GOLDIE PRIME MODULES WITH CS CONDITION

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With all of my love, I would like to dedicate this thesis to my parents and my family. Finally, I would like to give it as a gift to my small family.

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ABSTRACT

D. V. Huynh et al. (2000) found out the symmetry of the Goldie and CS conditions on prime rings. It was shown that a prime ring is right Goldie right CS with right uniform dimension greater than one if and only if it is left Goldie left CS with left uniform dimension greater than one. A semiprime ring is right Goldie left CS if and only if it is left Goldie, right CS. In this thesis, we apply D. V. Huynh et al. results to the class of prime and semiprime Goldie modules.

KEY WORDS : GOLDIE MODULES/ CS MODULES/ PRIME MODULES
SEMIPRIME MODULES/ UNIFORM DIMENSION

23 pages.

CONTENTS

CHAPTER I

INTRODUCTION

Throughout this text, all rings are associative (not necessarily commutative) with identity, and all modules are unitary.

Prime Goldie and semiprime Goldie rings have been investigated extensively for a long time. The class of semiprime Goldie rings is very special with a beautiful structure and plenty of nice properties. They have been deeply investigated by many authors such as A. W. Goldie (1960, [6]), R. Gordon (1971, [4]), A. W. Chatters and C. R. Hajarnavis (1980, [2]), T. Y. Lam (1999, [9]).

D. V. Huynh et al. (2000, [8]) studied prime and semiprime (right) Goldie rings in connection with the CS condition. It was shown that a prime ring is right Goldie, right CS with right uniform dimension greater than one if and only if it is left Goldie, left CS with left uniform dimension greater than one. A semiprime ring is right Goldie left CS if and only if it is left Goldie, right CS.

N. V. Sanh et al. (2010, [12]) introduced the primeness in module theory. It is a Morita invariant in module categories and is absolutely compatible with primeness on rings. N. V. Sanh et al. also pay attention on prime and semiprime Goldie modules, for example, [14, 13].

1.1 About the Thesis

In this work, we adopt the notions of primeness defined by N. V. Sanh. Then, we generalize the results of D. V. Huynh et al. to the case of prime and semiprime modules. Our main theorems will give the like-symmetry of the Goldie and CS conditions in prime and semiprime modules as a general case of prime and semiprime rings.

The thesis is divided into four chapters. All of our work is in the third one. The first chapter is an introduction with literature review and list of symbols. The next one summarizes the theoretical background of rings and modules relative to our work. The most significant section of the thesis is the chapter III which shows the main results of our work. Finally, chapter IV is for conclusion.

1.2 List of Symbols

$\hookrightarrow, \overset{\oplus}{\hookrightarrow}$	Submodule (ideal), direct summand, respectively
$\overset{0}{\hookrightarrow}$	Small (superfluous) submodule (ideal)
$\overset{*}{\hookrightarrow}$	Essential (large) submodule (ideal)
$\mathbf{r}_X(Y), \ell_X(Y)$	Right, left annihilator of Y in X, respectively
$\mathrm{Span}(X)$	The spanning submodule of X
$\mathrm{Hom}_R(M, N)$	The group of all homomorphisms from M to N
$\mathrm{End}(M_R)$ or $\mathrm{End}(M)$	The endomorphism ring of a module M
$\mathrm{Mod}(R)$	The category of all right $R-$modules
M/X	The quotient (factor) module of M modulo X

CHAPTER II
THEORETICAL BACKGROUND

Throughout this note, all rings are associative with identity and all modules are unitary. For a right (resp. left) $R-$module M, we will denote M_R (resp. ${}_RM$).

This chapter essentially provides a theoretical background on rings and modules relative to my work to serve for the next chapters. For further details and other materials, we refer the text books and articles in the bibliography.

Let R be an associative ring with identity and M, N be right $R-$modules. $\mathrm{Hom}_R(M, N)$ (or simply $\mathrm{Hom}(M, N)$) is the additive group of all homomorphisms from M to N. We denote $S = \mathrm{End}(M_R)$, the ring of all homomorphisms from M to M. If $M = R_R$, we can see that $\mathrm{End}(R_R)$ is isomorphic to R. For an $R-$homomorphism $f \in \mathrm{Hom}(M, N)$, kernel (resp. image) of f is denoted by $\mathrm{Ker}(f)$ (resp. $\mathrm{Im}(f) = f(M)$).

2.1 Annihilators

Let S, R be rings with identities and ${}_SM_R$ be a left $S-$ right $R-$bimodules. Then, for subsets $T \subset S, X \subset M, I \subset R$:

(1) *Right annihilator* of X in R is the set

$\mathbf{r}_R(X) = \{r \in R | xr = 0, \forall x \in X\}$.

(2) *Right annihilator* of T in M is the set

$\mathbf{r}_M(T) = \{m \in M | tm = 0, \forall t \in T\}$.

(3) *Left annihilator* of X in S is the set

$\ell_S(X) = \{s \in S | sx = 0, \forall x \in X\}$.

(4) *Left annihilator* of I in M is the set

$\ell_M(I) = \{m \in M | mk = 0, \forall k \in I\}$.

(5) If $T = M$, then $\mathbf{r}_R(M)$ is called the annihilator of M in R. If $\mathbf{r}_R(M) = 0$, then M is called a *faithful* module.

For singletons $\{x\} \subset M, \{t\} \subset S, \{r\} \subset R$, we simply write their annihilators as $\mathbf{r}_R(x), \mathbf{r}_M(t), \ell_S(x), \ell_M(r)$. For a ring R, regarding $I \subset R, x \in R$, we denote $\mathbf{r}(I), \ell(I)$, $\mathbf{r}(x), \ell(x)$, for short. If annihilating is made many times, then we write, for instance, $\mathbf{r}_M \ell_S(X)$, instead of $\mathbf{r}_M(\ell_S(X))$. Readers can easily look for more details of annihilators in [1], [2], [11]. It is routine to verify the following properties on annihilators.

Proposition 2.1 *Let S, R be rings with identities and ${}_SM_R$ be a left $S-$ right $R-$bimodule. Then, for subsets $T \subset S, X \subset M, I \subset R$:*

(1) $\mathbf{r}_R(X)$ is a right ideal of R and $X\mathbf{r}_R(X) = 0$;

(2) $\mathbf{r}_M(T)$ is a submodule of M_R and $T\mathbf{r}_M(T) = 0$;

(3) $\ell_S(X)$ is a left ideal of S and $\ell_S(X)X = 0$;

(4) $\ell_M(I)$ is a submodule of ${}_SM$ and $\ell_M(I)I = 0$;

(5) If $S = \mathrm{End}(M_R)$, then for some $f \in S, \mathbf{r}_M(f)$ is exactly the kernel of f, i.e. $\mathbf{r}_M(f) = \mathrm{Ker}(f)$.

By the claim (5) of the proposition 2.1, if $S = \mathrm{End}(M_R)$ and for some $T \subset S$, we sometimes write $\mathrm{Ker}(T) := \mathbf{r}_M(T) = \bigcap_{t \in T} \mathrm{Ker}(t)$.

Definition 2.2 *A submodule X of M is called an M-annihilator if $X = \mathbf{r}_M(T) = \mathrm{Ker}(T)$ for some subset $T \subset S$. Especially, a right ideal I of a ring R is called a right (resp. left) annihilator if $I = \mathbf{r}(K)$ (resp. $I = \ell(K)$) for some subset $K \subset R$.*

Proposition 2.3 *(refer to [1], 2.)*

Let $M = M_R$ be a right $R-$module. Then, for arbitrary subsets X, Y and submodules A, B of M :

(1) If $X \subseteq Y$, then $\mathbf{r}_R(Y) \subseteq \mathbf{r}_R(X)$;

(2) $\mathbf{r}_R(X) = \mathbf{r}_R \ell_M \mathbf{r}_R(X)$

(3) $\mathbf{r}_R(A + B) = \mathbf{r}_R(A) \cap \mathbf{r}_R(B)$;

(4) $\mathbf{r}_R(A) + \mathbf{r}_R(B) \subseteq \mathbf{r}_R(A \cap B)$.

Similar properties are hold for left modules and left annihilators.

The ACC and the DCC.

The ACC and *the DCC* stand for *the ascending chain condition* and *the descending chain condition*, respectively. Considering the following chains of submodules of a right $R-$module M :

(1) $A_1 \subseteq A_2 \subseteq ... \subseteq A_n \subseteq ...$

(2) $B_1 \supseteq B_2 \supseteq ... \supseteq B_m \supseteq ...$

The ascending chain (1) (resp. the descending chain (2)) is said to be *stationary* if there is some $n_0 \in N$ (resp. $m_0 \in N$) such that $A_{n_0+i} = A_{n_0}$ (resp. $B_{m_0+i} = B_{m_0}$) for every $i \in N$.

A module satisfying the ACC (resp. the DCC) on the class of all submodules is called a *Noetherian* (resp. *Artinian*) module. For a ring, we can verify that the ACC on right annihilators is equivalent to the DCC on left annihilators.

2.2 Generating and Generators

Let $M = M_R$ be a right module and $X \subset M$. The *spanning* of X, denoted by Span(X) or XR, is the set $\{x_1r_1 + x_2r_2 + ... + x_nr_n, x_i \in X, r_i \in R, i = 1, 2, ..., n\}$. It is easy to check that Span$(X) = XR$ is the smallest submodule of M containing X. Thus, we also say Span(X) is the submodule *spanned (or generated)* by X. If X is a submodule of M, then Span$(X) = XR = X$.

Definition 2.4 *Let $M = M_R$ be a right module. Then:*

(1) M is called a finitely generated module if it is spanned (or generated) by a finite number of its elements. In particular, if M is generated by a single element, then it is called a cyclic module.

(2) For a right $R-$module N, we say that M generates N if $N = KM := \sum_{f \in K} \mathrm{Im}(f)$, where $K = \mathrm{Hom}_R(M, N)$.

(3) M is called a self-generator if it generates all of its submodules.

Proposition 2.5 *Let $M = M_R$ be a right module with $S = \mathrm{End}(M_R)$.*

(1) If M is a finitely generated module, then there exists a finite subset $X = \{x_1, x_2, ..., x_n\} \subset M$ such that $M = XR = \{x_1r_1 + x_2r_2 + ... + x_nr_n | r_i \in R, i = 1, 2, ..., n\}$. Especially, if M is a cyclic module, then $M = mR = \{mr | r \in R\}$ for some $m \in M$.

(2) For a submodule X, if M generates X, then there is a subset $K \subset S$ such that $X = \sum_{f \in K} fM$. Moreover, if $X \neq 0$ and M generates X, then $\mathrm{Hom}(M, X) \neq 0$.

Every *simple* module (having no proper submodule) is cyclic. Any ring R is cyclic as a right module over itself. If $M = mR$ is a cyclic module, then there is a canonical homomorphism $R \to mR, r \mapsto mr$ with the kernel is the right annihilator of m in R. A ring R generates all right $R-$modules, and hence it is a generator.

2.3 Projective and Quasi-projective Modules

Definition 2.6 *Let M, N be right $R-$modules.*

(1) N is said to be an $M-$projective module if for any submodule X of M, every homomorphism α from N to M/X can be lifted to a homomorphism φ from N to M.

$$\begin{tikzcd} & N \arrow[dl, "\varphi"'] \arrow[d, "\alpha"] & \\ M \arrow[r] & M/X \arrow[r] & 0 \end{tikzcd}$$

(2) M is called a quasi-projective (or self-projective) module if it is an $M-$projective module.

(3) M is called a projective module if it is $N-$projective for every module

$N \in \mathrm{Mod}(R)$.

A right R module F is called a *free* modules if it has a basis. Note that every ring is free and every free module is projective. Every module is a homomorphic image of a projective module.

Proposition 2.7 *(Refer to 18.4, (ii), of [15])*

Let M be a finitely generated, quasi-projective right R-module and $S = \mathrm{End}(M_R)$. Then, for every right ideal K of S, we have $K = \mathrm{Hom}(M, KM)$.

2.4 Closed Submodules and CS Modules

In this part, *closed* submodules and *complements* are introduced. These concepts are significant materials to build up our research in next two chapters. In several contexts, complements are actually an other terminology of closed submodules. By Zorn's Lemma, complement of a submodule always exists. For more details, we refer to [10] and [3].

Definition 2.8 *Let $M = M_R$ be a module.*

(1) A nonzero submodule X of M is called an essential submodule if it has nonzero intersection with any nonzero submodule of M. Then, we say that X is essential in M, denote $X \overset{*}{\hookrightarrow} M$, and M is an essential extension of X.

(2) A submodule Y of M is called a closed submodule if Y has no proper essential extension, i.e. $Y \overset{*}{\hookrightarrow} X \hookrightarrow M$ implies $Y = X$.

(3) M is called a uniform module if every nonzero submodule is essential in M.

Definition 2.9 *Let M be a right $R-$module and Y, a submodule of M. A submodule X of M is called a complement of Y in M if it is maximal with respect to $X \cap Y = 0$. A submodule X of M is called a complement if there exists a submodule Y such that X it is a complement of Y in M.*

Proposition 2.10 *Let X, Y be submodules of a module M with $X \cap Y = 0$. Then:*

(1) There exists a complement K of Y such that $X \subseteq K$;

(2) $K \oplus Y \overset{*}{\hookrightarrow} M$;

(3) K is closed in M.

Proposition 2.11 *Let X be a submodule of a module M and Y, a complement of X. Then, X is a closed submodule if and only if it is a complement of Y.*

For a submodule X of M, the *closure* of X is a closed submodule Y of M such that $X \overset{*}{\hookrightarrow} Y$.

Proposition 2.12 *(Properties of complements)*

Let A, B, C be submodules of a module M with $A \subset B$. Then:

(1) There exists a closed submodule X of M such that $C \overset{*}{\hookrightarrow} X$.

(2) C is closed in M if and only if for any submodule X of $M, C \hookrightarrow X \overset{*}{\hookrightarrow} M$ implies $X/C \overset{*}{\hookrightarrow} M/C$.

(3) If B is closed in M, then B/A is closed in M/A.

(4) If A is closed in B and B is closed in M then A is closed in M.

CS Rings and CS Modules.

Definition 2.13 *A module is called a CS (or extending) module if every submodule is essential in a direct summand. In particular, a ring R is called a right CS ring if R_R is a CS module.*

It is easy to see that a module M is CS if and only if every closed submodule of M is a direct summand.

2.5 Uniform Dimension and Goldie Modules

Definition 2.14 *A module is said to have finite uniform dimension if it does not contain any direct sum of an infinite number of nonzero submodules. In particular,* *a ring R has finite right uniform dimension if R_R has finite uniform dimension.*

By the definition, a module has finite uniform dimension if it only has direct sums of finite numbers of nonzero submodules (if any).

Proposition 2.15 *Let M be a right R-module. Then:*

(1) If M has finite uniform dimension, then every nonzero submodule (if any) of M contains at least a uniform submodule.

(2) If $\{U_i\}_{i=1}^n$ and $\{V_j\}_{j=1}^m$ are two collections of uniform submodules of M such that $\oplus_{i=1}^n U_i$ and $\oplus_{j=1}^m V_j$ are both essential submodules of M, then $n = m$.

(3) M has finite uniform dimension if and only if there exists a family $\{U_i\}_{i=1}^n$ of uniform submodules of M such that $\oplus_{i=1}^n U_i \overset{*}{\hookrightarrow} M$.

The preceding theorem makes it possible to define the uniform dimension of a module. The natural number n in the theorem is unique, i.e. it is independent to any choice of collections of uniform submodules of M, and is called the *uniform dimension* of M, denoted by $\mathrm{u}-\dim(M)$. If such the number does not exist, i.e. if M contains a direct sum of an infinite number of nonzero submodules, then M is defined to have *infinite uniform dimension*, writing $\mathrm{u}-\dim(M) = \infty$. Uniform dimension is sometimes referred as *Goldie dimension* or *Goldie rank*.

Proposition 2.16 *Let M be a right $R-$module with finite uniform dimension. Then:*

(1) If N is a submodule of M, then $\mathrm{u}-\dim(N) \leq \mathrm{u}-\dim(M)$. The equality happens exactly when N is an essential submodule of M;

(2) If $M = M_1 \oplus ... \oplus M_n$, then $\mathrm{u}-\dim(M) = \mathrm{u}-\dim(M_1) + ... + \mathrm{u}-\dim(M_n)$.

Definition 2.17 *A module M is called a Goldie module if it has finite uniform dimension and the ACC on $M-$annihilators. In particular, a ring R is said to be right (resp. left) Goldie if R_R (resp. ${}_RR$) has finite uniform dimension and R has the ACC on right (resp. left) annihilators.*

2.6 Prime and Semiprime Modules

In this part, we introduce the notions of prime and semiprime submodules, prime and semiprime modules. They have been defined by N. V. Sanh et al. in [12].

Let R be an associative ring with identity. Let M be a right $R-$module and its endomorphism ring $S = \mathrm{End}(M_R)$.

A submodule X of M is called a *fully invariant* submodule if for any $s \in S$, we have $s(X) \subset X$. It is easy to see that the class of all fully invariant submodules of M is non-empty and closed under intersection and sum. In particular, a right ideal of R is a fully invariant submodule of R_R if it is a two-sided ideal.

For subsets $I, J \subset S$ and $X \subset M$, for convenience, we denote $I(X) = \sum_{f\in I} f(X), \mathrm{Ker}(I) = \bigcap_{f\in I} \mathrm{Ker}(f) = \mathbf{r}_M(I)$, and $IJ = \{\sum\limits_{1}^{n} x_i y_i | x_i \in I, y_i \in J, 1 \leq i \leq n, n \in N\}$.

Definition 2.18 *A fully invariant proper submodule X of M is called a prime submodule if for any ideal I of S, and any fully invariant submodule U of $M, I(U) \subset X$ implies either $I(M) \subset X$ or $U \subset X$.*

Especially, an ideal P of R is a prime ideal if for any ideals I, J of $R, IJ \subset P$ implies either $I \subset P$ or $J \subset P$.

A module M is called a prime module if 0 is prime in M.

A ring R is called a prime ring if R_R is a prime module.

Proposition 2.19 *(Theorem 1.2 of [12])*

Let X be a fully invariant proper submodule of M. Then the following conditions are equivalent:

(1) X is a prime submodule of M;

(2) For any right ideal I of S, any submodule U of M, if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$;

(3) For any $\varphi \in S$ and fully invariant submodule U of M, if $\varphi(U) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$;

(4) For any left ideal I of S and subset A of M, if $IS(A) \subset X$, then either $I(M) \subset X$ or $A \subset X$;

(5) For any $\varphi \in S$ *and for any* $m \in M$, *if* $\varphi(S(m)) \subset X$, *then either* $\varphi(M) \subset X$ *or* $m \in X$.

Moreover, if M *is quasi-projective, then the above conditions are equivalent to:*

(6) M/X *is a prime module.*

Definition 2.20 *A fully invariant proper submodule* X *of a right* $R-$*module* M *is called a semiprime submodule if it is an intersection of prime submodules of* M.

A module M *is called a semiprime module if 0 is semiprime. A ring* R *is called a semiprime ring if the module* R_R *is semiprime.*

By symmetry, R is a semiprime ring if ${}_RR$ is a semiprime module.

Proposition 2.21 *(Theorem 1.10 of [12])*

If X *is a prime (resp. semiprime) submodule of* M, *then* $I_X = \mathrm{Hom}(M, X)$ *is a prime (resp. semiprime) ideal of* S. *Conversely, if* M *is a self-generator and* I_X *is a prime (resp. semiprime) ideal of* S, *then* X *is a prime (resp. semiprime) submodule of* M.

Proposition 2.22 *(Theorem 2.4 of [12])*

If M *is a prime module, then its endomorphism ring* S *is a prime ring. Conversely, if* M *is a self-generator and* S *is a prime ring, then* M *is prime.*

Proposition 2.23 *(Theorem 3.3 of [13])*

Let M *be a quasi-projective, finitely generated right* R*-module which is a self-generator. If* M *is a Goldie module, then* $S = \mathrm{End}(M_R)$ *is a right Goldie ring. In this case, we have* $\mathrm{u} - \dim(M_R) = \mathrm{u} - \dim(S_S)$.

CHAPTER III
MAIN RESULTS

Firstly, we provide some technical lemmas. The first one is routine.

Lemma 3.1 *Let M be a right R-module with $S = \mathrm{End}(M_R)$. If M is a semiprime module, then S is a semiprime ring. Conversely, if M is a self-generator and S is a semiprime ring, then M is a semiprime module.*

Lemma 3.2 *Let M be a right R-module with $S = \mathrm{End}(M_R)$. Then, M satisfies the ACC (resp. DCC) on M-annihilators if and only if S has the ACC (resp. DCC) on right annihilators.*

Proof. Firstly, we assume that M satisfies the ACC on $M-$annihilators. Let $\mathbf{r}_S(X_1) \subseteq \mathbf{r}_S(X_2) \subseteq ... \subseteq \mathbf{r}_S(X_n) \subseteq ...(1)$ be an ascending chain of right annihilators of S, for an arbitrary family of subsets $\{X_n \subset S, n \in N\}$. Then, we have:

$\ell_S\mathbf{r}_S(X_1) \supseteq \ell_S\mathbf{r}_S(X_2) \supseteq ... \supseteq \ell_S\mathbf{r}_S(X_n) \supseteq ...(2)$, and

$\mathbf{r}_M\ell_S\mathbf{r}_S(X_1) \subseteq \mathbf{r}_M\ell_S\mathbf{r}_S(X_2) \subseteq ... \subseteq \mathbf{r}_M\ell_S\mathbf{r}_S(X_n) \subseteq ...(3)$.

The chain (3) is stationary by the hypothesis. Because of $\ell_S\mathbf{r}_M\ell_S\mathbf{r}_S(X_n) = \ell_S\mathbf{r}_S(X_n), n = 1, 2, ...$, we pass (3) to left annihilators in S to obtain (2). Therefore, (2) is also stationary. Similarly, we pass the chain (2) to right annihilators (in S) and then obtain (1). Hence, (1) stops somewhere, implying that S has the ACC on right annihilators.

Conversely, we suppose that S satisfies the ACC on right annihilators. Let $\mathbf{r}_M(X_1) \subseteq \mathbf{r}_M(X_2) \subseteq ... \subseteq \mathbf{r}_M(X_n) \subseteq ...(4)$ be an ascending chain of $M-$annihilators of M, where $X_n \subset S, n = 1, 2, ...$ Then:

$\ell_S\mathbf{r}_M(X_1) \supseteq \ell_S\mathbf{r}_M(X_2) \supseteq ... \supseteq \ell_S\mathbf{r}_M(X_n) \supseteq ...(5)$, and

$\mathbf{r}_S\ell_S\mathbf{r}_M(X_1) \subseteq \mathbf{r}_S\ell_S\mathbf{r}_M(X_2) \subseteq ... \subseteq \mathbf{r}_S\ell_S\mathbf{r}_M(X_n) \subseteq ...(6)$.

Since S has the ACC on right annihilators, (6) is stationary. We pass (6) to left annihilators in S to obtain (5), and convert (5) to right annihilators in M to obtain (4). As a consequence, the chain (5) stops at some index. Thus, (4) is also stationary, showing that M has the ACC on $M-$annihilators.

By the same argument, we can prove for the DCC case. The proof is now complete. □

Lemma 3.3 *Let $M = M_R$ be a finitely generated, quasi-projective right $R-$module which is a self-generator with $S = \mathrm{End}(M_R)$. Then:*

(1) X is a closed submodule of M if and only if $I_X = \{f \in S | f(M) \subseteq X\}$ is a closed right ideal of S;

(2) Conversely, K is a closed right ideal of S if and only if $KM = \sum_{s\in K} s(M)$ is a closed submodule of M.

Proof. (1). Assuming that X is a closed submodule of M, and $I_X \overset{*}{\hookrightarrow} K \hookrightarrow S_S$, for some right ideal K of S. Then, we have $X = I_X M \subseteq KM$, since M is a self-generator. For a nonzero submodule Y of $KM, Y = I_Y(M)$, where $I_Y = \{f \in S|f(M) \subseteq Y\} \neq 0$. Thus, $0 \neq I_X \cap I_Y$ and hence there exists $0 \neq s \in I_X \cap I_Y$ such that $s(M) \subset X, s(M) \subset Y$. Therefore, we have $0 \neq s(M) \subset (X \cap Y)$, proving $X \overset{*}{\hookrightarrow} KM$, and hence $X = KM$. This implies $I_X = K$, showing that I_X is closed in S.

Conversely, if I_X is closed in S and $X = I_X M \overset{*}{\hookrightarrow} A \hookrightarrow M$. Then, $I_X \subseteq I_A = \{f \in S|f(M) \subseteq A\}$. For right ideal $0 \neq K$ of S such that $K \subset I_A, 0 \neq KM \subseteq A$ and hence $KM \cap X \neq 0$. Since M is a self-generator, there is a nonzero $s \in S$ such that $s(M) \subseteq (KM \cap X)$, and hence, $s \in I_X$. Furthermore, since M is quasi-projective and finitely generated, by 18.4 of [15], $sS \subseteq \mathrm{Hom}(M, KM) = K$, and hence $s \in K$. Therefore, we have $0 \neq s \in I_X \cap K$, thus $I_X \overset{*}{\hookrightarrow} I_A$. This implies that $I_X = I_A$, since I_X is a closed right ideal. As a consequence, $X = A$, showing that X is closed in M.

(2). We ague similarly to (1). Note that if K is a right ideal of S, then $K = I_{KM} = \mathrm{Hom}(M, KM)$. $\square$

Lemma 3.4 *Let M_R be a finitely generated and quasi-projective module, which is a self-generator with $S = \mathrm{End}(M_R)$. Then:*

(1) U is a uniform submodule of M if and only if $I = I_U := \{f \in S | fM \subseteq U\}$ is a uniform right ideal of S;

(2) K is a uniform right ideal of S if and only if $KM = \sum_{f \in K} f(M)$ is a uniform submodule of M.

Proof. (1). Assuming that U is a uniform submodule of M, then $I = I_U := \{f \in S | fM \subseteq U\} = \mathrm{Hom}(M, U)$ and $U = IM$. Let X, Y be nonzero right ideals of S such that $X, Y \subseteq I$. Then, $0 \neq XM \cap YM = A \hookrightarrow U$. Since M is a self-generator, there exists $0 \neq s \in \mathrm{Hom}(M, A)$. Therefore, $s \in \mathrm{Hom}(M, XM) = X, s \in \mathrm{Hom}(M, YM) = Y$. Thus, $s \in X \cap Y$ showing that $I = I_U$ is a uniform right ideal of S.

We suppose that I_U is a uniform right ideal of S. For nonzero submodules A, B of M such that $A, B \subset U$, we have $0 \neq I_A = \mathrm{Hom}(M, A) \subset I_U, 0 \neq I_B = \mathrm{Hom}(M, B) \subset I_U$. Therefore, $I_A \cap I_B \neq 0$ and hence $A \cap B \neq 0$. This shows that U is uniform.

(2). By the same argument as that given in (1). $\square$

Lemma 3.5 *Let $M = M_R$ be a finitely generated, quasi-projective right $R-$module which is a self-generator with the endomorphism ring $S = \mathrm{End}(M_R)$. Then, X is a direct summand of M if and only if $I_X = \{f \in S | f(M) \subseteq X\}$ is a direct summand of S. In this case, $X = e(M)$ and $I_X = eS$ for some idempotent $e \in S$.*

Proof. The proof of this lemma is routine. $\square$

Lemma 3.6 *(Theorem 1 of [8])*

Let R be a prime ring. Then the following statements are equivalent:

(1) R is right Goldie, right CS with $\mathrm{u}-\dim(R_R) > 1$;

(2) R is left Goldie, left CS with $\mathrm{u}-\dim({}_RR) > 1$.

Lemma 3.7 *(Theorem 3 of [8])*

For a semiprime ring R, the following conditions are equivalent:

(1) R is left Goldie, right CS;

(2) R is right Goldie, left CS.

In this case, we have $R = \bigoplus_{1}^{n} R_i$, where each R_i is prime, right and left Goldie, right and left CS, for $i = 1, 2, ..., n$.

In the following, our main theorems are presented. They can be considered as applications of results of D. V. Huynh et al. in [8].

Theorem 3.8 *Let M be a finitely generated and quasi-projective right R-module, which is a self-generator and $S = \mathrm{End}(M_R)$, the endomorphism ring of M. Then, the following statements are equivalent:*

(1) M is prime, Goldie and CS with $1 < \mathrm{u}-\dim(M_R) < \infty$;

(2) S is prime, right Goldie and right CS with $1 < \mathrm{u}-\dim(S_S) < \infty$;

(3) S is prime, left Goldie and left CS with $1 < \mathrm{u}-\dim({}_SS) < \infty$.

In this case, S is a prime, right and left Goldie, right and left CS ring with $\mathrm{u}-\dim({}_SS) = \mathrm{u}-\dim(S_S) = \mathrm{u}-\dim(M_R) = n > 1$.

Proof. Firstly, we claim that M is a prime module if and only if S is a prime ring by theorem 2.4 of [12].

(1) ⇒ (2). By theorem 3.1 and 3.3 of [13], S is right Goldie with $\mathrm{u}-\dim(S_S) = \mathrm{u}-\dim(M_R) = n \geq 2$. It is clear that the closure of a uniform submodule is also uniform. According to [3], in order to prove that S is right CS, it is sufficient to show that every uniform closed right ideal of S is a direct summand, since S is of finite right uniform dimension.

Let X be a closed uniform right ideal of S. Then, XM is a closed uniform submodule of M, by lemmas 3.3 and 3.4. Therefore, X is a direct summand, because of the extending property of M. Thus, by lemma 3.5, $XM = eM$, for some $e = e^2 \in S$, and hence $X = eS$, a direct summand of S.

(2) ⇒ (1). Since S is right Goldie with $\mathrm{u}-\dim(S_S) = n > 1$, by theorem 3.1 of [13], we have $\mathrm{u}-\dim(M) = \mathrm{u}-\dim(S_S) = n$. Moreover, S has the ACC on right annihilators, then by lemma 3.2, M has the ACC on $M-$annihilators. Therefore, M is a Goldie module.

By lemmas 3.3 and 3.4, if U is a closed uniform submodule of M, then I_U is a closed uniform right ideal of S. Thus, by lemma 3.5, we have $I_U = eS$ for some idempotent $e \in S$, and hence $U = eS(M) = e(M)$ is a direct summand of M. Therefore, M is extending.

(2) ⇔ (3). It is straightforward by lemma 3.6. □

Remark 1. For a prime module $M = M_R$, if A is a (nonzero) direct summand of M which is fully invariant in M, then A is itself a prime module.

Proof. Let $S = \mathrm{End}(M_R)$ be the endomorphism ring of M.

Note that every direct summand of M is of the form $eM = e(M)$ for some $e = e^2 \in S$. If U is a fully invariant submodule of $A = eM$, then for very $f \in S, f' := f|_A : A \to A$, $f'(U) \subset U$ is satisfied. Therefore, U is also a fully invariant submodule of M.

We assume that M is prime. Let $t \neq 0$ be an endomorphism of A such that $t(U) = 0$. Then, $0 \neq te \in S$ and $te(U) \subset t(U) = 0$. This implies that $U = 0$, since M is prime. Thus, A is a prime module. □

According to [5] and [4], a *PWD (piecewise domain)* is a ring R with a complete set $\{e_1, ..., e_n\}$ of orthogonal idempotents with the property that $xy = 0$ implies $x = 0$ or $y = 0$ whenever $x \in e_tRe_k$ and $y \in e_kRe_j$. By proposition 1 of [4], a semiprime right Goldie, right CS ring is a PWD.

In remark 2 of [8], D. V. Huynh et al. gave a representation of a semiprime right Goldie right CS ring by a direct sum of prime right Goldie right CS rings. The following assertion can be considered as an adaptation.

Proposition 3.9 *Let $M = M_R$ be a finitely generated and quasi-projective module, which is a self-generator. If M is a semiprime, Goldie, CS module, then we have:*
$M = \bigoplus_1^n M_i$, *where each M_i is a prime, Goldie and CS module;*
$\mathrm{Hom}(M_i, M_j) = 0$ *whenever $i \neq j$;*
$S = \bigoplus_1^n S_i$, *where each $S_i = \mathrm{Hom}(M, M_i) \cong \mathrm{End}(M_i)$ is a prime, right Goldie right CS ring.*

Proof. It is clear that S is semiprime. By the proof of theorem 3.8, since M is a Goldie and CS module, S is right Goldie and right CS. Thus, by the remark 2 of [8], we get the representation $S = \bigoplus_1^n S_i$, where each $S_i = e_iS$ satisfies $\mathrm{Hom}(e_iS, e_jS) = 0$ whenever $j \neq i$. Thus, for every $f, g \in S$, if $i \neq j$, then $e_ife_jg = 0$. It is not difficult to verify that each S_i is an ideal of S (indeed, ${}_S[e_iS]_S = {}_{e_iS}[e_iS]_{e_iS}$).

It is obvious that S_i inherits the right Goldie, right CS properties from S. We will show that each S_i is prime. We consider $S_1 = e_1S$ only, and then deduce for other cases. We observe that for any $x \in S_1, x = e_1s$ for some $s \in S$, and $x = e_1s = e_1se_1$, because $1 - e_1 \notin S_1$. Let $xy = 0$ for some $x = e_1se_1, y = e_1te_1$ belonging to S_1, where $s, t \in S$. Then, we get either $x = e_1se_1 = 0$ or $y = e_1te_1 = 0$, since S is a PWD by proposition 1 of [4]. This shows that S_1 is a prime, right Goldie right CS ring.

We put $M_i = S_i(M) = e_i(M)$ for $i = 1, ..., n$. Then, $M = \bigoplus_1^m M_i$ follows immediately. Furthermore, we observe that for every $s \in S_i, s = e_ise_i = e_is = se_i$, and for every $f \in \mathrm{End}(M_i), fe_i = e_ife_i \in S_i$. Therefore, we can identify S_i with $\mathrm{End}(M_i)$ via the ring isomorphism $\mathrm{End}(M_i) \xrightarrow{\cong} S_i, f \mapsto fe_i$. Then, it is clear to state that each M_i is a prime, Goldie and CS module. The claims $\mathrm{Hom}(M_i, M_j) = 0$ whenever $i \neq j$ and $S_i = \mathrm{Hom}(M, M_i)$ are straightforward. The proof is now complete. $\square$

Theorem 3.10 *Let $M = M_R$ be a semiprime, finitely generated and quasi-projective module, which is a self-generator with $S = \operatorname{End}(M_R)$, the endomorphism ring of M. Then, the following conditions are equivalent:*

(1) S is left Goldie and M is CS;

(2) M is Goldie and S is left CS.

In this case, $M = \bigoplus_1^n M_i$, where each M_i is a prime, Goldie and CS-module; and $S = \bigoplus_1^n S_i$, where each $S_i = \operatorname{Hom}(M, M_i) \cong \operatorname{End}(M_i)$ is a prime, right and left Goldie, right and left CS ring, for $i \in \{1, 2, ..., n\}$.

Proof. By theorem 2.9 of [12], S is a semiprime ring.

(1)⇒(2). We deal with the proof by establishing two claims as followed.

Claim 1. M is a Goldie module. We suppose on the contrary that M contains an infinite direct sum $\bigoplus_1^\infty N_i$ of non-zero submodules of M. Then, $I_{N_i} = \{f \in S | f(M) \subset N_i\} = \operatorname{Hom}(M, N_i) \neq 0$, since M is a self-generator. It is easy to check that $I_{N_i} \cap \sum_{j \neq i} I_{N_j} = \operatorname{Hom}(M, N_i) \cap \sum_{j \neq i} \operatorname{Hom}(M, N_j) = \operatorname{Hom}(M, N_i) \cap \operatorname{Hom}(M, \sum_{j \neq i} N_j) = \operatorname{Hom}(M, N_i \cap \sum_{j \neq i} N_j) = 0$, because by assumption. Therefore, S contains $\bigoplus_1^\infty I_{N_i}$, a contradiction. Thus, M must be of finite Goldie dimension.

Since S is a semiprime left Goldie ring, by lemma 7.2.2 of [7], S has the DCC on left annihilators, and hence has the ACC on right annihilators. Thus, by lemma 3.2, M has the ACC on $M-$annihilators. It follows that M is a Goldie module. In this case, S is also a right Goldie right CS ring.

Claim 2. S is left CS. By proposition 3.9, since M is semiprime, Goldie and CS, M is a direct sum of prime, Goldie, CS modules, $M = \bigoplus_1^n M_i$, where $\operatorname{Hom}(M_i, M_j) = 0$; and $S = \bigoplus_1^n S_i$, where each $S_i = \operatorname{Hom}(M, M_i) \cong \operatorname{End}(M_i)$ is a prime, right and left Goldie right CS ring. If M_i is of dimension one, then M_i is a uniform module. Therefore, S_i is also right uniform, and hence is left uniform. Thus, S_i is right and left CS. If M_i has uniform dimension greater than one, then by theorem 3.8, S_i is left CS. This implies that $S = \bigoplus_1^n S_i$ is left CS.

(2)⇒(1). Since M is a Goldie module, S is a right Goldie ring and $\mathrm{u}-\dim(S_S) = \mathrm{u}-\dim(M_R)$. Being right Goldie and left CS, S is a left Goldie, right CS ring, by lemma 3.7.

In order to show that M is a CS module, we let U be a uniform closed submodule of M. Then, $I_U = \{f \in S | f(M) \subseteq U\}$ is a uniform closed right ideal of S by lemmas 3.3 and 3.4. Since S is right CS, I_U is a direct summand of S, i.e. $I_U = eS$ for some idempotent $e \in S$. Thus, $U = eM$ is a direct summand of M. This implies that M is uniform extending, thus it is extending by [3]. $\square$

CHAPTER IV
CONCLUSION

By two theorems in chapter III, we are successful to prove that for a finitely generated, quasi-projective right $R-$module M which is a self-generator and $S = \mathrm{End}(M_R)$, suppose that M is prime. Then M is Goldie CS with uniform dimension at least two if and only if its endomorphism ring is prime, left Goldie left CS with left uniform dimension at least two. In the case that M is semiprime, M is Goldie and S is left CS if and only if S is left Goldie and M is CS.

In further research, we wish to prove the absolute symmetry of the Goldie and CS conditions on prime and semiprime modules. More clearly, for example, for a given prime, finitely generated, quasi-projective right $R-$module M which is a self-generator, M_R is Goldie CS with uniform dimension at least two if and only if ${}_SM$ is Goldie CS with uniform dimension at least two, where S is the endomorphism ring of M. This is still an open problem.

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