



Original Article

A Chebyshev-Gauss collocation method for the numerical solution of ordinary differential equations

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Abstract

This paper presents a Chebyshev-Gauss collocation method to determine an approximate solution to the initial value problems of ordinary differential equations. We propose an algorithm to solve an ordinary differential equation on a single-interval domain and extend the algorithm to a multi-interval domain. We then generalize the algorithm to the system of ordinary differential equations including the Hamiltonian systems. Numerical results show that the proposed method gives a spectral accuracy. The comparison of our method to some related work is provided to show the accomplishment of the method.

Keywords: spectral methods, collocation methods, Chebyshev-Gauss collocation method, spectral accuracy, ordinary differential equations

1. Introduction

Ordinary differential equations occur mostly in problems in science and engineering. There are several numerical methods to solve the initial value problems of the ordinary differential equations. The classical methods such as the Euler method and explicit Runge-Kutta methods are known to provide numerical solutions with a low accuracy (Burden *et al.*, 2010), whereas the implicit Runge-Kutta methods give a high accuracy for the numerical results (Butcher, 1964, 1987; Guo, 2009). There are some other high accuracy methods for the ordinary differential equations proposed by Hairer *et al.* (1993), Lambert (1991) and Stuart *et al.* (1996).

Spectral methods have been successfully used to obtain the numerical solutions of ordinary and partial differential equations. The solutions of the methods are

approximated in forms of the expansion of higher-order polynomials (Canuto *et al.*, 2006; Hesthaven *et al.*, 2007; Kopriva, 2009; Shen *et al.*, 2011). The spectral collocation methods recently capture many researchers' interests as they give a spectral accuracy to the solutions. The smoother the exact solutions, the smaller the numerical errors are (Yang *et al.*, 2015).

The recent work for solving the ordinary differential equations using the spectral methods was proposed by Guo *et al.* (2007, 2008, 2009, 2010, 2012). They developed the Legendre-Gauss (Guo *et al.*, 2009) and designed Laguerre-Gauss and Laguerre-Radau type (Guo *et al.*, 2007, 2008) spectral collocation methods for the ordinary differential equations. Furthermore, Kanyamee *et al.* (2011) described the comparison of several spectral Galerkin and spectral collocation methods and symplectic methods for Hamiltonian systems. Wang *et al.* (2010, 2015) developed the Chebyshev-Gauss spectral collocation methods and developed the Legendre-Gauss collocation methods for nonlinear delay differential equations. E-Baghdady *et al.* (2015) designed a

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new Chebyshev spectral collocation method for solving a class of one-dimensional linear parabolic partial integro-differential equations.

In this work, we are interested in finding a numerical solution of the ordinary differential equations. We first propose the collocation method with $(N + 1)$ Chebyshev-Gauss points as the nodes. Then, we derive a new algorithm for solving an ordinary differential equation and a system of ordinary differential equations.

2. Materials and Methods

2.1 Chebyshev polynomials of the first kind on the interval

Chebyshev polynomials of the first kind, denoted $T_n(x)$, are the eigenfunctions of the singular Sturm-Liouville equation of the form

$$(1 - x^2)T_n''(x) - xT_n'(x) + n^2T_n(x) = 0, \quad x \in [-1, 1].$$

An alternative representation of the Chebyshev polynomial of degree n is given by

$$T_n(x) = \cos(n \arccos(x)).$$

Let $T_l(x)$ be the Chebyshev polynomial of degree l defined on the interval $[-1, 1]$. We define the shifted Chebyshev polynomials $T_{\tau,l}(t)$ on the interval $[0, \tau]$, with the transformation $x = \frac{2t}{\tau} - 1$, by

$$T_{\tau,l}(t) = T_l\left(\frac{2t}{\tau} - 1\right) = \cos\left(l \cos^{-1}\left(\frac{2t}{\tau} - 1\right)\right), \quad l = 0, 1, 2, \dots$$

The first few polynomials are illustrated as follows:

$$T_{\tau,0}(t) = \cos\left(0 \cos^{-1}\left(\frac{2t}{\tau} - 1\right)\right) = 1$$

$$T_{\tau,1}(t) = \cos\left(1 \cos^{-1}\left(\frac{2t}{\tau} - 1\right)\right) = \frac{2t}{\tau} - 1$$

$$T_{\tau,2}(t) = \cos\left(2 \cos^{-1}\left(\frac{2t}{\tau} - 1\right)\right) = \frac{8t^2}{\tau^2} - \frac{8t}{\tau} + 1.$$

By following a property of Chebyshev polynomials, we have the three-term recurrence relation for shifted Chebyshev polynomials

$$T_{\tau,l+1}(t) - 2\left(\frac{2t}{\tau} - 1\right)T_{\tau,l}(t) + T_{\tau,l-1}(t) = 0, \quad l \geq 1. \tag{2.1}$$

The shifted Chebyshev polynomials are also orthogonal on the interval $[0, \tau]$, i.e.

$$\int_0^\tau T_{\tau,l}(t)T_{\tau,m}(t)\omega(t)dt = \frac{\pi}{2}c_l\delta_{l,m}, \quad l \geq 0 \tag{2.2}$$

where $\omega(t) = t(\tau - t)^{-1/2}$, $c_0 = 2, c_l = 1$ and $\delta_{l,m}$ is the Kronecker symbol.

Consider any function $u \in L^2_\omega(0, \tau)$. A Chebyshev expansion of a function u is

$$u(t) = \sum_{l=0}^\infty \hat{u}_l T_{\tau,l}(t) \tag{2.3a}$$

where the expansion coefficients, \hat{u}_l , are constant. Multiplying both sides of (2.3a) by $T_{\tau,l}(t)\omega(t)$ and integrating with respect to t over the interval $[0, \tau]$ yields the expansion coefficients

$$\hat{u}_l = \frac{2}{\pi c_l} \int_0^\tau \frac{u(t)T_{\tau,l}(t)}{\sqrt{t(\tau - t)}} dt. \tag{2.3b}$$

Next, for any positive integer N , we consider the Chebyshev-Gauss quadrature. Let $\left\{ \left(x_j^N, \omega_j^N \right) \right\}_{j=0}^N$ be the Chebyshev nodes (x are the zeros of $T_{N+1}(x)$) and the corresponding weights on the interval $(-1,1)$. We define the shifted Chebyshev-Gauss nodes and the corresponding weights on $(0, \tau)$ as

$$t_{\tau,j}^N = \frac{\tau}{2} \left(x_j^N + 1 \right), 0 \leq j \leq N \text{ and } \omega_{\tau,j}^N = \frac{\tau}{2} \omega_j^N, 0 \leq j \leq N.$$

Let $\mathcal{P}_N(a, b)$ be the set of polynomials of degree at most N on $[a, b]$. According to the Gauss-type quadrature rule. The Gaussian quadrature is exact for all polynomials

$$p(x) \in \mathcal{P}_{2N+1}. \text{ As a results, for any } \phi \in \mathcal{P}_{2N+1}(-1, 1),$$

$$\int_{-1}^1 \frac{\phi(x)}{\sqrt{1-x^2}} dx = \frac{\pi}{N+1} \sum_{j=0}^N \phi(x_j^N).$$

Hence, for any $\phi \in \mathcal{P}_{2N+1}(0, \tau)$, we have

$$\begin{aligned} \int_0^\tau \frac{\phi(t)}{\sqrt{t(\tau-t)}} dt &= \int_{-1}^1 \frac{\phi\left(\frac{\tau}{2}(x+1)\right)}{\sqrt{1-x^2}} dx = \frac{\pi}{N+1} \sum_{j=0}^N \phi\left(\frac{\tau}{2}(x_j^N + 1)\right), \\ \int_0^\tau \frac{\phi(t)}{\sqrt{t(\tau-t)}} dt &= \frac{\pi}{N+1} \sum_{j=0}^N \phi\left(t_{\tau,j}^N\right). \end{aligned} \tag{2.4}$$

2.2 Discrete Chebyshev-Gauss expansion (Hesthaven *et al.*, 2007)

In the continuous $L^2_\omega(0, \tau)$ space, we define the inner product and L^2_ω -norm as

$$(u, v)_\tau = \int_0^\tau u(t)v(t) dt \text{ and } \|u\|_\tau = (u, u)_\tau^{1/2} \text{ for } u, v \in L^2_\omega(0, \tau).$$

For the discrete expansion, using the Chebyshev-Gauss quadrature formula, the discrete inner product and norm on $(0, \tau)$ is defined by

$$(u, v)_{\tau,N} = \frac{\pi}{N+1} \sum_{j=0}^N u\left(t_{\tau,j}^N\right)v\left(t_{\tau,j}^N\right) \text{ and } \|u\|_{\tau,N} = (u, u)_{\tau,N}^{1/2} \tag{2.5}$$

where $u, v \in L^2_\omega(0, \tau)$.

It follows from (2.4) that for any $\phi\psi \in \mathcal{P}_{2N+1}(0, \tau)$ and $\varphi \in \mathcal{P}_N(0, \tau)$,

$$\begin{aligned} (\phi, \psi)_\tau &= \int_0^\tau \phi(t)\psi(t)\omega(t) dt = \frac{\pi}{N+1} \sum_{j=0}^N \phi\left(t_{\tau,j}^N\right)\psi\left(t_{\tau,j}^N\right) \\ &= (\phi, \psi)_{\tau,N}, \end{aligned} \tag{2.6a}$$

and the two norms $\|\varphi\|_\tau^2$ and $\|\varphi\|_{\tau,N}^2$ coincide, i.e.

$$\|\varphi\|_\tau^2 = \int_0^\tau \varphi(t)\varphi(t)\omega(t) dt = \frac{\pi}{N+1} \sum_{j=0}^N \varphi\left(t_{\tau,j}^N\right)\varphi\left(t_{\tau,j}^N\right) = \|\varphi\|_{\tau,N}^2. \tag{2.6b}$$

Recall the Chebyshev expansion in (2.3), the truncated continuous expansion of u is considered as the projection of u on the finite dimensional space \mathcal{B}_{N+1} of the form

$$u(t) = \sum_{l=0}^{N+1} \hat{u}_l^N T_{\tau,l}(t) \tag{2.7}$$

where $\mathcal{B}_{N+1} = \text{span}\{x^k : 0 \leq k \leq N + 1\}$ with the coefficients

$$\hat{u}_l^N = \frac{2}{\pi c_l} \left(u, T_{\tau,l} \right)_\tau = \frac{2}{\pi c_l} \left(u, T_{\tau,l} \right)_{\tau,N}, \quad 0 \leq l \leq N \quad \text{and} \quad \hat{u}_{N+1}^N = \frac{2}{\pi c_{N+1}} \left(u, T_{\tau,N+1} \right)_\tau.$$

Let $\mathfrak{T}_{\tau,N} u$ be the discrete Chebyshev-Gauss expansion of any u in $L^2_\omega(0, \tau)$. Using the Chebyshev-Gauss quadrature, we define the discrete approximation of u

$$\mathfrak{T}_{\tau,N} u(t) = \sum_{l=0}^N \tilde{u}_l^N T_{\tau,l}(t) \tag{2.8}$$

where the discrete expansion coefficients are

$$\tilde{u}_l^N = \frac{2}{(N+1)c_l} \sum_{j=0}^N u(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N), \quad 0 \leq l \leq N.$$

This $\mathfrak{T}_{\tau,N} u \in \mathcal{P}_N(0, \tau)$ and it interpolates u at all the Gaussian quadrature points.

$$\mathfrak{T}_{\tau,N} u(t_{\tau,j}^N) = \sum_{l=0}^N \tilde{u}_l^N T_{\tau,l}(t_{\tau,j}^N) = \sum_{l=0}^N u(t_{\tau,l}^N) \ell_l(t_{\tau,j}^N) = u(t_{\tau,j}^N), \quad 0 \leq j \leq N$$

where $\ell_l(t)$ is the Lagrange polynomials based on the Chebyshev-Gauss nodes. From (2.6a) and (2.8), we have

$$(u, \phi)_{\tau,N} = \left(\mathfrak{T}_{\tau,N} u, \phi \right)_{\tau,N} = \left(\mathfrak{T}_{\tau,N} u, \phi \right)_\tau, \quad \forall \phi \in \mathcal{P}_{N+1}(0, \tau). \tag{2.9}$$

Using (2.9) and the above statement that $T_{\tau,l}(t) \in \mathcal{P}_N(0, \tau)$, the discrete expansion coefficients \tilde{u}_l^N in (2.8) can be written as

$$\tilde{u}_l^N = \frac{2}{\pi c_l} \left(\mathfrak{T}_{\tau,N} u, T_{\tau,l} \right)_\tau = \frac{2}{\pi c_l} \left(\mathfrak{T}_{\tau,N} u, T_{\tau,l} \right)_{\tau,N}, \quad 0 \leq l \leq N. \tag{2.10}$$

Next, we consider the relationship between the coefficients of the truncated continuous Chebyshev expansion in (2.7) and the discrete expansion in (2.8). For any $u \in \mathcal{P}_{N+1}(0, \tau)$, the coefficients \hat{u}_l^N and \tilde{u}_l^N determined in (2.7) and (2.10) gives

$$\tilde{u}_l^N = \frac{2}{\pi c_l} \left(\mathfrak{T}_{\tau,N} u, T_{\tau,l} \right)_{\tau,N} = \frac{2}{\pi c_l} \left(u, T_{\tau,l} \right)_{\tau,N} = \frac{2}{\pi c_l} \left(u, T_{\tau,l} \right)_\tau, \quad 0 \leq l \leq N.$$

Therefore, for any $u \in \mathcal{P}_{N+1}(0, \tau)$,

$$\hat{u}_l^N = \tilde{u}_l^N, \quad 0 \leq l \leq N. \tag{2.11}$$

The result from (2.11) gives the comparison of the discrete norm and the L^2_ω -norm of $u \in \mathcal{P}_{N+1}(0, \tau)$ as follows:

$$\begin{aligned} \|u\|_{\tau,N}^2 &= \left\| \mathfrak{T}_{\tau,N} u \right\|_{\tau,N}^2 = \left\| \mathfrak{T}_{\tau,N} u \right\|_\tau^2 = \sum_{l=0}^N \left(\hat{u}_l^N \right)^2 \int_0^\tau T_{\tau,l}^2(t) \omega(t) dt \\ \|u\|_{\tau,N}^2 &\leq \sum_{l=0}^{N+1} \left(\hat{u}_l^N \right)^2 \int_0^\tau T_{\tau,l}^2(t) \omega(t) dt = \int_0^\tau (u(t))^2 \omega(t) dt = \|u\|_\tau^2. \\ \|u\|_{\tau,N}^2 &\leq \|u\|_\tau^2. \end{aligned} \tag{2.12}$$

2.3 Chebyshev-Gauss collocation method

In this section, we introduce a Chebyshev-Gauss collocation method to obtain a numerical solution of ordinary differential equations. Consider the first-order ordinary differential equation on the interval $[0, \tau]$ of the form

$$\begin{aligned} \frac{d}{dt} X(t) &= f(X(t), t), \quad 0 < t \leq \tau \\ X(0) &= X_0. \end{aligned} \tag{2.13}$$

For the spectral collocation method, we find $X^N(t) \in \mathcal{P}_{N+1}(0, \tau)$ such that

$$\begin{aligned} \frac{d}{dt} X^N(t_{\tau,j}^N) &= f\left(X^N(t_{\tau,j}^N), t_{\tau,j}^N\right), \quad 0 \leq j \leq N \\ X^N(0) &= X_0, \end{aligned} \tag{2.14}$$

which implies the residual error vanishes at the collocation points $t_{\tau,j}^N, j = 0, \dots, N$, or

$$R_N\left(X^N(t_{\tau,j}^N), t_{\tau,j}^N\right) = \frac{d}{dt} X^N(t_{\tau,j}^N) - f\left(X^N(t_{\tau,j}^N), t_{\tau,j}^N\right) = 0.$$

In the Chebyshev collocation, we seek a solution $X^N(t) \in \mathcal{P}_{N+1}(0, \tau)$ of the form

$$X^N(t) = \sum_{l=0}^{N+1} \hat{X}_l^N T_{\tau,l}(t), \quad 0 < t \leq \tau. \tag{2.15}$$

As a result, we have that $X^N(t) T_{\tau,l}(t) \in \mathcal{P}_{2N+1}(0, \tau)$ for $0 \leq l \leq N$. Multiplying (2.15) by $T_{\tau,l}(t)\omega(t)$ and integrating the result over the interval $[0, \tau]$ together with (2.7), we obtain

$$\hat{X}_l^N = \frac{2}{\pi c_l} \left(X^N, T_{\tau,l}\right)_\tau = \frac{2}{\pi c_l} \left(X^N, T_{\tau,l}\right)_{\tau,N} = \frac{2}{(N+1)c_l} \sum_{j=0}^N X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N) \tag{2.16}$$

for any $0 \leq l \leq N$.

So far, we only obtain the coefficients $\hat{X}_l^N, 0 \leq l \leq N$. We still need to find the last coefficient \hat{X}_{N+1}^N . Considering $t = 0$ in

(2.15), $X_0 = \sum_{l=0}^{N+1} \hat{X}_l^N T_{\tau,l}(0)$. By using the property $T_{\tau,l}(0) = (-1)^l$ and (2.16), we obtain

$$\begin{aligned} \hat{X}_{N+1}^N &= (-1)^{N+1} X_0 + \sum_{l=0}^N (-1)^{N+l} \hat{X}_l^N \\ &= (-1)^{N+1} X_0 + \frac{2}{(N+1)} \sum_{l=0}^N \sum_{j=0}^N \frac{(-1)^{N+l}}{c_l} X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N). \end{aligned} \tag{2.17}$$

To derive the derivative term, $\frac{d}{dt} X^N$, for the equation, we consider the recurrence relation

$$\frac{d}{dt} T_{\tau,l}(t) = \left(\frac{l}{l-2}\right) \frac{d}{dt} T_{\tau,l-2}(t) + \frac{4(l)}{\tau} T_{\tau,l-1}(t).$$

Due to the nature the Chebyshev polynomials, we divide l into two different cases.

Case I l is even,

$$\begin{aligned} \frac{d}{dt} T_{\tau,2}(t) &= 2 \left(\frac{4}{\tau}\right) T_{\tau,1}(t) \\ \frac{d}{dt} T_{\tau,4}(t) &= \left[4 \left(\frac{4}{\tau}\right) T_{\tau,1}(t)\right] + \left[4 \left(\frac{4}{\tau}\right) T_{\tau,3}(t)\right] \\ \frac{d}{dt} T_{\tau,6}(t) &= \left[6 \left(\frac{4}{\tau}\right) T_{\tau,1}(t)\right] + \left[6 \left(\frac{4}{\tau}\right) T_{\tau,3}(t)\right] + \left[6 \left(\frac{4}{\tau}\right) T_{\tau,5}(t)\right] \\ \therefore \frac{d}{dt} T_{\tau,l}(t) &= \sum_{m=1}^{\frac{l}{2}} (l) \left(\frac{4}{\tau}\right) T_{\tau,l-(2m-1)}(t), \quad l = 2, 4, 6, \dots \end{aligned}$$

Case II l is odd,

$$\begin{aligned} \frac{d}{dt} T_{\tau,1}(t) &= \left(\frac{2}{\tau}\right) T_{\tau,0}(t) \\ \frac{d}{dt} T_{\tau,3}(t) &= \left[3\left(\frac{2}{\tau}\right) T_{\tau,0}(t)\right] + \left[3\left(\frac{4}{\tau}\right) T_{\tau,2}(t)\right] \\ \frac{d}{dt} T_{\tau,5}(t) &= \left[5\left(\frac{2}{\tau}\right) T_{\tau,0}(t)\right] + \left[5\left(\frac{4}{\tau}\right) T_{\tau,2}(t)\right] + \left[5\left(\frac{4}{\tau}\right) T_{\tau,4}(t)\right] \\ \therefore \frac{d}{dt} T_{\tau,l}(t) &= \left(\sum_{m=1}^{\frac{l-1}{2}} (l) \left(\frac{4}{\tau}\right) T_{\tau,l-(2m-1)}(t)\right) + \frac{2l}{\tau}, \quad l = 1, 3, 5, \dots \end{aligned}$$

Hence, we use (2.15)-(2.17) to obtain

$$\begin{aligned} \frac{d}{dt} X^N(t) &= \sum_{l=1}^{N+1} \hat{X}_l^N \frac{d}{dt} T_{\tau,l}(t) = \hat{X}_{N+1}^N \frac{d}{dt} T_{\tau,N+1}(t) + \sum_{l=1}^N \hat{X}_l^N \frac{d}{dt} T_{\tau,l}(t) \\ &= \left((-1)^{N+1} X_0 + \frac{2}{(N+1)} \sum_{l=0}^N \sum_{j=0}^N \frac{(-1)^{N+l}}{c_l} X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N) \right) \frac{d}{dt} T_{\tau,N+1}(t) \\ &\quad + \left(\sum_{l=1}^N \sum_{j=0}^N \frac{2}{(N+1)c_l} X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N) \right) \frac{d}{dt} T_{\tau,l}(t) \\ &= (-1)^{N+1} X_0 \frac{d}{dt} T_{\tau,N+1}(t) + \sum_{l=1}^N \left(\sum_{j=0}^N \frac{2}{(N+1)c_l} X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N) \right) \frac{d}{dt} T_{\tau,l}(t) \\ &\quad + \frac{2}{N+1} \left(\sum_{l=0}^N \sum_{j=0}^N \frac{(-1)^{N+l}}{c_l} X^N(t_{\tau,j}^N) T_{\tau,l}(t_{\tau,j}^N) \right) \frac{d}{dt} T_{\tau,N+1}(t). \end{aligned}$$

For simplicity, we let

$$\begin{aligned} a_{k,j}^N &= \frac{8}{N+1} \sum_{l=1}^N \frac{1}{c_l} T_{\tau,l}(t_{\tau,j}^N) \times \begin{cases} \left(\sum_{m=1}^{\frac{l}{2}} (l) T_{\tau,l-(2m-1)}(t_{\tau,k}^N) \right), & l \text{ is even} \\ \left(\sum_{m=1}^{\frac{l-1}{2}} (l) T_{\tau,l-(2m-1)}(t_{\tau,k}^N) \right) + \frac{l}{2}, & l \text{ is odd} \end{cases} \\ &\quad + 8 \left(\sum_{l=1}^N \frac{(-1)^{N+l}}{c_l} T_{\tau,l}(t_{\tau,j}^N) \right) \times \begin{cases} \left(\sum_{m=1}^{\frac{N+1}{2}} T_{\tau,N-(2m-2)}(t_{\tau,k}^N) \right), & N \text{ is odd} \\ \left(\sum_{m=1}^{\frac{N}{2}} T_{\tau,N-(2m-2)}(t_{\tau,k}^N) \right) + \frac{1}{2}, & N \text{ is even} \end{cases} \end{aligned} \tag{2.18a}$$

and

$$b_k^N = 4(N+1)(-1)^{N+1} \times \begin{cases} \sum_{m=1}^{\frac{N+1}{2}} T_{\tau, N-(2m-2)} \left(t_{\tau, k}^N \right), & N \text{ is odd} \\ \left(\sum_{m=1}^{\frac{N}{2}} T_{\tau, N-(2m-2)} \left(t_{\tau, k}^N \right) \right) + \frac{1}{2}, & N \text{ is even.} \end{cases} \tag{2.18b}$$

Therefore,

$$\frac{d}{dt} X^N \left(t_{\tau, k}^N \right) = \frac{1}{\tau} \sum_{j=0}^N a_{k,j}^N X^N \left(t_{\tau, j}^N \right) + \frac{X_0}{\tau} b_k^N, \quad 0 \leq k \leq N. \tag{2.19}$$

Substituting (2.19) into the left hand side of (2.14) yields the following matrix equation of (2.19)

$$A^N \mathbf{X}^N = (\tau) F^N \left(\mathbf{X}^N \right) - X_0 \mathbf{b}^N. \tag{2.20}$$

where A^N is the matrix with the entries $a_{k,j}^N, 0 \leq j, k \leq N$, given in (2.18a)

$$\mathbf{X}^N = \left(X^N \left(t_{\tau, 0}^N \right), X^N \left(t_{\tau, 1}^N \right), \dots, X^N \left(t_{\tau, N}^N \right) \right)^T, \quad \mathbf{b}^N = \left(b_0^N, b_1^N, \dots, b_N^N \right)^T,$$

$$F^N \left(X^N \right) = \left(f \left(X^N \left(t_{\tau, 0}^N \right), t_{\tau, 0}^N \right), f \left(X^N \left(t_{\tau, 1}^N \right), t_{\tau, 1}^N \right), \dots, f \left(X^N \left(t_{\tau, N}^N \right), t_{\tau, N}^N \right) \right)^T.$$

We solve this system for the solution $\mathbf{X}^N = \left(X^N \left(t_{\tau, 0}^N \right), X^N \left(t_{\tau, 1}^N \right), \dots, X^N \left(t_{\tau, N}^N \right) \right)^T$.

The last step of our algorithm is to determine $X^N(t)$ at the right end (or $X^N(\tau)$). This value will be used as the initial value of the consecutive interval when considering a domain decomposition. To compute $X^N(t)$, we use the values of $\left\{ X^N \left(t_{\tau, k}^N \right) \right\}_{k=0}^N$ which are obtained from (2.20) together with (2.17). Since $T_{\tau, l}(\tau) = 1$, we get

$$X^N(\tau) = \sum_{l=0}^{N+1} \hat{X}_l^N = \hat{X}_{N+1}^N + \sum_{l=0}^N \hat{X}_l^N$$

$$= (-1)^{N+1} X_0 + \left[\frac{2}{N+1} \sum_{l=0}^N \sum_{j=0}^N \frac{\left((-1)^{N+l} + 1 \right)}{c_l} X^N \left(t_{\tau, j}^N \right) T_{\tau, l} \left(t_{\tau, j}^N \right) \right]. \tag{2.21}$$

In the following theorems, we discuss the uniqueness of the solution in Theorem 2.1 and spectral accuracy of our method in Theorem 2.2. The proof of the theorems was completed in Yang *et al.* (2015). We will apply these theorems to our work.

Theorem 2.1. If $f(z, t)$ satisfies the following Lipschitz condition:

$$\left| f \left(z_1, t \right) - f \left(z_2, t \right) \right| \leq \gamma \left| z_1 - z_2 \right|, \quad \gamma > 0, \tag{2.22}$$

and $0 < \gamma\tau \leq \beta < \frac{1}{4}$, where β is a certain constant. Then the system (2.14) has a unique solution.

Proof See Yang *et al.* (2015).

Theorem 2.2. Assume that $f(z, t)$ fulfills the Lipschitz condition (2.22). Then for any $X \in H^r_{\omega^{\frac{r-3}{2}}}(0, \tau)$ with integers

$2 \leq r \leq N + 1$, we have

$$\|X - X^N\|_{L^2(0,\tau)}^2 \leq \frac{\tau}{2} \|X - X^N\|_{\tau,\omega} \leq c_\beta \tau^2 N^{4-2r} \int_0^\tau \omega^{\frac{r-3}{2}}(t) \left(\frac{d^r}{dt^r} X(t) \right)^2 dt. \tag{2.23}$$

$$|X(\tau) - X^N(\tau)| \leq c_\beta \tau^2 N^{4-2r} \int_0^\tau \omega^{\frac{r-3}{2}}(t) \left(\frac{d^r}{dt^r} X(t) \right)^2 dt \tag{2.24}$$

where c_β is a positive constant depending only on β .

Proof See Yang *et al.* (2015).

Theorem 2.2 shows a spectral accuracy of the Chebyshev-Gauss collocation methods. Since the theorem discussed in Yang *et al.* (2015) can be applied to any Chebyshev-Gauss collocation method, we can use it for our method. However, the algorithm to compute for X_n in Yang *et al.* (2015) is different from that in our algorithm. We will present the spectral accuracy of our method in the numerical result in section 3.

3. Results and Discussion

In this section, we present some numerical results to support the method in (2.20). We consider two errors, the discrete L^2 error and the point-wise absolute error, in order to compare the results obtained from the Chebyshev-Gauss collocation method, the Legendre-Gauss collocation method in Guo *et al.* (2009) and the Chebyshev-Gauss collocation method in Yang *et al.* (2015). For the Hamiltonian system, we compare the error in the total energy of the systems and CPU times of those three collocation methods.

For simplicity, we use the following notations,

- CGC : Chebyshev-Gauss collocation method (2.20).
- LCG : Legendre-Gauss collocation method in Guo *et al.* (2009).
- CGC-Yang : Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015).
- CGC-Yang1 : The simple iterative algorithm.
- The point-wise absolute error

$$E_{\tau,p} = |X(\tau) - X^N(\tau)|.$$

- The discrete L^2 error

$$E_{\tau,d} = \|X - X^N\|_{\tau,N}.$$

- The point-wise absolute error for systems of differential equations

$$E^N(t) = \sqrt{(p^N(t) - P(t))^2 + (q^N(t) - Q(t))^2}$$

- The maximum error in energy for systems of differential equations

$$E_H(\tau) = |H(P(0), Q(0)) - H(P(\tau), Q(\tau))|$$

The algorithms to solve for the solution are implemented using MATLAB and calculations are carried out with a computer Intel(R) Core(TM) i5-2410 CPU @ 2.30GHz RAM 4.00 GB.

3.1 Single-interval Domain

For the domain containing only one interval, we apply the algorithm in (2.20) directly. The scheme (2.20) is an implicit scheme. We apply an iterative method to solve the system.

3.2 Multi-interval Domain

For the domain decomposition, we break the domain $[0, T]$ into M subintervals where each of which is of length $\tau = \frac{T}{M}$. We first evaluate the solution $X_1^N(t)$ on the first subinterval $[0, \tau]$ with the given initial condition $X(0) = X_0$. Then, we compute the end point value of $X_1^N(\tau)$ and set it as the initial condition for the next subinterval. By continuing the process, the solution on the i -th interval can be evaluated by finding $X_i^N(t) \in \mathcal{P}_{2N+1}(0, \tau), 2 \leq i \leq M$ such that

$$\begin{aligned} \frac{d}{dt} X_i^N(t_{\tau,k}^N) &= f(X_i^N(t_{\tau,k}^N), \tau(i-1) + t_{\tau,k}^N), \quad 0 \leq k \leq N, 2 \leq i \leq M \\ X_i^N(0) &= X_{i-1}^N(\tau). \end{aligned} \tag{3.1}$$

The value of $X_i^N(t_{\tau,k}^N)$ is the local value for each subinterval. Patching all the solutions together with the global notation $X^N(t_{\tau,k}^N + \tau(i-1)) = X_i^N(t)$, we finally arrive with the numerical solution for the equation.

The following example demonstrates the solution of an initial value problem with single and multi-interval domains. The numerical results for the single domain are presented in Example 1 (Figure 1 to Figure 7) and the results with the multi-interval domains are presented in Example 1 (Figure 8), Example 2 and Example 3.

Example 1 We use scheme (2.20) to solve the problem

$$\begin{aligned} \frac{d}{dt} X(t) &= f(X(t), t), \quad 0 < t \leq \tau \\ X(0) &= 1. \end{aligned} \tag{3.2}$$

with $f(X(t), t) = e^{\left(\frac{1}{5} \sin X(t)\right)} + \frac{3}{2}(t+1)^{1/2} + 10 \cos(2t) - e^{\left(\frac{1}{5} \sin((t+1)^{3/2} + 5 \sin(2t))\right)}$.

The exact solution of this problem is

$$X(t) = (t+1)^{3/2} + 5 \sin(2t)$$

which oscillates and grows to infinity as t increases. The corresponding function on right side, $f(X(t), t)$, satisfies the Lipschitz condition as follows. Consider

$$f(X_1(t), t) - f(X_2(t), t) = e^{\left(\frac{1}{5} \sin X_1(t)\right)} - e^{\left(\frac{1}{5} \sin X_2(t)\right)}.$$

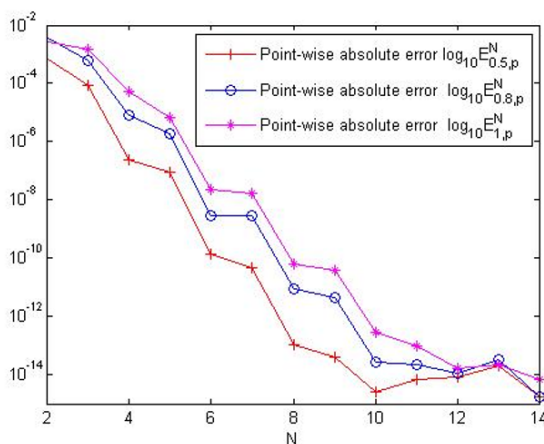


Figure 1. Point-wise absolute error of scheme (2.20) when at $\tau = 0.5, 0.8$ and 1 .

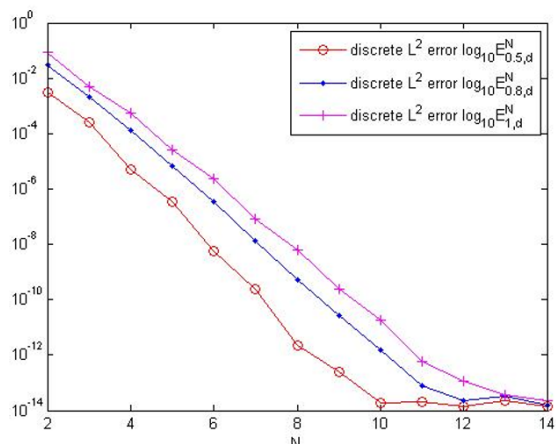


Figure 2. Discrete L^2 error of scheme (2.20) when at $\tau = 0.5, 0.8$ and 1 .

By the Mean Value Theorem, we have

$$\frac{e^{\left(\frac{1}{5}\sin X_1(t)\right)} - e^{\left(\frac{1}{5}\sin X_2(t)\right)}}{X_1(t) - X_2(t)} = \frac{1}{5} e^{\left(\frac{1}{5}\sin X_c(t)\right)} \left(\cos X_c(t)\right)$$

for some $X_c(t)$ between $X_1(t)$ and $X_2(t)$.

Since $|\sin X_c(t)| \leq 1$ and $|\cos X_c(t)| \leq 1$, it follows that

$$\left| e^{\left(\frac{1}{5}\sin X_1(t)\right)} - e^{\left(\frac{1}{5}\sin X_2(t)\right)} \right| \leq \frac{1}{5} e^{\left(\frac{1}{5}\right)} |X_1(t) - X_2(t)|.$$

Thus, $|f(X(t), t)|$ fulfills the Lipschitz condition with $\gamma \leq \frac{1}{5} e^{\left(\frac{1}{5}\right)}$. It follows from Theorems 2.1 and 2.2 that the equation (3.2) has a unique solution and has the error estimates in (2.23) and (2.24).

We implement the algorithm by using this function $f(X(t), t)$. The figures below illustrate the errors by the spectral collocation methods defined at the beginning of the section. In Figure 1, we plot the point-wise absolute error at $\tau = 0.5, 0.8$ and 1 with different values of N . We observe that the point-wise absolute error decreases as N increases and τ decreases. Furthermore, The errors oscillate between odd and even N . The rate of convergence when N is even is faster than the rate when N is odd.

In Figure 2, we present the discrete L^2 error at $\tau = 0.5, 0.8$ and 1 with various N . There is only slight oscillation for the discrete L^2 errors. The error decreases as N increases and τ decreases. In Figure 3 and Figure 4, we compare the Chebyshev-Gauss collocation method in (2.20) and the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015). The point-wise absolute error and the discrete L^2 error of two methods nearly coincide. The rate from both methods are of the same order.

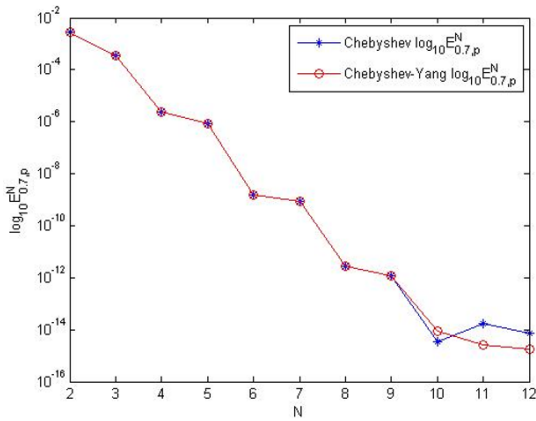


Figure 3. Point-wise absolute error of scheme (2.20) versus the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015).

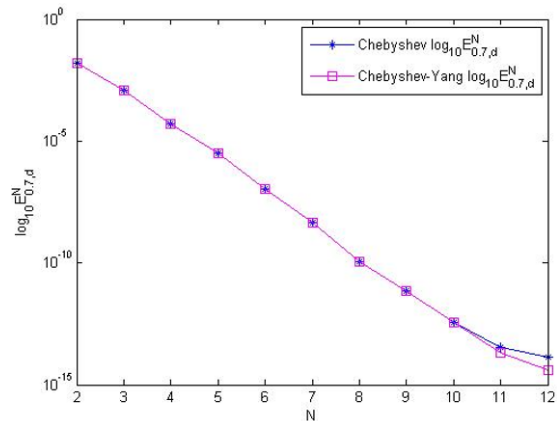


Figure 4. Discrete L^2 error of scheme (2.20) versus the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015).

The rate of convergence of the point-wise absolute errors and the discrete L^2 errors from the Chebyshev-Gauss collocation in (2.20) and the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015) shown in Figure 5 and 6 demonstrate the spectral accuracy. We plot the point-wise absolute errors of the two methods and estimate them by comparing with the function $e^{-3.1N}$ in Figure 5. It follows that convergence rate is of order $\mathcal{O}(e^{-3.1N})$. Similarly, as shown in Figure 6, the convergence rate of the discrete L^2 error is of order $\mathcal{O}(N^{-4.33\sqrt{N}})$.

In Figure 7, we compare the Chebyshev-Gauss collocation method in (2.20) with the Legendre-Gauss collocation method and the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015). We observe that the discrete L^2 error of

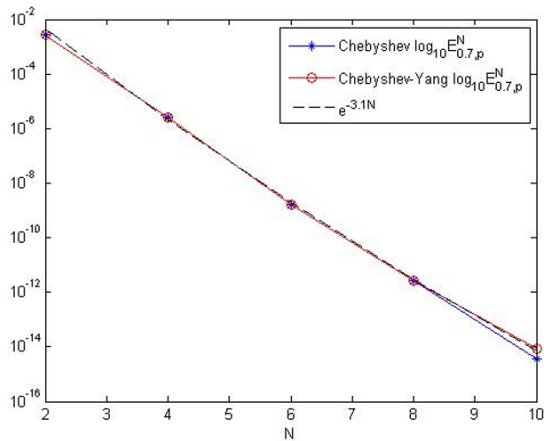


Figure 5. Point-wise absolute error of scheme (2.20) versus the Chebyshev-Gauss spectral collocation method and the convergence rate when N varies.

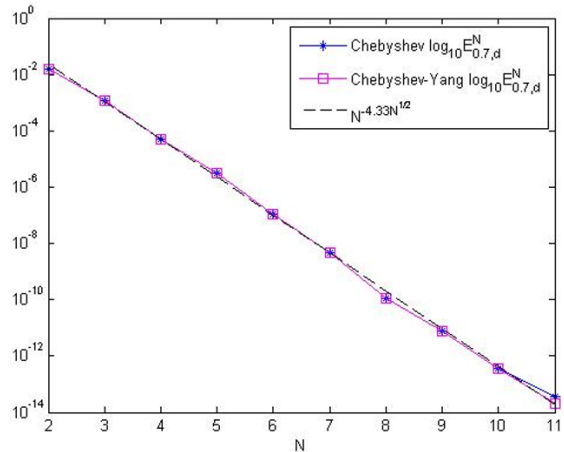


Figure 6. Discrete L^2 error of scheme (2.20) versus the Chebyshev-Gauss spectral collocation method and the convergence rate when N varies.

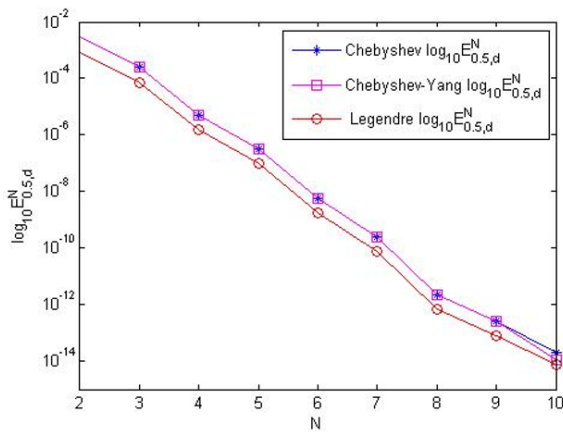


Figure 7. Discrete L^2 error of scheme (2.20) versus the Chebyshev-Gauss spectral collocation method and the Legendre-Gauss collocation method.

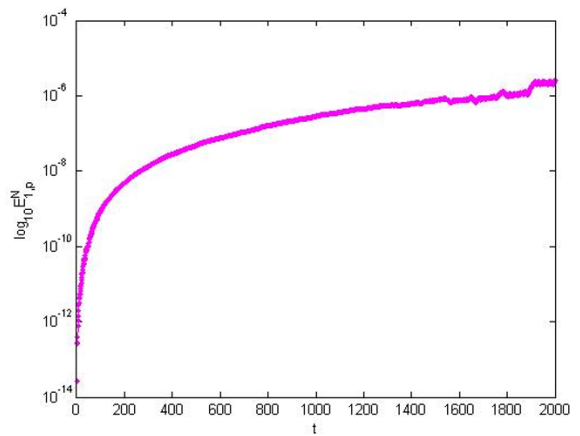


Figure 8. Point-wise absolute error of scheme (2.20) when $t = 2000$, $\tau = 1$ and $N = 17$.

the Legendre-Gauss collocation method decays slightly faster as N increases and the two Chebyshev-Gauss collocation methods are of the same rate.

Figure 8 shows the point-wise absolute error of the Chebyshev-Gauss collocation method in (2.20). We observe that the point-wise absolute error grows rapidly when the time is less than 200 seconds then it increases at a slower rate and slightly oscillates as t increases.

3.3 System of differential equations

For the system of differential equations, we denote the vectors

$$\bar{X}(t) = (X_1(t), X_2(t), \dots, X_n(t))^T$$

$$\bar{f}(\bar{X}(t), t) = (f^1(\bar{X}(t), t), f^2(\bar{X}(t), t), \dots, f^n(\bar{X}(t), t))^T.$$

We can apply the algorithm in (2.20) to the system of equations. The solution can be determined in a similar way. Consider the system

$$\frac{d}{dt} \bar{X}(t) = \bar{f}(\bar{X}(t), t), \quad 0 < t \leq \tau$$

$$\bar{X}(0) = \bar{X}_0. \tag{3.3}$$

In spectral collocation method, we approximate the solution of (3.3) as follows.

Find $\bar{X}^N(t) \in (\mathcal{P}_{N+1}(0, \tau))^n$ such that

$$\begin{aligned} \frac{d}{dt} \bar{X}^N(t_{\tau,j}^N) &= \bar{f}\left(\bar{X}^N(t_{\tau,j}^N), t_{\tau,j}^N\right), & 0 \leq j \leq N, 0 < t \leq \tau \\ \bar{X}^N(0) &= \bar{X}_0. \end{aligned} \tag{3.4}$$

We evaluate each $X_i(t)$ by applying (2.20) together with an iterative method.

In the examples below, we present the numerical solution of linear and nonlinear Hamiltonian systems. We are interested in the long-term behavior of the system. A good algorithm should preserve both the area (orbit) and the energy of the system (Kanyamee *et al.*, 2011).

Example 2 Consider the Hamiltonian system

$$\begin{aligned} p'(t) &= -q(t) + 1, & 0 < t \leq \tau \\ q'(t) &= p(t), & 0 < t \leq \tau \\ p(0) &= 1, & q(0) = 0. \end{aligned} \tag{3.5}$$

with the exact solution

$$p(t) = \cos(t) + \sin(t) \text{ and } q(t) = -\cos(t) + \cos^2(t) + \sin(t) + \sin^2(t).$$

The corresponding Hamiltonian function of this system is $H(p, q) = \frac{1}{2}p^2 + \frac{1}{2}q^2 - q$. The total energy of the system is $E = \frac{1}{2}p(0)^2 + \frac{1}{2}q(0)^2 - q(0) = \frac{1}{2}$.

Figure 9 represents the phase plots of $p^N(t)$ and $q^N(t)$ by using the Chebyshev-Gauss collocation method when $M = 10^5, \tau = 0.1$ and $N = 7$. The other two collocation methods also present the same orbit. We see that the orbit preserves the area as it does not shift from the exact solution when t is large.

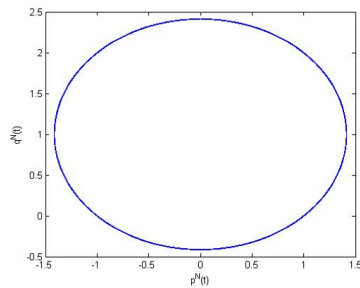


Figure 9. Phase plot p^N versus q^N when $M = 10^5, \tau = 0.1$ and $N = 7$.

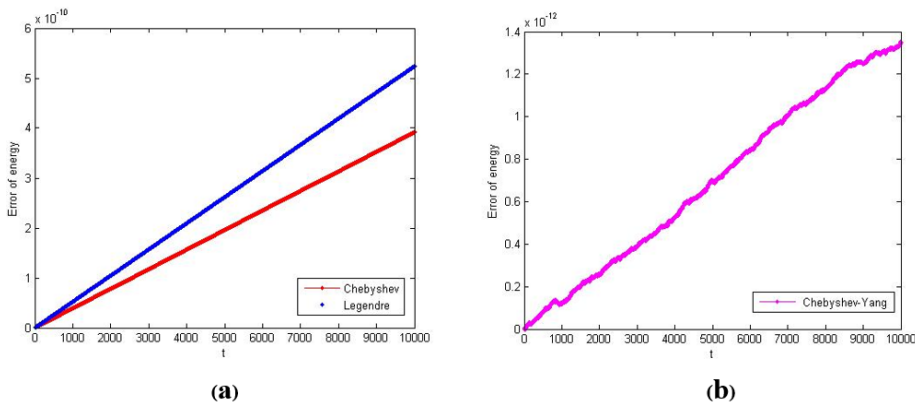


Figure 10. (a) Error in energy of scheme (3.4) versus the Legendre-Gauss collocation method (b) the Chebyshev-Gauss spectral collocation method when $M = 10^5, \tau = 0.1$ and $N = 7$.

In Figure 10(a) and 10(b), we compare the error in energy of the Chebyshev-Gauss collocation method in (3.4) with the Legendre-Gauss collocation method and the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015). The error in energy from the Chebyshev-Gauss collocation method in (3.4) is smaller than the error from the Legendre-Gauss collocation method. However, the errors are still larger than those from the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015). The three methods show a constant growth (with respect to the log scale) of the error, but the point-wise absolute error from Yang *et al.* (2015) slightly oscillates as t increases.

Table 1. Comparison of the errors and CPU times from the three collocation methods for the system (3.5).

	N	7	9	11	13	17	21
	t	0.1	0.25	0.5	1	2	4
$E^N(t)$	CGC	2.77e-11	1.77e-11	3.82e-11	2.48e-11	8.14e-11	2.62e-11
	LGC	3.70e-11	1.50e-11	3.44e-11	3.51e-11	5.05e-11	3.40e-11
	CGC-Yang1	1.15e-13	1.36e-13	2.33e-13	3.27e-12	1.03e-12	4.82e-12
$E_H(\tau)$	CGC	3.92e-11	2.48e-11	5.06e-11	3.48e-11	9.22e-11	3.21e-11
	LGC	5.24e-11	2.12e-11	4.70e-11	4.32e-11	3.26e-11	4.81e-11
	CGC-Yang1	1.36e-13	8.30e-14	4.54e-14	3.71e-13	6.26e-14	4.00e-13
CPU Times	CGC	15.60	25.20	42.07	87.14	221.91	659.08
	LGC	19.038	31.14	56.20	124.27	379.50	743.33
	CGC-Yang1	33.32	70.70	197.68	539.41	1.35e+3	2.15e+3

Table 1 represents the point-wise absolute error $E^N(t)$, the maximum error of energy $E_H(\tau)$ and the CPU times when $M = 10^4$ with different values of τ and N of the system (3.5). We have that the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015) preserves energy better and the point-wise absolute error is less than the other two methods. However, when we compared the CPU times, it takes much longer than the time taken from the other two methods. The Chebyshev-Gauss collocation method (3.4) gives the best CPU times.

Example 3 In this example, we consider the Henon–Heiles system given by

$$\begin{aligned}
 p'_1(t) &= -q_1(t) - 2q_1(t)q_2(t), & 0 < t \leq \tau \\
 p'_2(t) &= -q_2(t) - q_1^2(t) + q_2^2(t), & 0 < t \leq \tau \\
 q'_1(t) &= p_1(t), & 0 < t \leq \tau \\
 q'_2(t) &= p_2(t), & 0 < t \leq \tau
 \end{aligned}
 \tag{3.6}$$

with initial conditions $p_1(0) = 0.011$, $p_2(0) = 0$, $q_1(0) = 0.013$ and $q_2(0) = -0.4$.

The corresponding Hamiltonian function for this system is

$$H(p_1, p_2, q_1, q_2) = \frac{1}{2}(p_1^2 + p_2^2 + q_1^2 + q_2^2) + q_1^2 q_2 - \frac{1}{3} q_2^3.$$

The total energy of the system with respect to the initial conditions is

$$E = \frac{1}{2}(p_1^2(0) + p_2^2(0) + q_1^2(0) + q_2^2(0)) + q_1^2(0)q_2(0) - \frac{1}{3}q_2^3(0) = 0.1014.$$

Figure 11 and Figure 12 illustrate the phase plots of $p_1^N(t)$ and $q_1^N(t)$ and the phase plots of $q_1^N(t)$ and $q_2^N(t)$, respectively, using the Chebyshev–Gauss collocation method (3.4) when $M = 10^4$, $\tau = 0.1$ and $N = 7$. The other two collocation methods also present the same orbit.

Table 2 illustrates the maximum error of energy $E_H(\tau)$ and CPU times when $M = 10^4$ with different values of τ and N of the system (3.6). From the table, the method (3.4) and the Legendre-Gauss collocation method provide the error in energy of the same order. In this example, the Chebyshev-Gauss spectral collocation method in Yang *et al.* (2015) preserves energy better. However, when we compare the CPU times, the time from the Chebyshev-Gauss spectral collocation method in Yang

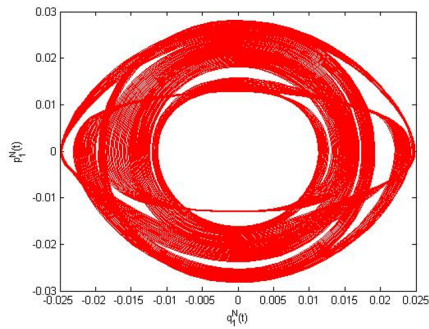


Figure 11. Phase plot q_1^N versus p_1^N when $M = 10^4$, $\tau = 0.1$ and $N = 7$.

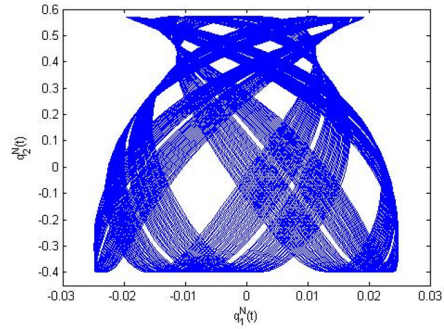


Figure 12. Phase plot q_1^N versus q_2^N when $M = 10^4$, $\tau = 0.1$ and $N = 7$.

Table 2. Comparison of the errors and CPU times from the three collocation methods for Henon Heiles (3.6).

	N	7	9	11	13	17	21
	t	0.1	0.25	0.5	1	2	4
$E_H(\tau)$	CGC	5.91e-12	1.10e-12	8.41e-12	9.30e-12	2.29e-11	1.85e-11
	LGC	3.45e-12	2.28e-11	9.48e-12	9.79e-12	4.19e-11	2.09e-11
	CGC-Yang1	1.21e-14	6.34e-15	2.39e-15	1.42e-14	2.73e-14	2.47e-12
CPU Times	CGC	10.69	16.34	26.19	54.59	158.19	290.49
	LGC	13.12	18.24	24.58	34.66	77.84	141.30
	CGC-Yang1	6.54	9.20	16.05	43.08	163.52	407.01

et al. (2015) grows as τ and N increases. The CPU times of the method (3.4) and the Legendre-Gauss collocation method are close to each other when τ and N are small.

4. Conclusions

In this work, we proposed the Chebyshev-Gauss collocation method to solve to the initial value problems of ordinary differential equations. We constructed the algorithm for an ordinary differential equation as well as for systems of ordinary differential equations in both single- and multi-interval domains. The numerical results support the theoretical result discussed in section 2. For a fixed τ , the error drops rapidly as N increases. This behavior is expected since we increase the number of collocation points. We compare the results to show the order of convergence of $\mathcal{O}(e^{-3.1\sqrt{N}})$ and $\mathcal{O}(N^{-4.33\sqrt{N}})$ in Figure 5 and 6 respectively. It shows that the method (2.20) possesses a spectral accuracy.

For fixed τ and N , with the scheme for a multi-interval domain, the error grows at a faster rate for a smaller t than a larger t . As t gets larger, the error increases due to the accumulation of errors from the subintervals. These errors may occur when we approximate the endpoint value of each subinterval and set it as the initial condition for the next subinterval. For the Hamiltonian systems, we have that the method (2.20) preserves both energy and the area. The CPU times for the method (2.20) are the best for the linear systems

and are comparable to the Legendre-Gauss collocation method and the Chebyshev-Gauss spectral collocation method in Yang et al. (2015) for nonlinear systems. One may improve the algorithm by designing the iterative methods for solving the implicit systems (2.20), especially when the coefficient matrix has a large condition number.

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