\ 5, & \text{if } x = 2 \\ x^2 + 6, & \text{if } x > 2 \end{cases}$. Sketch the graph of $h(x)$ and at which point is $h(x)$ discontinuous?

Students should figure out that the left hand limit is not equal to the right hand limit for $f(x)$ as $x \to 1$, and there is discontinuity at $x = 1$ as there is a jump. The function $g(x)$ is not continuous at $x = 0$ because $g(0) = 2$ but $\lim_{x \to 0} g(x) = \lim_{x \to 0} x + 1 = 1$. The function $h(x)$ is not continuous at $x = 2$ since $h(2) = 5$ but $\lim_{x \to 2^-} h(x) = \lim_{x \to 2^-} x^3 + 2 = 10$ and $\lim_{x \to 2^+} h(x) = \lim_{x \to 2^+} x^2 + 6 = 10$.

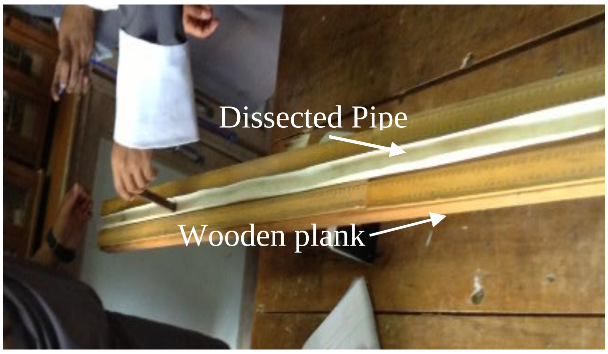
3.2.3 Differentiation (240 minutes)

The differentiation learning unit began with Exploration. Students performed the inclined plane activity in groups by recording the time taken by a ball to fall from the top of the inclined plane to each stopping point, and used the data to sketch a distance-time graph. Questions were used to introduce the concept of differentiation. Students also used the sketched graph to find the slope of the curve and relate the slope to the context of the inclined plane activity. They were introduced to the physical and algebraic concepts of differentiation to define the derivative of the function at a point and derivative of a function in Concept Introduction. Finally, students were asked to apply the concept learned to solve the problems in Concept Application.

Exploration *(90 minutes)*

Activity: An inclined plane activity

The inclined plane was designed using locally available materials by fixing a four-meter dissected pipe on a wooden plank as shown in Figure 3.24.



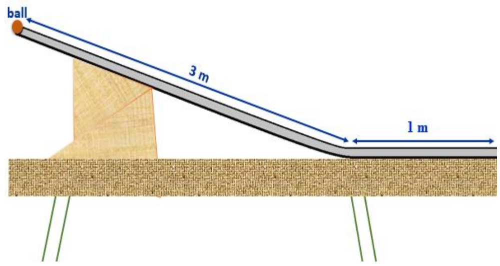


Figure 3.24 Inclined plane.

1) Students were provided with the following materials: a ball, a measuring tape/a ruler, a digital stop watch, and a worksheet.

2) Students began by discussing in groups how they could measure the time taken by the ball to fall from the top of the inclined plane to each stopping point on the inclined plane (Figure 3.25).

Figure 3.25 Students performing the inclined plane activity in a group.

3) On the inclined plane, students selected stopping points (at least five different points), measured the distance between the top of the inclined plane and each stopping point (Figure 3.26), and recorded the distances on the worksheet as shown in Table 3.4.

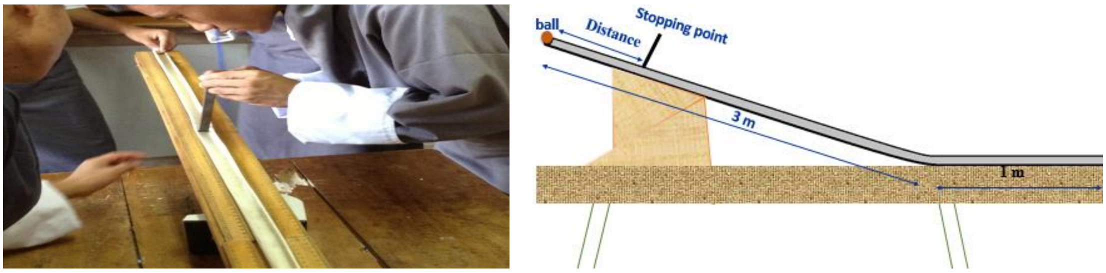


Figure 3.26 A stopping point on the inclined plane.

Table 3.4 Worksheet to record the time taken by the ball to fall to each stopping point during the inclined plane activity.

Distance (m)							
Trial No.	1						
	2						
	3						
	4						
	5						
Average time taken (s)							

4) At each stopping point, students released the ball from the top of the inclined plane and measured the time that the ball took to hit the stopping point (Figure 3.27), and repeated the experiment at least five times. Then, students recorded the time on the worksheet.

Figure 3.27 Students recording the time at a specific distance.

5) To help students minimize the error involved in measuring the time take by the ball, they were asked to find the average time taken by the ball for each distance.

6) Students sketched the graph of distances versus average times.

During the inclined plane activity, the data collected by students did not correspond to the expected results due to time measurement errors. To minimize the time measurement errors and align the data with the expected results, the teacher pooled all data from the groups. The pooled data were used to perform a regression analysis to find the acceleration (*a*) that minimized the sum of squared errors in the equation $t = \sqrt{2s/a} + e$ where *s* was distance, *t* was time, and *e* was the measurement error. The regression analysis yielded $a = 0.22$ m/s^2. For simplicity, the acceleration of 0.2 m/s^2 was used to form the equation $s = 0.1t^2$. The teacher sketched the graph of $s = 0.1t^2$ as shown in Figure 3.28 and gave it (without the equation) to students.

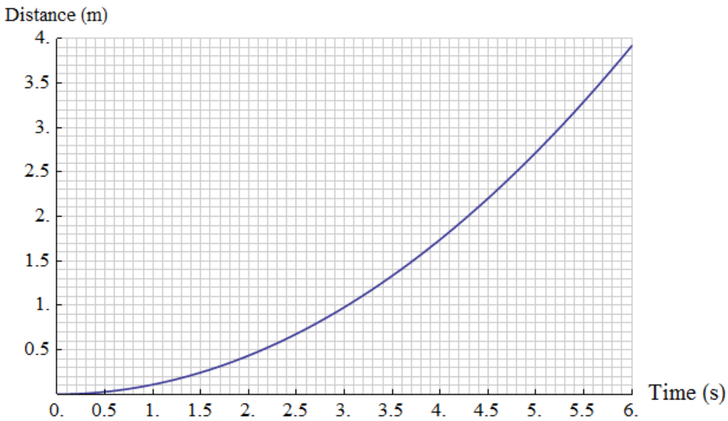


Figure 3.28 The graph of $s = 0.1t^2$.

7) Each student was asked to find the slope of the curve at a particular instant, for example, $t = 3.2$ seconds.

The graph of $s = 0.1t^2$ from the inclined plane activity should be explained to convince students that the true distance-graph should look like this if the time measurement errors were minimal. This should help students to relate the inclined plane activity to the graphical representation and vice versa.

The slope of the line at a particular instant was the speed of the ball at that instant. The unit of the slope should help students figure out that the slope represented the speed of the ball.

Concept Introduction (90 minutes)

Students were asked to answer the following questions from the sketched graph. This was to help students interpret the distance-time graph in relation to the inclined plane activity. Once students could interpret the graph in the context of the inclined plane activity, the term "differentiation" was introduced by the teacher.

i). What is the distance covered by the ball in 3 seconds?

ii). How long does it take for the ball to cover the distance of 2.5 m?

The ball covered 0.9 m in 3 seconds and it took 5 seconds to cover the distance of 2.5 m. Students should be able to figure out the time and the distance for the given distance and time using the graph of Figure 3.28.

iii). What is the unit of the slope?

iv). Regarding the unit, what does the slope of the curve indicate?

The unit of the slope is meters per second and it shows the speed of the ball. These questions were used to let students figure out that the slope of the distance-time graph was the speed of the ball.

v). What is the speed of the ball at 3.5 seconds?

vi). How do you find the speed of the ball at 3.5 seconds?

The speed of the ball is 0.7 m/s. Students should figure this out from the slope of the graph in Figure 3.28. Students should be able to explain how they found the speed of the ball by relating the slope of the graph to the inclined plane activity. The teacher also needed to guide students to find the slope of the graph.

The slopes are different at different points on the curve unlike the slope of a straight line which is constant along the line. There are two ways of finding the slope of the curve at a given point:

(i). Using the tangent line at that point as shown in Figure 3.29(a).

(ii). Taking points on the curve as shown in Figure 3.29(b).

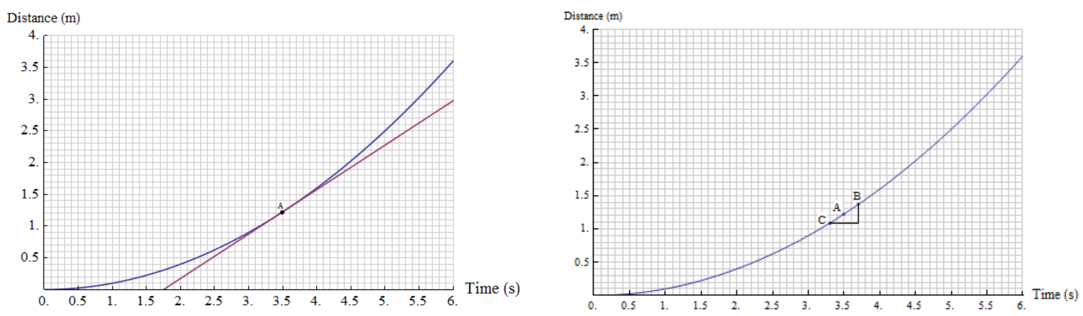
To find the slope of the curve using the tangent line, students should draw the tangent line that touched the curve at that point and calculate the slope of the tangent line by taking two points on the line. The teacher should guide them through finding slope of the tangent line precisely and accurately from the graph. The technique of taking two faraway points on the line closest to the intersections on the grid should give high-precision coordinates which in turn yield an accurate slope at the chosen point. For example, two faraway coordinates on the tangent line to find the slope at t = 3.5 seconds could be (2.2, 0.3) and (5.9, 2.9) which would give the slope of 0.70 m/s (the exact velocity, see figure 3.29(a)). Students needed to figure out the best techniques and skills to calculate the slope at the point from the graph accurately.

Students should also know another way to calculate the slope of the curve by selecting appropriate points on the curve and using them to find the slope accurately. The teacher should guide them through finding the slopes of the curve by asking probing questions rather than giving direct instruction. Students should figure out which points on the curve should be selected to find the slope at that point accurately. Ask students to find the slope of the curve at point A from the graph in Figure 3.29(b).

i). By taking points A and B.

ii). By taking points A and C.

iii). By taking points B and C.



(a) (b)

Figure 3.29 Finding the slope of the curve at a given point.

The slope of the curve at point A is 0.7 m/s by taking points B and C, 0.72 m/s by taking points A and B, and 0.68 m/s by taking points A and C. The teacher needed to tell students that they had to take two points, keeping the point where they wanted to find the slope in between the two chosen points. However, the teacher should confirm that keeping the point where they wanted to find the slope in the middle yielded the most accurate slope using algebraic differentiation in a later part of the lesson.

Students were given the equation of the distance-time graph from inclined plane activity $s(t) = 0.1t^2$ (see Figure 3.28). They were asked to find the average speed between two points and complete the table below. This was to let students see that the average speed approached a certain value (the instantaneous speed at t = 4 seconds) when the time interval approached zero.

Time (s)	4–4.0001	4–4.001	4–4.01	4–4.1	4–5
Distance (m)					
Average speed (m/s)					

From the above activity, students should figure out from the table that the average speed approached 0.8 m/s. Since an average speed was also the slope of the secant line that passed through the two points used in the calculation of that speed, the slope of a tangent was the limit of the slope of the secant line as the distance between the two points approached zero.

Another way to find the limit of the average speed was to employ the algebraic approach similar to the one used to find $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$. The average speed during the time interval starting at 4 seconds to t was given by $\frac{s(t) - 1.6}{t - 4}$. From the inclined plane activity, $s(t) = 0.1t^2$. That is, the average speed was the slope of the line passing through the points (t, $s(t)$) and (4, 1.6) as shown in Figure 3.30. The limit of the average speed was $\lim_{t \to 4} \frac{(0.1t^2 - 1.6)}{(t - 4)} = 0.1 \lim_{t \to 4} \frac{(t^2 - 16)}{(t - 4)} = 0.1 \lim_{t \to 4} (t + 4) = 0.1 \times 8 = 0.8$ m/s.

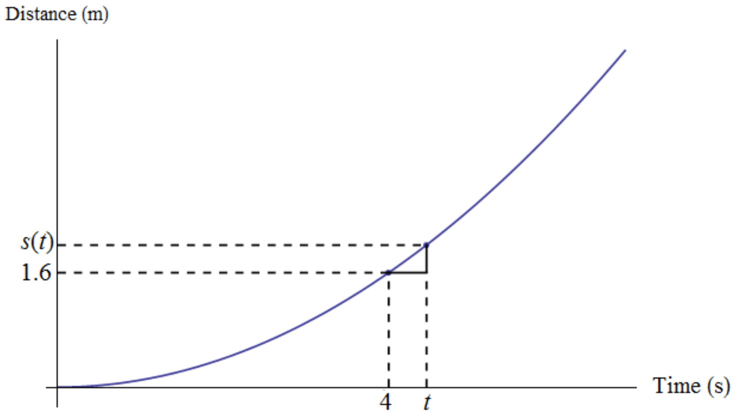


Figure 3.30 Finding the slope of the curve.

The teacher should introduce the term differentiation as a process of finding the slope of a line at a point, which was the speed at an instant *t* in this context. Students just learned that the slope of the secant line approached the slope of the tangent, which was also the slope of the line at the same point, as the time interval approaches zero. This concept of the derivative of a function at a point should be introduced to students, together with its definition.

Definition of the derivative of a function at a point
The derivative of a function $f(x)$ at $x = a$ is $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ provided the limit exists.

In this context, $x = t$ and $a = 4$. An equivalent expression for the derivative of $f(x)$ at $x = a$ is $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$. Students were asked to solve the following questions using both the numerical and algebraic approaches to familiarize themselves with the definition of the derivative of a function at a point.

i). Find the derivative at $x = 2$ of the function $f(x) = x^2 + 2x$ and show it on the graph.

ii). Find the speed of a car at $t = 5$ seconds if the distance (m) travelled by the car is given by the equation $s(t) = t^2 + 1$.

The derivative of $f(x)$ is 6 at $x = 2$ and the speed of the car is 10 m/s at $t = 5$ seconds.

Students were asked to generalize the definition of the derivative of a function at a point to define the derivative of a function using the context equation $s(t) = 0.1t^2$. The average speed during the time interval stating at $a = t_0$ to $x = t$ was given by $\frac{0.1t^2 - 0.1t_0^2}{t - t_0}$. To get the general formula for the derivative of the function, students needed to find $\lim_{t \to t_0} \frac{0.1(t^2 - t_0^2)}{(t - t_0)} = 0.1 \lim_{t \to t_0}(t + t_0) = 0.2t_0$. Since t_0 was arbitrary, the derivative of $0.1t^2$ is $02t$ which was the general form of the derivative of a function. This concept of the derivative of a function should be introduced to students, together with its definition.

Definition of the derivative of a function

A function $f(x)$ is differentiable if the derivative of $f(x)$ exists at every point in the domain of $f(x)$. The derivative of a differentiable function $f(x)$ is the map that sends every point in the domain of $f(x)$ to the derivative of $f(x)$ at that point.

The concept could be extended to find the derivative of a monomial using symbols and notations. Students were given some polynomials such as $(x^2 - x_0^2)$, $(x^3 - x_0^3)$, $(x^4 - x_0^4)$, $(x^5 - x_0^5)$, …, and asked to divide the given polynomials by $(x - x_0)$. Students should figure out the pattern as shown below.

$$\frac{(x^2 - x_0^2)}{(x - x_0)} = x + x_0$$

$$\frac{(x^3 - x_0^3)}{(x - x_0)} = x^2 + xx_0 + x_0^2$$

$$\frac{(x^4 - x_0^4)}{(x - x_0)} = x^3 + x^2x_0 + xx_0^2 + x_0^3$$

$$\frac{(x^5 - x_0^5)}{(x - x_0)} = x^4 + x^3x_0 + x^2x_0^2 + xx_0^3 + x_0^4$$

....

$$\frac{(x^n - x_0^n)}{(x - x_0)} = x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + ... + x^n x_0^{n-3} + xx_0^{n-2} + x_0^{n-1}$$

Taking the limits, they should find that

$$\lim_{x \to x_0} \frac{(x^n - x_0^n)}{(x - x_0)} = \lim_{x \to x_0}(x^{n-1} + x^{n-2}x_0 + x^{n-3}x_0^2 + ... + x^n x_0^{n-3} + xx_0^{n-2} + x_0^{n-1})$$

$$= nx_0^{n-1}.$$

Since x_0 was arbitrary, the derivative of a monomial x^n was nx^{n-1} which was the general form of differentiation.

Then students were asked to find the derivative of the $s(t) = 0.1t^2$ at t = 3.5 seconds using the general form of differentiation and cross check the answer with the one found using the graphical approach to calculate the slope at A using points B and C on the curve (see Figure 3.29(b)).

Students were asked to solve the following questions using both the numerical and algebraic approaches to familiarize themselves with the definition of the derivative of a function.

i). Find the derivative of $f(x) = 3x^2$ at $x = 1$.

ii). Graph the function $g(x) = \begin{cases} x+3, & x < -2 \\ (x+3)^2, & x \geq -2 \end{cases}$. Is $g(x)$ differentiable at $x = -2$?

The derivative of $f(x)$ is 6 at $x = 1$ and $g(x)$ is not differentiable at $x = -2$ since the left hand limit is not equal to the right hand limit, $\lim_{h \to 0} \frac{f(-2+h) - f(-2)}{h}$ does not exists.

Concept Application (60 minutes)

Students were asked to discuss in pairs and solve the following problems using the concepts they have learned from Concept Introduction I and II. The teacher moved around the class to ensure that students were able to solve the questions. The teacher should guide students by asking probing questions instead of giving direct answers.

i). Graph the function $f(x) = \begin{cases} 2x, & x < 1 \\ 2, & x \geq 1 \end{cases}$. Is $f(x)$ differentiable at x = 1?

ii). A ball thrown upward has height h in meters given by $h(t) = 10t - t^2$, where t is in seconds. What is the speed when t = 2 seconds?

iii). Find the derivative of the functions $g(x) = 3x + 1$ and $k(x) = 2x^2 + 2x$.

$f(x)$ is not differentiable at $x=1$. Since the left hand limit is not equal to the right hand limit, $\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$ does not exists. The speed of the ball is 6 m/s at $x=2$ seconds ($h'(t)=10-2t=10-2\times 2=6$ m/s). The derivative of $g(x)$ is 3 and that of $k(x)$ is $4x+2$.

3. 2.4 Integration (360 minutes)

In the integration learning unit, students were asked to find the area under the acceleration-time graph from the inclined plane activity and sketch the area on another graph in Exploration I, and introduced to the concept of definite integral in Concept Introduction I. Students were further asked to find and sketch the area under the speed-time graph in Exploration II, and the concept of integration were reiterated in Concept Introduction II to let them see the relationship between differentiation and integration. Finally, students solved problems by applying the concepts learned in Exploration and Concept Introduction in the Concept Application phase.

Exploration I (90 minutes)

Students already knew the slope of the speed-time graph from Concept Introduction of the differentiation learning unit. Students were asked to sketch the slope of the speed-time graph as shown in Figure 3.31. The slope of the speed-time graph represented the acceleration of the ball on the inclined plane.

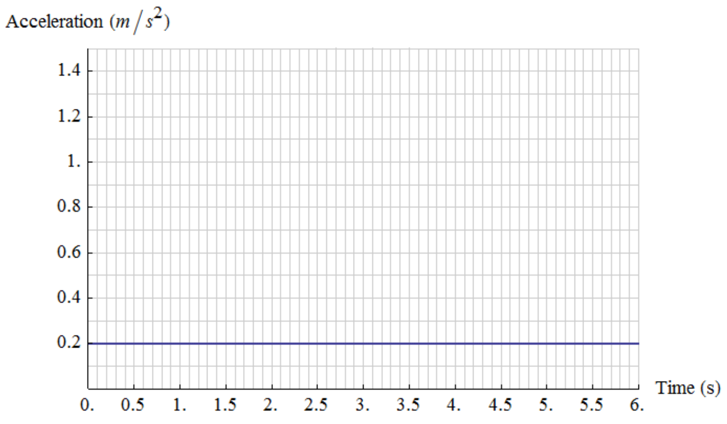


Figure 3.31 Acceleration-time graph.

1) Students divided the area under the acceleration-time graph in Figure 3.31 into six parts as shown in Figure 3.32.

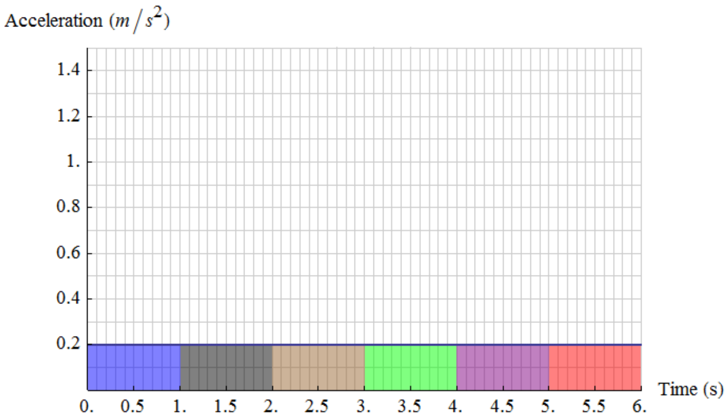


Figure 3.32 Acceleration-time graph showing the areas of six parts under the line.

2) Students were asked to find the area of each part and plot the accumulated area on another set of coordinates as shown in Figure 3.33.

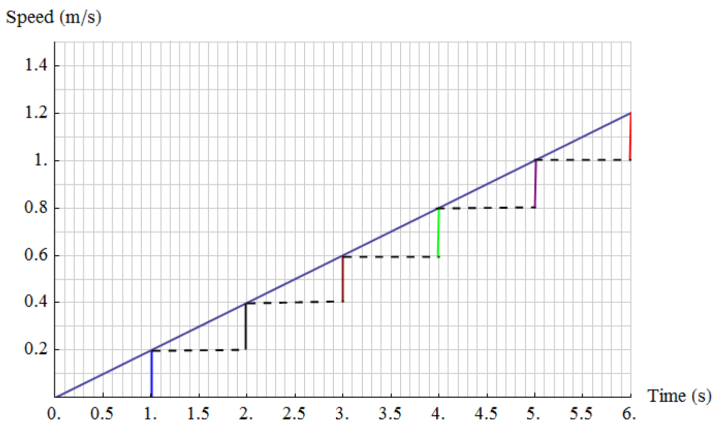


Figure 3.33 The accumulated area under the line of Figure 3.32.

When students found the area of each part under the line and sketch the accumulated area on another set of coordinates, they should figure out that it gave the speed-time graph. Moreover from the unit of the area (m/s), they should be able to figure out that it was the speed of the ball.

Concept Introduction I (60 minutes)

Students were asked to answer the questions regarding the activity in Exploration I before being introduced to the concept of integration.

i). What is the unit of the area when you find the area under the line?

ii). What does the unit of the area indicate?

The unit of the area under the line was m/s which indicated that the area was the speed of the ball on the inclined plane. The teacher could introduce the term definite integral as the area under the graph between a specific interval.

Informal definition of a definite integral

Given a function $f(x)$ defined in an interval [a, b]. The area bounded by $f(x)$, the x-axis, $x = a$, and $x = b$ is denoted by the definite integral $\int_a^b f(x)dx$.

In the definition, the $\int$ symbol is called the integral symbol, $f(x)$ is called the integrand, x (denoted by dx) is called the integration variable, a is the lower limit of the integral and b is the upper limit of the integral, and [a, b] is called the interval of integration.

In the previous example, let $a(t) = 0.2$ denote the acceleration function of the ball on the inclined plane. The teacher could give a couple of examples of the usage of this notation. For example, the total area under the acceleration-time graph in Figure 3.32 could be denoted by $\int_0^6 a(t)dt = \int_0^6 0.2dt = 6 \times 0.2 = 1.2$ and the area within a one-second interval from seconds 3 to 4 could be denoted by $\int_3^4 a(t)dt = 0.2$.

The teacher could then asked students to find the area under the graph from seconds 0 to t_0 where $0 < t_0 \leq 6$ and use the definite integral notation to represent the result $(\int_0^{t_0} a(t)dt = \int_0^{t_0} 0.2dt = 0.2t_0)$. Since t_0 was arbitrary, students should be able to generalize that the area under the graph from 0 to t was $0.2t$, which was the equation of the line in Figure 3.33 and also of the speed-time graph found in Concept Introduction of the differentiation learning unit.

Exploration II (90 minutes)

To let students work on another example of integration and see the relationship between differentiation and integration, students were asked to find the area under the speed-time graph and sketch the area on another graph.

1) Students divided the area under the speed-time graph into six parts (Figure 3.34).

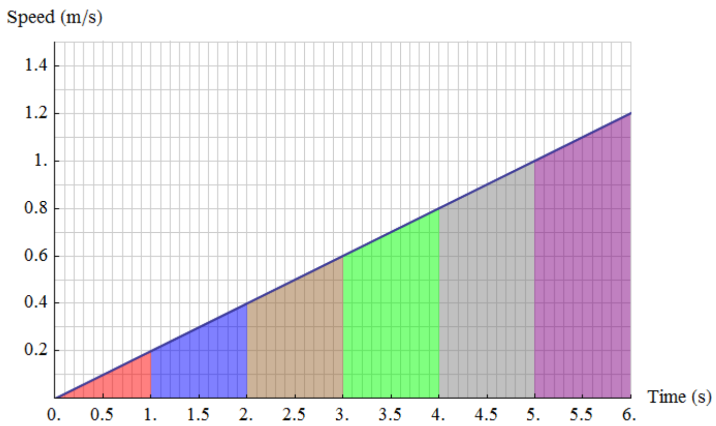


Figure 3.34 The speed-time graph showing the area under the line.

2) Students find the area of each part and simultaneously sketch the area of each part on another set of coordinates as shown in Figure 3.35–3.40.

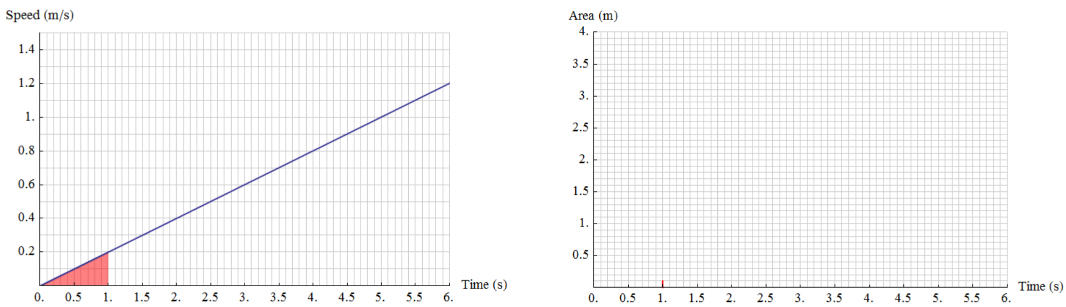


Figure 3.35 Area under the line and distance during the first hour.

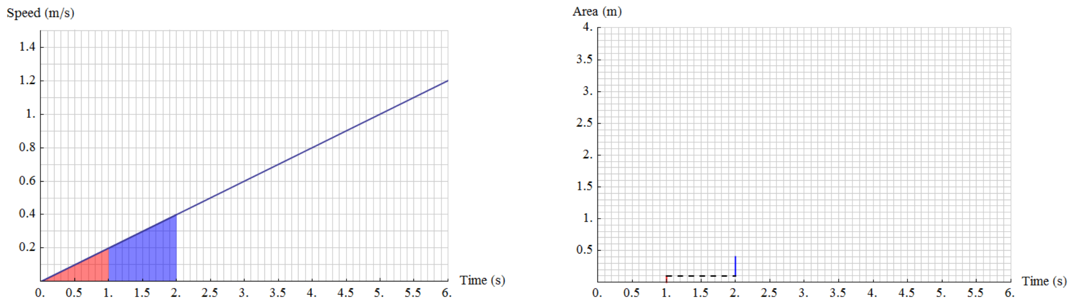


Figure 3.36 Area under the line and distance during the second hour.

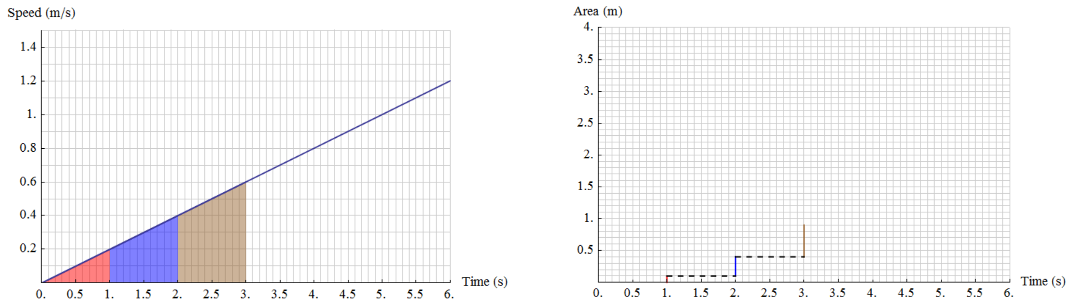


Figure 3.37 Area under the line and distance during the third hour.

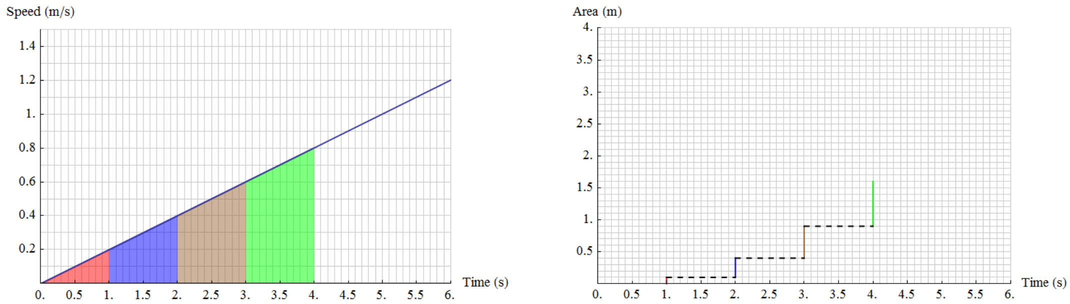


Figure 3.38 Area under the line and distance during the fourth hour.

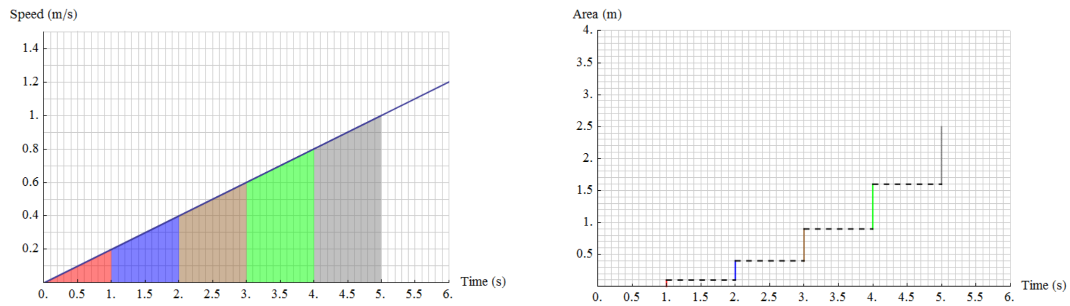


Figure 3.39 Area under the line and distance during the fifth hour.

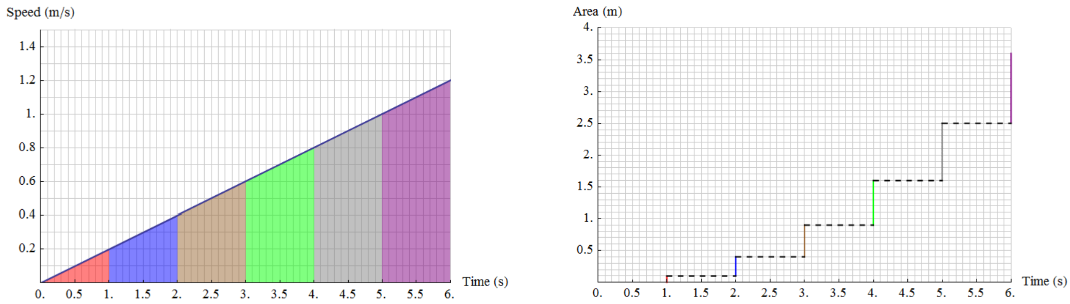


Figure 3.40 Area under the line and distance during the sixth hour.

Students should discover that the area under the speed-time graph was the distance covered by the ball in 6 seconds on the inclined plane. The area of each part indicated the distance travelled by the ball from 0 to 1, 1 to 2, 2 to 3, 3 to 4, 4 to 5, and 5 to 6 seconds respectively on the distance-time graph. They were then asked to find the area under the speed-time graph from 0 to t_0 seconds and represent the result with the definite integral notation $\int_0^{t_0} v(t)dt = \int_0^{t_0} 0.2tdt = t_0 \times 0.2t_0 / 2 = 0.1t_0^2$ (see Figure 3.41).

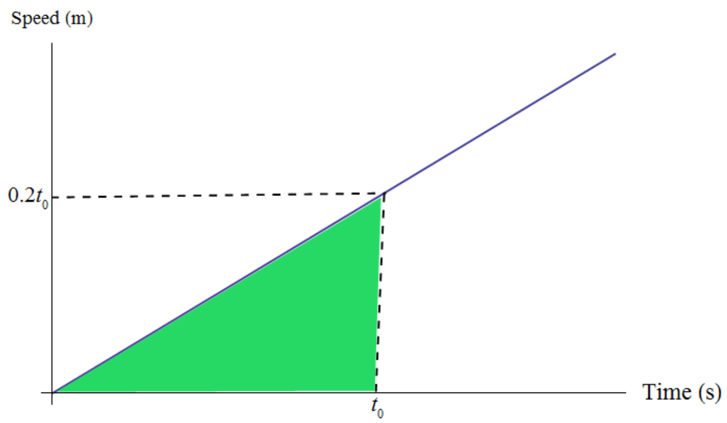


Figure 3.41 Finding the area under the line.

The result could then be generalized to find the equation of the distance-time graph. Students should be able to confirm that the resulting equation passed through all the points on the distance-time graph as shown in Figure 3.42 and coincided with the original distance-time relationship.

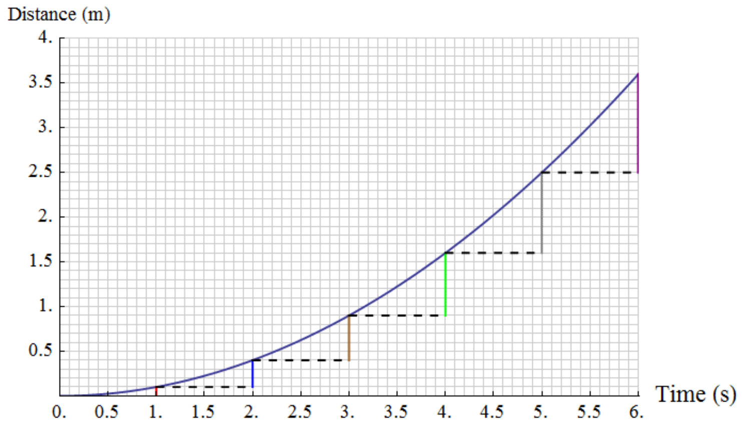


Figure 3.42 The distance-time graph.

Concept Introduction II (60 minutes)

From Exploration II, students should begin to have an idea about the relationship between differentiation and integration. This phase should help them formulate the idea more completely. Students were asked the following questions.

i) What do you get when you integrate the equation of the speed-time graph *v*(*t*) from 0 to t_0 ?

ii) What is the unit of the area in Figure 3.40? And what does the unit of the area tell you?

Students should be able to see that integrating the equation *v*(*t*) would give $0.1t_0^2$ and the unit of the area was in m (meters) which would indicate the speed of the ball in the context of the inclined plane activity. Then, students were also asked the following questions.

iii) What do you get when you find the slope of the equation on the distance-time graph $s(t) = 0.1t^2$?

iv) What was the unit of the slope in Figure 3.28? And what does the unit of the slope tell you?

Students should be able to see that finding the slope of the distance-time equation would give 0.2*t*. The unit of the slope in Figure 3.28 was m/s which indicate the speed of the ball.

Then, students were asked "do you see any relationship between the graphs in Figures 3.40 and 3.28 in terms of differentiation and integration in calculus? Explain."

Now, students should be able to see the relationship between differentiation and integration graphically, algebraically, and contextually from the activity and to conceptually understand that integration is the inverse process of differentiation. In order to introduce them to the algebraic method of using an anti-derivative to evaluate a definite integral, they were introduced to the following definition.

Definition of an indefinite integral

Given a function $f(x)$, an anti-derivative or an indefinite integral of $f(x)$ is any function $g(x)$ which is differentiable in the domain of $f(x)$ and whose derivative is $f(x)$.

Students were then asked the following questions.

i) In the inclined plane activity, what is an anti-derivative of $a(t) = 0.2$?

ii) What is an anti-derivative of $v(t) = 0.2t$?

They should be able to figure out that an anti-derivative of $a(t)$ was $v(t)$ and an anti-derivative of $v(t)$ was $s(t) = 0.1t^2$. To let them see the connection between an anti-derivative and a definite integral, the following questions were posed.

iii) What is the relationship between the definite integral $\int_2^4 a(t)dt$ and $v(t)$, an anti-derivative of $a(t)$?

iv) What is the relationship between the definite integral $\int_3^6 v(t)dt$ and $s(t)$, an anti-derivative of $v(t)$?

v) In general, if $g(x)$ is an anti-derivative of $f(x)$, how can we use $g(x)$ to evaluate the definite integral $\int_a^b f(x)dx$?

The teacher should ensure that students could see that $\int_2^4 a(t)dt = v(4) - v(2)$ and $\int_3^6 v(t)dt = s(6) - s(3)$, and came to the conclusion that, in general, $\int_a^b f(x)dx = g(b) - g(a)$.

Concept Application (60 minutes)

Students were asked to solve the following problems based on concepts learned in the Exploration and Concept Introduction phases.

i) Sketch the graph of $y = 3x^2$ and find the area under the curve $y = 3x^2$ between $x = 0$ and $x = 3$ on the *x*-axis.

ii) Evaluate the following integral. (a). $\int_0^3 2dx$. (b). $\int_0^2 2xdx$.

iii) A car is travelling 5 km/min on a smooth straight road for 5 minutes.

a) How far has a car travelled?

b) Represent the speed and distance travelled by a car on the graph

The area under the curve $y = 3x^2$ is 27 unit^2 between $x = 0$ and $x = 4$ the *x*-axis as shown in Figure 3.43. The car has travelled 25 km in 5 hours and students need to sketch the speed and distance travelled by the car on the graph as shown in Figure 3.44. This question is to let students see the relationship between differentiation and integration.

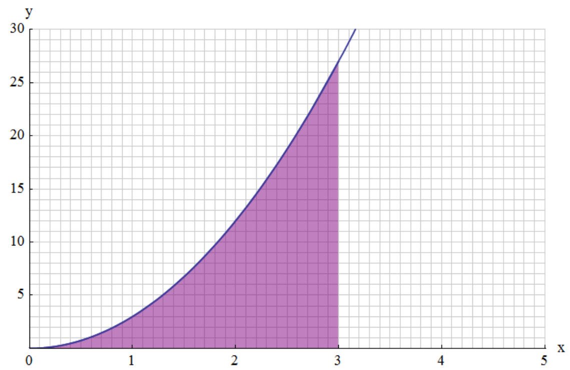


Figure 3.43 The area under the curve $y = 3x^2$.

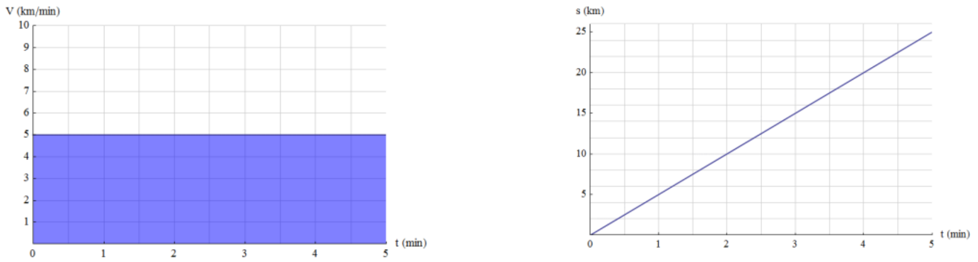


Figure 3.44 The speed of a car at 5km/min and the distance travelled by a car in 5 minutes.

3.3 Research Instruments

There are three research instruments used in this study. They are conceptual understanding tests, questionnaire, and semi-structured interview. The research tools are designed to answer the research questions as shown in Table 3.5.

Table 3.5 Research tools used to answer the research questions.

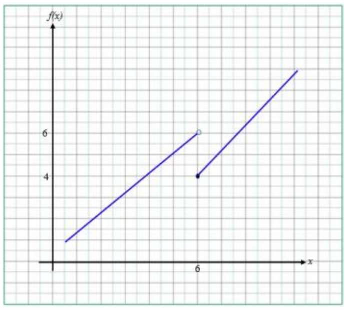
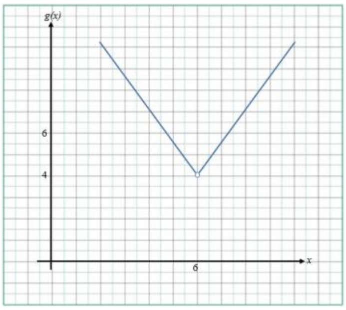
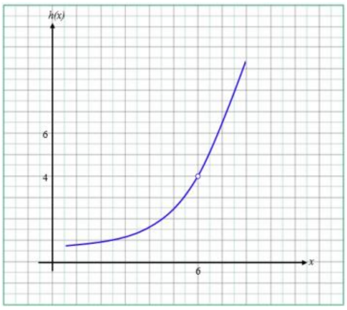
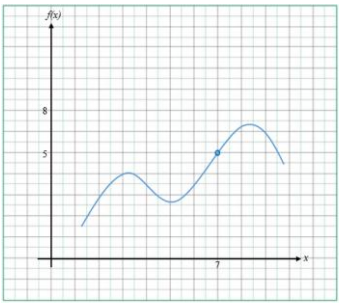
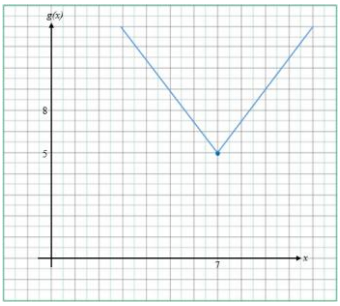
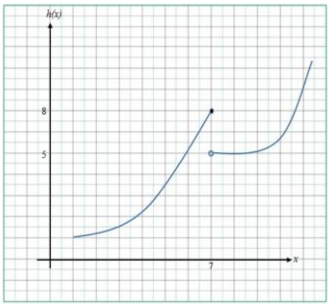
Research questions	Research tools
1) To what extent can the developed learning units enhance the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration?	• Conceptual understanding tests • Semi-structured interview
2) To what extent are the developed learning units more effective than the conventional instruction in improving students' understanding of the fundamentals of calculus and the relationship between differentiation and integration?	• Conceptual understanding tests
3) What are the students' attitudes towards the developed learning units?	• Questionnaire • Semi-structured interview

The pre-test and post-test were used to answer the first research question, and the semi-structured interview was also conducted for determining the students' deeper understanding of the fundamentals of calculus as well. To address the second research question, the scores of the pre-test and the post-test of the control and experimental groups were compared and analyzed. Finally, the questionnaire and interview were also used to find the students' attitudes towards the learning units.

3.3.1 Conceptual understanding tests

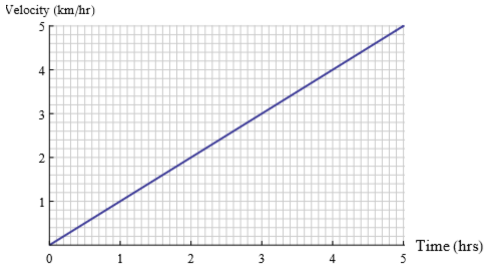
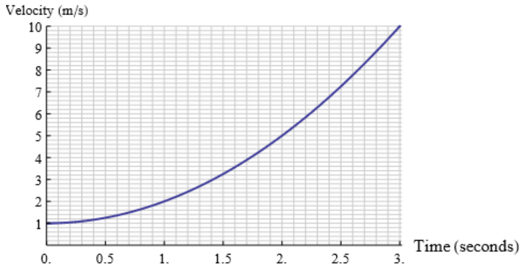
The conceptual understanding tests, the pre-test and the post-test, were used to evaluate the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration. The pre-test and post-test consist of five parallel open-ended questions. The pre-test was used to assess students' prior knowledge while the post-test was used to follow the enhancement of students' understanding of the fundamentals of calculus. The pre-test and post-test were conducted both in the experimental and control groups. Each pair of pre-test and post-test question is described in Table 3.6–3.10.

Table 3.6 The pre-test and post-test question 1.

Pretest
Below are the graphs of three functions $f(x)$, $g(x)$, and $h(x)$. Which of the following functions has the limit at 6? Why and why not?
  
Post-test
Below are the graphs of three functions $f(x)$, $g(x)$, and $h(x)$. Which of the following functions has the limit at 7? Why and why not?
  

Question 1s of the pre-test and post-test aimed to examine students' conceptual understanding of limits. The graphs have been changed in the post-test compared to those in the pre-test but their characteristics are similar. For the pre-test, the limits exist for the functions $g(x)$ and $h(x)$ at $x = 6$, but the limit of the function $f(x)$ does not exist at $x = 6$. Similarly for the post-test, the limits of the functions $f(x)$ and $g(x)$ exist at 7 and the limit of $h(x)$ does not exist at 7.

Table 3.7 The pre-test and post-test question 2.

<table>
<tr><th>Pre-test</th><th>Post-test</th></tr>
<tr><td>The graph below represents the relationship between the velocity and time of a moving object.

a) Find the derivative at $t = 3.5$ hours graphically from the velocity-time graph?
b) What does the derivative represent?</td><td>The graph below represents the relationship between the velocity and time of a moving object.

a) Find the derivative at $t = 2$ seconds graphically from the velocity-time graph?
b) What does the derivative represent?</td></tr>
</table>

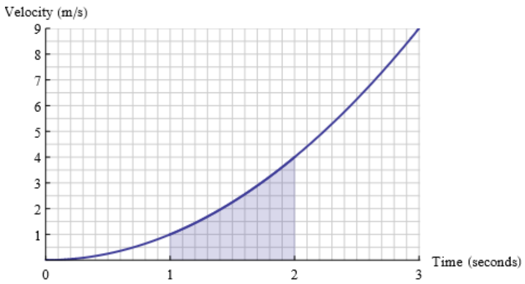
Question 2s of the pre-test and post-test were used to examine students' conceptual understanding of differentiation. In the pre-test, the derivative at $t = 3.5$ hours is 1 km/hr^2. Similarly in the post-test, the derivative at $t = 2$ seconds is 4 m/s^2.

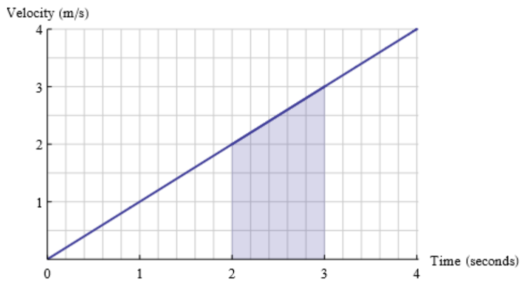
Table 3.8 The pre-test and post-test question 3.

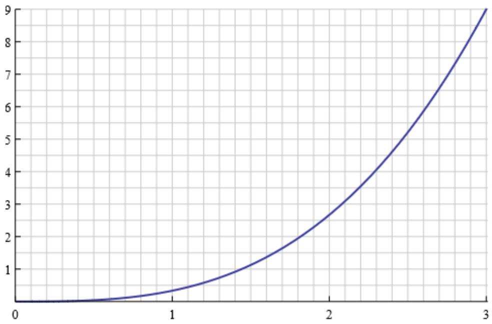
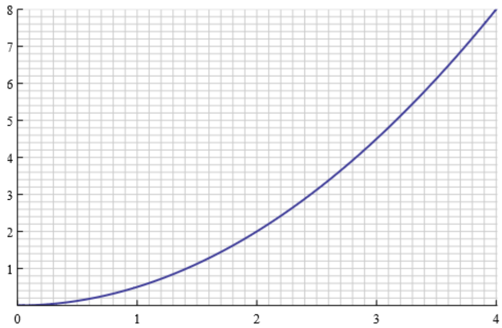
Pre-test	Post-test
Evaluate $\int_0^4 \lvert x-2 \rvert dx$ using the graph of $\lvert x-2 \rvert$?	Evaluate $\int_0^8 \lvert x-4 \rvert dx$ using the graph of $\lvert x-4 \rvert$?

Question 3s of the pre-test and post-test were used to examine students' conceptual understanding of integration. The definite integral of $\lvert x-2 \rvert$ from 0 to 4 would be 4 units^2. Similarly, 16 units^2 is the value of the definite integral of $\lvert x-4 \rvert$ from 0 to 8.

Table 3.9 The pre-test and post-test question 4.

<table>
<tr><th>Pre-test</th><th>Post-test</th></tr>
<tr><td>Graph 1.2 is an anti-derivative of Graph 1.1. Graph 1.1 is a velocity-time graph.</td><td>Graph 1.2 is an anti-derivative of Graph 1.1. Graph 1.1 is a velocity-time graph.</td></tr>
<tr><td>

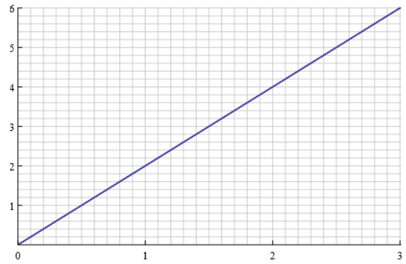
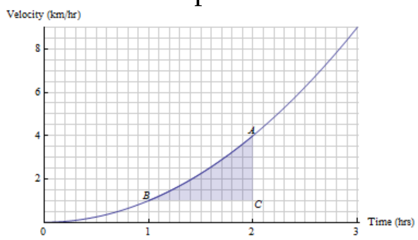
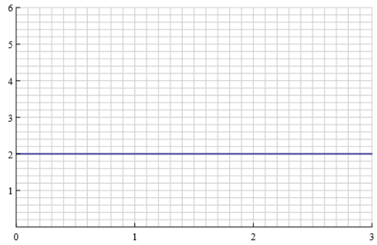
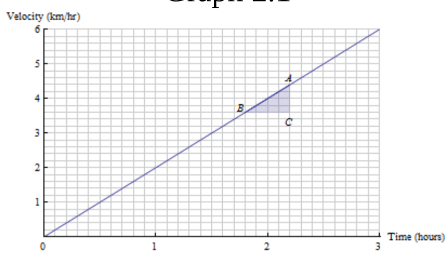
Graph 1.1</td><td>

Graph 1.1</td></tr>
<tr><td>
Graph 1.2</td><td>
Graph 1.2</td></tr>
<tr><td>a) What is the area of the shaded region under the line on Graph 1.1?
b) What is the unit of area of the shaded region under the line?
c) Show on Graph 1.2 the area of the shaded region under the line on Graph 1.1? What does it indicate on Graph 1.2?
d) What would be the unit of the slope of the line on Graph 1.2?
e) Label the unit of the axes on Graph 1.2 in relation to Graph 1.1?</td><td>a) What is the area of the shaded region under the line on Graph 1.1?
b) What is the unit of area of the shaded region under the line?
c) Show on Graph 1.2 the area of the shaded region under the line on Graph 1.1? What does it indicate?
d) What would be the unit of the slope of the line on Graph 1.2?
e) Label the unit of the axes on Graph 1.2 in relation to Graph 1.1?</td></tr>
</table>

Question 4s aimed to find out graphically and contextually how students related the integrals (areas under the lines) to the derivatives of the lines, and also to examine whether students could establish the relationship from integration to differentiation.

In the pre-test, the area of the shaded region of 2.33 m on Graph 1.1 represents the distance from $t=1$ to $t=2$ seconds on (the distance-time) Graph 1.2. Similarly in the post-test, the area of the shaded region of 2.5 m on Graph 1.1 represents the distance from $t=1$ to $t=2$ seconds on Graph 1.2.

Table 3.10 The pre-test and post-test question 5.

Pre-test	**Post-test**
Graph 2.1 is the derivative of Graph 2.2 which is a velocity-time graph.  Graph 2.1  Graph 2.2 a) What does the height *AC* of the triangle *ABC* on Graph 2.2 represent on the Graph 2.1? Show it on Graph 2.1 by shading it? b) What is the unit of the area you have just shaded on Graph 2.1? c) What is the unit of the slope of the line on Graph 2.2? d) Label the units of the axes on Graph 2.1 in relation to Graph 2.2.	Graph 2.1 is the derivative of Graph 2.2 which is a velocity-time graph.  Graph 2.1  Graph 2.2 a) What does the height *AC* of the triangle *ABC* on Graph 2.2 represent on Graph 2.1? Show it on Graph 2.1 by shading it? b) What is the unit of the area you have just shaded on Graph 2.1? c) What is the unit of the slope of the line on Graph 2.2? d) Label the units of the axes on Graph 2.1 in relation to Graph 2.2.

Question 5s aimed to find out graphically and contextually how students relate the derivatives (slopes of the lines) to the integrals (areas under the lines) of the lines, and also to examine whether students could establish the relationship from differentiation to integration. In the pre-test, the height AC of the triangle ABC on Graph 2.2 represents the area under the line (velocity in km/hr) from $t = 1$ to $t = 2$ hours on Graph 1.2 which is the acceleration-time graph. Similarly; in the post-test, the height AC of the triangle ABC on Graph 2.2 represents the area under the line (velocity in km/hr) from $t = 1.8$ to $t = 2.2$ hours on Graph 1.2 which is the acceleration-time graph.

3.3.2 Questionnaire

The questionnaire was developed to assess students' attitude towards the FC learning units. The questionnaire was administered to the students in the experimental group after the post-test. The questionnaire consists of 12 closed-ended Likert-type items and 3 open-end questions. To examine the effectiveness of the learning units, the 12 closed-ended Likert-type items were categorized into three themes as shown in Table 3.11.

Three activities in the learning units includes: (1) measuring the circumference of a circle using a rope, (2) the inclined plane, and (3) graphing activities. The first theme assessed whether the activities in the learning units were interesting and enriching for students, and also to find the effectiveness of the learning cycle approach compared to traditional teaching. The second theme on time aimed to find whether the allocated time was enough for the developed learning units. Third theme on teacher assessed whether the teacher was approachable and enthusiastic during the lesson, and teaching the lesson according to the level of students.

The three open-ended questions were to find out students' reason when they liked or disliked the activities in the learning units, and how to improve the activities if they disliked the activities.

Table 3.11 Categorization of questionnaire items into three themes.

Themes	**Sl. No**	**Items**
	1.	I like the activities in the calculus lessons.
	2.	I found the activities in the calculus lessons interesting.
	3.	It was too difficult to learn calculus by doing these activities.

Table 3.11 Categorization of questionnaire items into three themes (cont.)

Themes	Sl. No	Items
Activities in the learning units	4.	I find reading the textbook in detail is by itself sufficient for me to learn calculus.
	5.	I learn calculus better by reflecting on these activities instead of only by reading books and memorizing.
	6.	I look forward to solve more problems in calculus after the activities.
	7.	I will understand better if other topics in mathematics are taught using activities like the ones used in these calculus lessons.
Time	8.	The time was too short for the lessons.
	9.	The calculus lessons need more exercises until I understand and become fluent.
Teacher	10.	The mathematics teacher teaching calculus is enthusiastic in teaching calculus.
	11.	The mathematics teacher teaching calculus is encouraging and approachable during the lessons.
	12.	The mathematics teacher teaching calculus taught the lessons too fast and I could not follow the instruction.

3.3.3 Semi-structured interview

In this study, the semi-structured interview (see Appendix C) was used to find the in-depth information regarding the students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration, and also the students' attitudes towards the FC learning units. The objectives of the interview questions are shown in Table 3.12. Students in the experimental group were asked to volunteer for the interview. Since all students volunteered, only six were selected based on their mathematics performance in the mid-term of academic year 2013. Two high, two middle, and two low scoring students were selected. All dialogues in the interviewing process were audio recorded.

Table 3.12 Objectives of interview questions.

Question	Objectives
1	To find the students' attitudes towards the learning units.
2	To evaluate the students' conceptual understanding of limits.
3	To evaluate the students' conceptual understanding of differentiation and integration, and to find how students establish the relationship between differentiation and integration.

Question 1 asked the students about their attitudes towards the learning units, i.e. the learning units' interestingness or usefulness in developing the concepts, and also comments and suggestions for the improvement of the learning units. In question 2, the students' conceptual understanding of limits was determined again to cross check with the results from the conceptual understanding test. The students' conceptual understanding of differentiation and integration and how they establish the relationship between differentiation and integration were re-examined in question 3.

3.3.4 Content validity test (IOC index)

The content validity of the conceptual understanding tests was evaluated by using the Index of the Item-Objective Congruence (IOC) (Rovinelli & Hambleton, 1977; Turner & Carlson, 2003). The simplified IOC index can be calculated by using the formula

$$\mathrm{IOC}_k = \frac{\sum_{i=1}^{N} \frac{R_{ik} - (-1)}{2}}{N}, \text{ where}$$

IOC_k is the Index of the Item-Objective Congruence of item *k*,

R_{ik} is the score of item *k* from content expert *i*, and

N is the number of content experts.

In this study, four experts ($N = 4$) include two mathematics lecturers in the College of Education in Bhutan with more than 10 years of experiences in teaching calculus to pre-service teachers and two teachers from school A who had more than 15 years of experiences in teaching the introductory calculus course in the higher secondary level. The constructed table used by each expert during the item validation is shown in

Appendices E and F. Each expert assessed the agreement of each item with the stated objective for the item, and marked agree (+1 point) if the item agreed with its stated objective, not sure (0 point), or disagree (−1 point) if the item disagreed with its objective. The average of all the scores from the experts for one item was calculated.

For the interpretation, based on the suggestions of the research of Rovinelli and Hambleton (1977) and Turner and Carlson (2003), if there are four content experts, an item is acceptable if at least three out of four experts mark agree (+1 point) and the other mark disagree (−1 point) for the item and its objective. Obviously, at least 75% agreement of the content experts is required for an acceptable item. Therefore, the item with Index of Item-Objective Congruence $\geq$ 0.75 matches its stated objective. The average IOC for the conceptual tests was 0.89 and that for the questionnaire was 1.0, indicating that both were appropriate for this study.

3.4. Research Design

In order to address the research questions, the researcher employed quantitative and qualitative methods. The detail outline of the research design is shown in Figure 3.45.

The pre-test was conducted for both groups, and then one group was categorized as the experimental group while the other as the control group. The FC learning units were implemented to the students in the experimental group while the conventional teacher-centered and textbook-oriented instruction was used for the control group. Then, the post-test was conducted for both the groups after the intervention of the learning units. Six students were selected from the experimental group to be interviewed for determining their deeper understanding of the fundamentals of calculus.

The six students were selected based on their performance in the mathematics mid-term of the academic year 2013. Two students were high achievers, two students were middle achievers, and two students were low achievers in mathematics during the mid-term of the school. The questionnaire and interview were employed to investigate the students' attitudes towards the FC learning units.

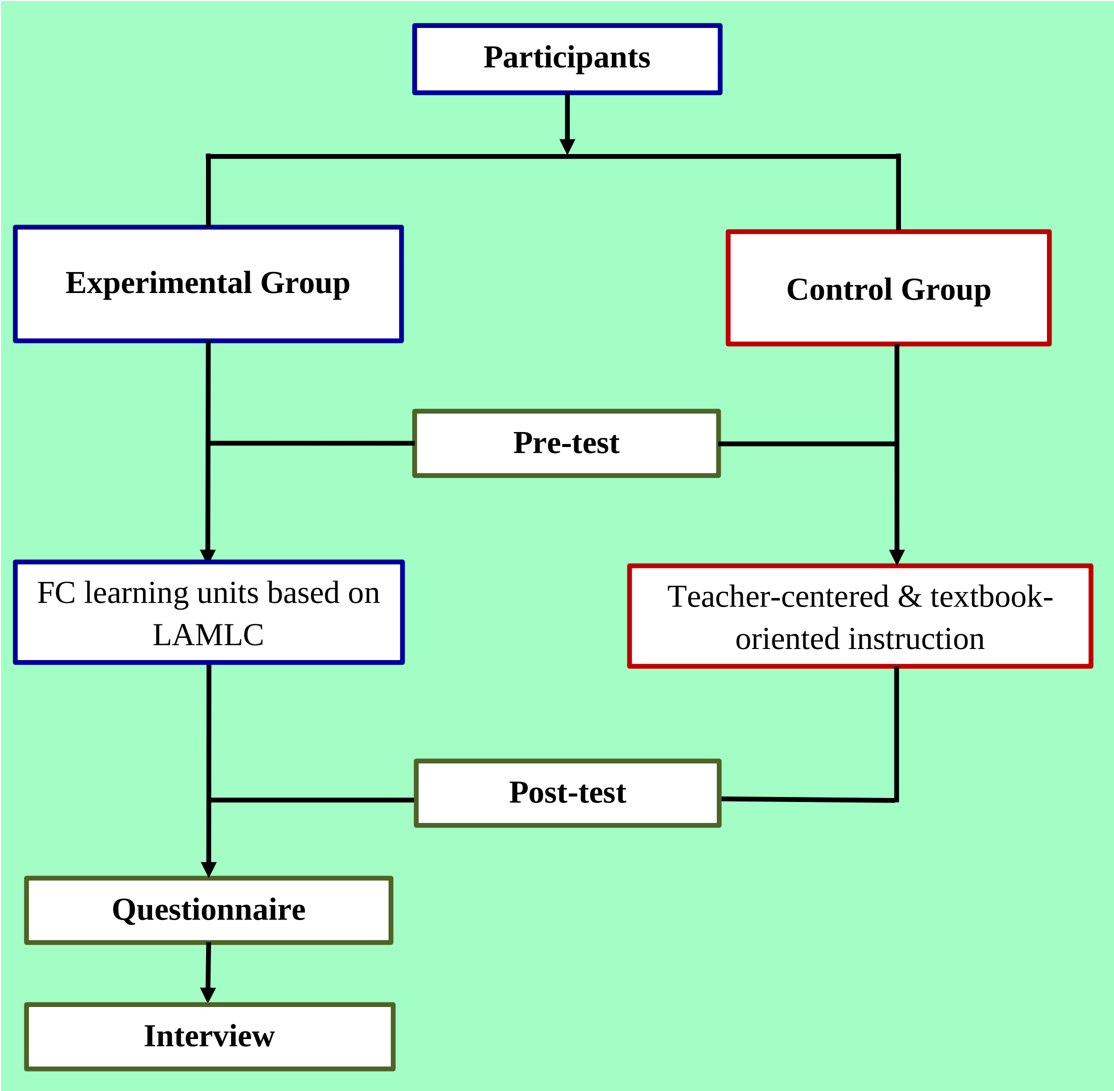


Figure 3.45 The outline of the research design for this study.

3.5 Implementation of the Fundamentals of Calculus (FC) Learning Units

To investigate the students' understanding of the fundamentals of calculus, the FC learning units were implemented to grade-11 students in one of the higher secondary schools in Bhutan following the normal schedule of the school. Each period lasted 60 minutes. In one week, there were 7 mathematics classes for the total of 420 minutes per week. The details of the implementation of the FC learning units are shown in Table 3.13.

Table 3.13 Time duration of each learning unit.

Topics	Duration (minutes)
Limits	240
Continuity of a function	120
Differentiation	240
Integration	360
Total	**960**

3.5.1 Population for the study

According to Policy and Planning Division (2013) statistical report, there were 53 higher secondary schools in Bhutan with 2,432 students enrolling in the grade-11 science stream (991 girls and 1,441 boys), 2,820 students (1,552 girls and 1,268 boys) in the arts stream, and 2,595 students (1,327 girls and 1,291 boys) in the commerce stream. All the schools throughout Bhutan followed the same syllabus and textbooks issued by DCRD. Almost all the teachers in the country followed the guidelines provided in the teacher's manuals to teach mathematics, and the grade-10 and grade-12 teachers followed the teaching manuals especially strictly as there were national examinations (board exams) at the end of the academic year for these two grade levels. Therefore, teaching methods used in teaching mathematics were almost uniform throughout the country. To compare the teaching methods applied in the Bhutanese context with the learning cycle approach, two schools in one district of Bhutan were selected for this study.

Based on the research design of the study, the sample size was calculated using the formula for the case of comparing the difference between two independent means (Δ) (Allen, 2011);

$$n\,/\,\text{group} = \frac{2\left(z_{\alpha/2} + z_{\beta}\right)^2}{\left(\frac{\sigma}{\Delta}\right)^2}\text{, where}$$

n = sample size,

$\sigma =$ standard deviation of the difference of mean (Δ), and

$\Delta =$ difference in mean between the two groups.

Using the above formula for a two-tailed test at $\alpha = 0.05$, $\beta = 0.2$, and the effect size of 0.5, the minimum sample size would be 63 students per group. In this study, 65 students each in the experimental and the control groups were selected.

3.5.2 Samples

The research sites were located in Samtse district in the south-west region of Bhutan as shown in Figure 3.46. The district had three higher secondary schools, two of which (school A and B in Figure 3.46) were selected as the research sites. The research study was carried out in Samtse district because it housed the home school (school A) of the researcher which made it convenient to arrange classes. Since school A was the home school of the researcher, the grade-11 science classroom from this school was chosen as the experimental group. School A had class levels 9 to 12 and school B had class levels 7 to 12 in school B. School A offered the science and commerce streams to students in grades 11 and 12 while school B offered only the science stream to students in grades 11 and 12. The focus of this study was grade-11 students studying science and mathematics as major subjects. Grade-11 students were chosen because the introductory calculus course began in grade 11.

The students in both schools had learned functions before the study was conducted. Both the schools followed the contents and syllabus issued by DCRD to teach functions. Moreover, students in grade 11 in both schools should have the same background in mathematics as they had followed the same mathematics curriculum issued by DCRD throughout their school years.

There were 65 students in school A and 70 students in school B. All the students in school A volunteered to participate in the study while only 65 students in school B did so. The five non-participating students were biological science students who were not required to study calculus. The medium of instruction was English as it was the main medium in the Bhutanese educational system. The participants were of mixed ability and gender. The details of the students who participated in this study are shown in Table 3.14.

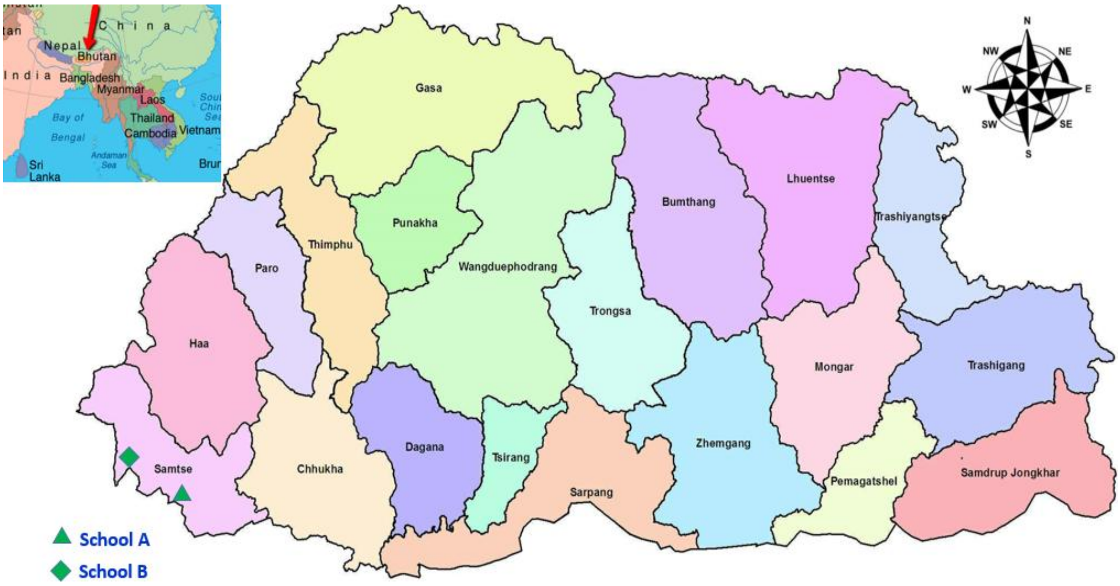


Figure 3.46 Map of Bhutan showing the district of research and the location of the schools in green shapes. (http://www.bhutanwonderfultravels.com/bhutan/about-bhutan/)

Table 3.14 Details of the students in the two schools.

School	Type	Section	Gender	
			Boys	Girls
A	Higher Secondary School	I	20	13
(Experimental group)	(Grade 9-12)	II	11	21
B	Higher Secondary School	III	22	11
(Control group)	(Grade 7-12)	IV	19	13

The teacher from school B who taught the students in the control group had more than 3 years of experience in teaching mathematics for grade 11. The researcher who taught the students in the experimental group had more than 3 years of experience in teaching mathematics for grade 11.

Table 3.15 shows the contents of the fundamentals of calculus taught in the control and the experimental groups using the textbook and learning cycle approach respectively. These contents were based on the syllabus issued by DCRD which was followed uniformly throughout Bhutan.

Table 3.15 Comparison of the contents of fundamentals of calculus taught in the control and the experimental group.

Control group	Experimental group
I. Limits 1) Notion and meaning of limits: Stated that the area of a polygon approached the area of a circle and used the number line to explain the meaning of $x \to a$. 2) The limits of a function: Used the numerical approach for a function such as $y = \frac{x^2-1}{x-1}$ and gave the definition of the limits of a function directly, and also introduced right hand and left hand limits using the numerical approach for functions such as $f(x) = \frac{\lvert x \rvert}{x}, x \neq 0$ and $f(x) = \lvert x \rvert,\ x \to 2$. 3) Algebraic rules of limits: Finding limits algebraically by the direct substitution and factorization methods.	**I. Limits** 1) Used an activity to measure the circumference of a circle by increasing the number of fixed points per rope and used the numerical and graphical approaches to give students the intuitive idea of limits. 2) Used the numerical and graphical approaches to find limits. Used probing questions to convey the validity of the factorization method. Gave the formal definition of limits and show the graphical meaning of the definition. 3) Questions were provided to apply the learned concepts.
II. Continuity of functions 1) The functions such as $y = 4x$ and $y = \begin{cases} = 3 \text{ for } x > 0 \\ = 1 \text{ for } x \leq 0 \end{cases}$ were used with graphs to give an intuitive idea of continuity of a function. 2) Definition of continuity with conditions and removal of discontinuity of a function.	**II. Continuity of functions** 1) The numerical and graphical approaches were used to find the continuity of functions. 2) The definition and conditions of continuity were defined. Series of activities and questions were provided to apply the learned concepts.

Table 3.15 Comparison of the contents of fundamentals of calculus taught in the control and the experimental group (cont.).

Control group	Experimental group
E.g. $f(x)=\begin{cases} =\dfrac{x^2-1}{x-1} & \text{when } x\neq 1 \\ =2 & \text{when } x=1 \end{cases}$.	
III. Differentiation 1) Meaning and geometrical interpretation of derivatives: Used the equation $y + dy = (x + dx)^2$ to interpret the geometrical meaning of dy/dx followed by definition of the derivative of a function and the derivative of a function at a point. 2) Differentiation from the definition: Used the algebraic approach to find the derivative of functions such as $y = x^3$, $y=\sqrt{x}$, etc. 3) Algebraic rules for derivative: Finding the derivatives of few functions using $\frac{d}{dx}(x^n)=nx^{n-1}$ together with the algebraic rules. 4) Gave the formulas for the derivatives of some standard functions.	**III. Differentiation** 1) Inclined plane activity was used to give the contextual meaning of derivative. Used graphs to find slopes and related to the algebraic approach of finding the derivative. 2) Based on contextual and graphical activities, the definition of the derivative of a function at a point was defined followed by that of the derivative of a function. 3) Questions were provided to apply the learned concepts.
IV. Integration 1) Introduced the standard form of integration: $\int x^n dx=\frac{x^{n+1}}{n+1}+c,\ n\neq -1$ and provided integration formulas comparable to differentiation formulas. 2) Integration as the inverse of differentiation in terms of the formulas.	**IV. Integration** 1) Used the context of the inclined plane activity and graphs to give the concepts of integration by finding the area under the graph and related to the definite integral. Then generalized the definite integral into the algebraic form and defined an indefinite integral.

Table 3.15 Comparison of the contents of fundamentals of calculus taught in the control and the experimental group (cont.).

Control group	Experimental group
3) Solve indefinite integrals such as $\int x^6 dx$, $\int \frac{1}{x^7} dx$, etc. followed by integration by simple substitution for simple polynomial functions.	2) Graphs were used to establish the relationship between differentiation and integration in the context of inclined activity and in algebraic context.

The contents taught in the control group were specifically selected from the syllabus of grade-11 mathematics in parallel with the FC learning units. However, some portions of the syllabus were left out during the research study and continued after the research study. The details of the syllabus is mentioned in chapter 2 (p. 25). The contents were basically taught following the textbook using mostly the algebraic method and the questions provided at the end of each topic were assigned as the exercise.

3.6 Data Collection

To collect the data for this study, the researcher visited the two higher secondary schools in Samtse district. In the process of data collection, administrative and ethical procedures were strictly followed to get the approval to conduct the study with the participants. Firstly, the researcher sought and got the approval from Mahidol University Institutional Review Board (MU-IRB) to carry out the data collection process. Then the approval letter issued by the Faculty of Graduate Study of the university was used to get the approval letter from Royal Civil Service Commission of Bhutan (RCSC). With the written approval letter from RCSC, the researcher approached the principals of the two schools to get the permission to conduct the research study in their schools. The principals, teachers concerned, and participants were explained about the purpose of the study. Confidentiality was maintained throughout the process of the research as well as after the research.

The pre-test was conducted for one hour to grade-11 science students of both the schools (the control group and the experimental group) simultaneously with help from concerned subject teachers of the schools. The developed learning units were implemented in school A (experimental group) for about 960 minutes (16 periods) and simultaneously in the school B, the conventional teacher-centered and textbook-oriented instruction was used to teach the fundamentals of calculus. After the implementation of the learning units, the post-test was conducted for one hour in both schools. Then the questionnaire was conducted for one hour for the experimental group followed by the interview of six volunteered students from the experimental group. The six students were asked for written consent before the interview was carried out.

3.7 Data Analysis

The collected data from the conceptual understanding tests, questionnaire and interview were analyzed both quantitatively and qualitatively. The results were combined together to answer the research questions. Here, the obtained data were analyzed from each instrument as described below.

3.7.1 Conceptual understanding tests

The researcher developed the marking criteria for the pre-test and post-test to evaluate students' scores (see Appendix A and B). The inferential *t* statistics and corresponding non-parametric statistics were used to analyze the students' pre-test and post-test scores within the groups and between the groups. Cohen's *d* (1988, 1992) was also used to find the effect size between the two groups.

Effect size (Cohen's d)

An effect size is a measure that describes the magnitude of the difference between two groups. Effect size is particularly valuable in best-practice research because they represent a standard measure by which all outcomes can be assessed. The measure commonly used to measure effect size is known as Cohen's *d*. Cohen's *d* is a standardized measure of effect size represented by the ratio of an estimate of the difference between two means expressed in standard deviation units. Cohen's *d* evaluates the size of the effect in a study independent of the scale. It also allows for the comparison of effect sizes from analysis to analysis (study to study) and form the basis of meta-analysis and power analysis (Cohen, 1992). Cohen (1988) proposed a rule of thumb for interpreting effect sizes as shown in Table 3.16.

Table 3.16 The rule of thumb for interpreting Cohen's *d*.

Cohen's ***d***	Effect Size
Below 0.20	Small
Between 0.20 and 0.80	Medium
Above 0.80	Large

Cohen's *d* calculation formula: $d = \frac{\overline{x}_1 - \overline{x}_2}{s}$ where

d = the effect size (Cohen's *d*),

$\overline{x}_1 =$ adjusted mean of the experimental group,

$\overline{x}_2 =$ adjusted mean of the control group, and

$s =$ pooled (combined) standard deviation given by

$s = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}}$ where

$s_1^2 = \frac{1}{n_1 - 1}\sum_{i=1}^{n_1}\left(x_{1,i} - \overline{x}_1\right)^2$ and $s_2^2 = \frac{1}{n_2 - 1}\sum_{i=1}^{n_2}\left(x_{2,i} - \overline{x}_2\right)^2$,

n_1 = sample size of the experimental group,

n_2 = sample size of the control group,

s_1 = variance of the experimental group, and

s_2 = variance of the control group.

3.7.2 Students' attitudes towards the learning unit

The Likert scale in the questionnaire included "1 = strongly disagree", "2 = disagree", "3 = neutral", "4 = agree", and "5 = strongly agree". Each item in the questionnaire was tabulated and interpreted using frequency analysis. Three open-ended questions were analyzed by grouping the students' responses into themes.

3.7.3 Semi-structured interview

The conversation between the interviewer and interviewee (six students) was transcribed and then analyzed to find the students' conceptual understanding of limits, differentiation, integration, and the relationship between differentiation and integration. The students' responses were also interpreted to find the students' attitudes towards the learning units by grouping into themes.

CHAPTER IV
RESULTS

Overview

This chapter reports the results of the implementation of the fundamentals of calculus learning units based on Lawson-Abraham's model of learning cycle. The results are described in two parts: (1) Students' understanding of the fundamentals of calculus and the relationship between differentiation and integration, and (2) Students' attitudes towards the FC learning units.

4.1 Conceptual Understanding Test Analysis

The pre-test was administered to 130 grade-11 students in the two schools and the students were categorized into an experimental group in one school and a control group in another school. Then, the experimental group was treated using the FC learning units while the control group was taught using conventional teacher-centered and textbook-oriented instruction. After the lessons in both the groups, the post-test was conducted in both the groups. The pre-test and post-test were analyzed to address the research questions: (1) To what extent can the contextual and graphing activity-based learning cycle units enhance the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration, and (2) To what extent have the experimental group students improved from the pre-test to the post-test compared to those in the control group?

Before conducting the inferential *t* tests, data normality tests were conducted using the Spiro-Wilk test of normality as shown in Table 4.1. The pre-test scores of the students from both schools A (experimental group) and B (control group) were not normally distributed (school A, $p = 0.001$ and school B, $p = 0.000$). The post-test scores of school A were normally distributed with p-value of 0.140 while the post-test scores of school B were not normally distributed with p-value of 0.000. The differences between pre-test and post-test scores of the students in school A were normally distributed (p = 0.219).

Table 4.1 Shapiro-Wilk test of normality for pre-test and post-test of school A and B.

School	Pre-test	Post-test	Difference (Post-test−Pre-test)
School A (Experimental group)	Not normal distribution (p = 0.001)	Normal distribution (p = 0.140)	Normal distribution (p = 0.219)
School B (Control group)	Not normal distribution (p = 0.000)	Not normal distribution (p = 0.000)	Not normal distribution (p = 0.006)

Therefore, the data which were not normally distributed (i.e. the pre-test scores of both groups, post-test of school B) were dealt with using non-parametric hypothesis tests: the Mann-Whitney U test to compare between the groups and the Wilcoxon signed-rank test to compare within the group. The data which were normally distributed (i.e. the difference between pre-test and post-test of school A) were dealt with using parametric hypothesis test: the paired-sample *t* test.

4.1.1 Pre-test and post-test results

The students' pre-test scores of school A and school B were compared to investigate the students' prior knowledge on the fundamentals of calculus and the relationship between differentiation and integration. The pre-test scores of students in school A and B were not normally distributed. Therefore the Mann-Whitney U test was used to analyze the pre-test scores of both schools as shown in Table 4.2

The means of the students' pre-test scores were 1.14 (SD = 0.81) for school A and 0.38 (SD = 0.57) for school B. According to the results from the Mann-Whitney U test for independent samples, the average ranks of the pre-test scores of the students were 67.10 for school A and 63.90 for school B. There was no significant difference between the scores (z = 0.54, p = 0.59) because the p-value of the test was greater than the significant p-value of the study (p = 0.05).

Table 4.2 Mann-Whitney U test result to investigate students' prior knowledge from pre-test scores of schools A and B.

School	N	Average Rank	Sum of Ranks	U	z	p-value (2-tailed)
A (Experimental Group)	65	67.10	4361.50	2008.50	0.54	0.59
B (Control group)	65	63.90	4153.50			

Thus, it was concluded that the students in both groups had similar prior knowledge in the fundamentals of calculus and the relationship between differentiation and integration. School A was categorized as the experimental group and school B as the control group.

To investigate the effect of the treatment to the experimental group, the analysis was conducted by comparing the students' pre-test and post-test scores. Since the differences between the pre-test and post-test scores of the students were normally distributed, the paired-sample *t* test was used for data analysis as shown in Table 4.3.

Table 4.3 Paired-sample *t* test result of students' performance in pre-test and post-test for the experimental group.

Test	N	Mean	SD	*t*	df	p-value (2-tailed)
Pre-test	65	1.14	0.81	26.59	64	0.000
Post-test	65	13.21	3.58			

The mean of the students' pre-test scores was 1.14 (SD = 0.81) and the post-test mean was 13.21 (SD = 3.58) for the experimental group with p-value of 0.000. Therefore, the result signified that the students' performance increased in the post-test compared to that in the pre-test (t = 26.59, p = 0.000) for the experimental group after the intervention.

To investigate the effect of the treatment to the control group, another analysis was conducted by comparing the students' pre-test and post-test scores. The differences between the pre-test and post-test scores were not normally distributed. Therefore, the Wilcoxon signed-rank test was used for data analysis as shown in Table 4.4.

Table 4.4 Wilcoxon signed-rank test result for students' performance in pre-test and post-test scores for the control group.

Difference between Post-test and Pre-test	N	Average Rank	Sum of Ranks	z	p-value (2-tailed)
Negative Rank	18	19.03	342.50	0.680	0.497
Positive Rank	21	20.83	437.50		
Equal	26				

For the students in the control group, the mean of the students' pre-test scores was 0.38 (SD = 0.57) and the post-test mean was 0.43 (SD = 0.56). According to the Wilcoxon signed-rank test, the sum of negative ranks of the scores was found to be 342.50 while the sum of positive ranks of their scores was 437.50 with z-value of 0.680 and p-value of 0.497.

This indicated that the result was not statistically significant as the p-value was higher than the significant p-value of the study (i.e. 0.05). The result suggested that the conventional teacher-centered and textbook-oriented instruction used to teach the fundamentals of calculus could not significantly improve students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration.

There was no significant difference between the pre-test scores of the students in the experimental and the control groups. The post-test scores of the students in both the groups were further analyzed to see the differences in the post-test scores of the students between the two groups after the intervention. The post-test scores of the students in the control group were not normally distributed while the post-test scores of the students in the experimental group were normally distributed. Therefore the non-parametric hypothesis test, the Mann-Whitney U test, was used for analysis as shown in Table 4.5.

The means of the students' post-test scores were 13.21 (SD = 3.58) for the experimental group and 0.43 (SD = 0.56) for the control group. According to the results from the Man-Whitney U test for independent samples, the average rank of the post-test scores of the students in the experimental group was 98.00 and that for the students in the control group was 33.00. The calculated z-value was 9.86 and the p-value was 0.00 (i.e. $p < 0.05$).

Table 4.5 Mann-Whitney U test result of students' performance in post-test for the experimental and the control groups.

<table>
<tr><th>Group</th><th>N</th><th>Average Rank</th><th>Sum of Ranks</th><th>U</th><th>Z</th><th>p-value (2-tailed)</th></tr>
<tr><td>Experimental</td><td>65</td><td>98.00</td><td>6370.00</td><td rowspan="2">0.00</td><td rowspan="2">9.86</td><td rowspan="2">0.00</td></tr>
<tr><td>Control</td><td>65</td><td>33.00</td><td>2145.00</td></tr>
</table>

This indicated that the result was statistically significant as the p-value was lower than the significant p-value of the study (i.e. 0.05). Moreover, the average ranks of the post-test scores of the students in the two groups signified that the every student in the experimental group out-performed every student in the control group in the post-test.

The means of the post-test scores of both groups were further analyzed using Cohen's *d* effect size. Cohen's *d* effect size measured the magnitude of the difference between the experimental and the control groups. It was found that Cohen's *d* effect size was 5.02 (experimental group: N = 65, Mean = 13.21, SD = 3.56 and control group: N = 65, Mean = 0.43, SD = 0.56).

The result indicated that there was a huge difference between the students' mean scores in the experimental and the control groups. The contextual and graphing activities–based learning units were more effective compared to the conventional teacher-centered and textbook-oriented instruction in enhancing the students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration. According to Cohen (1988), effect size of 0.80 or greater was considered "large".

4.1.2 Analysis of each post-test question for the students in the experimental group

The post-test consisted of five questions. The students in the experimental group were grouped into high, middle, and low achievers based on their mathematics scores in the mid-term of academic year 2013. The average scores in each question of the three groups are shown in Table 4.6.

Table 4.6 Average scores of high, middle, and low achievers in the experimental group.

Item	Full Score	High (N = 22)		Middle (N = 22)		Low (N = 21)	
		Mean	SD	Mean	SD	Mean	SD
1. Limits	3	2.75	0.69	2.48	0.87	2.14	1.05
2. Differentiation	3	1.86	0.94	1.68	0.87	1.33	0.76
3. Integration	3	2.77	0.51	2.20	1.00	2.26	0.86
4. Relationship: Integration $\rightarrow$ Differentiation	6	4.32	1.24	3.95	1.39	2.90	1.657
5. Relationship: Differentiation $\rightarrow$ Integration	5	3.89	1.39	2.95	1.45	2.26	1.63

The results from Table 4.1 indicated that the high achieving students performed better compared to the middle and low achieving students in each post-test question. The mean of each question reflected that even the low achieving students had performed reasonably well in the post-test. Since the post-test questions were open-ended, each question was further analyzed as follows.

Question 1: Question 1 of the post-test aimed to examine students' conceptual understanding of limits. The majority of the students justified that the limits of the function $f(x)$ and $g(x)$ existed and that of the function $h(x)$ did not exist by using the conditions of limits as shown in Figures 4.1–4.3.

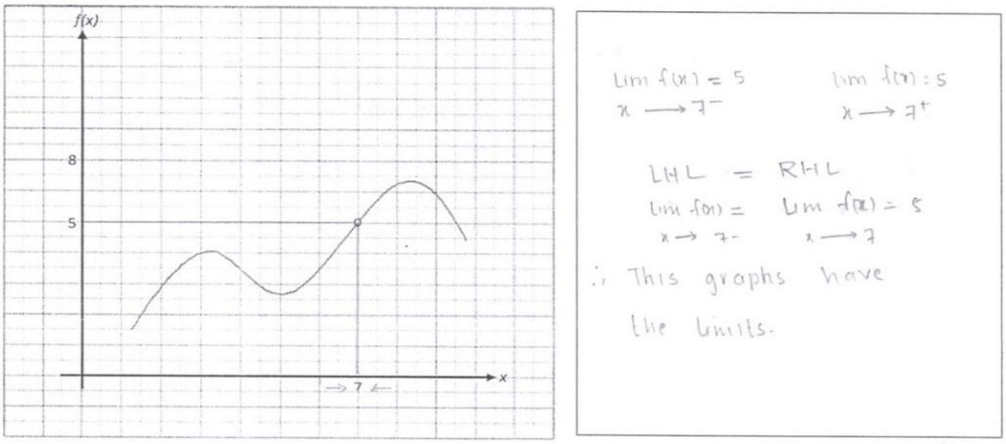


Figure 4.1 A student's answer to the graph of function $f(x)$.

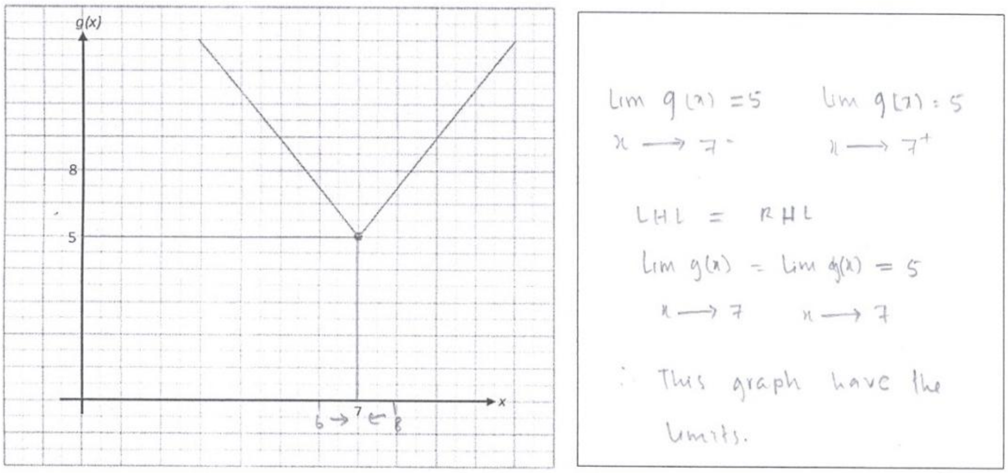


Figure 4.2 A student's answer to the graph of function $g(x)$.

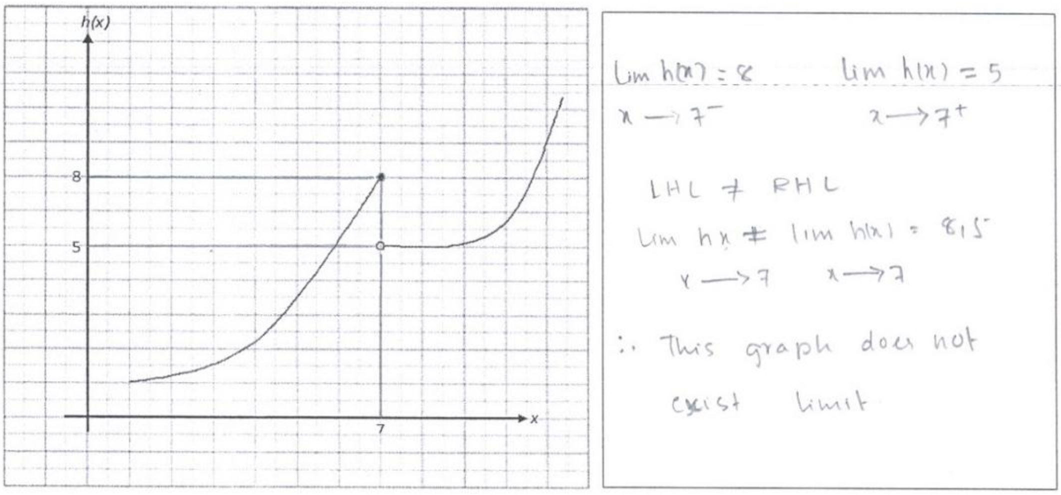


Figure 4.3 A student's answer to the graph of function $h(x)$.

A few students described the answers in sentences as shown in Figures 4.4–4.6.

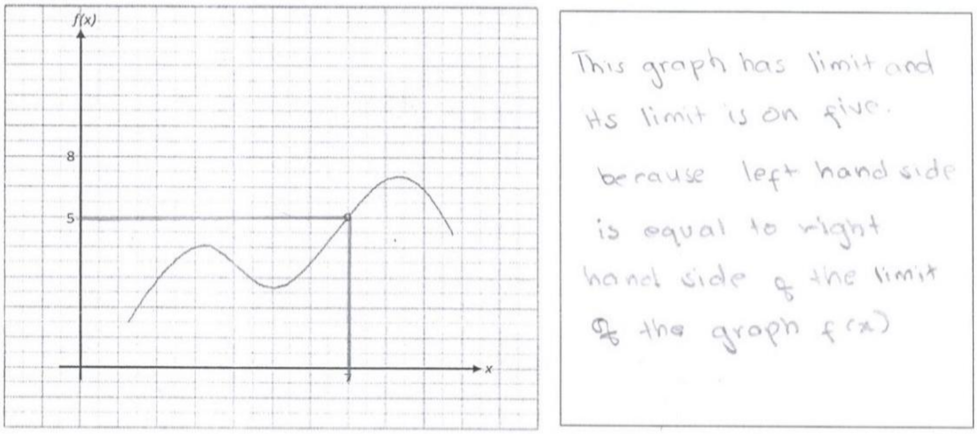


Figure 4.4 A student's answer to the graph of function $f(x)$.

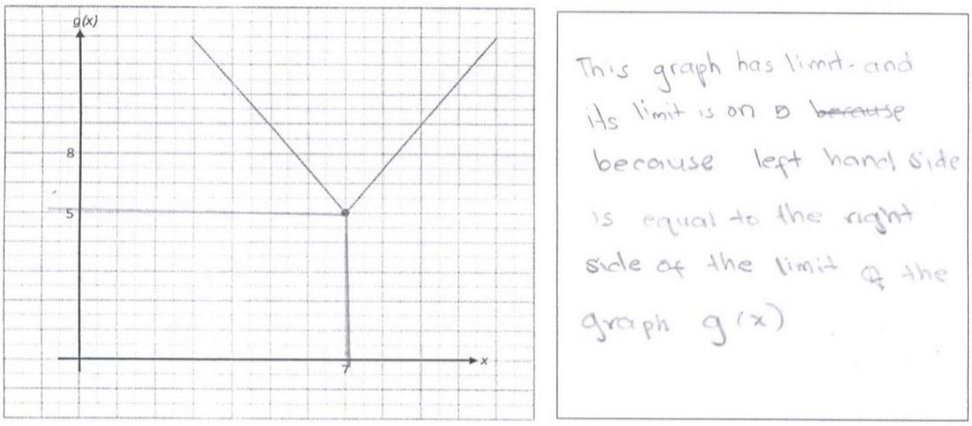


Figure 4.5 A student's answer to the graph of function $g(x)$.

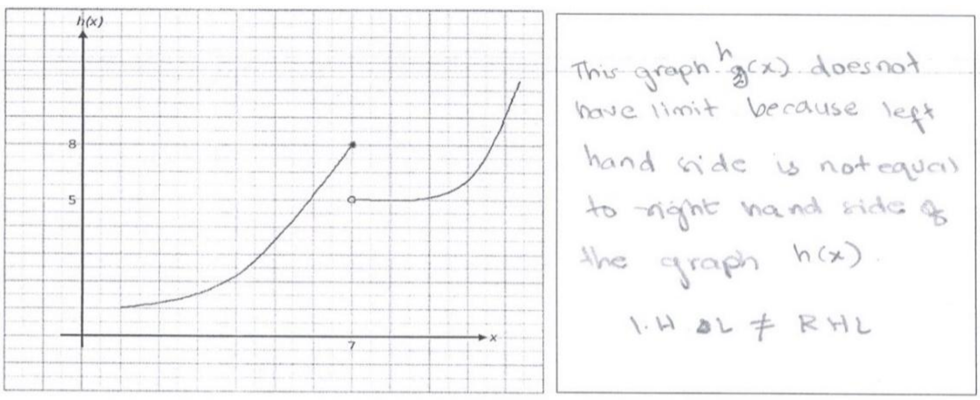
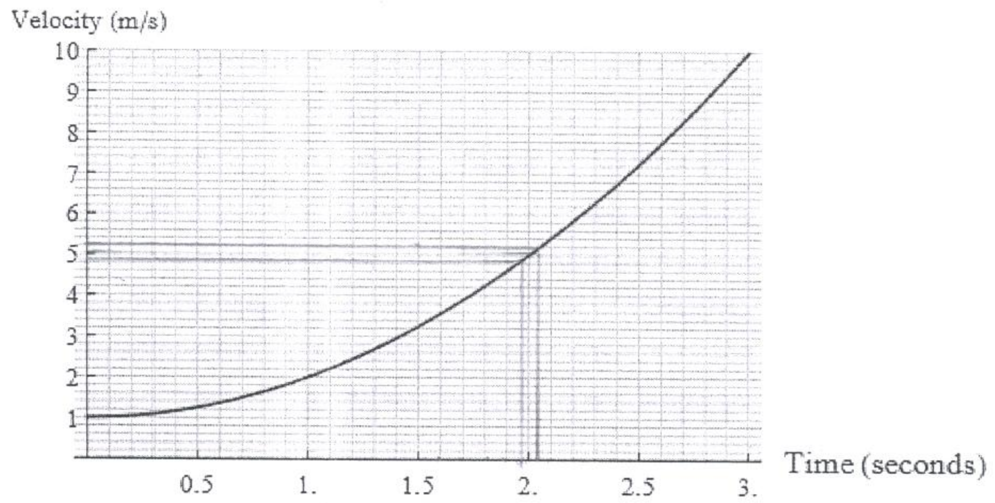


Figure 4.6 A student's answer to the graph of function $h(x)$.

These results indicated that the students could figure out the existence of the limits of the functions graphically as well as using the conditions of limits. The students had shown an intuitive idea of limits. However past studies (Cornu, 1991; Tall & Vinner, 1981) showed that students' intuitive ideas got conflicted when the formal definition was introduced. In this study, the students did not show the misconceptions of finding the limit of the function graphically even when the function $f(x)$ was not defined at $x = 7$. This indicated that the students had ideas about limits that were proof against the introduction of the formal definition of limits.

Question 2: Question 2 of the post-test aimed to examine students' conceptual understanding of differentiation. Most of the students used two points, one on the left and the other on the right of $t = 2$ seconds, to calculate the derivative from the graph as shown in Figure 4.7.



a) Find the derivative at $t = 2$ seconds graphically from the velocity-time graph?

derivative at $t=2$

$$\text{Slope} = \frac{5.2-4.8}{2.05-1.95}$$

$$= \underline{4}\ m/s^2$$

b) What does the derivative represent?

: Acceleration.

Figure 4.7 A student's answer to question 2 in the post-test.

The students used the concept of the slope of a secant line (average slope) to find the derivative at *t* = 2 seconds. They seemed to think that finding the average slope was the same as finding the derivative of a function even though there were certain similarities and differences between the two. With the concept of an average slope, a linear function is used but for a non-linear function, the tangent line is used to find the derivative. Usually in an algebra course, students learn the definition of the slope of a line as the rate of change of *y* with respect to *x*. It is also shown as the rise over the run and later students learn that a derivative is a rate of change. This might be the main cause for the use the concept of an average slope to find the derivative.

There were some students who lacked the reading skills of the graph although they had an idea about finding the derivative from the graph as shown in Figure 4.8.

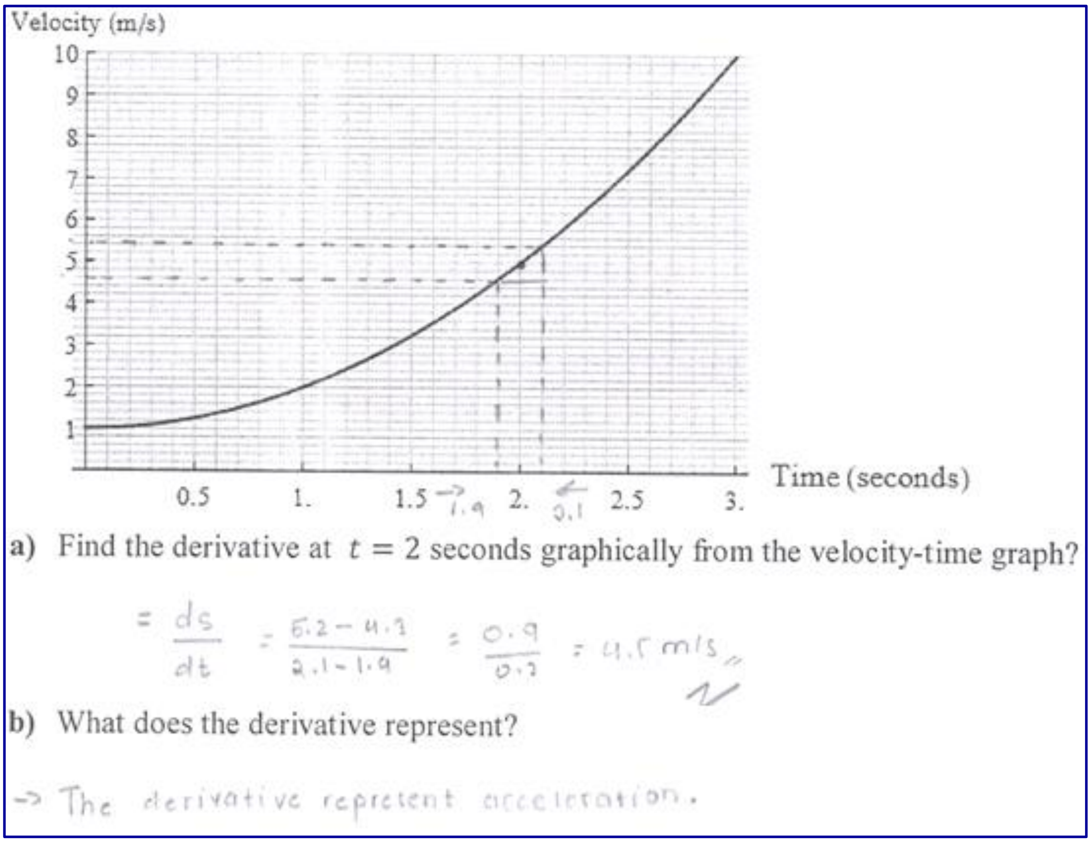


Figure 4.8 A student committing the reading errors from the graph.

Some students referred the algebraic formula for calculating the slope $\left[\frac{y_2 - y_1}{x_2 - x_1}\right]$ as shown in Figure 4.9. It seemed that they were more acquainted with use of formulas and some of them failed to interpret the derivative of the graph in the contextual situation.

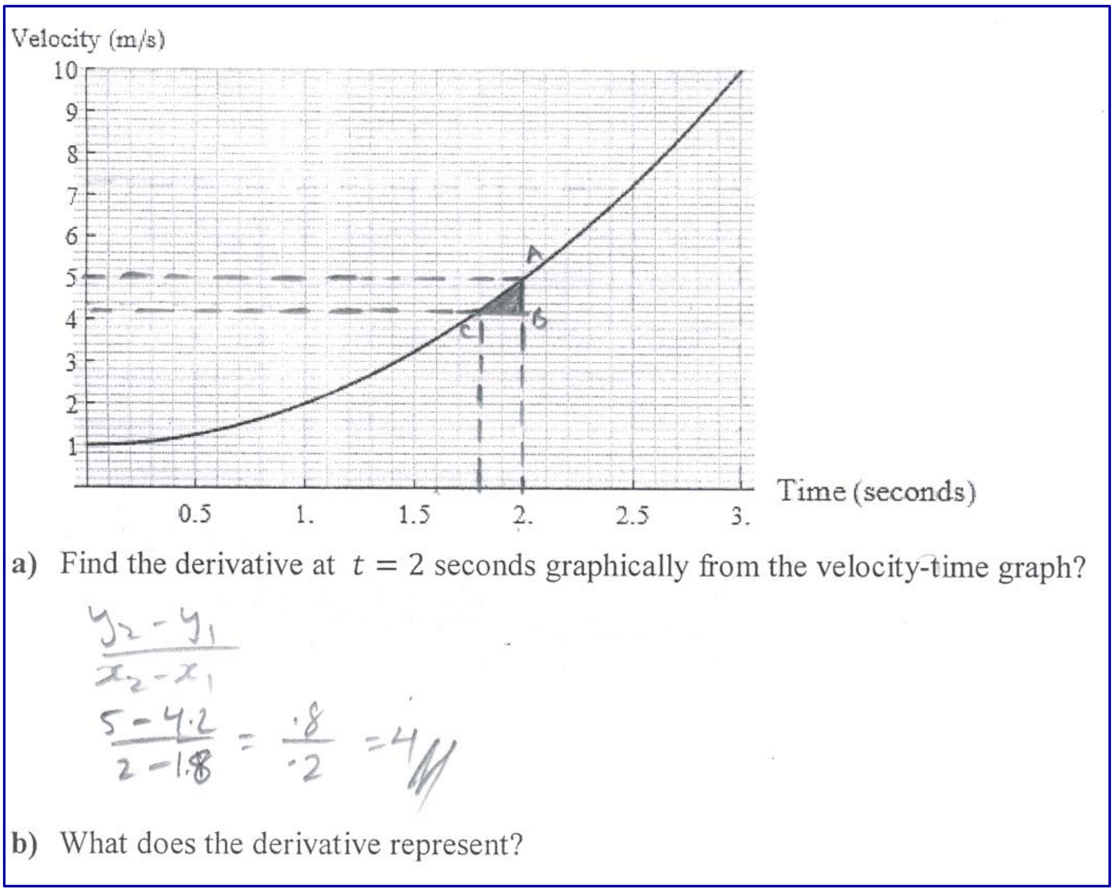


Figure 4.9 A student using the formula to calculate the derivative.

Question 3: Question 3 of the post-test aimed to examine students' conceptual understanding of integration. The students had to sketch the graph of $|x-4|$ to find $\int_0^8 |x-4|\,dx$ as shown in Figure 4.10. The majority of the students calculated the value of $|x-4|$ by taking the value of x from 0 to 8 to sketch the graph as shown in Figure 4.10 and then calculated the area under the graph. However some low achiever students sketched the graph of $|x-4|$ but could not find the area of the graph as shown in Figure 4.11.

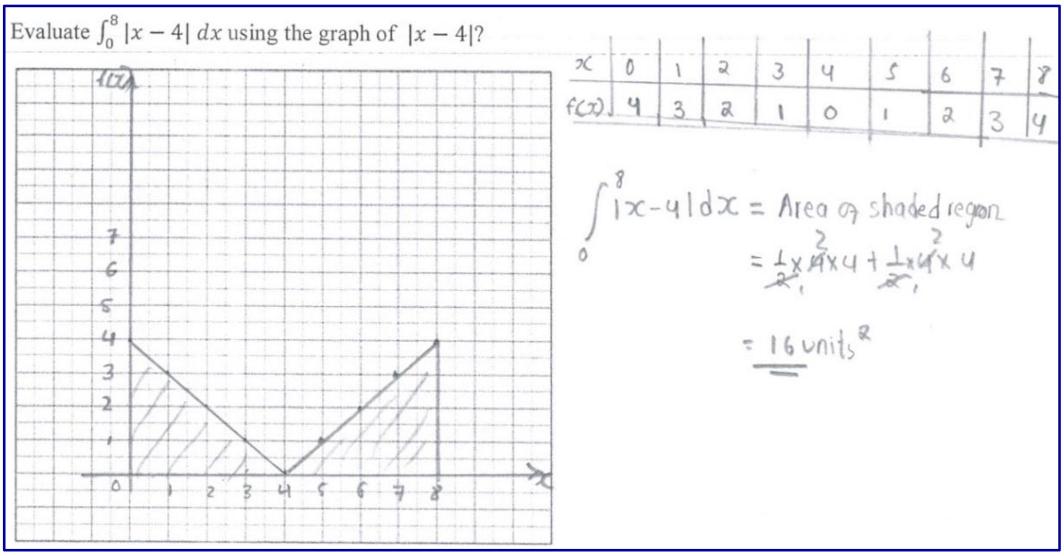


Figure 4.10 A student sketching the graph of $|x-4|$ to find $\int_0^8 |x-4|\,dx$ correctly.

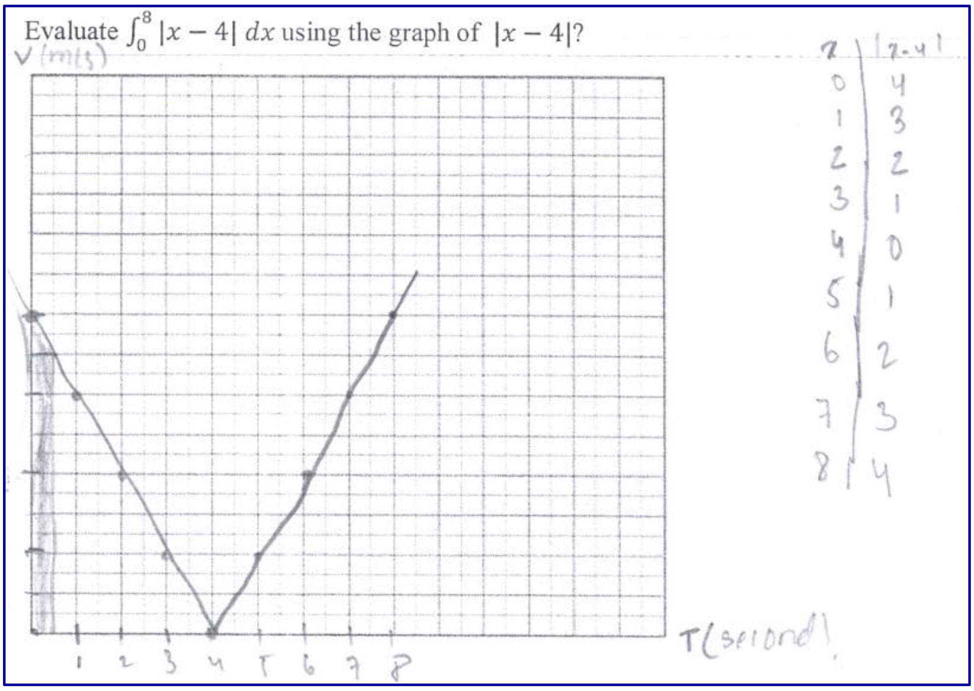
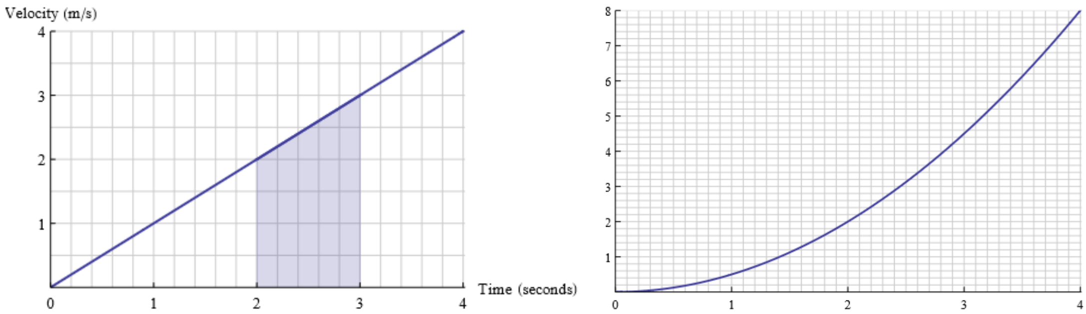


Figure 4.11 A student sketching the graph of $|x-4|$ to find $\int_0^8 |x-4|\,dx$ incorrectly.

Question 4: Question 4 aimed to find out graphically and contextually how students related the integral (area under the velocity-time graph) to the derivative of the graph, and also to examine whether students could establish the relationship from integration to differentiation. The students were given the graphs in Figure 4.12. Graph 1.1 was a velocity-time graph. Graph 1.2 was the anti-derivative of Graph 1.1.



Graph 1.1 Graph 1.2

Figure 4.12 Graph 1.2 is the anti-derivative of Graph 1.1 (Velocity-time).

Those students who possessed conceptual understanding of derivative and integral contextually and graphically were able to answer question 4 correctly as shown in Figure 4.13.

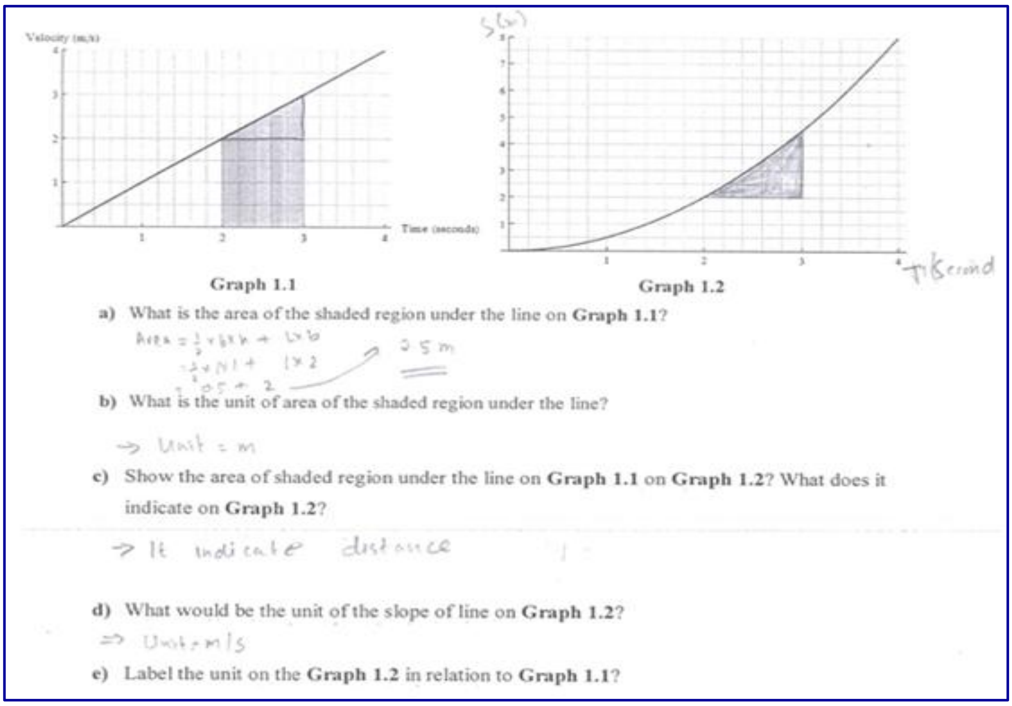


Figure 4.13 A student's correct answer to question 4.

However, the area under Graph of 1.1 did not make much sense if the student could not interpret the result on Graph 1.2. The students' common mistake was in finding the unit of the area under Graph 1.1 as shown in Figure 4.14.

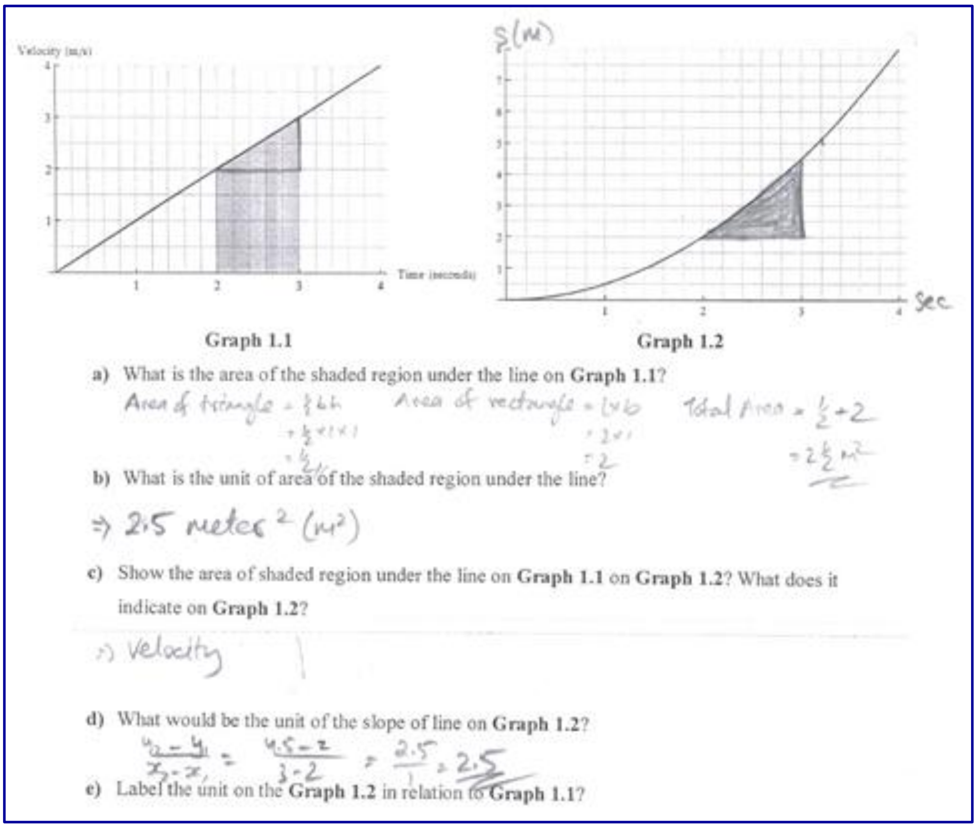
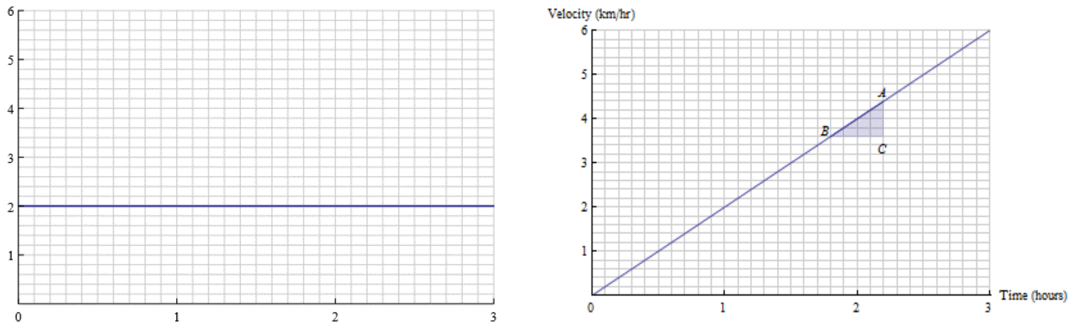


Figure 4.14 A student's incorrect answer to question 4.

The student could calculate the area under Graph 1.1 and represented the result on Graph 1.2 correctly. The student found the unit of the area incorrectly but labelled the Graph 1.2 correctly. The student also answered question (c) as 'velocity' and could not figure out the unit of the slope of Graph 1.2 but instead used the algebraic formula to calculate the slope of Graph 1.2. Similarly many students answered question (c) as 'acceleration' and could not figure out the unit of the slope of Graph 1.2. It seemed that these students used rote learning when the learning units were introduced and still lacked conceptual understanding of derivative and integral contextually and graphically. Moreover, the students seemed to lack the concepts of distance, velocity, and acceleration graphically.

Question 5: Question 5 aimed to find out graphically and contextually how students related the derivative (slope of the graph) to the integral (area under the graph) of the line, and also to examine whether students could establish the relationship from differentiation to integration. Students were asked the following questions with the graphs in Figure 4.15. Graph 2.1 was the derivative of Graph 2.2 which was a velocity-time graph.



Graph 2.1 Graph 2.2

Figure 4.15 Graph 2.1 is the derivative of Graph 2.2 (velocity-time).

Students who could relate the area under the graph (integral) to the derivative of the graph in question 4 could also easily figure out the reverse process from the derivative to integral. Figure 4.16 represents the work of a student who answered question 5 correctly.

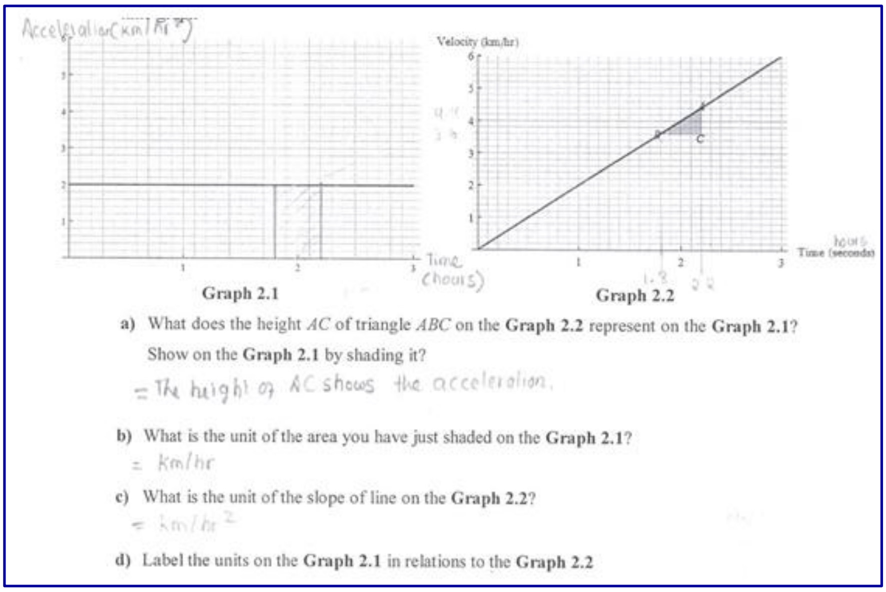


Figure 4.16 A student's correct answer to question 5.

There were a few students who had difficulty relating Graph 2.2 to Graph 2.1 contextually. The student's answer shown in Figure 4.17 indicates that he lacked the concepts of velocity and acceleration units. He could figure out that the height AC of the triangle ABC on Graph 2.2 was an acceleration and could represent the height AC on Graph 2.1. However, he labelled Graph 2.1 incorrectly.

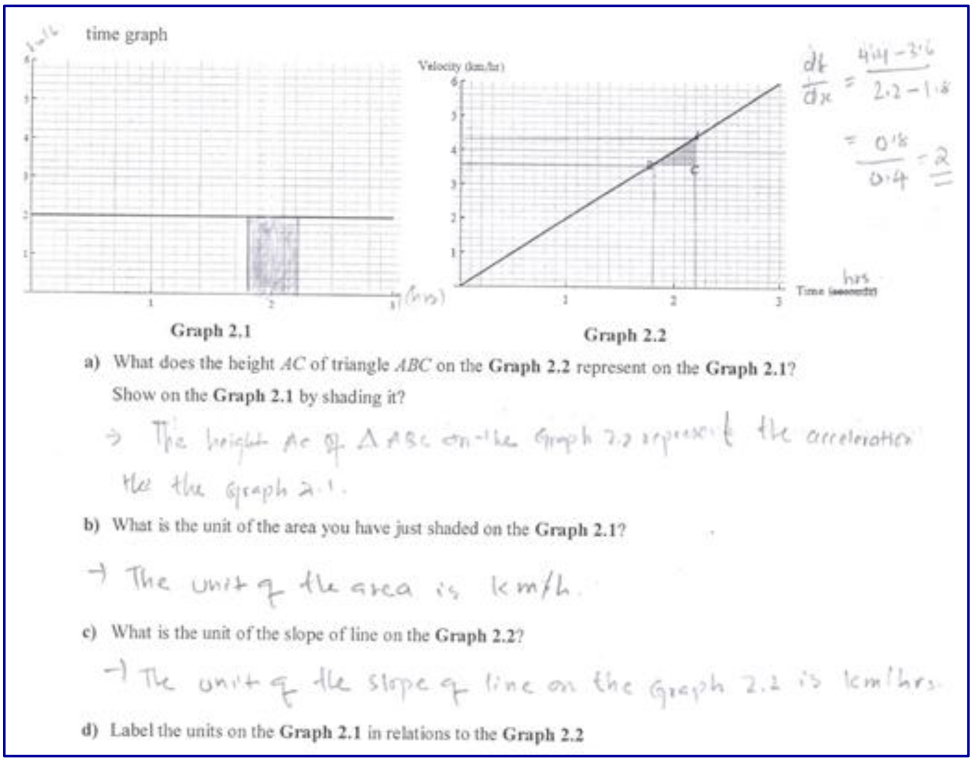


Figure 4.17 A student's incorrect answer to question 5.

The answers given by the student seemed to indicate the lack of the conceptual knowledge of motion even though he could figure out the graphical relationship.

The majority of the students could answer the conceptual questions correctly although there were a few students who still had difficulty interpreting the graphs contextually.

4.1.3 Students' semi-structured interview results

To validate and support the findings from the conceptual understanding test, six students from the experimental group were interviewed. The students' conceptual understanding of limits, differentiation, and integration was investigated followed by their ability to apply the concepts of differentiation and integration in explaining the relationship contextually, graphically, and algebraically between differentiation and integration. Each student was interviewed for about an hour. The interview was audio recorded and the graphs used for answering the interview questions were also analyzed. The six students consisted of the two highest scorers, the two middle scorers, and the two lowest scorers based on their mathematics mid-term scores of academic year 2013. The details of students who are interviewed is shown in Table 4.7.

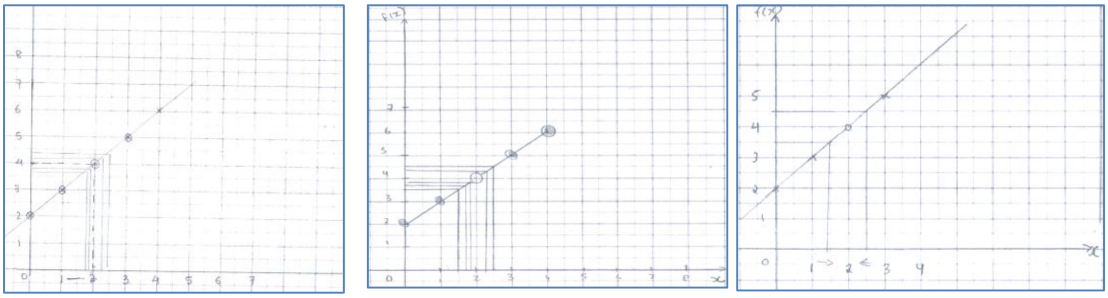
Table 4.7 The details of the six interviewed students from the experimental group.

	Student	**Gender**	**Mid-term score (100)**	**Pre-test score (20)**	**Post-test score (20)**
High Achiever	A	G	87	1	18
	B	G	77	1	15
Middle Achiever	C	G	34	1	19
	D	B	34	1	15
Low Achiever	E	B	9	0	12
	F	B	9	1	8

All the semi-structured interview responses from the six students were transcribed and analyzed as follows.

4.1.3.1 Limits

The function $f(x)=\dfrac{x^2-4}{x-2}$ was given to the students with graph papers, and they were asked to identify at which point the function was not defined. After a few minutes of algebraic calculation on the paper, the students found that the function was not defined at $x=2$. All the six students could figure out that the limit of the function does exist at the point $x=2$. When the students were asked what was the limit of the function when *x* approached 2? Five students could figure out that the limit of the function was 4 as *x* approached 2. Students A and B drew lines closer and closer to 2 on the horizontal *x*-axis, and then to the vertical *y*-axis which corresponded to 4 (see Figures 4.18 (a) and (b)). They tried to explain by drawing lines rather than speaking verbally.



(a) Student A (b) Student B (c) Student C

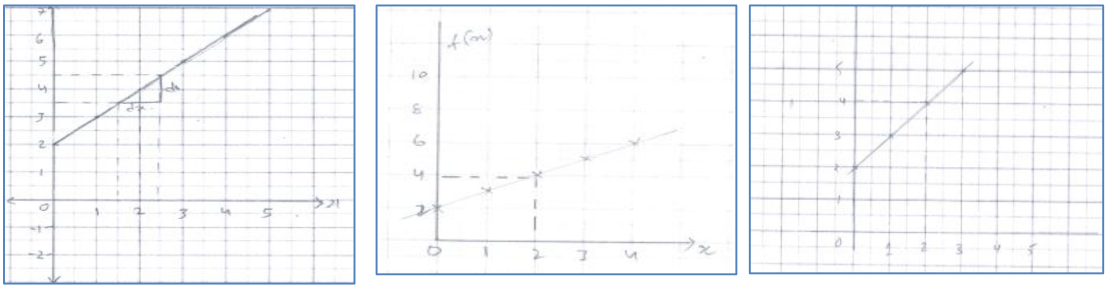
Figure 4.18 Limit of the function $f(x)=\dfrac{x^2-4}{x-2}$ sketched by students.

Students E and F also said that the limit of $f(x)=\dfrac{x^2-4}{x-2}$ existed as *x* approaches 2.

- Student E: *When we take x* = 2, *then it lies in f*(*x*). *It is 4, so limit exist. In order to find the limit, we have to see whether 2 is meeting with f*(*x*).
- Student F: *When x* = 2, *the line reaches to the limit of 4.*

Student D said "*No*" and later said that the limit was 4. When he was asked to explain, he drew a small triangle on the graph and seemed to try to find the derivative at $x=2$, and could not explain how he got the limit of the function. Rather he confused finding the limit with finding the derivative using a graph.

Students A, B, and C sketched the graph of the function correctly by indicating a gap at $x=2$ (see Figures 4.18 (a), (b), and (c)). When the function was not defined at $x=2$, however, students D, E, and F sketched the graph of the function $f(x)=x+2$ without a gap at $x=2$. They simplified the function $f(x)=\dfrac{x^2-4}{x-2}$ and ignored that the function was not defined at $x=2$ (Figures 4.19 (a), (b), and (c)).



(a) Student D (b) Student E (c) Student F

Figure 4.19 Limit of the function $f(x)=\dfrac{x^2-4}{x-2}$ sketched by students.

Although the function $f(x)=\dfrac{x^2-4}{x-2}$ was not defined at $x=2$, the limit of the function did exist as x approached 2. All the six students simplified the function $f(x)=\dfrac{x^2-4}{x-2}$ to $f(x)=x+2$ in order to sketch the graph. However, three students did not concern about the condition for the function when they sketched the graph of $f(x)=\dfrac{x^2-4}{x-2}$ and sketched the function without a gap at $x = 2$. Student D seemed to confuse the concept of limits with that of derivative when he drew a triangle on the line to explain how he found the limit of the function at $x = 2$. This result indicated that students D, E, and F did not have a clear concept of limits even though they had an intuitive idea of limits.

4.1.3.2 Differentiation and integration and their relationship

To investigate the students' conceptual understanding of differentiation and integration, and the relationship between differentiation and integration, the students were asked the following question.

The graph represents a car travelling from A to B and its distance s in kilometers from the starting point A is given by the function $s(t)=t^2$ where t is the time taken in hours as shown in Figure 4.20.

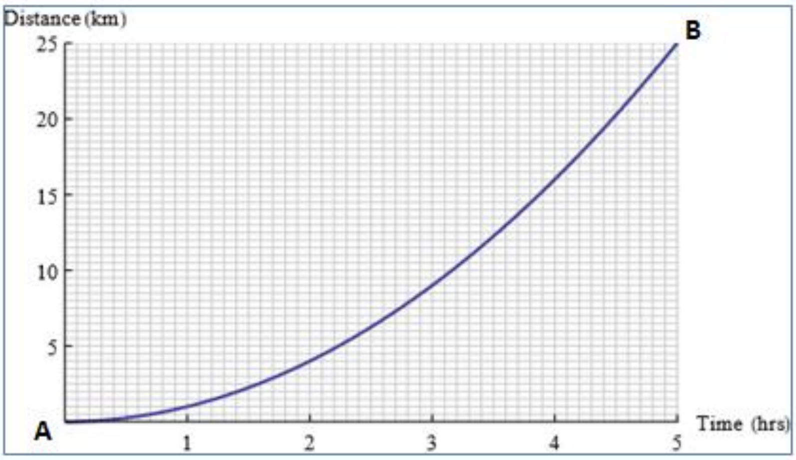


Figure 4.20 Graph of the function $s(t) = t^2$.

a) How far has the car travelled in 5 hours?

b) How long does the car take to travel 16 km?

c) What is the car's average velocity over; (i) the first 2 hours? (ii) the first 4 hours? (iii) between $t = 2$ hours and $t = 5$ hours?

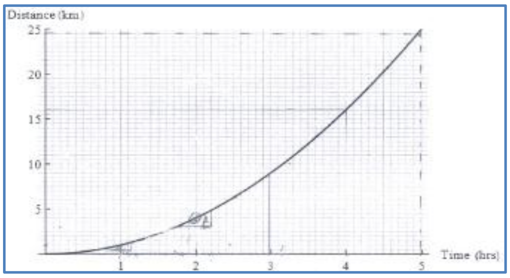
d) What is the velocity of the car at t = 2 hours? Explain how you get your answer.

e) Can you find the velocity of the car at t = 1, 2, 3, 4, and 5 hours and sketch the velocity of the car on another graph?

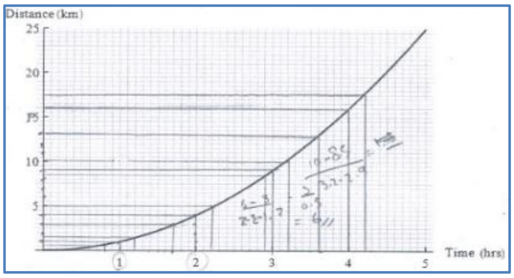
f) What is the area under the newly sketched graph in (e); (i) during the first 1 hour? (ii) during the first 2 hours? (iii) during the first 3 hours? (iv) during the first 4 hours? (v) during the first 5 hours?

g) Sketch the area under the graph on another graph. What does the newly sketched graph tells you regarding the graph of Figure 4.20?

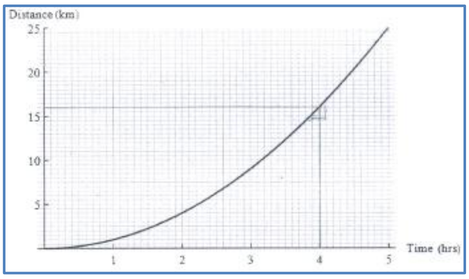
The first two questions (a) and (b) asked students to figure out whether they could read the graph to find the distance and time. All the students could answer that the car travelled the distance of 25 km in 5 hours and that the car took 4 hours to cover the distance of 16 km.



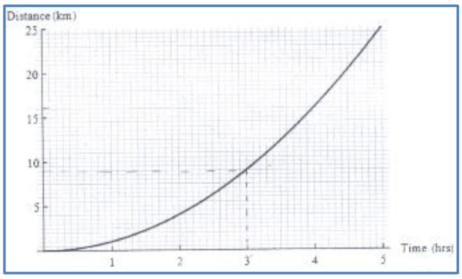
(a) Student A



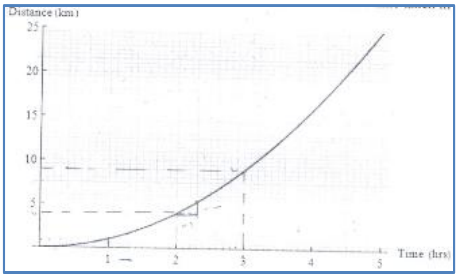
(b) Student B



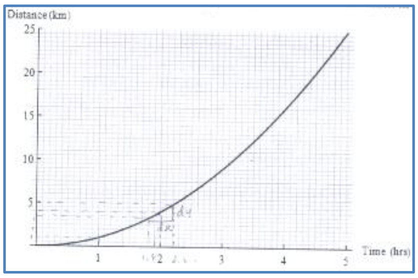
(c) Student C



(d) Student D



(e) Student E



(f) Student F

Figure 4.21 Students drawing the lines to find the distance and time from the graph.

When the students were asked to find the average velocity over the first 2 hours, the first 4 hours, and between *t* = 2 hours and *t* = 5 hours, students A and B used the velocity = distance over time ($v = \frac{s}{t}$) formula to find the average velocity. Students C, D, E and F used the slope formula ($\text{Slope} = \frac{\text{rise}}{\text{run}}$) to find the average velocity.

However when the students were asked to find the velocity of the car at $t = 2$ hours, students A and B found the velocity of the car by using the concept of slope (Figures 4.21 (a) and (b)). Student B also indicated that she used the concept of limit as well even though her drawing showed no sign of that. Students C, D, E, and F used the concept of slope as well as the equation of the line to find the velocity of the car at $t = 2$ hours as described below.

- Student A: *We have to keep this point* $t = 2$ *in the middle of the two points…then we find the slope of these two points.*
- Student B: *I used the concept of the limit which I drew line closer and closer to 2, then by using the formula of slope, rise over run. I found out the velocity.*
- Student C: *When I used the graph, I find the slope of the line taking small divisions* [points] *on the line keeping the point to find the slope between the two points. We* [I] *also find using differentiation* [using equation of the line].

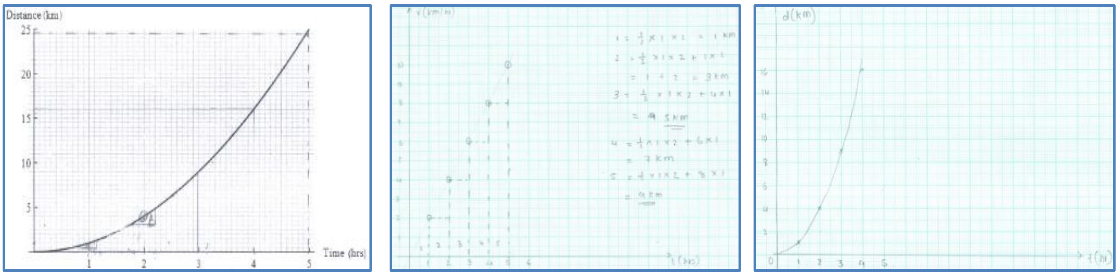
Student A was not very confident about finding the derivative at a point from the graph and thought that the algebraic formula for derivative always gave the correct answer. She used the equation of the line $s(t) = t^2$ to find the derivative at the point although she could explain how to calculate the slope from the graph without using the equation of the line. Student B could explain explicitly how to find the velocity at $t = 2$ by using the graph (Figure 4.21(b)) as well as the equation of the line. She was confident about using the graphical method to find the derivative rather than using the equation of the line.

Students C and D were able to choose precise two points on the graph to find the derivative at $t = 2$ and they took a lot of time to use the equation of the graph to find the derivative (Figures 4.21 (c) and (d) although they could do it correctly. They did not look very convinced when they used the equation of the line to find the velocity of the car. Students E and F calculated the velocity at $t = 2$ hours using both the graph and the equation of the line. They could choose the precise points on the graph to find the derivative but could not read the graph. Student E could differentiate the equation of the line in terms of variables *x* and *y* but could not use it to find the derivative at $t = 2$. Similarly student F tried to use the equation of the line to find the derivative but did it incorrectly. Students E and F seemed to memorize the steps during the learning process rather than to understand the concepts, and they also lacked the earlier graphing skills.

The above results indicated that some students were more confident in finding the derivative at the point using the graphical method. They expressed their understanding of the derivative easier using graphs than using the equation of the line. Most of the students could not relate the equation of the line to the context and explain although they knew the process of using the equation to find the derivative. Some students always looked for a short cut to get the end results. Student A was pre-occupied to get an end result rather understanding the concept. She could find the derivative at a point using the equation very easily.

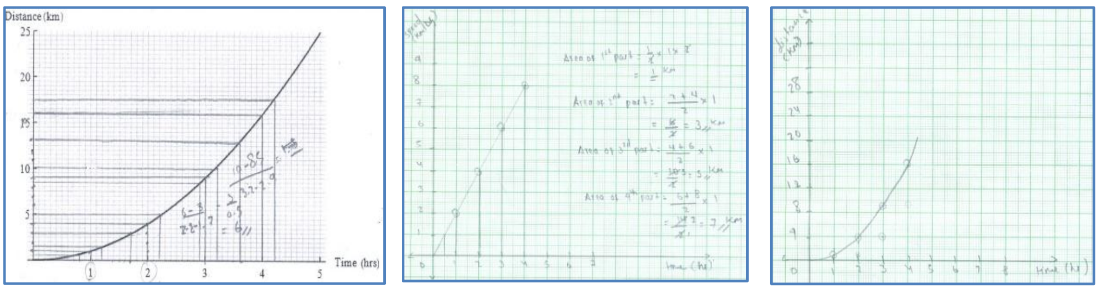
The students were also asked to find the velocity of the car at $t = 1$, 2, 3, 4, and 5 hours and sketch the velocity of the car. Four students could find the velocity of the car at $t = 1$, 2, 3, 4, and 5 hours. Student A used the equation of the line to find the velocity of the car at each instant of time while students B, C, and D used the graphical method. Students E and F tried to find the velocity of the car using the graphical method but could not perform the calculation correctly. They seemed to have partial idea about finding the derivative from the graph.

Then the students were asked to find the area under the graph of the velocity and sketch the area on another graph. Figures 4.22–4.25 shows the graph sketched by students A, B, C, and D after finding the velocity at each instant of time.



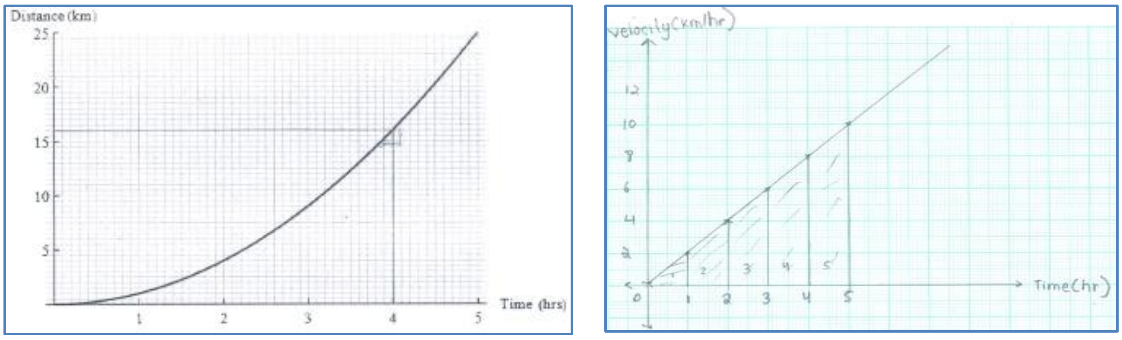
(a) (b) (c)

Figure 4.22 Student A: Finding the derivative and area under the line.



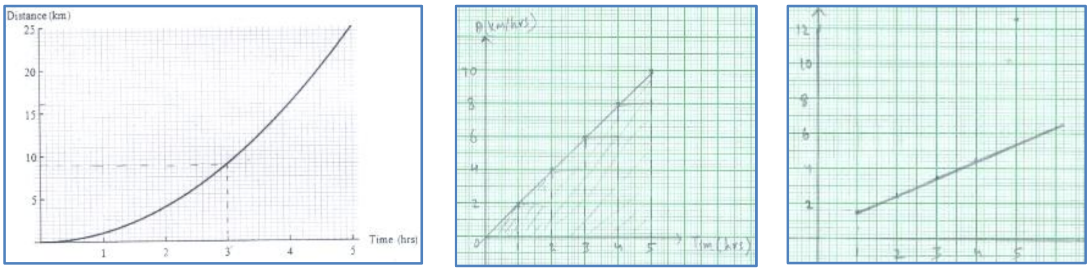
(a) (b) (c)

Figure 4.23 Student B: Finding the derivative and area under the line.



(a) (b)

Figure 4.24 Student C: Finding the derivative and area under the line.



(a) (b) (c)

Figure 4.25 Student D: Finding the derivative and area under the line.

Student A used the equation of the graph to find the velocity at each instant of time while students B, C, and D used the graphical method. Students E and F could not calculated the velocity of the car as they perform the calculation incorrectly. Finding the area of each part under the graph and plotting on another graph was not very difficult for students A, B, and C (Figures 4.22 (b) and (c)–4.24(b) and (c)). Student D had a wrong pre-occupied notion that the area under the graph would give the acceleration-time graph, and he sketched graph incorrectly (Figure 4.25(c)). This would be due to rote-learning during the implementation of the learning unit rather understanding the concept.

Then the students were asked to make a connection between the graph in Figure 4.20 and the graphs in Figures 4.22(b), 4.23(b), 4.24(b), and 4.25(b) in terms of the unit of the slope of the former and area under the later. The unit of the slope of graph in Figure 4.20 would be in km/hour and then finding the area under the graphs in Figures 4.22(b), 4.23(b), 4.24(b), and 4.25(b) would give back the distance in km as depicted by the graph in Figure 4.20. Students A, B, and C could explain the relationship between the graphs easily while student D who sketched the graph of the area incorrectly (Figure 4.25(c)) could not. Some of excerpts from the students' interview are as follows.

- Student A: *Finding the derivative of graph* [Figure 4.22(a)], *we get the velocity and when we find the area of the graph* [Figure 4.22(b)], *we get the graph* [Figure 4.22(c)], *the distance-time.*
- Student B: *The difference between the integration and differentiation is that integration is the anti-derivative* [inverse process] *of differentiation ... to find the differentiation, we used the slope …when we integrate… we used the area of the given* [under the curve]…

Once the students could relate the unit of the slope with the unit of the area under the graph contextually, they were reassured by the graphical picture. All the six students understood the concepts of differentiation and integration contextually and graphically, at least partially. However the low achieving students had difficulty establishing the relationship between differentiation and integration.

The results from the conceptual understanding test and interview showed that the contextual and graphing activities based on the learning cycle approach had enhanced the students' conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration.

4.2. Attitude Test Analysis

4.2.1. Questionnaire

The questionnaire was administered to 65 students in the experimental group for 60 minutes after the post-test to address the third research question: What are students' attitudes towards contextual and graphing activity-based learning cycle units? The questionnaire consisted of twelve closed-ended Likert-scale questionnaire and three open-ended questions (see Table 4.8). The open-ended questions were (1) which activity did they like the most and why? (2). which activity did they dislike the most and what improvement was needed for the activity they disliked? and (3) suggestions and comments regarding the activities in the learning units.

4.2.1.1 Likert-type item

The students' responses to each Likert-type item were analyzed by the frequency and mean of the students' responses to determine the students' attitudes towards the learning units as shown in Table.4.8.

Table 4.8 Students' responses to the questionnaire.

	Items	1	2	3	4	5	Mean	SD
1	I like the activities in learning calculus.	1	5	8	28	23	4.03	0.97
2	I found the activities in the calculus lessons interesting.	1	5	11	35	13	3.83	0.89
3	It was too difficult to learn calculus by doing these activities.	11	23	19	7	5	2.57	1.13
4	I find reading the textbook in detail is by itself sufficient for me to learn calculus.	31	16	12	3	3	1.93	1.13
5	I learn calculus better by reflecting on these activities instead of only by book and memorizing.	1	2	7	36	19	4.08	0.81
6	I look forward to solve more problems in calculus after the activities.	6	11	17	19	12	3.31	1.22
7	I will understand better if other topics in mathematics are taught using activities like the ones used in this calculus lessons.	2	7	14	26	16	3.72	1.05

Table 4.8 Students' responses to the questionnaire (cont.).

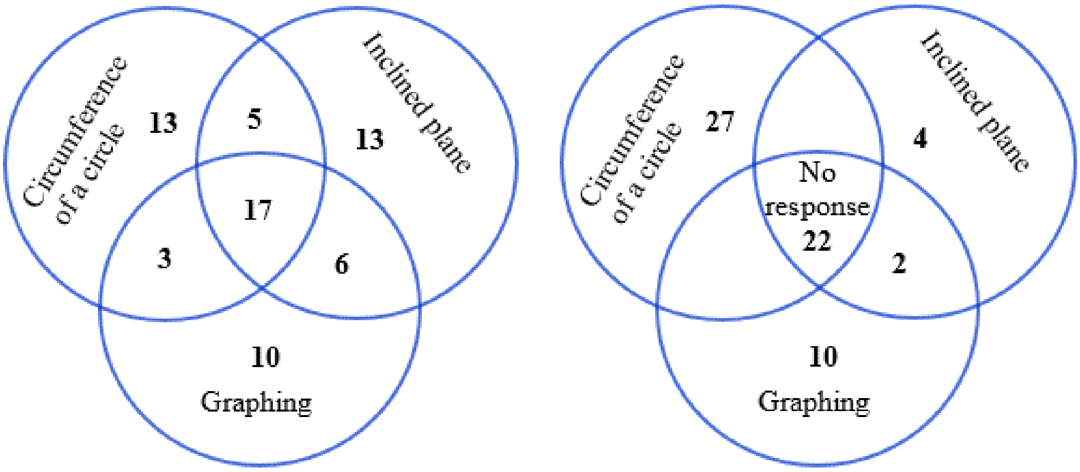
	Items	1	2	3	4	5	Mean	SD
8	The time was too short for the lessons.	4	4	13	26	18	3.77	1.11
9	The calculus lessons need more exercises until I understand and become fluent.	0	3	5	26	31	4.31	0.81
10	Mathematics teacher teaching calculus is enthusiastic in teaching calculus.	0	2	24	32	7	3.68	0.71
11	Mathematics teacher teaching calculus is encouraging and approachable during the lessons.	0	6	17	34	8	3.68	0.81
12	Mathematics teacher teaching calculus taught the lessons too fast and I could not follow the instruction.	17	18	15	11	4	2.49	1.23

In the closed-ended Likert-scale questionnaire, 51 out of 65 students liked the activities in the calculus lessons and 48 students found the activities in the learning units interesting and enriching. Thirty four students found that it was not very difficult to learn calculus by doing the activities and 47 students found that reading the textbook in detail was not sufficient for them to learn calculus. There were 55 students who responded that they learned calculus better by reflecting on the activities instead of only by reading books and memorizing, and 31 students looked forward to solve more problems in calculus after the activities. The majority of the students preferred to learn other topics in mathematics using activities like the ones used in the calculus lessons. There were also 44 students who found that the time was too short for the lessons in calculus and 57 students needed more exercises in the lessons.

The majority of the students felt that the time was too short for the lessons. The problem of time limitation was not under our control. We had to follow the mathematics curriculum issued by DCRD, which specified in detail how long each topic should be taught. However the majority of the students found that the instructor was enthusiastic in teaching calculus and encouraged the students in the learning process.

4.2.1.2 Open-ended questions

Figure 4.26 shows the number of students who liked and disliked the activities in the learning units.



(a) Liked activities (b) Disliked activities

Figure 4.26 Diagram showing the students' likes and dislikes of the activities.

The students mentioned how they felt during the activities and how the activities helped them in the learning process. For example, the students who liked the activity on measuring the circumference of a circle using a rope mentioned:

- *Experienced math practical* [activity] *for first time and got something idea* [limits].
- *Fun doing activity with my friends and our activity went on smoothly, it was first practical activity in mathematics*.
- *It was interesting* [activity] *and helped me understand well in that topic* [limit].

Students who liked the inclined plane activity mentioned:

- *It was a very suitable activity to understand the derivative* [differentiation] *and integration*.

Students who liked the graphing activity mentioned:

- *I can visualize the result through the graph. If the result matches with result i.e. algebraically, then I become more confident of the correctness. And graphing is least time taking and more accurate.*
- *I become more familiar and confident in doing graphing works. I enjoyed this* [graphing] *activity the most.*
- *I like* [the way] *was* [used] *to find slope. It is totally different from what the teacher taught me in lower classes.*

From students' excerpts, students enjoyed doing activities since these were their first practical mathematics activities which helped them in building up the concepts of limits, differentiation, and integration. The activities also helped students in the visualization (graphing) of the results and in building confidence in their accurate answers. Students also learned a way of calculating slope which was different from what they had learned in lower classes.

There are also few students who disliked the activities. The students who disliked the activities mentioned their specific problems and difficulties relating to the activities. For example:

- *The weather was not favorable* [too hot] *which resulted in poor concentration and lost interest to do activity.*
- *Teacher have to give main concepts of the* [inclined plane] *activity before. I did not get accurate reading.*
- *Time for the lesson* [activity] *was short, and could not get the exact answer.*
- *Time was too short to teach the vast syllabus of calculus*.
- *Calculus is very tough. Teacher should teach it slowly to understand more deeply*.

The activity concerning measuring the circumference of a circle using a rope could not be postponed and had to be conducted on the car parking, and on that particular day the weather was so hot and humid. The activities were designed in such a way that students had to explore by themselves to generate the concepts with minimum guidance from the teacher. Many students mentioned the limited time allotted for the activities and they wanted more examples and problems after the concepts were introduced.

The time limitation was not under our control. Due to the limitation of the time, most teachers gave priority to the width instead of the depth of knowledge by focusing on rote learning to cover all topics in the curriculum. The priority issue is universal, affecting even the most developed countries and it is not limited to mathematics. Major curriculum reform is needed for learning with understanding to become a norm rather than an exception.

4.2.2 Students' semi-structured interview results

To support and validate the students' responses to the questionnaire, six students from the experimental group were interviewed after the post-test. Four students found the class on calculus interesting and two of the students said:

- *It was very good.*
- *It was great.*

Students were further interviewed about in what ways the activities were helpful in learning calculus? Two students specifically mentioned the activity concerning measuring the circumference of a circle and the inclined plane activity as follows.

- *By doing the activity on measuring the circumference of a circle, I was able to have the mental picture about what actually limit means and through the inclined plane activity; I was able to know what actually derivative and integration is all about.*
- *When we draw a circle and measure with rope 6 or 7 times… it is related to limit and inclined plane is related to integration and differentiation.*

Four students did not specify the activities. Instead they described what they found helpful in the calculus class. For example:

- *It was new for us….we learned new things like, to find the limit, finding the derivative of the distance, we found out the velocity.*
- *I gained the knowledge about how to calculate the velocity and acceleration from the distance covered by any object on the graph.*

- *When we did the activities on graph, the answer that we got from the graph was as same as when we solved it algebraically*.

All the six students found that the class on calculus had interesting activities which were new to them and they never had practical activities in a mathematics class so far. However, two students mentioned that the time was too short for the lessons and the teacher taught the lessons very fast. The time constraint was the main factor that forced the teacher to rush to cover the topics in the allotted time which might compromise the quality of teaching. We had no control over the time and had to follow the directives and syllabus issued by DCRD which allotted specific time for each topic.

The overall analysis of the questionnaire and interview results revealed that the majority of the students found the activities in the learning units interesting and enriching. The activities also helped students to build the concepts in the calculus topics. Therefore we can conclude that the majority of the students had positive attitudes towards the developed learning units.

CHAPTER V
DISCUSSION AND CONCLUSIONS

Overview

This chapter summarizes and concludes the research findings. The discussions of the research findings and the contributions of research study to the field of mathematics education and to the teaching of mathematics are also discussed, followed by recommendations and suggestions for future studies.

5.1 Discussion of the Research Findings

This study was conducted to address the effectiveness of the contextual and graphing activity–based learning cycle units to enhance students' understanding of the fundamentals of calculus and the relationship between differentiation and integration compared to that of conventional teacher-centered and textbook-oriented instruction for grade-11 students in Bhutanese classrooms and to measure students' attitudes towards the learning units. There have been numerous studies done to improve the teaching of the fundamentals of calculus but there has been no concrete study done on teaching the fundamentals of calculus and the relationship between differentiation and integration using contextual and graphing activities based on the learning cycle approach.

Learning mathematics becomes concrete and it helps to visualize the concepts when real activities are used in the teaching and learning process (Doorman, 2002; Kwon, 2002). Students are motivated to learn when they are physically involved in the activities and get a real feel for the activities. There were three activities in the FC learning units to help students develop the intuitive ideas about limits, continuity, differentiation, and integration. They were the measuring the circumference, inclined plane, and graphing activities.

The measuring the circumference of a circle activity was designed to help students get the intuitive concepts of limits. The students had prior knowledge of finding the circumference of a circle using the formula $C = 2\pi r$. However, using a rope and taking fixed points on the circumference of a circle would give students the intuitive concepts of limits when they compare the result with the circumference of a circle calculated using the formula. This activity helped students to get a physical feel for limits before the graphing methods were introduced to teach the concepts of limits. The students could also visualize the limits intuitively from the graph when they sketched the graph of the number of fixed points per rope versus the perimeter of the polygon. To get better visualization from the graph, the students needed to take more fixed points per rope. During the activity, the students took only five fixed points per rope which were not very clear for visualizing that the perimeter of the polygon as the number of fixed points per rope increased approached the perimeter of the circle. The increase in the remaining length on the circumference was very small (in mm) as the number of fixed points per rope increased (see Table 3.3 in chapter 3). The students should be very careful and precise in measuring the remaining length after taking the fixed points on the circumference of the circle.

Galileo's inclined plane experiment to find the equation for a falling body by slowing down the motion of the falling body in 1638 was one of the greatest contributions to science (MacDougal, 2012; Straulino, 2008). Using the idea of Galileo's inclined plane, a locally available dissected pipe and a wooden plank were used to design the inclined plane experiment. Historically famous Galileo's inclined plane experiment has built the students' curiosity to learn calculus. However, revisiting the concepts of motion (distance, speed, and acceleration) was needed for the students to relate the inclined plane activity to the context of derivative and integral. The students should have an idea about the relationship between the time and the distance of a falling object in an ideal condition ($s \alpha\ t^2$) as they have already studied in physics. The time recorded by the students between the release of the ball from the top of the inclined plane and each stopping point deviated significantly from the theoretically expected time. This was an unexpected result. In fact to measure the time accurately, the inclined plane was set at minimal inclination (approximately 8 degrees) and students repeated the experiment for the minimum of five times for each stopping point. The major errors were committed by students when they were not able to time the starting and stopping of the ball accurately for each stopping point using a digital stop watch. Nonetheless, the students' recorded times for each stopping point were grouped together to align the recorded times with the expected time and the graph of distance versus time was sketch to teach the concepts of differentiation and integration.

A more accurate way of measuring time is to video record the inclined plane experiment. Mark the scale on the inclined plane and video record the release of the ball from the top to the end of the inclined plane and then ask the students to measure the time from the video recording. The video recording can save time and resources, and students can also use the video recording at home or anywhere if they can watch the video recording frame by frame.

Although the students should already be familiar with graphs, some of the students lacked the graphing skills. Some students had difficulty reading the graphs. Therefore, the specific areas where students commit errors in graphing should be investigated first and then a remedy course should be offered if necessary.

The pre-test scores of the students in the two schools were very low, which indicated that they had not learned the fundamentals of calculus and the relationship between differentiation and integration before. According to Department for Curriculum and Research Development's syllabus (2014), the introductory calculus course is in grade 11. During the time of the research study, the units on calculus had yet to start in both schools.

The post-test scores of the students in the control group were very low compared to those in the experimental group after the implementations of the FC learning units in the experimental group and of the conventional teacher-centered and textbook-oriented instruction in the control group. This result indicated that the students in the control group were more focused towards algebraic manipulation and they could not think outside the box to relate the algebraic ideas to graphs and contextual situations as opposed to those in the experimental group. In Bhutanese education, students usually study for examinations rather than conceptual understanding. Most teachers teach the contents of textbooks and students were hardly prepared to think beyond what they learn from textbooks. This result also indicated that the textbooks and syllabus need to change from grade 1, focusing towards conceptual understanding rather than algebraic manipulation. Besides, teachers also need professional supports to enhance their mathematics teaching skills. The courses for pre-service teachers in educational colleges must also be oriented towards preparing teachers to teach mathematics properly.

The pre-test and post-test scores of the students in the experimental group indicated that the learning units were effective in enhancing the conceptual understanding of the fundamentals of calculus. This study is consistent with the similar study conducted by Nevill and John (1998) where they tested the effects of a mathematical function plotter and an associated pedagogical approach in the teaching of elementary calculus to grade elevens. The treatment group was taught by the graphical computer-based approach while the control group was taught by a traditional limit-based approach. The results showed that the students in the treatment group were able to complete the standard algorithms of differentiation and interpret the results in a manner superior to those in the control group as the graphical approach placed the concepts in a context and thus enhancing the students' links between concept image and concept definition.

This study used contextually real experiments and graphing activities to teach the concepts of the fundamentals of calculus and the relationship between the differentiation and integration. The students could perform the activities to get the real feeling, use the graphs to visualize the results, and then relate to the algebraic methods. Another study by Orhun (2012) suggested that the real life situations and their graphical representations were needed by the students for better conceptual understanding of calculus which is in line with this study as contextually real and graphing activities were used to teach the concepts of calculus. Orhun investigated 102 high school students in grade 11 to find how they made the connection between the graphs of derived functions and some properties of the original functions. The results indicated that the students had difficulty in interpreting the graph of the derived function and did not use mathematical language to describe the graph of the derived function.

The overall results of this study indicated that the fundamentals of calculus learning units based on the learning cycle approach using contextual and graphing activities significantly outperformed the conventional teacher-centered and textbook-oriented instruction in teaching the fundamentals of calculus. Through the contextually real activities, the students explored the concepts by involving themselves in the activities and representing the collected data from the activities with graphs and then progressing to algebraic symbols and notations. Unlike the traditional teaching methods, Lawson-Abraham's model of learning cycle approach consists of three phases. Each phase helps the students to learn from a hands-on, exploratory approach by involving themselves in the activities where students invent new concepts. Then, the students re-evaluate the experiences they have encountered and accommodate them into their existing schemata. The whole process of Lawson-Abraham's model of learning cycle approach puts the students at the center of the learning experience.

Group activities lead to better interactions and development but success depends on group members who are competent and have desire to try and complete the activities. There were some groups that progressed with enthusiasm, while others were stuck and had to be prompted more during the instructional periods. The instructor guided the students by giving purposeful hints and suggestions. This was done for individual groups, thus addressing their specific needs and stages of learning.

The majority of the students who participated in the study said that they needed more time in the activities as well as more practice to solve problems. This is consistent with other studies (Klymchuk & Zverkova, 2001; Klymchuk, Zverkova, Gruenwald, & Sauerbier, 2010) where more than 500 university students from 9 countries also indicated that they found it difficult to move from the real world to the mathematical world because of limited time and practice in the application of the tasks. Practice is certainly one of the ways that help students to progress from novices to experts. According to Schwalbach and Dosemagen (2000), using concrete examples from physics classrooms helped students develop richer understanding of semantic as well as procedural knowledge in calculus which is similar to the finding of this study. In their study, students used data generated by the movement of a pendulum, the motion of an electronic cars on a flat horizontal surface, and a car's movement down a ramp to learn the physics concepts of position, speed, and acceleration and make the physics concepts' connections to the calculus concepts of function, differentiation, and integration.

5.2 Conclusions of the Research Study

In introductory calculus courses, a lot of attention is paid to how to do the calculations and manipulations of formulas instead of why and how they work. Students are usually taught calculus by means of what Ryan (1992) described as 'the rush to rule' where the meaning is ignored and students operate on a purely mechanical level, pushing symbols and notations on paper. This is the main problem that students face in conceptual understanding of calculus. Calculus originated from the study of motion which is realistic in nature with rich history and experiences common to all human beings. Students are hardly taught calculus using realistic or experimentally real situations and visualization tools. Although, there have been numerous studies done to teach the concepts of the fundamentals of calculus but there has been no concrete study done to improve the teaching of the relationship between differentiation and integration. Differentiation and integration are taught separately mostly using an algebraic approach which makes it difficult to visualize the relationship between differentiation and integration. Students just spell out the relationship by following what is written in textbooks—"Integration is the inverse process of differentiation"—and cannot explain how and why as it is difficult to visualize the relationship seeing only algebraic symbols and notations.

The pilot study was conducted for first-year undergraduate students who have already learned the introductory calculus course. The pre-test scores of the students indicated that the students encountered difficulty in establishing the relationship between differentiation and integration contextually and graphically. The learning unit on the relationship between differentiation and integration (R-DI) was designed using contextual examples and graphing activities based on Lawson-Abraham's model of learning cycle (LAMLC) and taught for one hour. The post-test scores and the answer to the attitude survey questionnaire of the students indicated that the students found the activities interesting and enriching probably because they took active roles in the lesson and felt motivated to learn calculus, hence leading to better performance. However, to really understand the relationship between differentiation and integration, students obviously need to understand both differentiation and integration which are themselves based on more fundamental concepts like limits and continuity of a function. Therefore, coherent lessons on the fundamentals of calculus based on the learning cycle approach using contextual and graphing activities were developed.

The R-DI learning unit based on Lawson-Abraham’s model of learning cycle (LAMLC) was extended to develop the fundamentals of calculus (FC) learning units. The FC learning units consisted of four learning units based on LAMLC using contextual and graphing activities. The measuring circumference and graphing activities were designed to teach the concept of limits. The graphs of functions were also used to teach the concepts of continuity and discontinuity of a function. The inclined plane experiment and graphing activities were designed to teach the concepts of differentiation, integration, and their relationship. The FC learning units were taught to 65 grade-11 students in the experimental group for about 960 minutes while textbook and teacher–centered instruction was use in the control group with 65 students in another school.

The five parallel open-ended pre-test and post-test questions were designed to investigate the students’ conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration. The twelve closed-ended Likert-type and three open-ended questions were used to find the students’ attitudes towards the developed learning units. Six students from the experimental group were interviewed to support and validate the findings from the conceptual tests and the questionnaire. The pre-test and post-test were administered to the students in both groups.

The data collected from the conceptual understanding tests and questionnaire were analyzed using inferential and descriptive statistics. The pre-test scores of the students in the two schools were very low and not significantly different which indicated that the students in both schools had the same level of background knowledge in the fundamentals of calculus. However, the post-test scores of the students in the experimental group were very high compared to those of the students in the control group with huge effect size. This result indicated that the developed FC learning units helped students to better enhance the conceptual understanding of the fundamentals of calculus and the relationship between differentiation and integration compared to the traditional teaching approach used for the students in the control group. The semi-structured interview of the six students showed that the majority of them understood the concepts of limits, differentiation, and integration, graphically and contextually. Moreover they could also figure out the relationship between differentiation and integration graphically and contextually even though the low achieving students had difficulty establishing the relationship between differentiation and integration.

The results from the questionnaire and the semi-structure interview of the students showed that the majority of the students found the activities in the learning units enriching and interesting. The students mentioned that the activities in the learning units helped them to understand the concepts of limits, differentiation, integration, and the relationship between differentiation and integration better as they were physically involved in the activities. The majority of the students had also mentioned that it was the first time in any mathematics class that they were experiencing the experimentally real activities and enjoyed doing the activities in groups as opposed to sitting and solving questions from the textbooks.

The findings from the post-test showed that the students in the experimental group outperformed those in the control group. Therefore, we could conclude that the contextual and graphing activity–based learning cycle units enhanced the students' understanding of the fundamentals of calculus and the relationship between differentiation and integration. The majority of the students' positive responses to the questionnaire and semi-structured interview showed that the students had positive attitudes towards the contextual and graphing activity–based learning cycle units.

Learning by doing is an unreplaceable approach consistent with the constructivist theory of learning. As said in Chinese proverb: "Tell me and I will forget. Show me and I may remember. Involve me and I will understand." When students are physically and mentally involved in the activities, they get the real feel of what they are doing which helps them retain what they have learned. In this study, the activities are designed to help students understand the calculus concepts better by involving them in the experimentally real activities.

5.3 Contributions of the Research Study

1) Teaching Calculus: The concepts of calculus are always taught beginning from functions, limits, continuity, differentiation, and integration at the end. The relationship between differentiation and integration is not stressed. Therefore, teaching students the relationship between differentiation and integration contextually and graphically would further enhance their concepts of differentiation and integration.

2) Teaching Mathematics: A mathematics classroom is rarely made realistic by involving students in activities that are experimentally real to students. The learning of mathematics becomes more concrete and attractive when experimentally real activities are used. Students become motivated and interested and enjoy learning mathematics when they are involved in the activities.

3) Mathematics Education: The learning cycle approach was originally developed and used in teaching science. However, the learning cycle approach can also be used in teaching mathematics as well. This study might set a platform for teaching calculus or mathematics as a whole using the learning cycle approach.

5.4 Recommendations for Future Study

The study should be conducted with more schools to ensure the effectiveness of the FC learning units.

The time allocated for the activities in the learning unit may be kept tentative to suit the time needed by the students for different activities. Some of the activities can be completed earlier than the time specified in the learning units and some may take more time, based on the ability of the students.

The process of learning is an important aspect of the study. Therefore, designing a formative assessment to follow students' learning process would help in evaluating more authentic learning outcomes.

In this study, the instructional learning units were developed using locally available materials and resources to fit with students' learning atmosphere. However, with the advancement of technology and computers, the concepts of the fundamentals of calculus and the relationship between differentiation and integration can be further developed in the form of multimedia provided that students are familiar with computers.

Otherwise, the developed multimedia can be used as supplementary in the teaching of calculus concepts. The concepts of calculus such as limits, differentiation, and integration are very difficult to visualize but using a computer of program would help visualize even a tiny point.

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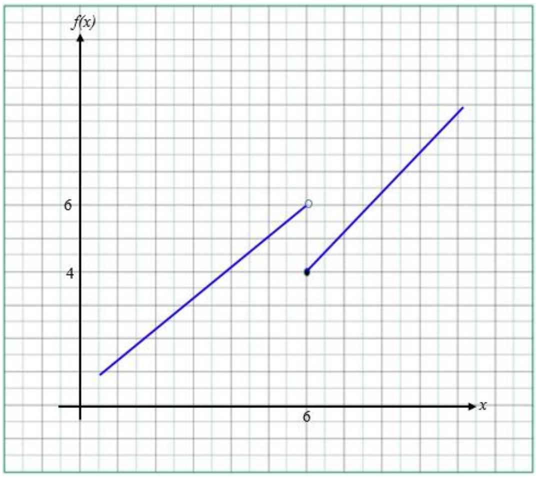
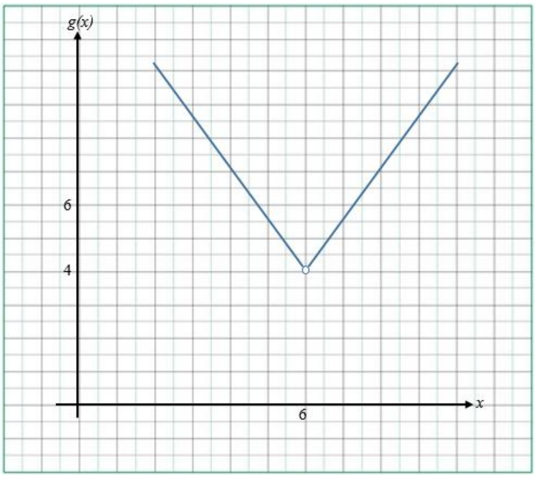
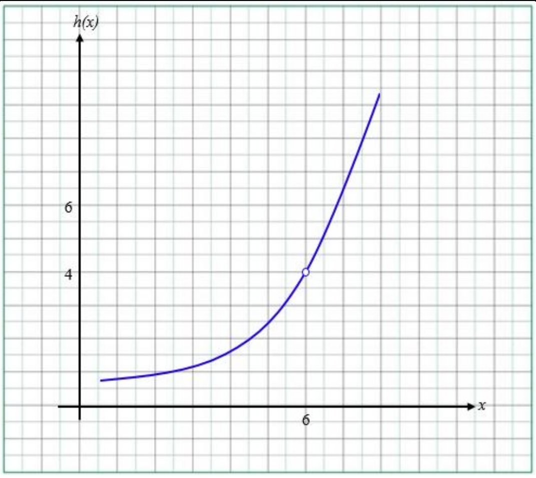
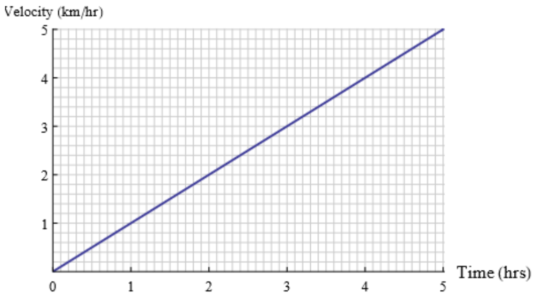
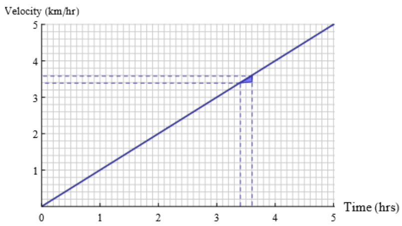
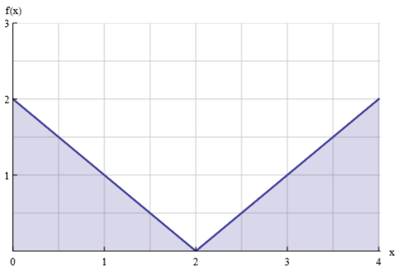
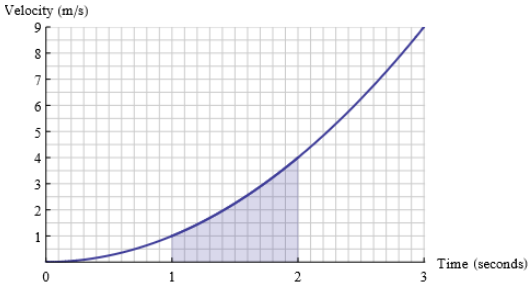
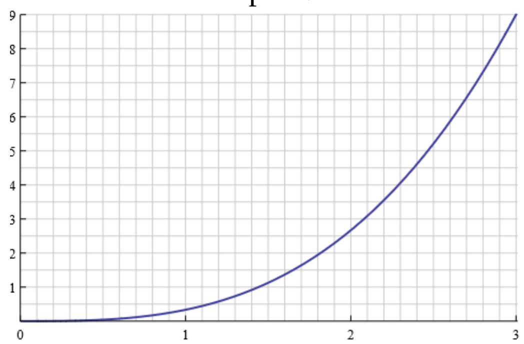
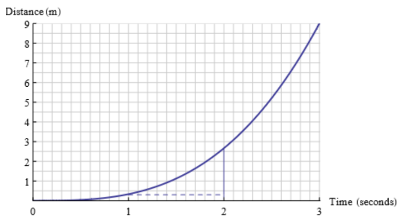
APPENDICES

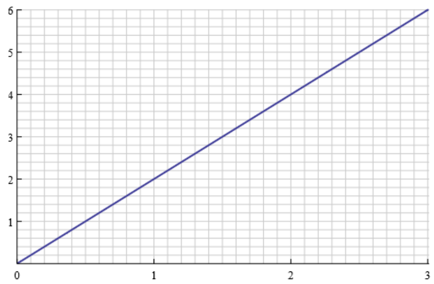
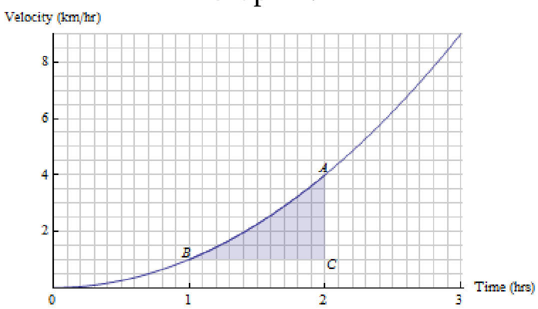
APPENDIX A

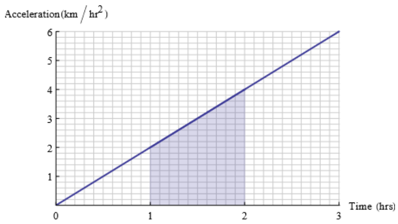
MARKING CRITERIA FOR PRE-TEST

Questions	Solutions	Marks
Question 1 Below are the graphs of three functions $f(x)$ $g(x)$ and $h(x)$. Which of the following functions has the limit at 6? Why and why not?  	As x approaches 6 from the left, $f(x)$ approaches 6. Thus, $\lim_{x\to6^-} f(x)=6$ As x approaches 6 from the right, $f(x)$ approaches 4. Thus, $\lim_{x\to6^+} f(x)=4$ Since LHL $\neq$ RHL, $\lim_{x\to6} f(x)$ does not exists.	0.5 0.5
 	As x approaches 6 from the left, $g(x)$ approaches 4. Thus, $\lim_{x\to6^-} g(x)=4$ As x approaches 6 from the right, $g(x)$ approaches 4. Thus, $\lim_{x\to6^+} g(x)=4$ Since LHL = RHL, $\lim_{x\to6} g(x)$ exists.	0.5 0.5

Questions	Solutions	Marks
	As x approaches 6 from the left, $h(x)$ approaches 4. Thus, $\lim_{x\to 6^-} h(x) = 4$ As x approaches 6 from the right, $h(x)$ approaches 4. Thus, $\lim_{x\to 6^+} h(x) = 4$ Since LHL = RHL, $\lim_{x\to 6} h(x)$ exists.	0.5 0.5
Question 2 The graph below represents the relationship between velocity and time of a moving object.  a) Find the derivative at $t = 3.5$ hours graphically from the velocity-time graph? b) What does the derivative represent?	Showing on the graph  a) The derivative at $t = 3.5$ $= \frac{3.6-3.4}{3.6-3.4} = 1$ km/hr^2 The derivative represents the acceleration.	1 1 1
Question 3 Evaluate $\int_0^4 \lvert x-2 \rvert\, dx$ using the graph of $\lvert x-2 \rvert$?	Sketching the graph  Shading the area under the line	1 1

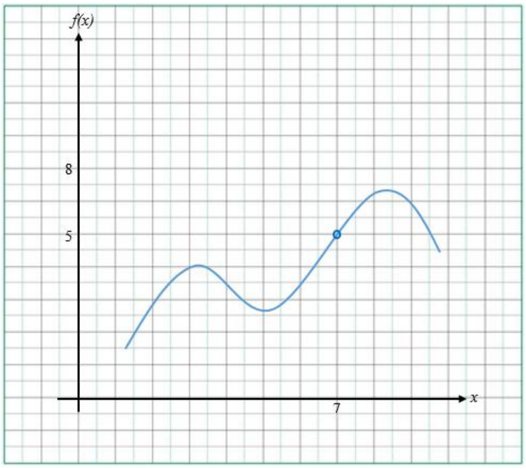
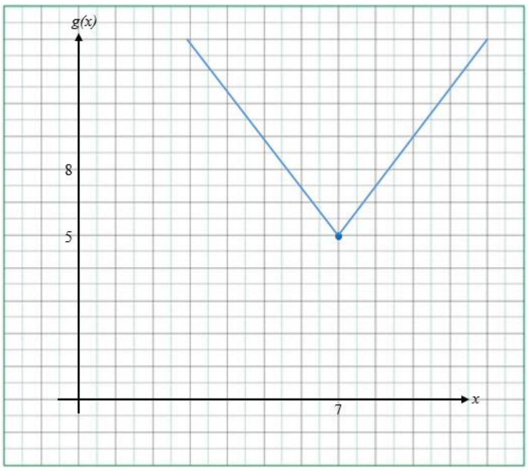
Questions	Solutions	Marks
	Finding the area under the line Area of 2 triangles under the line $2\left[\frac{1}{2}\times 2\times 2\right]=4$ units^2.	1
Question 4 Graph 1.2 is the anti-derivative of Graph 1.1. Graph 1.1 is a velocity-time graph.  Graph 1.1  Graph 1.2 a) What is the area of the shaded region under the line on Graph 1.1? b) What is the unit of area of the shaded region under the line? c) Show on Graph 1.2 the area of shaded region under the line on Graph 1.1? What does it indicate on Graph 1.2? d) What is the unit of area of the shaded region under the line on Graph 1.1?	a) The area of shaded region = 2.33 m.	1
	b) Meters.	1
	c) The derivative or distance from t = 1 to 2 seconds Showing the shaded region on the graph 1.2  	1 1
	d) Meter per second.	1

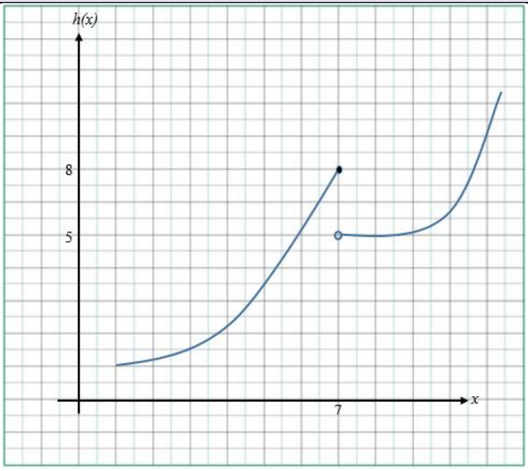
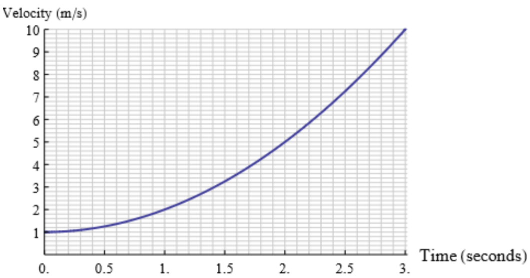
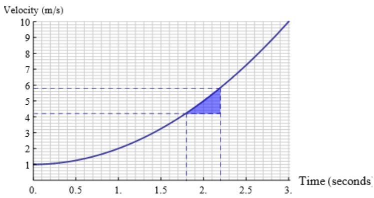
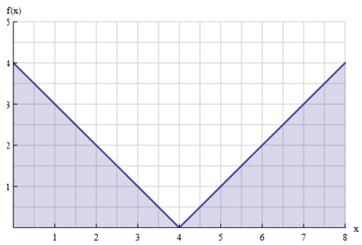
<table>
<tr><th>Questions</th><th>Solutions</th><th>Marks</th></tr>
<tr><td>d) What would be the unit of the slope of the line on Graph 1.2?
e) Label the unit of the axes on Graph 1.2 in relation to Graph 1.1?</td><td>e) Horizontal axis: Time in seconds and Vertical axis: Distance in meters.</td><td>1</td></tr>
<tr><td rowspan="4">Question 5
Graph 2.1 is the derivative of Graph 2.2. Graph 2.2 which is a velocity-time graph

Graph 2.1


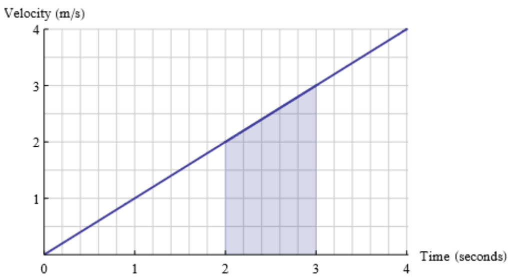
Graph 2.2
a) What does the height AC of triangle ABC on the Graph 2.2 represent on the Graph 2.1? Show it on Graph 2.1 by shading it?
b) What is the unit of the area you have just shaded on the Graph 2.1?
c) What is the unit of the slope of the line on Graph 2.2?
d) Label the units of the axes on Graph 2.1 in relations to the Graph 2.2</td><td>a) The area under the line from $t =$ 1 to 2 hours.
Showing it on the Graph 2.1

</td><td>1
1</td></tr>
<tr><td>b) The unit of the area of the shaded region on Graph 2.1 is kilometer per hour.</td><td>1</td></tr>
<tr><td>c) The unit of the slope of the line on Graph 2.2 is kilometer per hour squared.</td><td>1</td></tr>
<tr><td>d) Horizontal axis: Time in kilometer.
Vertical axis: Acceleration in kilometer per hour squared.</td><td>1</td></tr>
</table>

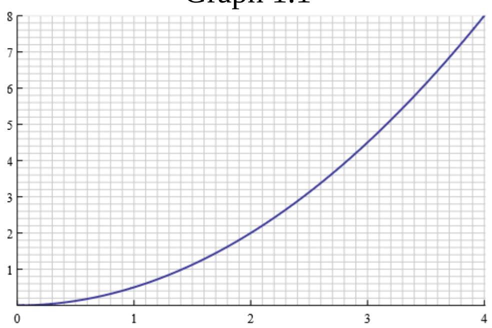
APPENDIX B

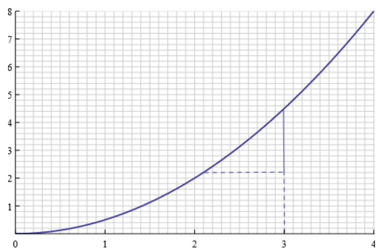
MARKING CRITERIA FOR POST-TEST

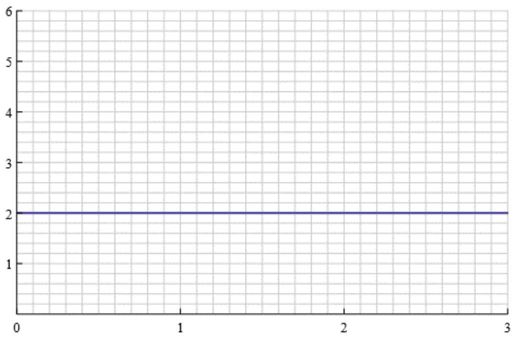
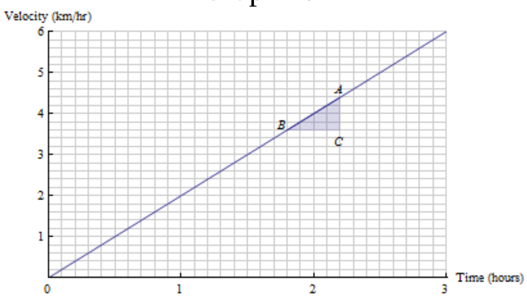
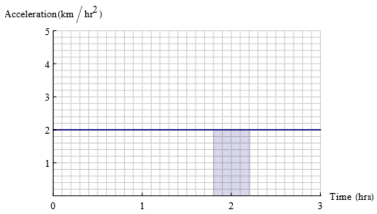
Questions	Solutions	Marks
Question 1 Below are the graphs of three functions $f(x)$, $g(x)$ and $h(x)$. Which of the following functions has the limit at 7? Why and why not? 	As x approaches 7 from the left, $f(x)$ approaches 5. Thus, $\lim_{x\to 7^-} f(x)=5$ As x approaches 7 from the right, $f(x)$ approaches 5. Thus, $\lim_{x\to 7^+} f(x)=5$ Since LHL = RHL, $\lim_{x\to 7} f(x)=5$ exists.	0.5 0.5
	As x approaches 7 from the left, $g(x)$ approaches 5. Thus, $\lim_{x\to 7^-} g(x)=5$ As x approaches 7 from the right, $g(x)$ approaches 5. Thus, $\lim_{x\to 7^+} g(x)=5$ Since LHL = RHL, $\lim_{x\to 7} g(x)=5$ exists.	0.5 0.5

Questions	Solutions	Marks
 	As x approaches 7 from the left $h(x)$ approaches 5. Thus, $\lim_{x\to 7^-} h(x)=8$ As x approaches 7 from the right, $h(x)$ approaches 8. Thus, $\lim_{x\to 7^+} h(x)=8$ Since LHL $\neq$ RHL, $\lim_{x\to 7} h(x)$ does not exists.	0.5 0.5
Question 2 The graph below represents the relationship between velocity and time of a moving object.  a) Find the derivative at $t=2$ seconds graphically from the velocity-time graph? b) What does the derivative represent?	Showing on the graph  a) The derivative at $t=2$ seconds $=\frac{5.4-4.6}{2.1-1.9}$ $=\frac{0.8}{0.2}=4$ m/s^2 b) The derivative represents the acceleration.	1 1 1
Question 3 Evaluate $\int_0^8 \lvert x-4\rvert\,dx$ using the graph of $\lvert x-4\rvert$?	Sketching on the graph  Shading the area under the line	1 1

<table>
<tr><th>Questions</th><th>Solutions</th><th>Marks</th></tr>
<tr><td></td><td>Finding the area under the line.
Area of 2 triangles under the line $= 2\left[\frac{1}{2}\times 4\times 4\right] = 16$ units^2.</td><td>1</td></tr>
<tr><td rowspan="4">Question 4
Graph 1.2 is the anti-derivative of Graph 1.1. Graph 1.1 is a velocity-time graph.


Graph 1.1


Graph 1.2
a) What is the area of the shaded region under the line on Graph 1.1?
b) What is the unit of area of the shaded region under the line?
c) Show on Graph 1.2 the area of shaded region under the line on Graph 1.1? What does it indicate?</td><td>a) Area of the rectangle $= 2\times 1 = 2$ m.
Area of the triangle $= \frac{1}{2}\times 1\times 1 = 0.5$ m.
The area of shaded region $= 2 + 0.5 = 2.5$ m.</td><td>1</td></tr>
<tr><td>b) Meter.</td><td>1</td></tr>
<tr><td>c) The derivative or distance from $t =$ 2 to 3 seconds.
Showing the shading region on the graph 1.2.
</td><td>1
1</td></tr>
<tr><td>d) Meter per second.</td><td>1</td></tr>
</table>

Questions	Solutions	Marks
d) What would be the unit of the slope of the line on Graph 1.2? e) Label the unit of the axes on Graph 1.2 in relation to Graph 1.1?	e) Horizontal axis: Time in seconds and Vertical axis: Distance in meter.	1
Question 5 Graph 2.1 is the derivative of Graph 2.2 which is a velocity-time graph  Graph 2.1  Graph 2.2 a) What does the height *AC* of triangle *ABC* on Graph 2.2 represent on Graph 2.1? Show it on Graph 2.1 by shading it?	a) The area under the line from =1.8 hours to 2.2. hours. Showing it on the Graph 2.1  	1 1
b) What is the unit of the area you have just shaded on the Graph 2.1?	b) The unit of the area of the shaded region on Graph 2.1 is kilometer per hour.	1
c) What is the unit of the slope of the line on Graph 2.2?	c) The unit of the slope of the line on Graph 2.2 is kilometer per hour squared.	1
d) Label the units of the axes on Graph 2.1 in relation to Graph 2.2	d) Horizontal axis: Time in hours. Vertical axis: Acceleration in kilometer per hour squared.	1

APPENDIX C
INTERVIEW PROTOCOL

Question I: Learning Units

i). How do you think was the class on calculus?

a) How was it different from the other class?

b) Can you give some reason?

ii). Tell me about activities that you did in class?

a) How was the learning from activities different from the normal class in mathematics?

b) What did you learn from the activities regarding the concept of limit, derivatives and integration?

c) Can you explain me in detail?

iii). Which activities you disliked the most?

a) Can you give some reasons?

b) For the activities you disliked, could you give comments to make these activities better?

Question II: Limits

Given the function $f(x) = \frac{x^2 - 4}{x - 2}$

a) At which point the function is not defined?

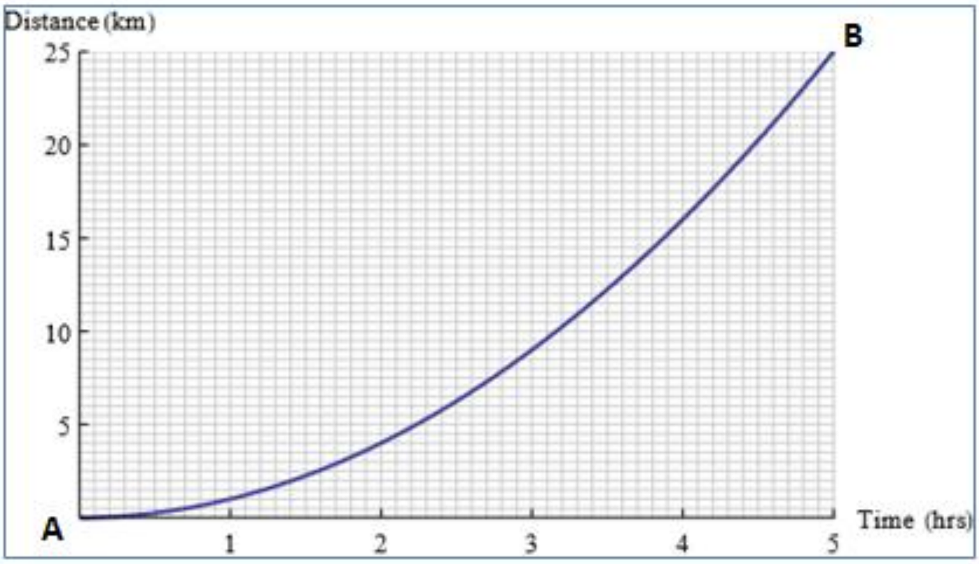
b) Sketch the graph of the function $f(x) = \frac{x^2 - 4}{x - 2}$

c) Does the limit exists at the point the function is not defined?

d) What is the limit of the function when x approaches 2 if the function is not defined at x =2?

Question III: Differentiation and Integration.

The graph 1(a) below represents a car travels from place A to place B, and its distance *s* kilometers from the starting point A is given by the function, $s(t) = t^2$ where t is the time taken in hours.



Graph 1(a)

Answer the following questions.

a) How far has the car travelled in 5 hours?

b) How long does the car takes to travel 16 km?

c) What is the car's average velocity over; (i) during the first 2 hours? (ii) during the first 4 hours? (iii) between t = 2 hours and t = 5 hours.

d) What is the velocity of the car at t = 2 hours? Explain how you get your answer.

e) Can you find the velocity of the car at t = 2, 3, 4, and 5 hours and sketch the velocity of the car on another graph 1(b).

f) What is the area under the newly sketched graph in (e); (i) during the first 1 hour? (ii) during the first 2 hours? (iii) during the first 3 hours? (iv) during the first 4 hours? (v) during the first 5 hours?

g) Sketch the area under the curve on another graph 1(c). What does the area under the graph tells you regarding the graph 1(a)?

APPENDIX D
ATTITUDE QUESTIONNAIRE

In order to describe the characteristics of your class as a whole, we need your responses to the following items.

1. What is your gender? ☐ Male ☐ Female

2. How old are you?

☐ Below 15 years old ☐ 15 years old ☐ 16 years old

☐ 17 years old ☐ 18 years old ☐ Above 18 years old

Directions.

The questions below are designed to identify your attitudes towards the learning cycle unit used in teaching of the fundamentals of calculus and the relationship between differentiation and integration. The item scale has 5 possible responses. The responses range from 1 to 5: *1 = Strongly Disagree, 2 = Disagree, 3 = Neither Agree nor Disagree, 4 = Agree and 5 = Strongly Agree*. Please read each statement. From the 5-point scale, mark the response with tick mark **(√)** that most clearly represents your degree of agreement or disagreement with that statement. Try not to think too deeply about each response; there are no correct or incorrect answers. Please respond to all of the statements.

Please use the 5-point scale to indicate your agreement or disagreement with each statement by ticking (√) the appropriate response		**1**	**2**	**3**	**4**	**5**
1	I like the activities in the calculus lessons.					
2	I found the activities in the calculus lesson interesting.					
3	It was too difficult to learn calculus by doing these activities.					
4	I find reading the textbook in detail is by itself sufficient for me to learn calculus.					

Please use the 5-point scale to indicate your agreement or disagreement with each statement by ticking (√) the appropriate response		**1**	**2**	**3**	**4**	**5**
5	I learn calculus better by reflecting on these activities instead of only by book and memorizing.					
6	I look forward to solve more problems in calculus after the activities.					
7	I will understand better if other topics in mathematics are taught using activities like the ones used in this calculus lessons.					
8	The time was too short for the lessons.					
9	The calculus lessons need more exercises until I understand and become fluent.					
10	The mathematics teacher teaching calculus is enthusiastic in teaching calculus.					
11	The mathematics teacher teaching calculus is encouraging and approachable during the lessons.					
12	The mathematics teacher teaching calculus taught the lessons too fast and I could not follow the instruction.					

1) Which activity you liked the most? Why?

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2) Which activity you disliked the most? What improvement is needed for the activity you disliked?

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3) Any other suggestions and comments:

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APPENDIX E
VALIDITY TEST FOR QUESTIONNAIRE (IOC INDEX)

Instructions

This table is for the experts to consider the validity between questions and the students' attitudes test towards the learning units. If you are sure that the questions can test the students' attitudes towards the learning units**,** please tick (√) for (+1) in the appropriate box. If you are not sure, please tick (√) of (−1), and if you are uncertain, tick (√) of (0).

Objectives	**Questions**	**Results**		
		+1	**0**	**-1**
To find how students feel towards the activities in the learning units.	I like the activities in the calculus lessons.			
	I found the activities in the calculus lesson interesting.			
	It was too difficult to learn calculus by doing these activities.			
	I find reading the textbook in detail is by itself sufficient for me to learn calculus.			
	I learn calculus better by reflecting on these activities instead of only by book and memorizing.			
	I look forward to solve more problems in calculus after the activities.			

Objectives	Questions	Results		
		+1	0	-1
	I will understand better if other topics in mathematics are taught using activities like the ones used in this calculus lessons.			
To find the time required for the activities in the learning units.	The time was too short for the lessons.			
	The calculus lessons need more exercises until I understand and become fluent.			
To find how students feel towards the teacher in the process of teaching the learning units.	The mathematics teacher teaching calculus is enthusiastic in teaching calculus.			
	The mathematics teacher teaching calculus is encouraging and approachable during the lessons.			
	The mathematics teacher teaching calculus taught the lessons too fast and I could not follow the instruction.			

Comments and Suggestions:

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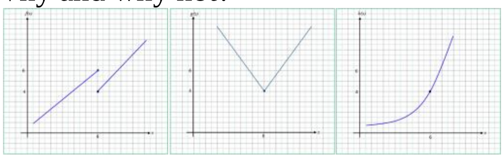
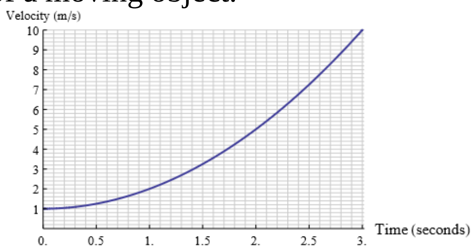
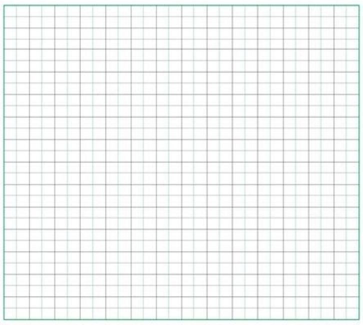
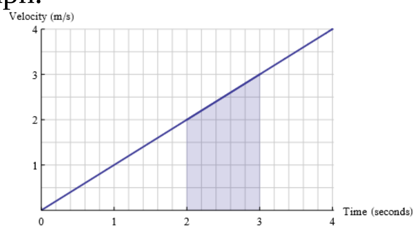
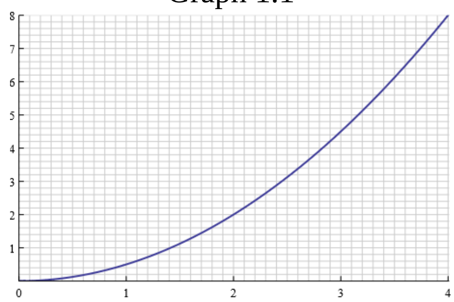
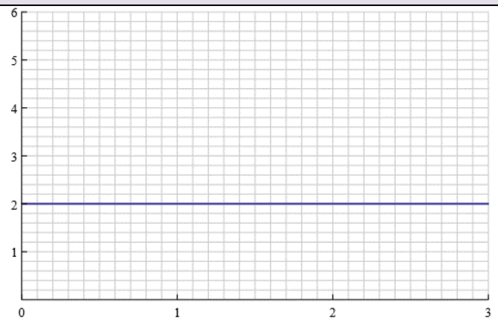
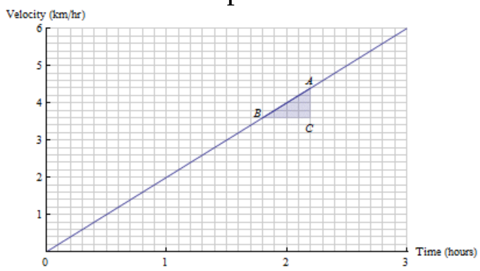
APPENDIX F
CONTENT VALIDITY TEST (IOC INDEX)

Instructions

This table is for the experts to consider the validity between questions and the students' conceptual test of the fundamentals of calculus (limits, differentiation and integration) and the relationship between differentiation and integration according to calculus units of grade-11 mathematics syllabus. If you are sure that the questions can test the students' conceptual understanding of the fundamentals of calculus, and the relationship between differentiation and integration accordingly to calculus units of grade-11 mathematics syllabus**,** please tick (√) for (+1) in the appropriate box. If you are not sure, please tick (√) of (−1) and if you are uncertain, tick (√) of (0).

Objectives	Questions	Topics	Results		
			+1	**0**	**-1**
To find out students' conceptual understanding of limit from the sketched graph.	Below are the graphs of three functions $f(x)$, $g(x)$ and $h(x)$. Which of the following functions has the limit at 7? Why and why not? 	Limits			
To find out student's conceptual understanding of the derivative from the graph relating to a particular context.	The graph below represents the relationship between velocity and time of a moving object.  a) Find the derivative at $t = 2$ seconds graphically from the velocity-time graph?	Differentiation			

Objectives	Questions	Topics	Results		
			+1	0	-1
	b) What does the derivative represent?				
To find out students understanding of integration in terms of area under the graph and also to see how students connect the algebraic method of integration to graphical form.	Evaluate $\int_0^8 \lvert x-4 \rvert dx$ using the graph of $\lvert x-4 \rvert$? 	Integration			
To find out graphically how students relate integral (area under the graph) to the derivative of the graph and also to see the relationship from integration to differentiation.	Graph 1.2 is the anti-derivative of Graph 1.1. Graph 1.1 is a velocity-time graph.  Graph 1.1  Graph 1.2 a) What is the area of the shaded region under the line on Graph 1.1? b) What is the unit of area of the shaded region under the line? c) Show on Graph 1.2 the area of shaded region under the line on Graph 1.1? What does it indicate? d) What would be the unit of the slope of line on Graph 1.2? e) Label the unit of the axes on Graph 1.2 in relation to Graph 1.1?	Relationship between differentiation and integration			
To find out graphically how	Graph 2.1 is the derivative of Graph 2.2 which is a velocity-time graph	Relationship			

Objectives	Questions	Topics	Results		
			+1	0	-1
students relate derivative to integral and also see the relationship from differentiation to integration.	 Graph 2.1  Graph 2.2 a) What does the height *AC* of triangle *ABC* on Graph 2.2 represent on Graph 2.1? Show it on Graph 2.1 by shading it? b) What is the unit of the area you have just shaded on the Graph 2.1? c) What is the unit of the slope of the line on Graph 2.2? d) Label the units of the axes on Graph 2.1 in relation to Graph 2.2	between differentiation and integration			

Comments and Suggestions:

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………………………………………………………………………………………

Signature:…..……………………………………………..

Name:……………….………………………………...

Position:………........……………………………….

APPENDIX G
IOC INDEX CALCULATION

I. Conceptual Understanding Test

Items	$\frac{R_{ik}-(-1)}{2}$				$IOC_k = \frac{\sum_{i=1}^{N} \frac{R_{ik}-(-1)}{2}}{N}$	Interpretation
	E1	E2	E3	E4		
1	1	1	1	1	1	Appropriate
2	1	1	1	1	1	Appropriate
3	1	0.5	1	1	0.85	Appropriate
4	1	0.5	1	1	0.85	Appropriate
5	1	0.5	0.5	1	0.75	Appropriate
Average IOC					**0.89**	**Appropriate**

*** E1, E2, E3 and E4 stands for Expert 1, 2, 3, and 4**

II. Questionnaire

Items	$\frac{R_{ik}-(-1)}{2}$				$IOC_k = \frac{\sum_{i=1}^{N} \frac{R_{ik}-(-1)}{2}}{N}$	Interpretation
	E1	E2	E3	E4		
1	1	1	1	1	1	Appropriate
2	1	1	1	1	1	Appropriate
3	1	1	1	1	1	Appropriate
4	1	1	1	1	1	Appropriate
5	1	1	1	1	1	Appropriate
6	1	1	1	1	1	Appropriate
7	1	1	1	1	1	Appropriate
8	1	1	1	1	1	Appropriate
9	1	1	1	1	1	Appropriate
10	1	1	1	1	1	Appropriate
11	1	1	1	1	1	Appropriate
12	1	1	1	1	1	Appropriate
Average IOC					**1**	**Appropriate**

APPENDIX H
IRB APPROVAL LETTER

COA. No. 2013/070.1908

Certificate of Approval
Mahidol University Institutional Review Board (MU-IRB)

Protocol No.: MU-IRB 2013/087.0908

Title of Project: Developing Contextual-and-Graphing-Activity-Based Learning Cycle Unit to Enhance Students' Understanding of the Fundamentals of Calculus and the Relationship between Differentiation and Integration

Principal Investigator: Mr.Kinley

Affiliation: Instituted for Innovative Learning

Approval includes:
1) MU-IRB Submission form version date 16 August 2013
2) Research Protocol version date 16 August 2013
3) Participant Information Sheet version date 16 August 2013
4) Participant Information Sheet for Interview version date 16 August 2013
5) Informed Consent form version date 9August 2013
6) The Attitude Survey Questionnaires version date 9August 2013
7) Interview Guideline version date 9August 2013
8) Questionnaire for Pre-test and Post-test version date 16 August 2013

Mahidol University Institutional Review Board is in full compliance with International Guidelines for Human Research Protection such as Declaration of Helsinki, The Belmont Report, CIOMS Guidelines and the International Conference on Harmonization in Good Clinical Practice (ICH-GCP)

Date of Approval: 19 August 2013

Date of Expiration: 18 August 2014

Signature of MU-IRB Chair: .... 19 August 2013
(Professor Shusee Visalyaputra) version date

Signature of Institute Representative: .... 19 August 2013
(Professor Prasit Palittapongarnpim) version date
Vice President for Research

Office of the President, Mahidol University, 999 Phuttamonthon 4 Rd., Salaya, Phuttamonthon District, Nakhon Pathom 73170. Tel. (662) 8496223-5 Fax. (662) 8496223

COA MU-IRB version date 31 January 2013

BIOGRAPHY

NAME	Kinley
DATE OF BIRTH	6 March 1980
PLACE OF BIRTH	Themdangbi, Mongar, Bhutan
INSTITUTIONS ATTENDED	Sherubtse College, Kanglung, Bhutan ISC Class XII Science (2000-2001) Bachelor of Education (Secondary) (2003-2005) Sherubtse College, Royal University of Bhutan, Bhutan Master of Science (Science and Technology Education) (International Postgraduate Program) (2012-2014)
SCHOLARSHIP	Thailand International Development Cooperation Agency (TICA) Ministry of Foreign Affairs, Thailand
POSITION AND OFFICE	Teacher Samtse Higher Secondary School Samtse, Bhutan (2006-2012)
HOME ADDRESS	Themdangbi, Mongar Geog Mongar, Bhutan Mobile No.:+ 975-77684080 Email: kinleay@hotmail.com

PRESENTATION AND PUBLICATION

Kinley, Laosinchai, P & Wongkia, W. (2014). Employing contextual examples and graphing activities to enhance students' understanding of the relationship between differentiation and integration in calculus. In *Proceedings of the 2nd ASEAN plus Three Graduate Research Congress (2ndAGRC) on Research and Innovation.* (pp. 509-516). Nakhon Pathom, Thailand: Faculty of Graduate Studies, Mahidol University.