สารบัญ

บทคัดย่อภาษาไทย

ในงานวิจัยเรื่องนี้ เราศึกษาพีชคณิตพัวส์ซอง A ที่มีตัวก่อกำหนด 3 ตัว ภายใต้ความสัมพันธ์ที่แตกต่างจากความสัมพันธ์บน T ใน [13] เราได้ทำการจำแนกพัวส์ซองมอดูลเชิงเดียวที่มีมิติจำกัดบน A โดยการพิจารณา J/J^2 ที่มีโครงสร้างของพีชคณิตลี เมื่อ J คือไอดีลใหญ่ที่สุดของ A ที่เป็นพัวส์ซอง เราพิสูจน์ได้ว่า A มีไอดีลใหญ่ที่สุดที่เป็นพัวส์ซองเพียง 2 ตัวเท่านั้น กล่าวคือ J_1 และ J_2 พิจารณากรณี J_1/J_1^2 พัวส์ซองมอดูลเชิงเดียวที่มีมิติจำกัดซึ่งถูกกำจัดโดย J_1 จะมี 1 มิติ สำหรับกรณีของ J_2/J_2^2, พีชคณิตพัวส์ซอง A เป็น 1-homogeneous นั่นคือสำหรับแต่ละจำนวนเต็มบวก d จะมีพัวส์ซองมอดูลเชิงเดียวที่มีมิติ d อยู่เพียงหนึ่งมอดูลเท่านั้น

บทคัดย่อภาษาอังกฤษ (Abstract)

In this research, we study a Poisson algebra A with three generators but having three relations different from the relations of T in [13]. We classify the finite-dimensional simple Poisson modules over A by considering its Lie structure J/J^2, where J is a Poisson maximal ideal of A. We show that there are only two Poisson maximal ideals J_1 and J_2. For the case J_1/J_1^2 , the finite dimension Poisson modules annihilated by J_1 is one-dimensional. For the case J_2/J_2^2, A is 1-homogeneous, that is, there is one d-dimensional simple Poisson module for each positive integer d.

คำนำ

นักคณิตศาสตร์ในกลุ่มสาขาพีชคณิตสนใจศึกษา การจำแนกมอดูลที่มีมิติจำกัดบนพีชคณิตที่เราสนใจ คณะวิจัยได้ทำการศึกษางานบางส่วนจากวิทยานิพนธ์ในระดับปริญญาเอกเรื่อง Reversible skew Laurent polynomial rings, rings of invariants and related rings ซึ่งในวิทยานิพนธ์นี้ได้ศึกษาริงของอินแวเรียนที่อยู่ในรูปแบบ T/pT เมื่อ T คือพีชคณิตที่มีตัวก่อกำหนด 3 ตัว ภายใต้ความสัมพันธ์ 3 ข้อ และ p คือสมาชิกศูนย์กลางในมุมมองของพีชคณิตโลคอลไลซ์เอนเวลลอบปิ้ง T คือ ริงพหุนามเสมือนแบบซ้ำ พร้อมกับอัตสัณฐานและเดอริเวชัน แต่ไม่ได้ปรากฏในกรณีของควอนตัมทอรัส ซึ่ง T ถูกแทนด้วย T_q เมื่อ q ไม่ใช่ root of unity จากการศึกษาข้างต้นประกอบกับการปรึกษากับ Prof. David A. Jordan จาก University of Sheffield, UK คณะวิจัยเห็นว่ามีความน่าสนใจที่เราจะศึกษาพีชคณิตพัวส์ซองมีตัวก่อกำเนิด 3 ตัว ภายใต้ความสัมพันธ์ที่อยู่ในอีกรูปแบบหนึ่งของความสัมพันธ์บนการจำแนกมอดูลที่มีมิติจำกัดที่ปรากฏในวิทยานิพนธ์ดังกล่าวข้างต้นนั้นได้ใช้เทคนิคการคำนวณที่ตรงไปตรงมา เนื่องจากความสัมพันธ์ใน T_q มีความซับซ้อนไม่มาก แต่ในงานวิจัยของคณะวิจัยชิ้นนี้ได้ศึกษาความสัมพันธ์ซึ่งมีความซับซ้อนกว่า ทำให้เทคนิคที่ปรากฏในวิทยานิพนธ์นี้ใช้ไม่ได้ผล จึงต้องศึกษาเพื่อหาเทคนิคใหม่ซึ่งพบว่าต้องอาศัยไอดีลใหญ่ที่สุดของพีชคณิตพัวส์ซองช่วยในการแก้ปัญหา ซึ่งได้ผลลัพธ์ที่กระชับและไม่ต้องผ่านการคำนวณมาก ดังที่ผู้อ่านจะได้เห็นในรายงานผลการวิจัยต่อไป

กิตติกรรมประกาศ

งานวิจัยเรื่อง "ซิมเพิลพัวส์ซองมอดูลและสมบัติบางประการ" สำเร็จลุล่วงไปเกือบทั้งหมด ทั้งนี้ ในส่วนที่เหลือ คณะผู้วิจัยจะยังคงดำเนินการต่อให้แล้วเสร็จ งานวิจัยนี้ได้รับการสนับสนุนจากงบประมาณแผ่นดินประจำปีงบประมาณ 2553 มหาวิทยาลัยอุบลราชธานี โดยได้รับการประเมินข้อเสนอจากสำนักงานคณะกรรมการวิจัยแห่งชาติ (วช) คณะผู้วิจัยขอขอบพระคุณไว้ ณ โอกาสนี้ ขอขอบคุณ Prof. David A. Jordan จาก University of Sheffield ที่ให้ข้อแนะนำในงานวิจัยที่กำลังเป็นที่สนใจอยู่ในวงการ Non-Commutative Algebra จนนำไปสู่หัวข้อวิจัยนี้ ขอบคุณภาควิชาคณิตศาสตร์ สถิติ และคอมพิวเตอร์ คณะวิทยาศาสตร์ มหาวิทยาลัยอุบลราชธานี ที่สนับสนุนและให้โอกาสในการใช้เวลาส่วนหนึ่งในการปฏิบัติงานประจำ เพื่อศึกษางานวิจัยนี้ ท้ายนี้ขอบคุณผู้ทรงคุณวุฒิ ผู้เชี่ยวชาญ ที่ได้สละเวลาอันมีค่าของท่านอ่านและให้ข้อเสนอแนะกับงานวิจัย ซึ่งคณะผู้วิจัยขอน้อมรับ และนำมาปรับปรุงงานของตนเองต่อไป

คณะผู้วิจัย

มิถุนายน 2559

บทสรุปผู้บริหาร

งานวิจัยเรื่องนี้มีแนวคิดที่จะจำแนก finite-dimensional simple Poisson modules บนพีชคณิต $A = \mathbb{C}[x, y, z]$ ที่มีตัวก่อกำเนิด 3 ตัว พร้อมด้วยความสัมพันธ์

$$\begin{aligned} xy - qyx &= (q-1)(yx + x + y + z), \\ yz - qzy &= (q-1)(zy + x + y + z), \\ zx - qxz &= (q-1)(zx + x + y + z) \end{aligned}$$

เมื่อ q ไม่ใช่ root of unity ซึ่งเป็นการต่อยอดจากวิทยานิพนธ์ของ [13] ซึ่งได้ทำการจำแนก finite-dimensional simple Poisson modules บนพีชคณิต $A = \mathbb{C}[x, y, z]$ พร้อมด้วยความสัมพันธ์

$$\begin{aligned} xy - qyx &= (q-1)z, \\ yz - qzy &= (q-1)x, \\ zx - qxz &= (q-1)y \end{aligned}$$

และผลลัพธ์ที่ได้จากวิทยานิพนธ์เรื่องนี้คือ A มี Poisson maximal ideal ทั้งหมด 5 ตัวดังนี้

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x+1)A + (y+1)A + (z+1)A, \\ J_3 &= (x+1)A + (y-1)A + (z-1)A, \\ J_4 &= (x-1)A + (y+1)A + (z-1)A \quad \text{และ} \\ J_5 &= (x-1)A + (y-1)A + (z+1)A. \end{aligned}$$

เครื่องมือที่ใช้ในการจำแนก finite-dimensional simple Poisson modules ในวิทยานิพนธ์นี้ คือ เริ่มต้นศึกษา Poisson modules annihilated by J_1 โดยใช้เทคนิคการการลดรูปตัวแปรที่ซับซ้อนลง ก่อนที่จะดำเนินการจำแนก finite-dimensional simple Poisson modules และผลลัพธ์ที่ได้ในกรณีของ J_1 คือ

"สำหรับแต่ละ $d \geq 1$, มี d-dimensional simple Poisson modules over A annihilated by J_1 เพียง 1 ตัว"

และดำเนินการศึกษาอีกกรณีที่สำคัญคือ Poisson modules annihilated by J_2 โดยใช้เทคนิคเดียวกัน และผลลัพธ์ที่ได้ในกรณีของ J_2 คือ

"สำหรับแต่ละ $d \geq 1$, มี d-dimensional simple Poisson modules over A annihilated by J_2 เพียง 1 ตัว"

แต่ในการศึกษา Poisson modules annihilated by J_3, J_4 และ J_5 เราใช้ความรู้เรื่อง Poisson Automorphism ช่วยในการเชื่อมต่อผลลัพธ์จาก J_2 ไปยัง $J_i, i = 3, 4, 5$ และผลลัพธ์สำหรับแต่ละ J_i นั้น คือ

"สำหรับแต่ละ $d \geq 1$, มี d-dimensional simple Poisson modules over A annihilated by J_2 เพียง 1 ตัว"

จึงสรุปได้ว่า "สำหรับแต่ละ $d \geq 1$, มี d-dimensional simple Poisson modules over A annihilated by J_2 ทั้งหมด 5 ตัว"

ในส่วนของงานวิจัยเรื่องนี้เราศึกษาพีชคณิต A ที่มีความสัมพันธ์ที่ซับซ้อนขึ้นทำให้ไม่สามารถ
ใช้เทคนิคที่กล่าวไว้ในวิทยานิพนธ์นี้ได้ จึงต้องทำการหาเครื่องมือใหม่และพบว่าเครื่องมือที่ปรากฏใน [9] สามารถช่วยแก้ปัญหาได้ซึ่งผลลัพธ์ที่ได้คือ A มี Poisson maximal ideal 2 ตัวดังนี้

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x+3)A + (y+3)A + (z+3)A. \end{aligned}$$

ในกรณีของ J_1 เราทำการศึกษา $\mathcal{G}(J_1) = J_1/J_1^2$ ซึ่งมีโครงสร้างของ
พีชคณิตลี เราตั้งสมมติฐานว่า $\mathcal{G}(J_1)$ เป็น Heisenberg algebra ดังได้อธิบายไว้ในหัวข้อ 2.5.2 และได้ทำการตรวจสอบดังนี้

พีชคณิตลี $\mathcal{G}(J_1) = J_1/J_1^2$ มีมิติ 3 พร้อมด้วย bracket

$$[x, y] = x + y + z, \;\; [y, z] = x + y + z, \;\; [z, x] = x + y + z$$

และลำดับต่อไปพิจารณา derived algebra $\mathcal{G}(J_1)' = [\mathcal{G}(J_1), \mathcal{G}(J_1)]$ ของ $\mathcal{G}(J_1)$ เราจะได้ว่า $\mathcal{G}(J_1)'$ ก่อกำเนิดโดย $x + y + z$ ดังนั้น $\mathcal{G}(J_1)'$ มีมิติเป็น 1 หลังจากนั้นทำการตรวจสอบว่า $x + y + z$

เป็นสมาชิกศูนย์กลางของ $\mathcal{G}(J_1)$ จากการตรวจสอบดังกล่าวข้างต้นทำให้ทราบว่า $\mathcal{G}(J_1)$ เป็น Heisenberg algebra ซึ่งนำไปสู่ผลลัพธ์ที่ว่าทุก ๆ finite-dimensional simple Poisson A-modules มี one-dimensional โดยใช้ [6, Corollary 1.3.13] และ [9, Theorem 4]

สำหรับกรณีของ $\mathcal{G}(J_2) = J_2/J_2^2$ จะพิจารณาทำนองเดียวกันแต่ต้องเปลี่ยนตัวแปรดังนี้

$$u = x + 3, v = y + 3 \text{ และ } w = z + 3$$

และได้ bracket ใหม่คือ

$$[u, v] = w - 2u - 2v, \ \ [v, w] = u - 2v - 2w, \ \ [w, u] = v - 2u - 2w$$

และ derived algebra $\mathcal{G}(J_2') = [\mathcal{G}(J_2), \mathcal{G}(J_2)] = \mathcal{G}(J_2)$ และเราแสดงได้ว่า $\{u, v, w\}$ เป็นอิสระเชิงเส้น ดังนั้น $\mathcal{G}(J_2)$ มีมิติ 3 ทำให้ได้ว่า $\mathcal{G}(J_2) \cong sl_2$

จากผลลัพธ์ที่เป็นที่แพร่หลาย (ศึกษาจาก [7] หรือ [8]) จะได้ สำหรับแต่ละ $d \geq 1$, sl_2 มี d-dimensional simple Poisson module annihilated by J_2 เพียงชุดเดียวเท่านั้น
และจาก [9, Theorem 4] และนำไปสู่ผลลัพธ์ที่ว่า มี finite-dimensional simple Poisson A-modules เพียงหนึ่งตัวสำหรับแต่ละมิติ

เนื้อหางานวิจัย

บทที่ 1

Introduction

In this research, we classify finite dimensional simple Poisson modules for some Poisson algebra by using the different method of Sasom's thesis [13]. In Sasom's thesis [13], we have seen the classification of the finite-dimensional simple Poisson T_q-modules where T_q is a $\mathbb{C}$-algebra with three generators U, V, W subject to the relations

$$\begin{aligned} UV - qVU &= (1-q^2)W \\ VW - qWV &= q^{-1}(1-q^2)U \\ WU - qUW &= q^{-1}(1-q^2)V. \end{aligned}$$

If $q^2 \neq 1$ then we shall change the generators to $X,\ Y,\ Z$ and these three relations become

$$XY - qYX = Z, \quad YZ - qZY = X, \quad ZX - qXZ = Y, \tag{1.1}$$

where $X = \frac{q^{1/2}}{1-q^2}U, \quad Y = \frac{q^{1/2}}{1-q^2}V,$ and $Z = \frac{q}{1-q^2}W$.

Sasom [13] studied Poisson algebra related to T_q and modified the generators so that the relations become

$$xy - qyx = (q-1)z, \quad yz - qzy = (q-1)x, \quad zx - qxz = (q-1)y,$$

where $q \neq 1$ but when $q = 1$ it is a commutative polynomial algebra. Replacing q by an indeterminate. Thus we have seen the algebra T generated by $x,\ y,\ z,\ t,\ t^{-1}$ subject to the relations

$$xy - tyx = (t-1)z, \quad yz - tzy = (t-1)x, \quad zx - txz = (t-1)y,$$

and

$$xt = tx, \quad yt = ty, \quad zt = tz, \quad tt^{-1} = 1 = t^{-1}t.$$

In this algebra, t is a central non-unit non-zero-divisor such that $T/(t-1)T$ is commutative and isomorphic to $A := \mathbb{C}[x, y, z]$. This induces a Poisson bracket $\{-, -\}$ on A such that,

$$\{\overline{\alpha}, \overline{\beta}\} = \overline{(t-1)^{-1}[\alpha, \beta]}.$$

for $\alpha, \ \beta \in T$. This implies the Poisson brackets,

$$\{x, y\} = yx + z, \quad \{y, z\} = zy + x, \quad \{z, x\} = xz + y.$$

Thus there are five Poisson maximal ideals J_i, $1 \leq i \leq 5$, of A for the Poisson bracket:

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x+1)A + (y+1)A + (z+1)A, \\ J_3 &= (x+1)A + (y-1)A + (z-1)A, \\ J_4 &= (x-1)A + (y+1)A + (z-1)A \quad \text{and} \\ J_5 &= (x-1)A + (y-1)A + (z+1)A. \end{aligned}$$

For $d \geq 1$, the Poisson algebra A has precisely one d-dimensional simple Poisson module annihilated by each J_i.

In our research, we use the same method of construction a Poisson algebra but different method for classifying finite dimensional Poisson simple modules which were not as hard to find as the finite dimensional simple modules over the deformation. If we change an indeterminate x's to X's and so on and then write $x = (q-1)X$ and so on then the relations should become

$$\begin{aligned} xy - yx &= (q-1)yx + (q-1)z + (q-1)y + (q-1)x \\ yz - zy &= (q-1)zy + (q-1)z + (q-1)y + (q-1)x \\ xz - zx &= (q-1)zx + (q-1)z + (q-1)y + (q-1)x. \end{aligned}$$

If we treat q like an indeterminate and factor out $q-1$ we get commutative polynomials and this gives a Poisson bracket on $\mathbb{C}[x, y, z]$ with

$$\begin{aligned} \{x, y\} &= yx + x + y + z, \\ \{y, z\} &= zy + x + y + z, \\ \{z, x\} &= xz + x + y + z. \end{aligned}$$

There are two Poisson maximal ideals for this Poisson algebra as the following

$$\begin{aligned} J_1 &= xA + yA + zA, \\ J_2 &= (x + 3)A + (y + 3)A + (z + 3)A. \end{aligned}$$

We classify the Poisson simple modules by using the method in [9]. We find that there is one of the Poisson maximal, the Lie algebra $\mathcal{G}(J_1)$ is soluble so all its finite-dimensional simple modules are 1-dimensional. The Lie algebra $\mathcal{G}(J_2)$ is isomorphic to sl_2. So we find that there are infinitely many one-dimensional simple Poisson module related to J_1 and one finite-dimensional simple Poisson module for each dimension related to J_2. The material in Chapter 2 contains the preliminary standard material that is applied elsewhere. Chapter 3 is original. This algebra studied in this research is suggested by D. A. Jordan and use the same method as his paper [9] to classify finite-dimensional simple Poisson modules.

บทที่ 2

Preliminaries

This chapter contains some of the background material that will be used throughout this thesis. The main topics are algebras, Lie algebras, derived algebras, low-dimensional Lie algebras, homomorphisms Poisson algebras and Poisson modules. For more basic knowledge, we can also study in [1], [3], [10], [11] and [12].

2.1 Algebras

An **algebra** over a field $\mathbb{F}$ is a vector space A over $\mathbb{F}$ together with a bilinear map,

$$A \times A \to A, \quad (x, y) \mapsto xy.$$

We say that xy is the **product** of x and y.

The algebra A is said to be **associative** if

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in A$$

and **unital** if there is an element 1_A in A such that $1_A x = x = x 1_A$ for all non-zero elements of A.

Definition 2.1.1. A **bilinear map or bilinear form** is a map from cartesian product of vector spaces to some other vector space. Let $\alpha : U \times V \to W$ be a bilinear map. Then α satisfies

i. $\alpha(x+y,z) = \alpha(x,z) + \alpha(y,z)$ for all $x, y \in U, z \in V$,

ii. $\alpha(x,y+z) = \alpha(x,y) + \alpha(x,z)$ for all $x \in U, y, z \in V$,

iii. $\alpha(cx,y) = c\alpha(x,y) = \alpha(x,cy)$ for all $c \in F$ and all $x \in U, y \in V$.

Theorem 2.1.2 (Hilbert's Nullstellensatz Theorem). *Let $R = \mathbb{F}[x_1, x_2, \ldots, x_n]$ be the polynomial ring over $\mathbb{F}$ in the $n(> 0)$ indeterminates $x_1, x_2, \ldots, x_n$. The ideal M is a maximal ideal if and only if there exist $a_1, a_2, \ldots, a_n$ such that $M = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.*

Proof. See [14, Theorem 14.6]. □

2.2 Lie Algebras

Let F be a field. A **Lie algebra** over F is an F-vector space L, together with a bilinear map, the **Lie bracket**

$$L \times L \to L, \quad (x,y) \mapsto [x,y]$$

such that

(L1) $[x,x] = 0$ for all $x \in L$

(L2) (Jacobi Identity) $[x,[y,z]] + [y,[z,x]] + [z,[x,y]] = 0$ for all $x, y, z \in L$.

We call $[x,y]$ a **commutator of x and y**.

As the Lie bracket $[-,-]$ is linear, we have

$$0 = [x+y, x+y] = [x,y] + [y,x].$$

This implies, $[x,y] = -[y,x]$ for all $x, y, z \in L$.

Example 2.2.1. Any vector space V endowed with the identically zero Lie bracket becomes a Lie algebra. Such Lie algebras are called abelian. Any one-dimensional Lie algebra over a field is abelian, by the antisymmetry of the Lie bracket.

Definition 2.2.2. Let L be a Lie algebra. We define a **Lie subalgebra of** L to be a vector space $K \subseteq L$ such that

$$[x, y] \in K, \quad \text{for all } x, y \in K.$$

We also define an **ideal** of a Lie algebra L to be a subspace I of L such that

$$[x, y] \in I, \quad \text{for all } x \in I, y \in L.$$

An ideal is always a subalgebra. On the other hand, a subalgebra need not be an ideal of L.

Example 2.2.3. Let $gl(n, F)$ be the vector space of all $n \times n$ matrices over F and $b(n, F)$ be an upper triangular matrices in $gl(n, F)$. A $b(n, F)$ is a subalgebra of $gl(n, F)$ but provided $n \geq 2$, is not an ideal of L.

Example 2.2.4. Let $\mathbb{R}$ be a set of real numbers. The Heisenberg algebra $H_3(\mathbb{R})$ is a three-dimensional Lie algebras generated by elements x, y and z with Lie brackets

$$[x, y] = z, \;\; [x, z] = 0, \quad [y, z] = 0.$$

It is explicitly realized as the space of 3×3 strictly upper-triangular matrices, with the Lie brackets given by the matrix commutator,

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

We have $[x, y] = xy - yx = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $[z, x] = zx - xz = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $[y, z] = yz - zy = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

The Lie algebra L is itself an ideal of L. At the other extreme, $\{0\}$ is an ideal of L. We call these the **trivial ideals** of L.

An important example of an ideal which frequently is non-trivial is the **centre** of L, defined by

$$Z(L) := \{x \in L : [x, y] = 0 \text{ for all } y \in L\}.$$

We know precisely when $L = Z(L)$ as this is the case if and only if L is abelian.

Remark 2.2.5. Let L be a Lie algebra. The Lie bracket is associative, that is, $[x, [y, z]] = [[x, y], z]$ for all $x, y, z \in L$, if and only if for all $a, b \in L$ the commutator $[a, b]$ lies in $Z(L)$.

2.3 Homomorphisms

If L_1 and L_2 are Lie algebras over a filed F, then we say that a map $\varphi : L_1 \to L_2$ is **homomorphism** if

i. φ is a linear map

ii. $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$.

Notice that in this equation the first Lie bracket is taken in L_1 and the second bracket is taken in L_2.

We say that φ is an **isomorphism** if φ is also bijective.

An extremely important homomorphism is the **adjoint homomorphism.** If L is a Lie algebra, define

$$\text{ad} : L \to gl(L)$$

by $(\text{ad}\, x)(y) := [x, y] \quad \text{for all} \quad x, y \in L$.

It follows from the bilinearity of the Lie bracket that the map $\text{ad}\, x$ is linear for each $x \in L$.

For the same reason, the map $x \mapsto \text{ad}\, x$ is itself linear.

Definition 2.3.1. Let A be an algebra over a field $\mathbb{F}$. A **derivation** of A is an $\mathbb{F}$-linear map $D : A \to A$ such that

$$D(ab) = aD(b) + D(a)b \text{ for all } a, b \in A.$$

Let $\operatorname{Der} A$ be the set of derivations of A. This set is closed under addition and scalar multiplication and contains the zero map. Hence $\operatorname{Der} A$ is a vector space of $gl(A)$. Moreover, $\operatorname{Der} A$ is a Lie subalgebra of $gl(A)$.

2.4 Derived Algebras

In order to find how many essentially different, that is, non-isomorphic, Lie algebras there are in order to classify them. The basic way is to understand the low-dimensions which we are going to do in this topic. Another reason to study the low-dimensional Lie algebras is that they often appear as subalgebras of the larger Lie algebras. We shall have a look at the Lie algebras of dimensions $1, 2$ and 3. All the material in this topic is from [2].

First of all, we consider the abelian Lie Algebras. For $n \in \mathbb{N}$, there is an abelian Lie algebras L of $\dim n$ where for all $x, y \in L$. We also know that if L_1 and L_2 are abelian algebras over the same field, then $L_1 \cong L_2$ if and only if they have the same dimension [2, Exercise 1.11]. Hence the case of Abelian Lie algebras is solved completely. From now on, we focus only on non-abelian Lie algebras. We know that Lie algebras of different dimensions cannot be isomorphic.

Before we continue to study the further results, we introduce the important algebra called **derived algebras.** Let I and J be ideals of a Lie algebra , we can define a product of ideals. Let

$$[I, J] = \operatorname{Span}\{[x, y] \mid x \in I, y \in J\}.$$

Then $[I, J]$ is an ideal of L [[2], 2.1]. When we take $I = J = L$, we obtain $[L, L]$ denoted by $L^{'}$.

As above, $L^{'}$ is the derived algebra of L. Then

$$\begin{aligned} L^{'} = [L, L] &= \operatorname{Span}\{[x, y] \mid x \in I, y \in J\} \\ &= \text{Span of the commutators of elements of } L. \end{aligned}$$

Example 2.4.1. Let $gl(n, F)$ be the vector space of all $n \times n$ matrices over F and $sl(n, F)$ be the subspace of $gl(n, F)$ consisting of all matrices of traces 0. Then

i. $gl(n,F)' = [gl(n,F), gl(n,F)] = sl(n,F)$

ii. $sl(n,F)' = [sl(n,F), sl(n,F)] = sl(n,F)$

2.5 Low-Dimensional Lie algebras

2.5.1 Dimension 1 and Dimension 2

Notice that for any one dimensional Lie Algebras is abelian. Now we consider the case of dimension 2. Suppose L is a non-abelian Low Algebras of $\dim 2$ over the field $\mathbb{F}$. Claim that L' cannot be more than $1-\dim$. Let $\{x,y\}$ be a basis of L, then L' is spanned by $[x,y]$. And $L' \neq 0$ otherwise L would be abelian. Hence L' must have dimension 1. Now, take $0 \neq x \in L'$ and extend it in any way to a vector space basis $\{x,\tilde{y}\}$ of L. Then $[x,\tilde{y}] \in L'$ and $[x,\tilde{y}] \neq 0$ otherwise L would be abelian. So there is $0 \neq \alpha \in \mathbb{F}$ such that $[x,\tilde{y}] = \alpha x$. This scalar factor does not contribute anything to the structure of L, so if we replace $\tilde{y}$ with $y = \alpha^{-1}\tilde{y}$, then $[x,\tilde{y}] = x$. Follow these ideas, we have the next Theorem

Theorem 2.5.1. *Let $\mathbb{F}$ be a field. Up to isomorphism, there is a unique $2-\dim$ non-abelian Lie Algebra over $\mathbb{F}$. This Lie algebra has a basis $\{x,y\}$ such that its Lie bracket is described by $[x,y] = x$. The centre of this Lie algebra is 0.*

We have shown that if a 2-dim non-ableian Lie algebra exists, then it must have basis $\{x,y\}$ with $[x,y] = x$. Note that the bracket $[x,y] = x$ induces that $[x,y] = 0$ and $[x,y] = -[y,x]$.

2.5.2 Dimension 3

If L is a non-abelian $3-\dim$ Lie algebra over a field $\mathbb{F}$, then the derived algebra L' is non-zero. Hence $\dim L' = 1$ or $\dim L' = 2$ or $\dim L' = 3$. And also the centre of L, $Z(L)$, is a proper ideal of L (since L is a non-abelian so there are other elements than 0). We organize our search by relating L' to $Z(L)$. For the Lie algebras of dimension 3, there are 4 cases as following:

i. The Heisenberg Algebra

ii. Another Lie Algebras where $\dim L^{'} = 1$

iii. Lie Algebras with a 2-dim Derived Algebra

iv. Lie Algebras where $L = L^{'}$

Now, we study for each cases,

i. The Heisenberg Algebra

Assume that $\dim L^{'} = 1$ and $L^{'} \subseteq Z(L)$. We show that there is a unique Lie algebra and it has a basis f, g, z where $[f, g] = z = Z(L)$. This Lie algebra is known as the **Heisenberg algebras.**

Take any $f, g \in L$ such that $[f, g] \neq 0$ as we assume that $\dim L^{'} = 1$, the commutator $[f, g]$ spans $L^{'}$. We also assume that $L^{'} \subseteq Z(L)$, so $[f, g]$ commutes with all elements in L. Now let

$$L := [f, g]$$

To show that f, g, z are linearly independent, let $\alpha, \beta, \gamma \in F$ be such that $\alpha f + \beta g + \gamma z = 0$. Consider $[\alpha f + \beta g + \gamma z, g] = 0$. Then $\alpha[f, g] + \beta[g, g] + \gamma[z, g] = 0$. Since $[g, g] = 0$ and $[f, g] = z \in Z(L)$, $\alpha[f, g] = 0$. But $[f, g] \neq 0$ so $\alpha = 0$. Analogously, we can show that $\beta = 0$ and $\gamma = 0$. Hence f, g, z form a basis of L.

Example 2.5.2. As before, all other Lie brackets are already fixed. In this case, we observe that the Lie algebra of strictly upper triangular 3×3 matrices over $\mathbb{F}$ has this form if one takes the basis, that is $[e_{12}, e_{23}] = e_{13}$. Moreover $L^{'} \subseteq Z(L)$.

ii. Another Lie Algebras where $\dim L^{'} = 1$

To consider these Lie algebras, we need to know the concept of the direct sum. Suppose L_1 and L_2 are Lie algebras. Let $L = \{(x_1, x_2) \mid x_i \in L_i\}$ be the direct sum of their underlying vector spaces. Define

$$[(x_1, x_2), (y_1, y_2)] = ([x_1, y_1], [x_2, y_2]),$$

then L becomes a Lie algebra, the direct sum of L_1 and L_2. As for vector space, we denote the direct sum of Lie algebras L_1 and L_2 by $L_1 = L_1 \oplus L_2$.

In this case, we consider the Lie algebra with $\dim L^{'} = 1$, and $L^{'}$ is not contained in the centre $Z(L)$ of L. We use the direct sum construction to give one such Lie algebra

Namely, we use $L_1 = L_1 \oplus L_2$, where L_1 is $2-\dim$ and a non-abelian (the one we found in Theorem 2.5.1) and L_2 is $1-\dim$, that is $L_2^{'} = 0$. Then

$$L^{'} := L_1^{'} \oplus L_2^{'} = L_1^{'} \oplus 0 = L_1^{'}$$

Then $L^{'}$ is $1-\dim$. Moreover $Z(L) = Z(L_1) \oplus Z(L_2) = 0 \oplus L_2 = L_2$. Hence $L^{'}$ and $Z(L)$ is $1-\dim$, so $L^{'}$ is not contained in $Z(L) = L_2$.

As the above idea, we have the next Theorem.

Theorem 2.5.3. *Let F be any field. There is a unique $3-\dim$ Lie algebra over F such that $L^{'}$ is $1-\dim$ and is not contained in $Z(L)$. This Lie algebra is the direct sum of the $2-\dim$ non-abelian Lie algebra with the $1-\dim$ Lie algebra.*

Proof. See [2, Theorem 3.2]. ☐

iii. Lie Algebras with a 2-dim Derived Algebra

Suppose that $\dim L = 3$ and $\dim L^{'} = 2$. We shall see that, over $\mathbb{C}$ at least, there are infinitely many non-isomorphic such Lie algebras.

Take a basis of $L^{'}$, say $\{y, z\}$, and extend it to a basis of L, say by x. To understand the Lie algebra L, we need to understand the structure of $L^{'}$ as a Lie algebras in its own right and how the linear map $\operatorname{ad} x : L \to L$ act on $L^{'}$.

Lemma 2.5.4. (a) The derived algebra $L^{'}$ is abelian.

(b) The linear map $\operatorname{ad} x : L^{'} \to L^{'}$ is an isomorphism.

Proof. See [2, Section 3.2.3]. ☐

iv. Lie Algebras where $L^{'} = L$

Suppose L is a complex Lie algebra of $\dim 3$ such that $L^{'} = L$. We already know one example, namely, $L = sl(2, \mathbb{C})$. We shall show that, up to isomorphism, it is the only one! For more detail, read [2, Section 3.2.4].

บทที่ 3

Poisson Algebras and Poisson modules

In this chapter, we shall determine definition of Poisson algebras, Poisson modules and Lie algebra $\mathcal{G}(J)$ where J is a Poisson maximal ideal in order to classify the finite-dimensional simple Poisson modules.

3.1 Definitions and Notations

Throughout A will be a finitely generated commutative algebra over $\mathbb{C}$.

Definition 3.1.1. A *Poisson bracket* on A is a Lie algebra bracket $\{-,-\}$ satisfying the Leibniz rule

$$\{ab, c\} = a\{b, c\} + \{a, c\}b \quad \text{for all } a, b, c \in A.$$

The pair $(A, \{-,-\})$ is called a *Poisson algebra.*

Definition 3.1.2. A subalgebra B of A is a *Poisson subalgebra* of A if $\{b, c\} \in B$ for all $b, c \in B$.

Definition 3.1.3. An ideal I of a Poisson subalgebra A is a *Poisson ideal* if $\{i, a\} \in I$ for all $i \in I$ and all $a \in A$.

Definition 3.1.4. If I is a Poisson ideal of A then A/I is a Poisson algebra in the obvious way: $\{a + I, b + I\} = \{a, b\} + I$.

Definition 3.1.5. A Poisson algebra A is said to be *simple* if its only Poisson ideals are (0) and A.

Definition 3.1.6. Let P be an ideal of a Poisson algebra A. Then P is a *Poisson prime ideal* if P is both a prime ideal and a Poisson ideal. It follows from [?, 3.3.2] that this is equivalent to saying that P is a Poisson ideal and, for all Poisson ideals $I,\ J \subseteq A$,

$$IJ \subseteq P \quad \text{implies that } I \subseteq P \text{ or } J \subseteq P.$$

Definition 3.1.7. By *maximal Poisson ideal*, we shall mean a Poisson ideal I of A such that if J is a Poisson ideal and $I \nsubseteq J$ then $J = A$. An ideal I of a Poisson algebra A is said to be a *Poisson maximal ideal* if I is a maximal ideal of A and also a Poisson ideal. For example, let $A = \mathbb{C}[x,y]$ which is a Poisson algebra with the Poisson bracket $\{x,y\} = 1$. Then 0 is a maximal Poisson ideal but is not a Poisson maximal ideal.

Definition 3.1.8. Let R be a commutative $\mathbb{C}$-algebra and let $h \in R$. Let A be an $R-$algebra and suppose that h is not a zero divisor in A, and that $\overline{A} := A/hA$ is a commutative $\mathbb{C}$-algebra. Then there is a Poisson bracket $\{,\}$ on $\overline{A}$ such that $\{\overline{a},\overline{b}\} = \overline{h^{-1}[a,b]}$ for all $\overline{a} = a + hA$ and $\overline{b} = b + hA$. Following [1, III.5.4], we call A a *quantization* of the Poisson algebra $\overline{A}$.

There is more than one definition of Poisson module in the literature. We shall use the one introduced by D. R. Farkas [4].

Definition 3.1.9. Let A be a commutative Poisson algebra with Poisson bracket $\{-,-\}$. We shall say that an A-module M is a *Poisson module* if there is a bilinear form $\{-,-\}_M : A \times M \to M$ such that

i. $\{a,a'm\}_M = \{a,a'\}m + a'\{a,m\}_M$;

ii. $\{aa',m\}_M = a\{a',m\}_M + a'\{a,m\}_M$;

iii. $\{\{a,a'\},m\}_M = \{a,\{a',m\}_M\}_M - \{a',\{a,m\}_M\}_M$;

for all $a,a' \in A$ and all $m \in M$.

A submodule N of a Poisson module M is called a *Poisson submodule* if $\{a,n\}_M \in N$, for all $a \in A$, $n \in N$.

Definition 3.1.10. Let N be a left module over a ring R. Give any subset $X \subseteq N$, the **annihilator** of X is the set

$$\operatorname{ann}_R(X) = \{r \in R : rx = 0 \text{ for all } x \in X\},$$

which is a left ideal of R.

Lemma 3.1.11. Let A be a Poisson algebra and M be a Poisson $A-$module.

i. The annihilator $\operatorname{ann}_A(M)$ is a Poisson ideal of A;

ii. if M is a simple Poisson module then $\operatorname{ann}_A(M)$ is a Poisson prime ideal of A;

iii. if M is a finite-dimensional simple Poisson module then $\operatorname{ann}_A(M)$ is a Poisson maximal ideal of A.

Proof. See[13, Lemma 4.1.1]. □

Lemma 3.1.12. Let $A = \mathbb{C}[x_1, x_2, \ldots, x_n]$ with a Poisson bracket $\{-,-\}$. Let $V = \operatorname{Sp}(x_1, x_2, \ldots, x_n)$ and let M be an A-module. Suppose that there is a bilinear form $\{-,-\}_M : V \times M \to M$. Extend this to a bilinear form $\{-,-\}_M : A \times M \to M$ using Definition 3.1.9(ii) and $\{1, m\}_M = 0$. If Definition 3.1.9(i) and (iii) hold, for all $m \in M$, whenever $a = x_i$, and $a' = x_j$ for $1 \leq i < j \leq n$ then Definition 3.1.9(i) and (iii) hold for all $a, a' \in A$.

Proof. See[13, Lemma 4.1.2]. □

3.2 Lie algebra $\mathcal{G}(J)$ where J is a Poisson maximal ideal

Let I and J be Poisson ideals of a Poisson algebra A. Then IJ is a Poisson ideal of A. Of course I and J are Lie subalgebra of A under $\{-,-\}$. If $I \subseteq J$, then I is a Lie ideal of J and J/I is a Lie algebra. In particular, J/J^2 is a Lie algebra.

Studying Poisson modules, one natural way to find Poisson modules is, for I and J are Poisson ideals of A with $I \subseteq J$, the factor J/I is a Poisson module with $\{a, j + I\}_{J/I} = \{a, j\}_J + I$. We can check that $\{-,-\}_{J/I}$ is well-defined, and all the axioms for a Poisson module are hold. By above, J/I is also a Lie algebra. Every Poisson subalgebra of J/I is a Lie ideal, so if J/I is simple as a Lie algebra, then it is simple as a Poisson module. If A is affine and J is a Poisson maximal ideal, so that $A = J + \mathbb{C}$, then the converse is also true because every Lie ideal of J/I is then a Poisson A-submodule.

If I and J be Poisson ideals of a Poisson algebra A, then I/IJ and J/IJ are Poisson modules.

The following is the main result of [9]. We use this result to tackle the later research problems. He proves the result giving a method to determine the finite-dimensional simple Poisson modules over any affine Poisson algebra (that is, a Poisson algebra that is finitely generated as a $\mathbb{C}$-algebra)

If J is a Poisson maximal ideal of A, then J/J^2 has a Lie algebra structure. It is shown that there is a bijection, *preserving dimension*, between the isomorphism classes of finite-dimensional simple Poisson A-module and pairs $(J, \widehat{M})$ when J is a Poisson maximal ideal of A and $\widehat{M})$ is an isomorphism classes of finite dimensional modules over the lie algebra J/J^2. In this bijection, the simple Poisson modules in a class corresponding to the pair $(J, \widehat{M})$ are annihilated by J.

Let M be a Poisson module over a Poisson algebra A and let $S \subseteq M$. In the module sense, we denote the annihilator of S in A by $\mathrm{ann}_A(S)$. And we denote

$$\mathrm{Pann}_A(S) = \{a \in A : \{a, m\}_M = 0 \text{for all} m \in S\}.$$

Lemma 3.2.1. Let A be an affine Poisson algebra and let M be a Poisson A-module. Let $J = \mathrm{ann}_A(M)$.

i. J is a Poisson ideal of A.

ii. If M is simple, then J is a prime ideal of A.

iii. If M is finite-dimensional and simple, then J is a maximal ideal of A.

iv. $\mathbb{C} + J^2 \subseteq \mathrm{Pann}_A(M)$

Proof. See [9, Lemma 3.1]. □

Remark 3.2.2. If A, M and J are as in 3.1 and I is a Poisson ideal such that $I \subseteq J^2$ then, by 3.1(iv), M becomes a Poisson A/I -module where $(a+I)m = am$ and $\{a+I, m\}_M = \{a, m\}_M$.

Notation 3.2.3. Let A be an affine Poisson algebra. For a finite-dimensional simple Poisson A-module M, let $\widehat{M}$ denote its isomorphism class. Similarly, for a Lie algebra $\mathcal{G}$ and a finite-dimensional simple $\mathcal{G}$-module N, let $\widehat{N}$ denote its isomorphism class.

Theorem 3.2.4. *Let A be an affine generated Poisson algebra.*

i. *Let M be a finite-dimensional simple Poisson A-module and let $J = \mathrm{ann}_A(M)$. There is a simple module M^* for the Lie algebra $\mathcal{G}(J)$ such that $M^* = M$, as $\mathbb{C}$-vector space, and $[j + J^2, m]_{M^*} = \{j, m\}_M$ for all $j \in J$ and $m \in M$.*

ii. *Let J be a Poisson maximal ideal A of and let N be a finite-dimensional simple $\mathcal{G}(J)$-module. There exist a simple Poisson A -module N' and a Lie homomorphism $f : A \to \mathcal{G}(J)$ such that $N' = {}^f N$ as a Lie module over A and $\mathrm{ann}_A(N') = J$.*

iii. *For all finite-dimensional simple Poisson modules M, $M^{*'} = M$. For all Poisson maximal ideals J of A and all finite-dimensional simple $\mathcal{G}(J)$-modules N, $N{*}' = N$.*

iv. *(iv) The procedure in (i) and (ii) establish a bijection Γ from the set of isomorphism classes of finite-dimensional simple Poisson module over$A$ to the set of pairs $(J, \widehat{N})$, where J is a Poisson maximal ideal of A and N is a finite-dimensional simple $\mathcal{G}(J)$-module, given by $\Gamma(\widehat{M}) = (\mathrm{ann}_A(M), \widehat{M^*})$.*

Proof. See[9, Theorem 1]. □

บทที่ 4

Finite-dimensional simple Poisson Modules

In this chapter, we classify the finite-dimensional simple Poisson modules over a Poisson algebra A.

4.1 Poisson Algebra A

In this section, we consider the Poisson algebra generated by three generators x, y, and z with three relations, say A.

Let S be the $\mathbb{C}$-algebra generated by $x,\ y,\ z,\ q$ and q^{-1} subject to the relations

$$\begin{aligned} xy - qyx &= (q-1)(x+y+z), \quad (4.1)\\ yz - qzy &= (q-1)(x+y+z), \quad (4.2)\\ zx - qxz &= (q-1)(x+y+z) \quad \text{and} \quad (4.3) \end{aligned}$$

$$xq = qx, \quad yq = qy, \quad zq = qz, \quad qq^{-1} = 1 = q^{-1}q. \quad (4.4)$$

Then q is a central element of S. Let

$$A := S/(q-1)S \simeq \mathbb{C}[x, y, z],$$

which is a commutative polynomial algebra. The induced Poisson bracket on A is such that

$$\begin{aligned}\{x,y\} &= \tfrac{1}{q-1}[x,y] = \tfrac{1}{q-1}(xy-yx)\\ &= \tfrac{1}{q-1}(qyx-yx+(q-1)(x+y+z))\\ &= yx+x+y+z.\end{aligned}$$

Similarly, we obtain

$$\{y,z\} = zy+x+y+z, \quad \{z,x\} = xz+x+y+z.$$

Hence, these are the Poisson blacket of A

$$\{x,y\} = yx+x+y+z, \tag{4.5}$$
$$\{y,z\} = zy+x+y+z, \tag{4.6}$$
$$\{z,x\} = xz+x+y+z. \tag{4.7}$$

In the next lemma, we find the Poisson maximal ideals of A for this Poisson bracket. First of all, let J be a Poisson maximal ideal of A. Since $A$is a commutative polynomial ring over $\mathbb{C}$, by Theorem 2.1.2, $J = \langle x-a, y-b, z-c\rangle$ for suitable $a,\ b,\ c \in \mathbb{C}$.

Lemma 4.1.1. In the above Poisson algebra A, there are only two Poisson maximal ideals of A. They are:

$$\begin{aligned}J_1 &= xA+yA+zA,\\ J_2 &= (x+3)A+(y+3)A+(z+3)A\end{aligned}$$

Proof. Let $J = (x-a)A+(y-b)A+(z-c)A$ be a Poisson maximal ideal of A, for all $a,b,c \in \mathbb{C}$. Since J is a Poisson ideal, $\{x,J\} \subseteq J$, $\{y,J\} \subseteq J$, and $\{z,J\} \subseteq J$. Observe that

$$J \supseteq \{x,y-b\} = \{x,y\} - \{x,b\} = \{x,y\} = yx+x+y+z, \tag{4.8}$$
$$J \supseteq \{y,z-c\} = \{y,z\} - \{y,c\} = \{y,z\} = zy+x+y+z, \tag{4.9}$$
$$J \supseteq \{z,x-c\} = \{z,x\} - \{z,c\} = \{z,x\} = xz+x+y+z. \tag{4.10}$$

By the above three equations, we have

$$\begin{aligned} ab + a + b + c &= 0, && (4.11)\\ bc + a + b + c &= 0, && (4.12)\\ ac + a + b + c &= 0. && (4.13) \end{aligned}$$

It induces that

$$\begin{aligned} ab - ba &= 0 \quad \text{i.e} \quad b = 0 \quad \text{or} \quad a = c,\\ ab - ac &= 0 \quad \text{i.e} \quad a = 0 \quad \text{or} \quad b = c,\\ bc - ac &= 0 \quad \text{i.e} \quad c = 0 \quad \text{or} \quad a = b. \end{aligned}$$

There are two cases to be considered.

i. **Case 1.** $b = 0$. Consider
By the equation (4.11), we have $a = -c$. Then Substitute this in the equation(4.13), it must give $c = 0$. Hence it induces that $a = 0$.

ii. **Case 2.** $b \neq 0$ **and** $a = c$.

$$\begin{aligned} \text{The equation (4.11) gives} \quad ab + 2a + b &= 0, \quad \text{and}\\ \text{the equation (4.13) gives} \quad a^2 + 2a + b &= 0. \end{aligned}$$

These give $0 = a^2 - ab = a(a - b)$. Thus $a = 0$ or $a = b$. If $c = a = 0$, then it is forced by the equations (4.11), (4.12) and (4.13) that $b = 0$.

Hence for both cases, it can be concluded that there are two possible solutions:

i. $a = b = c = 0$,

ii. $a = b = c$, where $a \neq 0, b \neq 0$ and $c \neq 0$.

If $a = b = c = 0$ then we have $J_1 = xA + yA + zA$. If $a = b = c$, where $a \neq 0$, $b \neq 0$ and $c \neq 0$ and we have $a^2 + 3a = 0$. This implies that $a = 0$ or $a = -3$. But $a \neq 0$, then we have $a = -3$. Thus we have $J_2 = (x + 3)A + (y + 3)A + (z + 3)A$. Therefore that result holds. □

We shall present the concepts of the proof for the next two sections as they need more accuracy to be made in the way.

4.2 Finite-dimensional simple Poisson module over $\mathcal{G}(J_1)$

The Sketch proof to the results 4.2 is in the following steps.

i. Consider the Lie algebra $\mathcal{G}(J_1) = J_1/J_1^2$, where $J_1 = xA + yA + zA$.

ii. The Lie algebra has dimension 3 and its bracket becomes

$$[x, y] = x + y + z, \;\; [y, z] = x + y + z, \;\; [z, x] = x + y + z.$$

iii. Consider the derived algebra $\mathcal{G}(J_1)' = [\mathcal{G}(J_1), \mathcal{G}(J_1)]$ of $\mathcal{G}(J_1)$. We obtain that $\mathcal{G}(J_1)'$ is generated by $x + y + z$ and it has dimension 1.

iv. It is routine to check that $x + y + z$ is in the centre of $\mathcal{G}(J_1)$.

v. By all the properties i-iv and theorem in Low dimension, we conclude that $\mathcal{G}(J_1)$ is the Heisenberg algebra.

vi. Then we use the result proved in [6, Corollary 1.3.13] that every finite-dimensional simple $\mathcal{G}(J_1)$-module N is one-dimensional and annihilated by $\mathcal{G}(J_1)' = [\mathcal{G}(J_1), \mathcal{G}(J_1)]$.

vii. Therefore, by [9, Theorem 4], there is a unique finite-dimensional simple Poisson A-modules for each dimension.

4.3 Finite-dimensional simple Poisson module over $\mathcal{G}(J_2)$

The Sketch proof to the results 4.3 is in the following steps.

i. Consider the Lie algebra $\mathcal{G}(J_2) = J_2/J_2^2$, where $J_2 = (x + 3)A + (y + 3)A + (z + 3)A$.

ii. To simplify these, we shall replace $u = x + 3$, $v = y + 3$ and $w = z + 3$. Then $J_2 = uA + vA + wA$

iii. The bracket of Lie algebra $\mathcal{G}(J_2)$ becomes

$$[u,v] = w - 2u - 2v, \;\; [v,w] = u - 2v - 2w, \;\; [w,u] = v - 2u - 2w.$$

iv. Consider the derived algebra $\mathcal{G}(J_2^{'}) = [\mathcal{G}(J_2), \mathcal{G}(J_2)]$ of $\mathcal{G}(J_2)$, we obtain $\mathcal{G}(J_2^{'}) = \mathcal{G}(J_2)$.

v. Show that $\{u, v, w\}$ is the linearly independent set. Then $\mathcal{G}(J_2)$ has dimension 3.

vi. By theorem in Low dimension, $\mathcal{G}(J_2) \cong sl_2$.

vii. It is well known result that, see [7] or [8] that for each $d \geq 1$, $\mathcal{G}(J_2) \cong sl_2$ has a unique d-dimensional simple Poisson module annihilated by J_2.

viii. Again, by [9, Theorem 4], every finite-dimensional simple Poisson A-modules is one-dimensional.

ภาคผนวก

คณะผู้วิจัยกำลังดำเนินการเขียนบทความวิจัยที่ได้ผลลัพธ์จากโครงการวิจัยนี้โดยคาดว่าจะแล้วเสร็จภายในเดือนสิงหาคม 2559 และเมื่อได้รับการตีพิมพ์เรียบร้อยจะรีบดำเนินการรายงานแก่สำนักงานคณะกรรมการวิจัยแห่งชาติ (วช) และมหาวิทยาลัยอุบลราชธานีต่อไป

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