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## APPENDICES

## APPENDIX A

Constants  $\mathbf{a}_m$  and  $\Gamma_{mi}$  appearing in equations (2.37) and (2.38) are given explicitly by

$$\mathbf{a}_m = \left\{ \bar{\eta}_j \left( 1 - \bar{\eta}_j \bar{P}_j + \bar{\eta}_j \bar{k}_{2j} \right) \mathbf{r}_m^3 + \left( 1 - \bar{\eta}_j^2 \bar{k}_{2j} \right) \mathbf{r}_m \right\} \quad (\text{A.1})$$

$$\Gamma_{11} = \left[ (\mathbf{a}_3 \mathbf{a}_4 - \mathbf{a}_2 \mathbf{a}_4) \mathbf{e}^{(\tau_2 + \tau_3)} + (\mathbf{a}_2 \mathbf{a}_3 - \mathbf{a}_3 \mathbf{a}_4) \mathbf{e}^{(\tau_2 + \tau_4)} - \mathbf{a}_2 \mathbf{a}_3 + \mathbf{a}_2 \mathbf{a}_4 \right] / \Omega \quad (\text{A.2})$$

$$\Gamma_{21} = \left[ (\mathbf{a}_1 \mathbf{a}_4 - \mathbf{a}_3 \mathbf{a}_4) \mathbf{e}^{(\tau_1 + \tau_3)} + (\mathbf{a}_3 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_3) \mathbf{e}^{(\tau_1 + \tau_4)} + \mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_1 \mathbf{a}_4 \right] / \Omega \quad (\text{A.3})$$

$$\Gamma_{31} = \left[ (\mathbf{a}_2 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_4) + (\mathbf{a}_1 \mathbf{a}_2 - \mathbf{a}_2 \mathbf{a}_4) \mathbf{e}^{(\tau_1 + \tau_4)} + (\mathbf{a}_1 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_2) \mathbf{e}^{(\tau_2 + \tau_4)} \right] / \Omega \quad (\text{A.4})$$

$$\Gamma_{41} = \left[ (\mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_2 \mathbf{a}_3) + (\mathbf{a}_2 \mathbf{a}_3 - \mathbf{a}_1 \mathbf{a}_2) \mathbf{e}^{(\tau_1 + \tau_3)} + (\mathbf{a}_1 \mathbf{a}_2 - \mathbf{a}_1 \mathbf{a}_3) \mathbf{e}^{(\tau_2 + \tau_3)} \right] / \Omega \quad (\text{A.5})$$

$$\Gamma_{12} = \left[ (\mathbf{a}_2 - \mathbf{a}_3) \mathbf{e}^{(\tau_2 + \tau_3)} + (\mathbf{a}_4 - \mathbf{a}_2) \mathbf{e}^{(\tau_2 + \tau_4)} + \mathbf{a}_3 - \mathbf{a}_4 \right] / \Omega \quad (\text{A.6})$$

$$\Gamma_{22} = \left[ (\mathbf{a}_3 - \mathbf{a}_1) \mathbf{e}^{(\tau_1 + \tau_3)} + (\mathbf{a}_1 - \mathbf{a}_4) \mathbf{e}^{(\tau_1 + \tau_4)} - \mathbf{a}_3 + \mathbf{a}_4 \right] / \Omega \quad (\text{A.7})$$

$$\Gamma_{32} = \left[ (\mathbf{a}_1 - \mathbf{a}_2) + (\mathbf{a}_4 - \mathbf{a}_1) \mathbf{e}^{(\tau_1 + \tau_4)} + (\mathbf{a}_2 - \mathbf{a}_4) \mathbf{e}^{(\tau_2 + \tau_4)} \right] / \Omega \quad (\text{A.8})$$

$$\Gamma_{42} = \left[ (\mathbf{a}_2 - \mathbf{a}_1) + (\mathbf{a}_1 - \mathbf{a}_3) \mathbf{e}^{(\tau_1 + \tau_3)} + (\mathbf{a}_3 - \mathbf{a}_2) \mathbf{e}^{(\tau_2 + \tau_3)} \right] / \Omega \quad (\text{A.9})$$

$$\Gamma_{13} = \left[ (\mathbf{a}_2 \mathbf{a}_4 - \mathbf{a}_2 \mathbf{a}_3) \mathbf{e}^{\tau_2} + (\mathbf{a}_2 \mathbf{a}_3 - \mathbf{a}_3 \mathbf{a}_4) \mathbf{e}^{\tau_3} + (\mathbf{a}_3 \mathbf{a}_4 - \mathbf{a}_2 \mathbf{a}_4) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.10})$$

$$\Gamma_{23} = \left[ (\mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_1 \mathbf{a}_4) \mathbf{e}^{\tau_1} + (\mathbf{a}_3 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_3) \mathbf{e}^{\tau_3} + (\mathbf{a}_1 \mathbf{a}_4 - \mathbf{a}_3 \mathbf{a}_4) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.11})$$

$$\Gamma_{33} = \left[ (\mathbf{a}_1 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_2) \mathbf{e}^{\tau_1} + (\mathbf{a}_1 \mathbf{a}_2 - \mathbf{a}_2 \mathbf{a}_4) \mathbf{e}^{\tau_2} + (\mathbf{a}_2 \mathbf{a}_4 - \mathbf{a}_1 \mathbf{a}_4) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.12})$$

$$\Gamma_{43} = \left[ (\mathbf{a}_1 \mathbf{a}_2 - \mathbf{a}_1 \mathbf{a}_3) \mathbf{e}^{\tau_1} + (\mathbf{a}_2 \mathbf{a}_3 - \mathbf{a}_1 \mathbf{a}_2) \mathbf{e}^{\tau_2} + (\mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_2 \mathbf{a}_3) \mathbf{e}^{\tau_3} \right] / \Omega \quad (\text{A.13})$$

$$\Gamma_{14} = \left[ (\mathbf{a}_3 - \mathbf{a}_4) \mathbf{e}^{\tau_2} + (\mathbf{a}_4 - \mathbf{a}_2) \mathbf{e}^{\tau_3} + (\mathbf{a}_2 - \mathbf{a}_3) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.14})$$

$$\Gamma_{24} = \left[ (\mathbf{a}_4 - \mathbf{a}_3) \mathbf{e}^{\tau_1} + (\mathbf{a}_1 - \mathbf{a}_4) \mathbf{e}^{\tau_3} + (\mathbf{a}_3 - \mathbf{a}_1) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.15})$$

$$\Gamma_{34} = \left[ (\mathbf{a}_2 - \mathbf{a}_4) \mathbf{e}^{\tau_1} + (\mathbf{a}_4 - \mathbf{a}_1) \mathbf{e}^{\tau_2} + (\mathbf{a}_1 - \mathbf{a}_2) \mathbf{e}^{\tau_4} \right] / \Omega \quad (\text{A.16})$$

$$\Gamma_{44} = \left[ (\mathbf{a}_3 - \mathbf{a}_2) \mathbf{e}^{\tau_1} + (\mathbf{a}_1 - \mathbf{a}_3) \mathbf{e}^{\tau_2} + (\mathbf{a}_2 - \mathbf{a}_1) \mathbf{e}^{\tau_3} \right] / \Omega \quad (\text{A.17})$$

$$\begin{aligned} \Omega = & 2(\mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_1 \mathbf{a}_4 - \mathbf{a}_2 \mathbf{a}_3 + \mathbf{a}_2 \mathbf{a}_4) + (-\mathbf{a}_1 \mathbf{a}_2 + \mathbf{a}_1 \mathbf{a}_4 + \mathbf{a}_2 \mathbf{a}_3 - \mathbf{a}_3 \mathbf{a}_4) \left[ \mathbf{e}^{(\tau_1 + \tau_3)} + \mathbf{e}^{(\tau_2 + \tau_4)} \right] \\ & + (\mathbf{a}_1 \mathbf{a}_2 - \mathbf{a}_1 \mathbf{a}_3 - \mathbf{a}_2 \mathbf{a}_4 + \mathbf{a}_3 \mathbf{a}_4) \left[ \mathbf{e}^{(\tau_1 + \tau_4)} + \mathbf{e}^{(\tau_2 + \tau_3)} \right] \end{aligned} \quad (\text{A.18})$$

## APPENDIX B

The shape functions  $\psi_i(\bar{x})$  and  $\hat{\psi}_i(\bar{x})$  for a special case without the elastic lateral restraint (i.e.  $k_{11}=0$  and  $k_{21}=0$ ). The governing differential equation (2.27) simply reduces to

$$r^4 + 2\bar{\omega}r^2 = 0 \quad (B.1)$$

where

$$\bar{\omega} = \frac{\bar{P}_j}{2(1 - \bar{\eta}_j \bar{P}_j)} \quad (B.2)$$

The general solution of buckling shape  $\bar{v}(\bar{x})$  and  $\beta(\bar{x})$  takes the form

$$\bar{v}(\bar{x}) = C_1 e^{r_1 \bar{x}} + C_2 e^{r_2 \bar{x}} + C_3 \bar{x} + C_4 \quad (B.3)$$

$$\beta(\bar{x}) = \tilde{C}_1 e^{r_1 \bar{x}} + \tilde{C}_2 e^{r_2 \bar{x}} + C_3 \quad (B.4)$$

where  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are arbitrary constants,  $r_1$  and  $r_2$  are distinct roots of the characteristic equation (B.1), and

$$\tilde{C}_m = \left\{ \bar{\eta}_j (1 - \bar{\eta}_j \bar{P}_j) r_m^3 + r_m \right\} C_m \quad (B.5)$$

By enforcing essential boundary conditions (2.34) along with using the relation (B.5), it leads to the same form of buckling shapes as that shown in equation (2.35) - (2.36) but the shape functions  $\psi_i(\bar{x})$  and  $\hat{\psi}_i(\bar{x})$  are given differently by

$$\psi_i(\bar{x}) = \sum_{m=1}^2 \Gamma_{mi} e^{r_m \bar{x}} + \Gamma_{3i} \bar{x} + \Gamma_{4i} \quad (B.6)$$

$$\hat{\psi}_i(\bar{x}) = \sum_{m=1}^2 \tilde{a}_m \Gamma_{mi} e^{r_m \bar{x}} + \Gamma_{3i} \quad (B.7)$$

where constants  $\tilde{a}_m$  and  $\Gamma_{mi}$  are given explicitly by

$$\tilde{a}_m = \left\{ \bar{\eta}_j (1 - \bar{\eta}_j \bar{P}_j) r_m^3 + r_m \right\} \quad (B.8)$$

$$\Gamma_{11} = \left[ \tilde{a}_2 (1 - e^{\bar{c}_2}) \right] / \Omega \quad (B.9)$$

$$\Gamma_{21} = [\tilde{a}_1(e^{\epsilon^1} - 1)]/\Omega \quad (\text{B.10})$$

$$\Gamma_{31} = [\tilde{a}_1\tilde{a}_2(e^{\epsilon^2} - e^{\epsilon^1})]/\Omega \quad (\text{B.11})$$

$$\Gamma_{41} = [(\tilde{a}_1\tilde{a}_2 - \tilde{a}_2)e^{\epsilon^1} + (\tilde{a}_1 - \tilde{a}_1\tilde{a}_2)e^{\epsilon^2} + \tilde{a}_2 - \tilde{a}_1]/\Omega \quad (\text{B.12})$$

$$\Gamma_{12} = [(1 - \tilde{a}_2)e^{\epsilon^2} - 1]/\Omega \quad (\text{B.13})$$

$$\Gamma_{22} = [1 + (\tilde{a}_1 - 1)e^{\epsilon^1}]/\Omega \quad (\text{B.14})$$

$$\Gamma_{32} = [\tilde{a}_1e^{\epsilon^1} - \tilde{a}_2e^{\epsilon^2} + \tilde{a}_2 - \tilde{a}_1]/\Omega \quad (\text{B.15})$$

$$\Gamma_{42} = [(1 - \tilde{a}_1)e^{\epsilon^1} + (\tilde{a}_2 - 1)e^{\epsilon^2}]/\Omega \quad (\text{B.16})$$

$$\Gamma_{13} = [\tilde{a}_2(e^{\epsilon^2} - 1)]/\Omega \quad (\text{B.17})$$

$$\Gamma_{23} = [\tilde{a}_1(1 - e^{\epsilon^1})]/\Omega \quad (\text{B.18})$$

$$\Gamma_{33} = [\tilde{a}_1\tilde{a}_2(e^{\epsilon^1} - e^{\epsilon^2})]/\Omega \quad (\text{B.19})$$

$$\Gamma_{43} = [\tilde{a}_1(e^{\epsilon^1} - 1) + \tilde{a}_2(1 - e^{\epsilon^2})]/\Omega \quad (\text{B.20})$$

$$\Gamma_{14} = [1 + \tilde{a}_2 - e^{\epsilon^2}]/\Omega \quad (\text{B.21})$$

$$\Gamma_{24} = [e^{\epsilon^1} - \tilde{a}_1 - 1]/\Omega \quad (\text{B.22})$$

$$\Gamma_{34} = [\tilde{a}_2(1 - e^{\epsilon^1}) + \tilde{a}_1(e^{\epsilon^2} - 1)]/\Omega \quad (\text{B.23})$$

$$\Gamma_{44} = [\tilde{a}_1 - \tilde{a}_2 - e^{\epsilon^1} + e^{\epsilon^2}]/\Omega \quad (\text{B.24})$$

$$\Omega = 2(\tilde{a}_2 - \tilde{a}_1) + (\tilde{a}_1 + \tilde{a}_2)[e^{\epsilon^1} + e^{\epsilon^2}] + \tilde{a}_1\tilde{a}_2[e^{\epsilon^1} - e^{\epsilon^2}] \quad (\text{B.25})$$

Since the shape function  $\psi_i(\bar{x})$  and  $\bar{\psi}_i(\bar{x})$  are given in terms of exponential and linear functions, all elements stiffness matrices  $\mathbf{K}_{bi}$ ,  $\mathbf{K}_{si}$ , and  $\mathbf{K}_{gi}$  can readily be obtained in an explicit form via the direct integration. Entries of these matrices are given by

$$[K_{bi}]_{mn} = E_i I_i \sum_{j=1}^2 \sum_{k=1}^2 \tilde{a}_j \tilde{a}_k r_j r_k \Gamma_{jm} \Gamma_{kn} \left( e^{(r_j+r_k)L} - 1 \right) / (r_j+r_k) \quad (\text{B.26})$$

$$[K_{si}]_{mn} = \lambda_i G_i A_i \sum_{j=1}^2 \sum_{k=1}^2 (r_j - \tilde{a}_j)(r_k - \tilde{a}_k) \Gamma_{jm} \Gamma_{kn} \left( e^{(r_j+r_k)L} - 1 \right) / (r_j+r_k) \quad (\text{B.27})$$

$$\begin{aligned}
[K_{gi}]_{mn} = & P_i \sum_{j=1}^2 \sum_{k=1}^2 r_j r_k \Gamma_{jm} \Gamma_{kn} \left( e^{(r_j+r_k)L} - 1 \right) / (r_j+r_k) + \sum_{j=1}^2 \Gamma_{jm} \Gamma_{3n} \left( e^{r_j L} - 1 \right) \\
& + \sum_{k=1}^2 \Gamma_{kn} \Gamma_{3m} \left( e^{r_k L} - 1 \right) + \Gamma_{3m} \Gamma_{3n}
\end{aligned}
\tag{B.28}$$

## APPENDIX C

The shape functions  $\psi_i(\bar{x})$  and  $\hat{\psi}_i(\bar{x})$  for a special case without the elastic lateral restraint (i.e.  $k_{1i}=0$  and  $k_{2i}=0$ ) and shear deformation. The governing differential equation (2.27) simply reduces to

$$\frac{d^4 v}{dx^4} + k^2 \frac{d^2 v}{dx^2} = 0 \quad (C.1)$$

where

$$k = \sqrt{\frac{P}{EI}} \quad (C.2)$$

The general solution of buckling shape  $\bar{v}(\bar{x})$  takes the form

$$\bar{v}(\bar{x}) = C_1 + C_2 \bar{x} + C_3 \cos(\gamma \bar{x}) + C_4 \sin(\gamma \bar{x}) \quad (C.3)$$

where  $C_1$ ,  $C_2$ ,  $C_3$  and  $C_4$  are arbitrary constant and  $\gamma = kL$ . By enforcing essential boundary conditions (2.34), it leads to the same form of buckling shapes as that shown in equation (2.35) but the shape functions  $\psi_i(x)$  are given differently by

$$\psi_1(\bar{x}) = \frac{1}{\Gamma_2 + 2\Gamma_1} \{ \Gamma_1 + \Gamma_2 - \Gamma_2 \bar{x} + \Gamma_1 \cos(\gamma \bar{x}) + \Gamma_3 \sin(\gamma \bar{x}) \} \quad (C.4)$$

$$\psi_2(\bar{x}) = \frac{1}{\gamma(\Gamma_2 + 2\Gamma_1)} \{ \Gamma_4 + \Gamma_1 \gamma \bar{x} - \Gamma_4 \cos(\gamma \bar{x}) + (\Gamma_1 + \Gamma_2) \sin(\gamma \bar{x}) \} \quad (C.5)$$

$$\psi_3(\bar{x}) = \frac{1}{\Gamma_2 + 2\Gamma_1} \{ \Gamma_1 + \Gamma_2 \bar{x} - \Gamma_1 \cos(\gamma \bar{x}) - \Gamma_3 \sin(\gamma \bar{x}) \} \quad (C.6)$$

$$\psi_4(\bar{x}) = \frac{1}{\gamma(\Gamma_2 + 2\Gamma_1)} \{ \Gamma_5 + \Gamma_1 \gamma \bar{x} - \Gamma_5 \cos(\gamma \bar{x}) - \Gamma_1 \sin(\gamma \bar{x}) \} \quad (C.7)$$

where  $\Gamma_1 = \cos \gamma - 1$ ,  $\Gamma_2 = \gamma \sin \gamma$ ,  $\Gamma_3 = \sin \gamma$ ,  $\Gamma_4 = \sin \gamma - \gamma \cos \gamma$  and  $\Gamma_5 = \gamma - \sin \gamma$ . Note that the buckling shape  $\beta(\bar{x})$  can be obtained by taking derivative of (2.35) with respect to  $\bar{x}$ . Since the shape function  $\psi_i(\bar{x})$  are given in terms of trigonometric and linear functions, all elements stiffness matrices  $\mathbf{K}_{bi}$  and  $\mathbf{K}_{gi}$  can readily be obtained in an explicit form via the direct integration. Entries of these matrices are given by

$$[K_b]_{11} = \frac{\gamma^3 [2\Gamma_1\Gamma_3(1-\cos^2\gamma) + \cos\gamma\sin\gamma(\Gamma_1^2 - \Gamma_3^2) + \beta(\Gamma_1^2 + \Gamma_3^2)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.8)$$

$$[K_b]_{12} = \frac{\gamma^2 [(1-\cos^2\gamma)(\Gamma_1^2 + \Gamma_1\Gamma_2) + (\cos^2\gamma - 1)\Gamma_3\Gamma_4 - \cos\gamma\sin\gamma(\Gamma_1\Gamma_3 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_3)]}{2(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{\gamma^2 [\gamma(\Gamma_1\Gamma_3 - \Gamma_1\Gamma_4 + \Gamma_2\Gamma_3)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.9)$$

$$[K_b]_{13} = -[K_b]_{11} \quad (C.10)$$

$$[K_b]_{14} = \frac{\gamma^2 [(\cos^2\gamma - 1)(\Gamma_1^2 + \Gamma_3\Gamma_5) + \Gamma_1\cos\gamma\sin\gamma(\Gamma_3 - \Gamma_5) - \gamma\Gamma_1(\Gamma_3 + \Gamma_5)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.11)$$

$$[K_b]_{22} = \frac{\gamma [2\Gamma_4(\cos^2\gamma - 1)(\Gamma_1 + \Gamma_2) - \cos\gamma\sin\gamma(\Gamma_1^2 + 2\Gamma_1\Gamma_2 + \Gamma_2^2 - \Gamma_4^2)]}{2(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{\gamma [\gamma(\Gamma_1^2 + 2\Gamma_1\Gamma_2 + \Gamma_2^2 + \Gamma_4^2)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.12)$$

$$[K_b]_{23} = -[K_b]_{12} \quad (C.13)$$

$$[K_b]_{24} = \frac{\gamma [(1-\cos^2\gamma)\Gamma_1\Gamma_4 + (\cos^2\gamma - 1)(\Gamma_1\Gamma_5 + \Gamma_2\Gamma_5) - \gamma(\Gamma_1^2 + \Gamma_1\Gamma_2 - \Gamma_4\Gamma_5)]}{2(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{\gamma [\cos\gamma\sin\gamma(\Gamma_1^2 + \Gamma_1\Gamma_2 + \Gamma_4\Gamma_5)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.14)$$

$$[K_b]_{33} = [K_b]_{11} \quad (C.15)$$

$$[K_b]_{34} = -[K_b]_{14} \quad (C.16)$$

$$[K_b]_{44} = \frac{\gamma [2\Gamma_1\Gamma_5(1 - \cos^2\gamma) + \cos\gamma\sin\gamma(\Gamma_5^2 - \Gamma_1^2) + \gamma(\Gamma_1^2 + \Gamma_5^2)]}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.17)$$

$$[K_g]_{11} = \frac{2\gamma\Gamma_1\Gamma_3(\cos^2\gamma - 1) + \gamma\cos\gamma\sin\gamma(\Gamma_3^2 - \Gamma_1^2) + \gamma^2(\Gamma_1^2 + \Gamma_3^2)}{2(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{2(\Gamma_2^2 + 2\Gamma_1\Gamma_2(1 - \cos\gamma) - 2\Gamma_2\Gamma_3\sin\gamma)}{2(\Gamma_2 + 2\Gamma_1)^2} \quad (C.18)$$

$$[K_g]_{12} = \frac{\gamma [(\cos^2\gamma - 3)(\Gamma_1^2 + \Gamma_1\Gamma_2) + (1 - \cos^2\gamma)\Gamma_3\Gamma_4] + 2\cos\gamma(\Gamma_2\Gamma_4 + \gamma\Gamma_1^2)}{2\gamma(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{\gamma\cos\gamma\sin\gamma(\Gamma_1\Gamma_3 + \Gamma_1\Gamma_4 + \Gamma_2\Gamma_3) - 2\sin\gamma(\Gamma_1\Gamma_2 + \Gamma_2^2 - \gamma\Gamma_1\Gamma_3) - 2\Gamma_2\Gamma_4}{2\gamma(\Gamma_2 + 2\Gamma_1)^2} \\ + \frac{\gamma^2(\Gamma_1\Gamma_3 - \Gamma_1\Gamma_4 + \Gamma_2\Gamma_3)}{2\gamma(\Gamma_2 + 2\Gamma_1)^2} \quad (C.19)$$

$$[K_g]_{13} = -[K_g]_{11} \quad (C.20)$$

$$[K_g]_{14} = \frac{-\gamma[(1+\cos^2\gamma)\Gamma_1^2+(\cos^2\gamma-1)\Gamma_3\Gamma_5]-\gamma^2(\Gamma_1\Gamma_3+\Gamma_1\Gamma_5)-2\Gamma_2(\Gamma_5+\gamma\Gamma_1)}{2\gamma(\Gamma_2+2\Gamma_1)^2} \\ + \frac{\gamma\cos\gamma\sin\gamma(\Gamma_1\Gamma_5-\Gamma_1\Gamma_3)+2\cos\gamma(\Gamma_2\Gamma_5+\gamma\Gamma_1^2)+2\Gamma_1\sin\gamma(\Gamma_2+\gamma\Gamma_3)}{2\gamma(\Gamma_2+2\Gamma_1)^2} \quad (C.21)$$

$$[K_g]_{22} = \frac{2[(3-\cos^2\gamma)\Gamma_1\Gamma_4+(1-\cos^2\gamma)\Gamma_2\Gamma_4]+\cos\gamma\sin\gamma(\Gamma_1^2+2\Gamma_1\Gamma_2+\Gamma_2^2-\Gamma_4^2)}{2\gamma(\Gamma_2+2\Gamma_1)^2} \\ + \frac{\gamma(3\Gamma_1^2+2\Gamma_1\Gamma_2+\Gamma_2^2+\Gamma_4^2)-4\Gamma_1\Gamma_4\cos\gamma+4\sin\gamma(\Gamma_1^2+\Gamma_1\Gamma_2)}{2\gamma(\Gamma_2+2\Gamma_1)^2} \quad (C.22)$$

$$[K_g]_{23} = -[K_g]_{12} \quad (C.23)$$

$$[K_g]_{24} = \frac{[(1+\cos^2\gamma)\Gamma_1\Gamma_4+(3-\cos^2\gamma)\Gamma_1\Gamma_5+(1-\cos^2\gamma)\Gamma_2\Gamma_5]-2\Gamma_1\cos\gamma(\Gamma_4+\Gamma_5)}{2\gamma(\Gamma_2+2\Gamma_1)^2} \\ + \frac{\gamma(\Gamma_1^2-\Gamma_1\Gamma_2+\Gamma_4\Gamma_5)-\cos\gamma\sin\gamma(\Gamma_1^2+\Gamma_1\Gamma_2+\Gamma_4\Gamma_5)+2\Gamma_1\Gamma_2\sin\gamma}{2\gamma(\Gamma_2+2\Gamma_1)^2} \quad (C.24)$$

$$[K_g]_{33} = [K_g]_{11} \quad (C.25)$$

$$[K_g]_{34} = -[K_g]_{14} \quad (C.26)$$

$$[K_g]_{44} = \frac{2\Gamma_1\Gamma_5(1+\cos^2\gamma)+\cos\gamma\sin\gamma(\Gamma_1^2-\Gamma_5^2)+\gamma(3\Gamma_1^2+\Gamma_5^2)-4\Gamma_1\Gamma_5\cos\gamma-4\Gamma_1^2\sin\gamma}{2\gamma(\Gamma_2+2\Gamma_1)^2} \quad (C.27)$$

## BIOGRAPHY

Mr. Nidvichai Watcharakorn was born in 1987, at Ramathibodi Hospital, Bangkok. He graduated from Suankularb Wittayalai School in 2004, and enrolled in B.Eng. and M. Eng. courses at Department of Civil Engineering Faculty, Chulalongkorn University. The major in his master's degree is structural civil engineering, which includes studies and researches in structural mechanics and with advanced mathematical techniques. This thesis aims to investigate the flexural buckling load of structures.



