Songklanakarin J. Sci. Technol.
39 (3), 275-291, May - Jun. 2017

SJST
http://www.sjst.psu.ac.th

Original Article

Fuzzy $\Lambda_r^{g_X}$-sets and generalization of closed sets in generalized fuzzy topological spaces

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Received: 11 January 2016; Accepted: 26 May 2016

Abstract

The aim of this paper is to introduce fuzzy $\Lambda_r^{g_X}$-set and to study some of its generalized forms. The structure constituted by fuzzy $\Lambda_r^{g_X}$-set is also included in the treatise. Moreover we define generalized fuzzy g_X-closed set and fuzzy $\Lambda_r^{g_X}$-closed set, which are the weaker forms of fuzzy g_X-closed set. The above two weaker forms are independent to each other but jointly they imply fuzzy g_X-closed set. Finally, in addition, as the application of fuzzy $\Lambda_r^{g_X}$-closed set, we study fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuity and fuzzy $(\Lambda_r^{g_X}gf - g_X, g_Y)$-continuity and a few of their properties. In general, fuzzy contra (g_X, g_Y)-continuity does not imply fuzzy (g_X, g_Y)-continuity, but this paper provides one condition under which the fact is true.

Keywords: fuzzy $\Lambda_r^{g_X}$-set, fuzzy Λ^{g_X}-closed set, fuzzy $\Lambda_r^{g_X}$-closed set, $\Lambda_r^{g_X}$gf-g_X- closed set, fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuity

1. Introduction

Maki (1986) introduced the concept of Λ-set, which is a feeble form of open set in a topological space. And this Λ-set gives a new direction in the field of topological space. Thereafter, different authors gave many approach of Λ-set in various manners. In the literature, different types of Λ-sets, namely Λ_S-sets(2000), pre-Λ(2002), Λsp-sets(2004), Λ_m-sets(2005), Λ_b-sets(2006), Λ_α-sets(2007), Λ_δ-sets(2009), Λ_r-sets(2011) exist. Using the concept of Λ-sets, Arenas *et al.* (1997) initially defined the λ-closed set, which is an intersection of a Λ-set and a closed set. Shafei and Zakari (2006) introduced fuzzy Λ-set in fuzzy settings. Fuzzy Λ_b-set has been studied by Aslim and Günel (2009) in fuzzy topological spaces.

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Császár (2002) introduced the notions of generalized neighborhood systems and generalized topological spaces. Chetty (2008) extended the concept of generalized topological space in fuzzy environment and named it generalized fuzzy topological space. Caldas *et al.* (2008) introduced Λ_g-closed set in a topological space, which is a weaker form of closed sets and stronger form of g-closed sets in topological spaces.

Balasubramanian *et al.* (2014) introduced the above concept in fuzzy environment. Ekici and Roy (2011) extended the concept of Λ-sets in generalized topological space and named it Λ_μ-sets, and they have shown that (X, Λ_μ) is an Alexandroff space. The concept of Λ-sets in generalized fuzzy environment was studied by Bhattacharya *et al.* in 2015, and named it fuzzy generalized Λ^{g_X}-set.

The aim of our paper is to continue the research work in the similar direction but using a different approach. Here, we study fuzzy Λ^{g_X}-closed set. In the literature we have seen that every closed set is Λ-closed set but the converse may not always be true. This paper presents two conditions where the converse is also true.

In section 3 we define fuzzy $\Lambda_r^{g_X}$-set and fuzzy $V_r^{g_X}$-set . Here we prove that the collection of fuzzy $V_r^{g_X}$-set forms a fuzzy supra topology.

We study the concept of fuzzy $\Lambda_r^{g_X}$-closed set in generalized fuzzy topological spaces and construct one equivalent form of closed set with the help of fuzzy g_X-semi closed set in section 4.

Section 5 is devoted to study $\Lambda_r^{g_X}$gf-g_X- closed set which is a weaker form of fuzzy g_X-closed set. Caldas *et al.*(2008) have shown one equivalent condition with the help of locally closed set but we formulate the same equivalent condition with the help of fuzzy $\Lambda_r^{g_X}$-closed set which is a weaker form of fuzzy g_X-locally closed set.

In section 6, we study the application of fuzzy $\Lambda_r^{g_X}$-closed set by introducing fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuity and establish some related results.

2. Preliminaries

Let X be a nonempty set and g_X be a collection of fuzzy subsets of X. Then g_X is called a generalized fuzzy topology on X iff $0_X \in g_X$ and $G_i \in g_X$ for $i \in I \neq \phi$ implies $G = V_{i\in I} G_i \in g_X$. The pair (X, g_X) is called a generalized fuzzy topological space (GFTS, for short) on X. The elements of g_X are called fuzzy g_X-open set and their complements are called fuzzy g_X-closed set. The collection of all fuzzy g_X- open sets is denoted by GFO(X, g_X) and the collection of all fuzzy g_X-closed sets is denoted by GFC(X, g_X). The fuzzy g_X-closure of a subset λ of X, denoted by $c_{g_X}(\lambda)$, is the intersection of all fuzzy g_X-closed sets containing λ. The interior of λ, denoted by $i_{g_X}(\lambda)$, is the union of fuzzy g_X-open sets contained in λ. The complement of a fuzzy set λ is denoted by λ'.

2.1 Definition (Balasubramanian *et al.*, 1997)

Let X be a fuzzy topological space. A fuzzy set λ in X is called a generalized fuzzy closed set (in short, gfc) iff $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy open.

2.2 Definition (Bhattacharya *et al.*, 2015)

Let A be a fuzzy subset of a GFTS (X, g_X).Then the kernel of A is denoted by $\Lambda^{g_X}(\mathrm{A})$ or ker(A) is defined by

$$\Lambda^{g_x}(\mathrm{A}) = \begin{cases} \wedge \quad \{\mathrm{B} \in \mathrm{GFO}(\mathrm{X}, \mathrm{g_x}) : \mathrm{A} \leq \mathrm{B}\}, \text{if there exist } \mathrm{B} \in \mathrm{GFO}(X, g_X) \\ 1_{\mathrm{X}}, \quad \text{otherwise.} \end{cases}$$

2.3 Definition (Bhattacharya *et al.*, 2015)

Let A be a fuzzy subset of a GFTS (X, g_X). Then a fuzzy set A is defined by

$$\mathrm{V}^{g_x}(\mathrm{A}) = \begin{cases} \vee \quad \{\mathrm{B} \in \mathrm{GFC}(\mathrm{X}, \mathrm{g_x}) : \mathrm{B} \leq \mathrm{A}\}, \text{if there exist } \mathrm{B} \in \mathrm{GFC}(\mathrm{X}, \mathrm{g_x}) \\ 0_{\mathrm{X}}, \quad \text{otherwise.} \end{cases}$$

2.4 Definition (Ganster and Reilly, 1989)

A subset S of X is said to be locally closed if $S = U \cap F$, where U is open and F is closed.

2.5 Definition (Balasubramanian *et al.*, 2014)

A fuzzy set μ of a fuzzy topological space (X, τ) is called Λ-generalized fuzzy closed (briefly Λgf-closed) if $cl(\mu) \leq \beta$ whenever $\mu \leq \beta$ and β is λf-open.

2.6 Definition (Ekici, 2012)

Let (X,μ) be a GTS and $S \subset X$. Then
(1) S is said to be μ-dense if $c_\mu(S) = X$,
(2) (X,μ) is said to be submaximal if each μ-dense subset of (X,μ) is a μ-open set.

2.7 Definition (Abdel Monsef *et al.*, 1987)

A fuzzy space (X, FS) is said to be a fuzzy supra topological space if 0, 1 are the members of the fuzzy space and also arbitrary union of the members of the set are the members of the set.

2.8 Definition (Balasubramanian and Sundaram, 1997)

A map $f : X \to Y$ is called generalized fuzzy continuous (in short gf-continuous) if the inverse image of every fuzzy closed set in Y is gf-closed in X.

2.9 Definition (Caldas *et al.*, 2007)

A function $f : (X,\tau) \to (Y,\sigma)$ is called (λ,α)-continuous if $f^{-1}(V)$ is a (λ,α) open subset of X for every open subset V of .

3. Fuzzy $\Lambda_r^{g_X}$-set and Fuzzy $V_r^{g_X}$-set

3.1 Definition

Let (X,g_X) be a generalized fuzzy topological space. A fuzzy set λ is called a fuzzy g_X-regular open if $\lambda = i_{g_X}(c_{g_X}(\lambda))$ and a fuzzy g_X-regular closed if $\lambda = c_{g_X}(i_{g_X}(\lambda))$.

The collection of all fuzzy g_X-regular open (resp. fuzzy g_X-regular closed) sets in (X,g_X) is denoted by GFRO (X,g_X)(resp. GFRC(X,g_X)).

3.2 Definition

Let (X,g_X) be a GFTS and μ be a fuzzy subset of a GFTS (X,g_X). Then the regular kernel or r-kernal of μ is denoted by $\Lambda_r^{g_X}(\mu)$ and is defined by

$$\Lambda_r^{g_X}(\mu) = \begin{cases} \wedge \{\lambda \in \mathrm{GFRO}(X,g_X) : \mu \le \lambda\}, \text{if there exist } \lambda \in \mathrm{GFRO}(X,g_X) \text{ such that } \mu \le \lambda; \\ 1_X, \quad \text{otherwise.} \end{cases}$$

3.3 Definition

Let μ be a fuzzy subset of a GFTS (X,g_X). Then $V_r^{g_X}(\mu)$ of a fuzzy set μ is defined by

$$V_r^{g_X}(\mu) = \begin{cases} \vee \{\lambda \in \mathrm{GFRC}(X,g_X) : \lambda \le \mu\}, \text{if there exist } \lambda \in \mathrm{GFRC}(X,g_X) \text{ such that } \lambda \le \mu; \\ 0_X, \quad \text{otherwise.} \end{cases}$$

3.4 Lemma

Let (X,g_X) be a GFTS and $\lambda, \mu_j (j \in I)$ be the fuzzy subsets of X. Then the following properties hold:
(i) $\mu \le \Lambda_r^{g_X}(\mu)$,
(ii) $\mu \le \lambda$ implies that $\Lambda_r^{g_X}(\mu) \le \Lambda_r^{g_X}(\lambda)$,
(iii) $\Lambda_r^{g_X}\left(\Lambda_r^{g_X}(\mu)\right) = \Lambda_r^{g_X}(\mu)$,
(iv) if $\lambda \in \mathrm{GFR0}(X,g_X)$ then $\lambda = \Lambda_r^{g_X}(\lambda)$,

(v) $\vee\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}\leq \Lambda_r^{g_X}\left(\vee\left\{\mu_j : j\in J\right\}\right)$,

(vi) $\Lambda_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)\leq \wedge\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}$,

(vii) $\Lambda_r^{g_X}\left(1_X-\mu\right)=1_X-V_r^{g_X}(\mu)$.

Proof. Here (i),(ii),(iii) and (iv) are obvious.

To prove (v)

Let $\mu=\vee\left\{\mu_j : j\in J\right\}\geq \mu_j$, $\forall j\in J$. Then from (ii), we have $\Lambda_r^{g_X}\left(\mu_j\right)\leq \Lambda_r^{g_X}\left(\mu\right), \forall j\in J$.

This implies that $\vee\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}\leq \Lambda_r^{g_X}\left(\mu\right)$.

Then $\vee\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}\leq \Lambda_r^{g_X}\left(\vee\{\mu_j : j\in J\}\right)$.

To prove (vi)

Let us suppose that $\mu=\wedge\left\{\mu_j : j\in J\right\}\leq \mu_j$, $\forall j\in J$,

so from (ii) $\Lambda_r^{g_X}\left(\mu\right)\leq \Lambda_r^{g_X}\left(\mu_j\right), \forall j\in J$.

That is $\Lambda_r^{g_X}\left(\mu\right)\leq \wedge\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}$.

Hence $\Lambda_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)\leq \wedge\left\{\Lambda_r^{g_X}\left(\mu_j\right): j\in J\right\}$.

To prove (vii)

$$1_X-V_r^{g_X}\left(\mu\right)=1_X-\vee\{\lambda:\lambda\leq\mu, \lambda\in GFRC\left(X,g_X\right)\}$$
$$=\wedge\{1_X-\lambda : 1_X-\mu\leq 1_X-\lambda, 1_X-\lambda\in GFRO\left(X,g_X\right)\}$$
$$=\Lambda_r^{g_X}\left(1_X-\mu\right).$$

By using the above lemma one can easily prove the following lemma.

3.5 Lemma

Let (X,g_X) be a GFTS and λ, $\mu_j\,(j\in I)$ be the fuzzy sets of *X*. Then the following properties hold:

(i) $V_r^{g_X}\left(\mu\right)\leq\mu$,

(ii) $\mu\leq\lambda$ implies that $V_r^{g_X}\left(\mu\right)\leq V_r^{g_X}\left(\lambda\right)$,

(iii) $V_r^{g_X}\left(V_r^{g_X}\left(\mu\right)\right)=V_r^{g_X}\left(\mu\right)$,

(iv) if $\mu\in GFRC\left(X,g_X\right)$ then $\mu=V_r^{g_X}\left(\mu\right)$,

(v) $V_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)\leq\wedge\left\{V_r^{g_X}\left(\mu_j\right): j\in J\right\}$,

(vi) $\vee\left\{V_r^{g_X}\left(\mu_j\right): j\in J\right\}\leq V_r^{g_X}\left(\vee\left\{\mu_j : j\in J\right\}\right)$.

3.6 Remark

In (v) and (vi) of lemma 3.4, the equality may not hold as shown in the following examples.

3.7 Example

Let $X=\left\{a,b,c\right\}, g_X=\{0_X,1_X,\lambda,\mu,\lambda\vee\mu\}$

where $\lambda=\{\left(a,0.5\right)\left(b,0.3\right),\left(c,0.6\right)\}$, $\mu=\{\left(a,0.5\right),\left(b,0.7\right),\left(c,0.3\right)\}$ and $\lambda\vee\mu=\{\left(a,0.5\right),\left(b,0.7\right),\left(c,0.6\right)\}$.

Then $\Lambda_r^{g_X}(\lambda)=\lambda$, $\Lambda_r^{g_X}(\mu)=\mu$, $\Lambda_r^{g_X}(\lambda)\vee\Lambda_r^{g_X}(\mu)=\lambda\vee\mu$ but $\Lambda_r^{g_X}(\lambda\vee\mu)=1_X$.
So $\Lambda_r^{g_X}(\lambda)\vee\Lambda_r^{g_X}(\mu)\neq\Lambda_r^{g_X}(\lambda\vee\mu)$.

3.8 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,1_X,\lambda,\mu,\lambda\vee\mu\}$ where $\lambda=\{(a,0.4)(b,0.7),(c,0.3)\}$, $\mu=\{(a,0.5),(b,0.2),(c,0.4)\}$ and $\lambda\vee\mu=\{(a,0.5),(b,0.7),(c,0.4)\}$. Let $\delta=\{(a,0.6),(b,0.5),(c,0.3)\}$, $\gamma=\{(a,0.5),(b,0.7),(c,0.07)\}$ then $\Lambda_r^{g_X}(\delta)=1_X$, $\Lambda_r^{g_X}(\gamma)=1_X$, $\Lambda_r^{g_X}(\delta)\wedge\Lambda_r^{g_X}(\gamma)=1_X$. but $\Lambda_r^{g_X}(\delta\wedge\gamma)=\lambda\vee\mu$.

Therefore, $\Lambda_r^{g_X}(\delta\Lambda\gamma)\neq\Lambda_r^{g_X}(\delta)\wedge\Lambda_r^{g_X}(\gamma)$.

3.9 Definition

In a GFTS (X,g_X), a fuzzy subset λ is called a fuzzy $\Lambda_r^{g_X}$ -set (resp. fuzzy $V_r^{g_X}$ -set) if $\lambda=\Lambda_r^{g_X}(\lambda)$ (resp. $\lambda=V_r^{g_X}(\lambda)$).

3.10 Proposition

In a GFTS (X,g_X) the following properties are satisfied:

(i) 0_X and 1_X are fuzzy $\Lambda_r^{g_X}$ -set and fuzzy $V_r^{g_X}$ -set.
(ii) Arbitrary intersection of fuzzy $\Lambda_r^{g_X}$ -sets is fuzzy $\Lambda_r^{g_X}$ -set.
(iii) Arbitrary union of fuzzy $V_r^{g_X}$ -sets is fuzzy $V_r^{g_X}$ -set.

Proof. (i) The proof is obvious and hence omitted.

(ii) Let $\{\mu_j : j\in J\}$ be a family of fuzzy $\Lambda_r^{g_X}$ -sets in a GFTS (X,g_X). Then by using lemma 3.4 (vi) and definition 3.9 $\Lambda_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)\leq\wedge\left\{\Lambda_r^{g_X}\left(\mu_{\mathrm{j}}\right):\mathrm{j}\in\mathrm{J}\right\}=\wedge\{\mu_j : j\in J\}$. But from lemma 3.4 (i) we know that $\wedge\{\mu_j : j\in J\}\leq\Lambda_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)$. Therefore we have $\wedge\{\mu_j : j\in J\}=\Lambda_r^{g_X}\left(\wedge\{\mu_j : j\in J\}\right)$.

(iii) Let $\{\mu_j : j\in J\}$ be a family of fuzzy $V_r^{g_X}$ - sets in a GFTS (X,g_X).Then by using lemma 3.5 (vi) and definition 3.9 $V_r^{g_X}\left(\vee\{\mu_j : j\in J\}\right)\geq\vee\left\{V_r^{g_X}\left(\mu_{\mathrm{j}}\right):\mathrm{j}\in\mathrm{J}\right\}=\vee\{\mu_j : j\in J\}$. But from lemma 3.5 (i) we have that $V_r^{g_X}\left(\vee\{\mu_j : j\in J\}\right)\leq\vee\{\mu_j : j\in J\}$ and hence $V_r^{g_X}\left(\vee\{\mu_j : j\in J\}\right)=\vee\{\mu_j : j\in J\}$.

3.11 Remark

The collection of all fuzzy $\Lambda_r^{g_X}$ -set (resp. $V_r^{g_X}$ -set) is denoted by $\Lambda_r^{g_X}$ (resp. $V_r^{g_X}$) in GFTS (X,g_X).

3.12 Remark

Finite union of fuzzy $\Lambda_r^{g_X}$ -set is not fuzzy $\Lambda_r^{g_X}$ -set.

3.13 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,1_X,\lambda,\mu,\lambda\vee\mu\}$ where $\lambda=\{(a,0.5)(b,0.3),(c,0.6)\}$, $\mu=\{(a,0.5),(b,0.7),(c,0.3)\}$, $\lambda\vee\mu=\{(a,0.5),(b,0.7),(c,0.6)\}$. Then $\Lambda_r^{g_X}(\lambda)=\lambda$ and $\Lambda_r^{g_X}(\mu)=\mu$. Here λ and μ are fuzzy $\Lambda_r^{g_X}$ -set. But $\lambda\vee\mu$ is not fuzzy $\Lambda_r^{g_X}$ -set since $\Lambda_r^{g_X}(\lambda\vee\mu)=1_X\neq\lambda\vee\mu$.

3.14 Remark

Finite intersection of fuzzy $V_r^{g_X}$-set is not fuzzy $V_r^{g_X}$-set.

3.15 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,1_X,\lambda,\mu,\lambda\vee\mu\}$ where $\lambda=\{(a,0.5)(b,0.3),(c,0.6)\}$, $\mu=\{(a,0.5),(b,0.7),(c,0.3)\}$, $\lambda\vee\mu=\{(a,0.5),(b,0.7),(c,0.6)\}$. Let $\delta=\{(a,0.5),(b,0.7),(c,0.4)\}$, $\gamma=\{(a,0.5),(b,0.3),(c,0.07)\}$. Here δ and γ are fuzzy $V_r^{g_X}$ -set. But $\delta\wedge\gamma$ is not fuzzy $V_r^{g_X}$ -set since $\delta\wedge\gamma=\{(a,0.5),(b,0.3),(c,0.4)\}$ and $V_r^{g_X}\ \delta\wedge\gamma)=0_X$.

It is known that if arbitrary intersections of open sets of *X* are open in *X*, then *X* is called an Alexandroff space.

Ekici and Roy(2011) have shown that the collection (X,Λ_μ) and (X,V_μ) form Alexandroff spaces.

But in this paper both the collection $(X,\Lambda_r^{g_X})$ and $(X,V_r^{g_X})$ do not form Alexandroff spaces.

3.16 Remark

Here $(X,V_r^{g_X})$ forms generalized fuzzy topological space, rather we emphasize that the structure is always a fuzzy supra topological space.

3.17 Remark

Every fuzzy $\Lambda_r^{g_X}$ -set is fuzzy Λ^{g_X} -set.

3.18 Remark

However, converse of the above remark may not be true.

3.19 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,\lambda,\mu\}$ where $\lambda=\{(a,0.3)(b,0.6),(c,0.2)\}$, $\mu=\{(a,0.4),(b,0.7),(c,0.3)\}$. Then $\Lambda^{g_X}(\lambda)=\lambda$ and $\Lambda_r^{g_X}(\lambda)=\{(a,0.4),(b,0.7),(c,0.3)\}\neq\lambda$.

It implies that λ is fuzzy Λ^{g_X}-set but not fuzzy $\Lambda_r^{g_X}$-set.

3.20 Definition

A fuzzy set λ in a GFTS (X,g_X) is called generalized fuzzy g_X-closed (briefly gf-g_X-closed) if $c_{g_X}(\lambda)\le\mu$ whenever $\lambda\le\mu$ and $\mu\in\mathrm{GFO}(X,g_X)$.

3.21 Proposition

If λ is gf-g_X-closed set and $\lambda\le\Lambda_r^{g_X}(\lambda)\le c_{g_X}(\lambda)$ then $\Lambda_r^{g_X}(\lambda)$ is gf-g_X-closed.

Proof. Let λ be gf-g_X-closed set and $\lambda\le\Lambda_r^{g_X}(\lambda)\le c_{g_X}(\lambda)\le\mu$. It implies that $c_{g_X}(\lambda)\le c_{g_X}(\Lambda_r^{g_X}(\lambda))\le c_{g_X}(\lambda)$. Therefore, $c_{g_X}(\Lambda_r^{g_X}(\lambda))=c_{g_X}(\lambda)$. So $\Lambda_r^{g_X}(\lambda)$ is gf-g_X-closed set.

3.22 Proposition

If λ be any fuzzy g_X-closed set then $i_{g_X}(\lambda)$ is a fuzzy $\Lambda_r^{g_X}$ -set.

Proof. Let λ be any fuzzy g_X-closed set then $i_{g_X}(\lambda)$ is a fuzzy g_X- regular open set. So $i_{g_X}(\lambda)=\Lambda_r^{g_X}(i_{g_X}(\lambda))$. Therefore, $i_{g_X}(\lambda)$ is a fuzzy $\Lambda_r^{g_X}$ -set.

3.23 Proposition

If λ be any fuzzy g_X-open set in X then there exist a fuzzy $\Lambda_r^{g_X}$-set μ and fuzzy g_X- regular open set δ such that $\lambda\le\delta\wedge\mu$.

Proof. Let λ be any fuzzy g_X-open set in X.
Then $\lambda \le c_{g_X}(\lambda) \Rightarrow i_{g_X}(\lambda) \le i_{g_X} c_{g_X}(\lambda) \Rightarrow i_{g_X}(\lambda) \wedge \mu \le i_{g_X} c_{g_X}(\lambda) \wedge \mu \Rightarrow \lambda \le \delta \wedge \mu$ where δ is fuzzy g_X- regular open set.

4. Fuzzy $\Lambda_r^{g_X}$-closed set

In this section we study fuzzy Λ^{g_X}-closed, fuzzy $\Lambda_r^{g_X}$-closed sets and their properties. In general, fuzzy $\Lambda_r^{g_X}$-closed set is not fuzzy g_X-closed set but in this section we construct a condition under which the fact is true.

4.1 Definition

A fuzzy set λ of a GFTS (X, g_X) is called fuzzy Λ^{g_X}-closed if $\lambda = \mu \wedge \delta$, where μ is a fuzzy Λ^{g_X}-set and δ is fuzzy a g_X-closed set.

A fuzzy set λ of a GFTS (X, g_X) is called fuzzy Λ^{g_X}-open set if its complement is a fuzzy Λ^{g_X}-closed set.

4.2 Lemma

For a fuzzy subset λ of a GFTS (X, g_X) the following conditions are equivalent:

(i) λ is fuzzy Λ^{g_X}-closed;

(ii) $\lambda = \mu \wedge c_{g_X}(\lambda)$ where μ is a fuzzy Λ^{g_X}-set;

(iii) $\lambda = \Lambda^{g_X}(\lambda) \wedge c_{g_X}(\lambda)$.

Proof. (i)$\Rightarrow$(ii) Let λ be any fuzzy Λ^{g_X}-closed set. Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy Λ^{g_X}-set and d is a fuzzy g_X-closed set. Since $\lambda \le \delta$, it implies that $c_{g_X}(\lambda) \le \delta$ and $\lambda = \mu \wedge \delta \ge \mu \wedge c_{g_X}(\lambda) \ge \lambda$. Therefore, we have $\lambda = \mu \wedge c_{g_X}(\lambda)$.

(ii)$\Rightarrow$(iii) Let $\lambda = \mu \wedge c_{g_X}(\lambda)$, where μ is a fuzzy Λ^{g_X}-set. Since, $\lambda \le \mu$ it implies that $\Lambda^{g_X}(\lambda) \le \mu$ and $\lambda = \mu \wedge c_{g_X}(\lambda) \ge \Lambda^{g_X}(\lambda) \wedge c_{g_X}(\lambda) \ge \lambda$. Therefore, we have $\lambda = \Lambda^{g_X}(\lambda) \wedge c_{g_X}(\lambda)$.

(iii)$\Rightarrow$(i) It is obvious and hence omitted.

From the definition of fuzzy g_X-closed we draw a conclusion that fuzzy Λ^{g_X}-closed set and gf-g_X-closed set are generalized form of fuzzy g_X-closed but they are independent concepts as shown in the following examples.

4.3 Remark

Every fuzzy Λ^{g_X}-closed set is not gf-g_X-closed set.

4.4 Example

Let $X = \{a, b, c\}$,
$g_X = \{\{0_X, 1_X, (a, 0.5), (b, 0.3), (c, 0.6)\}$, $\{(a,0.5), (b,0.7), (c,0.3)\}$, $\{(a,0.5), (b,0.7), (c,0.6)\}\}$ and
$\lambda = \{(a, 0.5)(b, 0.3), (c, 0.6)\}$. Here λ is fuzzy Λ^{g_X}-closed set but not gf-g_X-closed set.

4.5 Remark

Every gf-g_X-closed set is not fuzzy Λ^{g_X}-closed.

4.6 Example

Let $X = \{a, b, c\}$, $g_X = \{0_X, 1_X \{(a, 0.5), (b, 0.3), (c, 0.4)\}\}$. Let us suppose that $\mu = \{(a, 0.5), (b, 0.7), (c, 0.3)\}$. Here μ is gf-g_X-closed set but not fuzzy Λ^{g_X}-closed.

4.7 Proposition

A gf-g_X-closed set is fuzzy g_X-closed set iff it is fuzzy Λ^{g_X}-closed set.

Proof. Suppose a gf-g_X-closed set is closed then it is obviously a fuzzy Λ^{g_X}-closed set.
Conversely, let λ be gf-g_X-closed set. So it implies that $c_{g_X}(\lambda) \le \mu$ whenever $\lambda \le \mu$ and μ is fuzzy g_X-open. But $c_{g_X}(\lambda) \le \Lambda^{g_X}(\lambda)$. Again λ is fuzzy Λ^{g_X}-closed set so $\lambda = \Lambda^{g_X}(\lambda) \wedge c_{g_X}(\lambda) = c_{g_X}(\lambda)$. Therefore λ is fuzzy g_X-closed.

4.8 Definition

A fuzzy set d of a GFTS (X, g_X) is called a fuzzy g_X-locally closed set if there exist a fuzzy g_X-open set λ and fuzzy g_X-closed set μ such that $\delta = \lambda \wedge \mu$.

4.9 Remark

Every fuzzy g_X- locally closed is fuzzy Λ^{g_X}-closed set.

4.10 Definition

A fuzzy set λ of a GFTS (X, g_X) is called fuzzy g_X-dense if $c_{g_X}(\lambda) = 1_X$.

4.11 Definition

A GFTS (X, g_X) is said to be fuzzy g_X-submaximal if each fuzzy g_X-dense subset of (X, g_X) is a fuzzy g_X-open set.

4.12 Theorem (Ekici ,2012)

Let (X, μ) be a GTS where $c_\mu(\phi) = \phi$. Then the following properties are equivalent:
(i) (X, μ) is a submaximal GTS,
(ii) Each subset of (X, μ) is a μ-locally closed set.
(iii) Each subset of μ-dense subset (X, μ) is a intersection of a μ-open set and a μ-closed set in (X, μ).
The above theorem is also true for GFTS which is stated as follows:

4.13 Theorem

Let (X, g_X) be a GFTS where $c_{g_X}(0_X) = 0_X$. Then the following properties are equivalent:
(i) (X, g_X) is a fuzzy g_X-submaximal space,
(ii) Each fuzzy subset of (X, g_X) is a fuzzy g_X- locally closed,
(iii) Each fuzzy subset of fuzzy g_X-dense subset (X, g_X) is a intersection of a g_X-open set and a g_X-closed set in (X, g_X).

4.14 Proposition

In a fuzzy g_X-submaximal space every fuzzy subset is fuzzy Λ^{g_X}-closed set when $c_{g_X}(0_X) = 0_X$.

Proof. Let (X, g_X) be a fuzzy g_X-submaximal space and we know that every fuzzy subset λ of a fuzzy g_X-submaximal space can be expressed as intersection of fuzzy g_X-open set μ and fuzzy g_X-closed set δ. It means that μ is fuzzy Λ^{g_X}-set and therefore $\lambda = \mu \wedge \delta$. It implies that λ is fuzzy Λ^{g_X}-closed set.

4.15 Proposition

In a fuzzy g_X-submaximal space (X, g_X) every fuzzy Λ^{g_X}-open set is fuzzy g_X-open if there does not exist any fuzzy g_X-regular closed set other than 1_X.

Proof. Let λ be any fuzzy Λ^{g_X}-open set then λ can be expressed as $\lambda = \mu \vee \delta$, where μ be any fuzzy V^{g_X}-set and δ be any fuzzy g_X-open set. Here $c_{g_X}(\lambda) = c_{g_X}(\mu \vee \delta) \geq c_{g_X}(\mu) \vee c_{g_X}(i_{g_X}(\delta)) \geq c_{g_X}(\mu) \vee 1_X = 1_X$. Therefore $c_{g_X}(\lambda) = 1_X$. So λ is fuzzy g_X-open since in fuzzy g_X-submaximal space every fuzzy g_X-dense set is fuzzy g_X-open.

4.16 Definition

A fuzzy subset λ of a GFTS (X, g_X) is called fuzzy $\Lambda_r^{g_X}$ closed if $\lambda = \mu \wedge \delta$ where μ is fuzzy $\Lambda_r^{g_X}$- set and δ is fuzzy g_X-closed set.

The complement of fuzzy $\Lambda_r^{g_X}$-closed set is called fuzzy $\Lambda_r^{g_X}$-open set.

4.17 Lemma

For a fuzzy subset λ of a GFTS (X, g_X) the following conditions are equivalent:

(i) λ is fuzzy $\Lambda_r^{g_X}$-closed;

(ii) $\lambda = \mu \wedge c_{g_X}(\lambda)$ where μ is a fuzzy $\Lambda_r^{g_X}$-set;

(iii) $\lambda = \Lambda_r^{g_X}(\lambda) \wedge c_{g_X}(\lambda)$.

Proof. (i)$\Rightarrow$(ii) Let λ be any fuzzy $\Lambda_r^{g_X}$-closed set. Then $\lambda = \mu \wedge \delta$, where μ is a fuzzy $\Lambda_r^{g_X}$-set and δ is fuzzy g_X-closed set. Since, $\lambda \leq \delta$ then it implies $c_{g_X}(\lambda) \leq \delta$ and $\lambda = \mu \wedge \delta \geq \mu \wedge c_{g_X}(\lambda) \geq \lambda$. Therefore, we have $\lambda = \mu \wedge c_{g_X}(\lambda)$.

(ii)$\Rightarrow$(iii) Let $\lambda = \mu \wedge c_{g_X}(\lambda)$, where μ is a fuzzy $\Lambda_r^{g_X}$-set. Since, $\lambda \leq \mu$ then it implies $\Lambda_r^{g_X}(\lambda) \leq \mu$ and $\lambda = \mu \wedge \delta \geq \Lambda_r^{g_X}(\lambda) \wedge c_{g_X}(\lambda) \geq \lambda$. Therefore, we have $\lambda = \Lambda_r^{g_X}(\lambda) \wedge c_{g_X}(\lambda)$.

(iii)$\Rightarrow$(i) It is obvious.

4.18. Remark

Every fuzzy $\Lambda_r^{g_X}$-closed set is a fuzzy $\Lambda_r^{g_X}$-closed set since every fuzzy $\Lambda_r^{g_X}$-set is a fuzzy Λ^{g_X}-set.

4.19 Remark

The converse of the above remark may not be true that can be verified in the example 3.19.

4.20 Remark

A gf-g_X-set is fuzzy g_X-closed set iff it is fuzzy $\Lambda_r^{g_X}$-closed set.

4.21. Proposition

If λ be fuzzy g_X-dense set and fuzzy $\Lambda_r^{g_X}$-closed set then λ is fuzzy $\Lambda_r^{g_X}$ set.

Proof. Let λ be any fuzzy g_X-dense set and fuzzy $\Lambda_r^{g_X}$-closed set then $\lambda = \Lambda_r^{g_X}(\lambda) \wedge c_{g_X}(\lambda) = \Lambda_r^{g_X}(\lambda) \wedge 1_X = \Lambda_r^{g_X}(\lambda)$. It implies that $\lambda = \Lambda_r^{g_X}(\lambda)$. Therefore, λ is $\Lambda_r^{g_X}$-set.

4.22 Definition

A GFTS (X, g_X) is said to be fuzzy g_X-$T_{\frac{1}{2}}$ space if every gf-g_X-closed set is fuzzy g_X-closed set.

4.23. Proposition

In a GFTS (X, g_X) the following conditions are equivalent:

(i) GFTS(X, g_X) is fuzzy g_X- $T_{\frac{1}{2}}$ space.

(ii) Every gf-g_X-closed set is fuzzy $\Lambda_r^{g_X}$-closed set.

Proof. (i)$\Rightarrow$ (ii) Given (X, g_X) be a fuzzy g_X- $T_{\frac{1}{2}}$ space. Hence by the definition of fuzzy g_X- $T_{\frac{1}{2}}$ space every gf-g_X-closed set is a fuzzy g_X-closed set. In lemma 4.17, it is shown that every fuzzy g_X-closed set is a fuzzy $\Lambda_r^{g_X}$-closed set. This implies that every gf-g_X-closed set is a fuzzy $\Lambda_r^{g_X}$-closed set.

(ii)$\Rightarrow$ (i) Using remark 4.20, it can be proved easily.

4.24 Definition

Let (X, g_X) be a generalized fuzzy topological space. A fuzzy set λ is called a fuzzy g_X-semi-open if $\lambda \le c_{g_X}\left(i_{g_X}(\lambda)\right)$ and a fuzzy g_X-semi-closed if $c_{g_X}(i_{g_X}(\lambda)) \le \lambda$.

4.25 Proposition

A fuzzy Λ^{g_X}-closed set is fuzzy g_X-closed set iff it fuzzy g_X-semi -closed set.

Proof. Suppose a fuzzy Λ^{g_X}-closed set is fuzzy g_X-closed set then it is obviously a fuzzy g_X-semi closed set.

Conversely, let λ be any fuzzy Λ^{g_X}-closed set and fuzzy g_X-semi-closed set. This implies $\lambda = \Lambda^{g_X}(\lambda) \wedge c_{g_X}(\lambda)$ and $i_{g_X}(c_{g_X}(\lambda)) \le \lambda$. Let $i_{g_X}(c_{g_X}(\lambda)) \le \lambda \le \delta_i, i \in I$ where δ_i' are fuzzy g_X-open sets. Then $c_{g_X}(\lambda) \le \wedge \delta_i$. This implies that $c_{g_X}(\lambda) \le \Lambda^{g_X}(\lambda)$. Therefore, $\lambda = c_{g_X}(\lambda)$, hence λ is fuzzy g_X- closed.

4.26 Remark

The above proposition is also true for fuzzy $\Lambda_r^{g_X}$-closed set.

4.27 Definition

A fuzzy point x_p is called a fuzzy $\Lambda_r^{g_X}$ cluster point of λ if for every fuzzy $\Lambda_r^{g_X}$-open set d containing x_p, $\lambda \wedge \delta \ne 0_X$.

4.28 Definition

We define fuzzy $\Lambda_r^{g_X}(c_{g_X}(\lambda))$ for any fuzzy set λ in GFTS (X, g_X) as follows:

$\Lambda_r^{g_X}\left(c_{g_X}(\lambda)\right) = \wedge\{\mu : \lambda \le \mu$ and μ is fuzzy $\Lambda_r^{g_X}$-closed set$\}$.

4.29 Proposition

If λ_i' are fuzzy $\Lambda_r^{g_X}$-closed for each $i \in I$, then $\wedge_{i\in I} \lambda_i$ is fuzzy $\Lambda_r^{g_X}$-closed set.

Proof. Suppose $\lambda = \wedge_{i\in I} \lambda_i$ and $x_p \in \Lambda_r^{g_X}\left(c_{g_X}(\lambda)\right)$. Then x_p is a fuzzy $\Lambda_r^{g_X}$ cluster point of λ. Then there exist a fuzzy $\Lambda_r^{g_X}$-open set δ containing x_p such that $\lambda \wedge \delta \ne 0_X$. That implies $(\wedge_{i\in I} \lambda_i) \wedge \delta \ne 0_X$. That implies $\lambda_i \wedge \delta \ne 0_X$ for each $i \in I$. If $x_p \notin \lambda$ then for each $i \in I$, $x_p \notin \lambda_i$. Since λ_i is fuzzy $\Lambda_{g_x}{}^{r}$-closed, $\lambda_i = \Lambda_r^{g_X}\left(c_{g_X}(\lambda_i)\right)$ and hence $x_p \notin \Lambda_r^{g_X}\left(c_{g_X}(\lambda_i)\right)$.

Thus x_p is not a fuzzy $\Lambda_r^{g_X}$-cluster point of λ_i. So there exist a fuzzy $\Lambda_r^{g_X}$-open set μ containing x_p such that $\lambda_i \wedge \mu = 0_X$. By this a contradiction as it gives that $x_p \in \lambda$. Therefore, $\Lambda_r^{g_X}$-cl(λ) $\leq \lambda$ and hence $\lambda = \Lambda_r^{g_X}\left(c_{g_X}(\lambda)\right)$. Thus $\wedge_{i\in I} \lambda_i$ is fuzzy $\Lambda_r^{g_X}$-closed set.

4.30 Remark

The finite union of fuzzy $\Lambda_r^{g_X}$-closed set need not be fuzzy $\Lambda_r^{g_X}$-closed set.

4.31 Example

Let $X = \{a,b,c\}$, $g_X = \{0_X, 1_X, \lambda, \mu, \lambda \vee \mu\}$ where $\lambda = \{(a,0.5)(b,0.3),(c,0.6)\}$, μ={(a,0.5),(b,0.7),(c,0.3)}, $\mu = \{(a,0.5),(b,0.7),(c,0.3)\}$. Then $\lambda \vee \mu = \{(a,0.5),(b,0.7),(c,0.6)\}$ and $\Lambda_r^{g_X}(\lambda) = \{(a,0.5)(b,0.3),(c,0.6)\}$. Here λ and μ are fuzzy $\Lambda_r^{g_X}$-set. Consequently it implies λ and μ is fuzzy $\Lambda_r^{g_X}$-closed set. But $\lambda \vee \mu$ is not a fuzzy $\Lambda_r^{g_X}$-closed set. Since $\Lambda_r^{g_X}(\lambda \vee \mu) = 1_X$, $\Lambda_r^{g_X}(\lambda \vee \mu) \wedge c_{g_X}(\lambda \vee \mu) = 1_X \neq \lambda \vee \mu$.

Therefore, the union of two fuzzy $\Lambda_r^{g_X}$-closed sets need not be fuzzy $\Lambda_r^{g_X}$-closed set.

5. $\Lambda_r^{g_X}$-Generalized Fuzzy g_X-Closed Sets

5.1. Definition

A fuzzy set λ of a GFTS (X, g_X) is called Λ^{g_X}- generalized fuzzy g_X-closed set (briefly Λ^{g_X}-gf- g_X-closed) if $c_{g_X}(\lambda) \leq \mu$, whenever $\lambda \leq \mu$ and μ is a fuzzy Λ^{g_X}-open set.

5.2 Definition

A fuzzy set λ of a GFTS (X, g_X) is called $\Lambda_r^{g_X}$- generalized fuzzy g_X -closed set (briefly $\Lambda_r^{g_X}$- gf- g_X-closed) if $c_{g_X}(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy $\Lambda_r^{g_X}$-open.

5.3 Remark

Every $\Lambda_r^{g_X}$- gf- g_X-closed set is Λ^{g_X}-gf- g_X-closed.

5.4 Proposition

In a GFTS (X, g_X),

(i) Every fuzzy g_X-closed set is a Λ^{g_X}-gf- g_X set.

(ii) Every Λ^{g_X}-gf-g_X-closed set is a gf-g_X-closed.

Proof. (i) Let λ be fuzzy g_X-closed set and μ be any fuzzy Λ^{g_X}-open set such that $\lambda \leq \mu$, $c_{g_X}(\lambda) = \lambda \leq \mu$. Thus $c_{g_X}(\lambda) \leq \mu$ and hence λ is Λ^{g_X}-gf-g_X-closed.

(ii) Since every fuzzy g_X-open is fuzzy Λ^{g_X}-open set so every Λ^{g_X}-gf-g_X-closed set is gf-g_X-closed as follows from the definition 5.1.

5.5 Remark

Proposition 5.4 is also holds if we consider $\Lambda_r^{g_X}$-gf-g_X-closed set instead of Λ^{g_X}-gf-g_X-closed set.

5.6 Remark

Every $\Lambda_r^{g_X}$-gf-g_X-closed set need not be a fuzzy g_X-closed set.

5.7 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,\{(a,1),(b,0),(c,0)\}\}$. Let us suppose $\mu=\{(a,1),(b,1),(c,0)\}$ and $\mu\le 1_X$, $c_{g_X}(\mu)=1_X\le 1_X$. Here μ is a $\Lambda_r^{g_X}$-gf- g_X-closed set but μ is not a fuzzy g_X-closed set .

5.8 Remark

Every gf-g_X-closed set need not be a $\Lambda_r^{g_X}$-gf-g_X-closed set.

5.9 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,1_X\ \{(a,0.5),(b,0.3),(c,0.4)\}\}$

Let us suppose $\mu=\{(a,0.5),(b,0.7),(c,0.3)\}$ and $\lambda=\{(a,0.),(b,0.7),(c,0.3)\}$.

Here $\mu\le 1_X, c_{g_X}(\mu)=1_X\le 1_X$. It implies that μ is a fuzzy g_X-closed set. But $\mu\le\lambda$ and $c_{g_X}(\mu)=1_X\not\le\lambda$ where λ is fuzzy $\Lambda_r^{g_X}$-open set. Therefore, μ is not $\Lambda_r^{g_X}$-gf- g_X-closed set.

5.10. Remark

Even union of any two fuzzy $\Lambda_r^{g_X}$-gf- g_X-closed sets need not be a $\Lambda_r^{g_X}$-gf- g_X-closed set.

5.11 Example

Let $X=\{a,b,c\}, g_X=\{0_X,\{(a,0.4),(b,0.7),(c,0.5)\}, \{(a,0.6),(b,0.3),(c,0.5)\}, \{(a,0.6),(b,0.7),(c,0.5)\}\}$ let $\lambda=\{(a,0.4),(b,0.7),(c,0.5)\}$, $\mu=\{(a,0.6),(b,0.3),((c,0.5)\}$ and $\delta=\{(a,0.6),(b,0.7),(c,0.5)\}$. Here $c_{g_X}(\lambda)=\lambda\le\lambda$ and $c_{g_X}(\mu)=\mu\le\mu$. Then $\lambda\vee\mu=\delta$ and $c_{g_X}(\lambda\vee\mu)=1_X\not\le\delta$. Therefore, λ and μ are two $\Lambda_r^{g_X}$-gf- g_X-closed set but $\lambda\vee\mu$ is not a $\Lambda_r^{g_X}$-gf- g_X-closed set.

5.12 Remark

Finite intersection of $\Lambda_r^{g_X}$-gf- g_X-closed set need not be a $\Lambda_r^{g_X}$-gf- g_X-closed set.

5.13 Example

Let $X=\{a,b,c\}, g_X=\{0_X,\{(a,1),(b,0),(c,0)\}\}$.

Here $\lambda=\{(a,1),(b,0),(c,0)\}$, $\mu=\{(a,1),(b,1),(c,0)\}$ and $\delta=\{(a,1),(b,0),((c,1)\}$. Also $\mu\le 1_X, c_{g_X}(\mu)=1_X\le 1_X$ and $\delta\le 1_X, c_{g_X}(\delta)=1_X\le 1_X$, $\mu\wedge\delta=\lambda, c_{g_X}(\mu\wedge\delta)=1_X\not\le\mu\wedge\delta$. Therefore, μ and δ are two $\Lambda_r^{g_X}$-gf- g_X-closed set but $\mu\wedge\delta$ is not a $\Lambda_r^{g_X}$-gf- g_X-closed set.

So we have studied two generalized forms of fuzzy g_X-closed set which are independent of each other as shown in the following examples:

5.14 Remark

Every fuzzy $\Lambda_r^{g_X}$-closed set need not be a $\Lambda_r^{g_X}$-gf- g_X-closed.

5.15 Example

Let $X=\{a,b,c\}$, $g_X=\{\{0_X,1_X,(a,0.5),(b,0.3),(c,0.6)\}, \{(a,0.5),(b,0.7),(c,0.3)\}, \{(a,0.5),(b,0.7),(c,0.6)\}\}$ and $\lambda=\{(a,0.5)(b,0.3),(c,0.6)\}$. Here $\Lambda_r^{g_X}(\lambda)=\lambda$. It implies that λ is a fuzzy $\Lambda_r^{g_X}$-closed set. Again $\lambda\le\lambda$ and $c_{g_X}(\lambda)=\{(a,0.5)(b,0.3),(c,0.7)\}\not\le\lambda$. Therefore, λ is not a $\Lambda_r^{g_X}$-gf- g_X-closed set.

5.16 Remark

Every $\Lambda_r^{g_X}$-gf- g_X-closed set need not be a fuzzy $\Lambda_r^{g_X}$-closed set.

5.17 Example

Let $X=\{a,b,c\}$, $g_X=\{0_X,1_X,\{(a,1)(b,0),(c,0)\}\}$ and $\lambda=\{(a,1)(b,1),(c,0)\}$. Here $\Lambda_r^{g_X}(\lambda)=1_X$ and $c_{g_X}(\lambda)=1_X$. Again $\lambda\le 1_X$ and $c_{g_X}(\lambda)=1_X\le 1_X$. Therefore, λ is $\Lambda_r^{g_X}$-gf-g_X-closed set. But $\Lambda_r^{g_X}(\lambda)\wedge c_{g_X}(\lambda)=1_X\ne\lambda$. It implies that λ is not a fuzzy $\Lambda_r^{g_X}$-closed set.

Caldas *et al.* (2008) shown one equivalent condition in Theorem 2.15 by using locally closed set in topological space. In the following theorem we show the same equivalent condition in GFTS (X,g_X) with the help of fuzzy $\Lambda_r^{g_X}$-closed which is a weaker form of fuzzy g_X-locally closed set.

5.18 Theorem

Let λ be any fuzzy $\Lambda_r^{g_X}$-closed set in a GFTS (X,g_X). For the fuzzy set λ the following properties are equivalent:

(i) λ is fuzzy g_X-closed set;

(ii) λ is $\Lambda_r^{g_X}$-gf-g_X-closed;

(iii) λ is gf-g_X-closed set.

Proof. (i) $\Longrightarrow$ (ii) Using proposition 5.4(i) the proof can be done easily.

(ii) $\Longrightarrow$ (iii) Using proposition 5.4(ii) it can be proved easily.

(iii) $\Longrightarrow$ (i) It can be proved by using proposition 4.7.

5.19 Remark

Theorem 5.18 is also true if we consider Λ^{g_X}-gf-g_X-closed set instead of fuzzy $\Lambda_r^{g_X}$-gf-g_X-closed set.

The fundamental relationship between the various types of sets considered in this section can be summarized in the following diagram:

$$\begin{tikzcd}
\text{Fuzzy } g_X\text{-closed} \arrow[r] \arrow[dd] & \text{fuzzy } \Lambda_r^{g_X}\text{-closed} \arrow[d] \\
& \text{fuzzy } \Lambda^{g_X}\text{-closed} \arrow[d, leftrightarrow] \\
\Lambda_r^{g_X}\text{-gf- } g_X\text{-closed} \arrow[r] & \text{gf-}g_X\text{-closed}
\end{tikzcd}$$

6. Fuzzy (Λ^{g_X}, g_Y)-Continuity and Fuzzy $(\Lambda_r^{g_X}, g_Y)$-Continuity

6.1 Definition

Let (X,g_X) and (Y,g_Y) be generalized fuzzy topological spaces (in short GFTS's). Then a function $f: X\to Y$ is said to be fuzzy (g_X,g_Y)-continuous if for each fuzzy g_X-open set λ in Y, $f^{-1}(\lambda)$ is fuzzy g_X-open in X.

6.2 Definition

A function f: $(X, g_X) \to (Y, g_Y)$ from a GFTS to another GFTS is called fuzzy contra (g_X, g_Y)-continuous if for each fuzzy g_X- open set λ in *Y*, $f^{-1}(\lambda)$ is fuzzy g_X- closed in *X*.

6.3 Definition

Let f: $(X, g_X) \to (Y, g_Y)$ be a function from a GFTS (X, g_X) into a GFTS (Y, g_Y) then f is called

(i) fuzzy (Λ^{g_X}, g_Y)-continuity if $f^{-1}(\mu)$ is fuzzy Λ^{g_X}-closed in (X, g_X) for each $\mu \in$ GFC(Y, g_Y),

(ii) fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuity if $f^{-1}(\mu)$ is fuzzy $\Lambda_r^{g_X}$-closed in (X, g_X) for each $\mu \in$ GFC(Y, g_Y),

(iii) generalized fuzzy (g_X, g_Y)-continuity if $f^{-1}(\mu)$ is gf-g_X-closed in (X, g_X) for each $\mu \in$ GFC(Y, g_Y),

(iv) fuzzy $\Lambda_r^{g_X}$ gf-g_X, g_Y)-continuity if $f^{-1}(\mu)$ is fuzzy $\Lambda_r^{g_X}$ gf-g_X-closed in (X, g_X) for each $\mu \in$ GFC(Y, g_Y).

6.4 Proposition

Every fuzzy (g_X, g_Y)-continuous function is a fuzzy (Λ^{g_X}, g_Y)-continuous function.

Proof. The proof is straight forward from the definition.

6.5. Remark

But the converse of the above theorem may not be true.

6.6 Example

Let $X = Y = \{a,b,c\}$, $g_X = \{0_X, \lambda_1\}$, where $\lambda_1(a) = 0.5$, $\lambda_1(b) = 0.7$, $\lambda_1(c) = 0.4$. and $g_Y = \{0_X, 1_X, \lambda_2, \lambda_3, \lambda_4\}$ where $\lambda_1 = \{(a,0.5)(b,0.3),(c,0.6)\}$, $\lambda_2 = \{(a,0.5),(b,0.7),(c,0.3)\}$, $\lambda_3 = \{(a,0.5),(b,0.7),(c,0.6)\}$. Here f: $(X, g_X) \to (Y, g_Y)$ defined by $f(a) = a$, $f(b) = b$, $f(c) = c$. Then f is fuzzy (Λ^{g_X}, g_Y)-continuous function but not fuzzy (g_X, g_Y)-continuous function. Since $f^{-1}(\lambda_1') = \lambda_2$ is a fuzzy Λ^{g_X}-closed set but not fuzzy g_X-closed set.

Generally fuzzy (Λ^{g_X}, g_Y)-continuity does not imply fuzzy (g_X, g_Y)- continuity but in our next theorem it is shown that the converse part is also true under the circumstances mentioned therein.

6.7 Proposition

A fuzzy (Λ^{g_X}, g_Y)-continuous function from *X* to *Y* is fuzzy (g_X, g_Y) continuous if every fuzzy Λ^{g_X}-closed set is fuzzy g_X-semi-closed in *X*.

Proof. Let f be any fuzzy (Λ^{g_X}, g_Y) continuous function from (X, g_X) to (Y, g_Y) and let $\mu \in$ GFC(Y, g_Y). Therefore, $f^{-1}(\mu)$ is fuzzy Λ^{g_X}-closed in (X, g_X) and by given condition every fuzzy Λ^{g_X}-closed set is fuzzy g_X-semi-closed in *X*. Then by using Proposition 4.25, $f^{-1}(\mu)$ is a fuzzy g_X-closed set in (X, g_X). Therefore, f is fuzzy (g_X, g_Y)-continuous.

6.8 Remark

The above theorem is true for fuzzy $\Lambda_r^{g_X}$-closed set as because of every fuzzy $\Lambda_r^{g_X}$-closed set is a fuzzy Λ^{g_X}-closed set.

6.9 Theorem

f: $(X, g_X) \to (Y, g_Y)$ be any function, where *X* is a fuzzy submaximal space and the only fuzzy g_X-regular closed set in *X* is 1_X, then the following conditions are equivalent:

(i) f is a fuzzy (g_X, g_Y)-continuous.

(ii) f is a fuzzy (Λ^{g_X}, g_Y)-continuous.

Proof. (i)$\Rightarrow$(ii) Let $\mu \in$ GFC(Y, g_Y). Since f is fuzzy (g_X, g_Y)-continuous. So $f^{-1}(\mu)$ is fuzzy g_X- closed in X. Thus $f^{-1}(\mu)$ is fuzzy Λ^{g_X}-closed in (X, g_X) since every fuzzy g_X-closed is fuzzy Λ^{g_X}-closed. Therefore, f is fuzzy (Λ^{g_X}, g_Y)-continuous.

(ii)$\Rightarrow$(i) Let $\mu \in$ GFC(Y, g_Y). Since f is fuzzy (Λ^{g_X}, g_Y)-continuous. Therefore $f^{-1}(\mu)$ is fuzzy g_X-closed set in (X, g_X) since in submaximal space every fuzzy Λ^{g_X}-closed is a fuzzy g_X-closed set. Therefore, f is fuzzy (g_X, g_Y)-continuous function.

6.10 Theorem

For any function, f: $(X, g_X) \to (Y, g_Y)$ the following conditions are equivalent:

(i) f is fuzzy (Λ^{g_X}, g_Y)-continuous,

(ii) $f^{-1}\left(i_{g_Y}(\lambda)\right) \le \Lambda^{g_X}\text{-}i_{g_X}(f^{-1}(\lambda))$ for any fuzzy subset λ of Y,

(iii) $\Lambda^{g_X}\text{-}c_{g_X}(f^{-1}(\lambda)) \le f^{-1}\left(c_{g_Y}(\lambda)\right)$ for any fuzzy subset λ of Y.

Proof. (i)$\Rightarrow$(ii) Let f be fuzzy (Λ^{g_X}, g_Y)-continuous and λ be any fuzzy subset of Y. Then $i_{g_Y}(\lambda) \le \lambda \Rightarrow f^{-1}\left(i_{g_Y}(\lambda)\right) \le f^{-1}(\lambda)$. So $f^{-1}\left(i_{g_Y}(\lambda)\right)$ is a fuzzy Λ^{g_X}-open set in (X, g_X). Therefore, $f^{-1}\left(i_{g_Y}(\lambda)\right) \le \Lambda^{g_X}\text{-}i_{g_X}(f^{-1}(\lambda))$.

(ii)$\Rightarrow$(iii) Let λ be any fuzzy subset of Y then $f^{-1}\{i_{g_Y}(Y-\lambda)\} \le \Lambda^{g_X}\text{-}i_{g_X}(f^{-1}(Y-\lambda)) \Rightarrow f^{-1}\{Y - c_{g_Y}(\lambda)\} \le \Lambda^{g_X}\text{-}i_{g_X}\left(X - f^{-1}(\lambda)\right) \Rightarrow X - f^{-1}\left(c_{g_Y}(\lambda)\right) \le X - \Lambda^{g_X}\text{-}c_{g_X}(f^{-1}(\lambda)) \Rightarrow \Lambda^{g_X}\text{-}c_{g_X}(f^{-1}(\lambda)) \le f^{-1}\left(c_{g_Y}(\lambda)\right)$.

(iii)$\Rightarrow$(i) let λ be any fuzzy g_Y-closed set in Y. So $f^{-1}(\lambda) = f^{-1}\left(c_{g_Y}(\lambda)\right) \ge \Lambda^{g_X}\text{-}c_{g_X}\left(f^{-1}(\lambda)\right) = f^{-1}(\lambda)$. It implies that $f^{-1}(\lambda)$ is a fuzzy Λ^{g_X}-closed in X. Therefore, f is a (Λ^{g_X}, g_Y)- continuous function.

6.11 Proposition

Every fuzzy contra (g_X, g_Y)-continuous function is a fuzzy (Λ^{g_X}, g_Y)-continuous function.

Proof. Let $\mu \in$ GFC(Y, g_Y). Since f is a fuzzy contra (g_X, g_Y)-continuous function, then $f^{-1}(\mu)$ is a fuzzy g_X-open set in X. So $\Lambda^{g_X}\left(f^{-1}(\mu)\right) \wedge c_{g_X}\left(f^{-1}(\mu)\right) = f^{-1}(\mu) \wedge c_{g_X}\left(f^{-1}(\mu)\right) = f^{-1}(\mu)$. Thus $f^{-1}(\mu)$ is a fuzzy Λ^{g_X}-closed set in (X, g_X). Therefore, f is fuzzy (Λ^{g_X}, g_Y)-continuous function.

Generally in a GTS as well as in GFTS contra (g_X, g_Y)-continuity does not imply (g_X, g_Y)-continuity. But in the following proposition we have study the relation between fuzzy contra (g_X, g_Y)-continuity and fuzzy (g_X, g_Y)-continuity.

6.12 Proposition

A fuzzy contra (g_X, g_Y)-continuous function is fuzzy (g_X, g_Y)- continuous if every fuzzy Λ^{g_X}-closed set is fuzzy g_X- semi-closed set in X.

Proof. It follows from proposition 6.7 and 6.11.

6.13 Proposition

Every fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuous function is a fuzzy (Λ^{g_X}, g_Y)-continuous function.

Proof. The proof is straight forward from the remark 4.18.

6.14 Remark

Then the converse of the above theorem may not be true in general.

6.15 Example

Let $X = \mathrm{Y} = \{a,b,c\}$, $g_X = \{0_X, \lambda_1\}$ where $\lambda_1(a) = 0.3$, $\lambda_1(b) = 0.6$, $\lambda_1(c) = 0.2$. and $g_Y = \{0_X, 1_X, \lambda_2, \lambda_3\}$, where $\lambda_2 = \{(a, 0.3)(b, 06), (c, 0.2)\}$, $\lambda_3 = \{(a, 0.4), (b, 0.7), (c, 0.3)\},\}$. Here f: $(X, g_X) \to (Y, g_Y)$ defined by $f(a) = a$,

$f(b)=b$, $f(c)=c$. Then f is a fuzzy (Λ^{g_X}, g_Y)-continuous function but not a fuzzy a $(\Lambda_r^{g_X}, g_Y)$-continuous function since $f^{-1}(\lambda_1)=\lambda_2$ is a fuzzy Λ^{g_X}-closed set but not a a fuzzy $\Lambda_r^{g_X}$-closed set in (X, g_X).

6.16. Theorem. Let f: $(X, g_X) \rightarrow (Y, g_Y)$ be a function. Then the following are equivalent.

(i) f is fuzzy (g_X, g_Y)-continuous function.

(ii) f is generalized fuzzy (g_X, g_Y)-continuous function and fuzzy $(\Lambda_r^{g_X}, g_Y)$ - continuous function.

Proof. (i) $\Rightarrow$ (ii) It is straight forward from the definition.

(i) $\Rightarrow$ (ii) Let $\mu \in$ GFC(Y, g_Y). Since f is generalized fuzzy (g_X, g_Y)- continuous function and fuzzy $(\Lambda_r^{g_X}, g_Y)$-continuous function. So $f^{-1}(\mu)$ is gf-g_X-closed set and fuzzy $\Lambda_r^{g_X}$-closed set. Therefore, $f^{-1}(\mu)$ is a fuzzy g_X-closed set using remark 4.20. Thus f is fuzzy (g_X, g_Y)-continuity.

6.17 Theorem

Every fuzzy ($\Lambda_r^{g_X}$gf-g_X, g_Y)-continuous function is generalized fuzzy (g_X, g_Y)-continuous function.

Proof. The proof is straight forward from the definition.

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